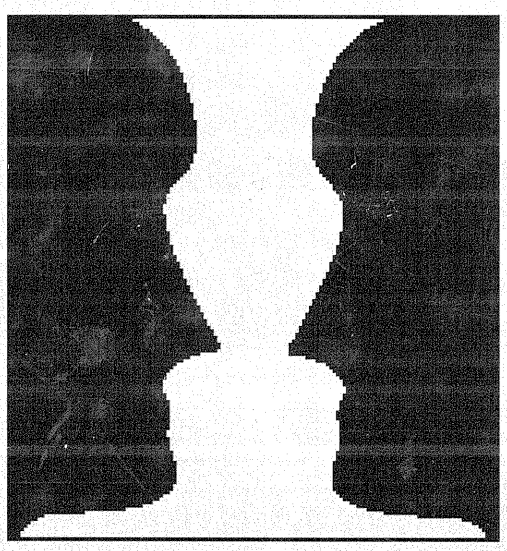
Quantum Field Theory

A Self-Contained Course

David ATKINSON
Porter Wear JOHNSON

Volume 2



Rinton Press

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This book is the second in a series of four volumes. The only prerequisites are a knowledge of partial differential equations, Maxwell's theory of electromagnetism, and the content of Volume 1. Special relativity and the covariance of Maxwell's equations are explained, as is the group theory of the Lorentz transformation. The Dirac equation is introduced and solved for the hydrogen atom, and the basic rules for obtaining Feynman graphs in relativistic scattering are derived and applied. This is done primarily in quantum electrodynamics; but an appendix sketches the extension to the electroweak theory. Each chapter is complemented by ten problems, and the student is advised to try them all by himself or herself before looking at our solutions in Volume 4. Physics can only be learned by thinking, writing, and worrying.

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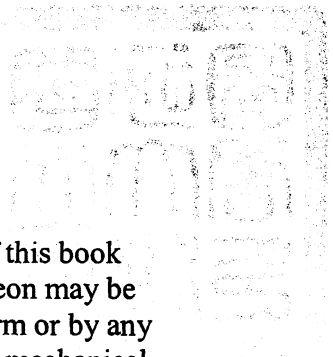

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Preface

This book is the second in a series of four volumes devoted respectively to

(1) Nonrelativistic Quantum Mechanics
(2) Relativistic Quantum Field Theory
(3) Exercises and problems, with fully worked out solutions, on the subjects treated in Volume 1.
(4) Exercises and problems, with fully worked out solutions, on the subjects treated in Volume 2.

These volumes are self-contained in the sense that the only prerequisites are a knowledge of partial differential equations and Maxwell's theory of electromagnetism. It is not necessary for the reader to have studied the special theory of relativity, the relativistic reformulation of Maxwell's equations, or to know anything about unbounded operators on Hilbert space, for all these subjects are developed in requisite detail at the various points where they are needed. On the other hand, it is advisable for the student to have covered the material of Volume 1 before starting on Volume 2.

Each chapter is complemented by ten problems, and the student is advised to try them all by himself or herself before looking at our solutions in Volume 4. It cannot be emphasized too strongly that the temptation for a student to look immediately at the worked-out answers is self-defeating. Look at the answers, certainly, but only after you have engaged in serious battle. Physics cannot be learned just by reading or listening, but only by thinking, writing and worrying.

This second volume, being an introduction to quantum field theory, employs canonical quantization methods exclusively, the path-integral formalism having been avoided. We have not made any attempt to cover all subjects of physical interest, but have made a choice appropriate for a course based on two hours of lectures and two hours of problem sessions during one undergraduate semester. The basic goal is the setting up of rules for drawing Feynman graphs and for calculating amplitudes from them. This is done primarily in quantum electro-

dynamics; but an appendix sketches the extension to the electroweak theory, and some Feynman rules are given for this extended Lagrangian.

Despite the plethora of books of instruction on these subjects, we hope that our work will fill a need, and we believe our approach to be pedagogically sound, given its attention to mathematical detail combined with physics, which resulted in the integration of the appropriate mathematical tools at the points in the text where they are needed. The mathematics is at the same time explicit but kept in check, so that the reader does not get bogged down in annoying generalizations that might distract him or her from the physics. At a number of points recourse is taken to the computer to solve transcendental equations; in most cases a simple program is given in *Mathematica* (which is a trademark of Wolfram Research Inc.), but for readers without access to this system, we indicate where the numerical results can be checked by means of a scientific calculator.

We wish to acknowledge the help we have had from reading the books on quantum field theory that are listed in the bibliography; and in particular we thank Dr. Mees de Roo, in Groningen, for a number of corrections.

The drawings on the covers of the volumes are ambiguous representations.

Volume 1: a duck, or is it a rabbit?
Volume 2: a vase, or are there two faces?
Volume 3: a young, or is it an old woman?
Volume 4: an American Indian, or is it an Eskimo?

The simultaneous existence of two pictures is perhaps the closest metaphor we can find to the fundamental mystery of quantum mechanics, namely the linear superposition of two aspects of reality, each of which separately can be pictured, and whose combination can barely be comprehended by the eye of introspection, but which can be apprehended by the power of mathematical language.

August 2002,

David Atkinson (The Netherlands)
Porter Wear Johnson (United States of America)

Contents

Chapter 1

Relativity and Quantization

1.1 Unbounded Operators on Hilbert Space

Quantum mechanics is plagued by infinities. Some of these infinities can be removed in a mathematically responsible manner; but some have not yet been handled in a way that satisfies the exigencies of mathematicians. In these two preliminary sections, a number of mathematical methods for dealing with these infinities will be sketched.

Many observables are represented by unbounded operators. For example, in configuration space the position operator of a particle in one dimension satisfies $q\psi(x) = x\psi(x)$, and there are some square-integrable functions, $\psi(x)$, for which $x\psi(x)$ is not square integrable. $\mathcal{L}^2$ is the Hilbert space of square-integrable functions (Problem 1.5); but the norm of q on $\mathcal{L}^2$ does not exist. The subspace of $\mathcal{L}^2$ on which the norm *does* exist is called the domain, $\mathcal{D}$, of q:

$$\mathcal{D}(q) = \left\{ \psi : \psi \in \mathcal{L}^2, \int_{-\infty}^{\infty} dx\, |x\psi(x)|^2 < \infty \right\} .$$

In configuration space, the momentum operator, p, satisfies $p\psi(x) = -i\psi'(x)$ (here we have set $\hbar = 1$). It is also unbounded, and its domain is

$$\mathcal{D}(p) = \left\{ \psi : \psi \in \mathcal{L}^2, \int_{-\infty}^{\infty} dx\, |\psi'(x)|^2 < \infty \right\} .$$

The Hermitian conjugate (or adjoint) of p, written $p^\dagger$, is defined by the relation, in Dirac's notation,

$$\langle \phi | p\psi \rangle = \langle p^\dagger \phi | \psi \rangle , \tag{1.1}$$

for all ψ in the domain of p, and all ϕ in its range. The range of p, which is the domain of $p^\dagger$, is the subspace spanned by $|p\psi\rangle$ as $|\psi\rangle$ runs over the domain of p.

In configuration space, Eq.(1.1) reads

$$\langle\phi|p\psi\rangle = -i\int_{-\infty}^{\infty} dx\,\phi^*(x)\psi'(x) = \langle p^\dagger\phi|\psi\rangle\,.$$

On the other hand,

$$\langle p\phi|\psi\rangle = \int_{-\infty}^{\infty} dx\,[-i\phi'(x)]^*\psi(x) = i\int_{-\infty}^{\infty} dx\,\phi^{*\prime}(x)\psi(x)\,,$$

and therefore p is Hermitian, as we expect:

$$\langle p\phi|\psi\rangle - \langle p^\dagger\phi|\psi\rangle = i\int_{-\infty}^{\infty} dx\,[\phi^*(x)\psi(x)]' = 0\,, \tag{1.2}$$

since the boundary terms vanish. A Hermitian operator, p, is said to be self-adjoint if the domain of p and that of $p^\dagger$ are the same, on condition that this domain is dense in $\mathcal{L}^2$, i.e., on condition that, for any $\psi \in \mathcal{L}^2$, there exists a sequence, $\{\psi_n,\, n = 1, 2, \dots\} \in \mathcal{D}(p)$, such that $\langle\psi_n - \psi|\psi_n - \psi\rangle \to 0$ as $n \to \infty$. It is the property of self-adjointness, rather than merely Hermiticity, that is essential for proving the reality of the eigenvalues. If the particle is confined to a finite part of the real line, say $[-a \le x \le a]$, then the integral in Eq.(1.2), over the finite interval, is

$$i\big[\phi^*(a)\psi(a) - \phi^*(-a)\psi(-a)\big]\,,$$

and this can be made to vanish by restricting the domain of p to be a space of periodic functions, expressible as a Fourier series:

$$\psi(x) = \sum_{n=-\infty}^{\infty} \psi_n\, \mathrm{e}^{i\pi nx/a}\,.$$

This domain is dense in $\mathcal{L}^2$, and the range of p is precisely the same space, since the derivative of the Fourier series is again a series of the same type, so p is self-adjoint on this domain (Problem 1.5).

The mere fact that some operators are unbounded is not a problem, although it does mean that some care is required. Some important operators *must* be unbounded, for *if* q and p were bounded operators, we could argue that

$$\mathrm{Tr}\big([q,p]\big) = \mathrm{Tr}\big(qp\big) - \mathrm{Tr}\big(pq\big) = 0\,,$$

and this is inconsistent with the basic commutation relation, $[q,p] = i$, from which the Heisenberg uncertainty relation is derived. This argument is invalid because, in the sum over the basis that is implied in the trace operation, some of the terms are undefined (i.e., infinite).

1.2 Rigged Hilbert Space

A second place where the Dirac formulation of quantum mechanics runs into difficulties is that, if we stick to Hilbert space, not all self-adjoint operators have a complete set of eigenvectors, i.e., a set that spans the space. Indeed, Hilbert space does not contain e^{ipx}, the configuration space eigenvector of the momentum operator, because it is not square integrable. Nor does it contain the distribution $\delta(x - x_0)$, the eigenvector of the position operator in configuration space. This problem has been resolved by Gel'fand and Vilenkin by extending Hilbert space in such a way that it does indeed include such eigenfunctions. Such an extension is called *rigged* Hilbert space, a picturesque way of indicating a space that is equipped like a galleon to sail into battle.

The triplet $\Omega < \mathcal{H} < \Omega^x$ is called a rigged Hilbert space, where $\mathcal{H}$ is a Hilbert space, consisting of all functions, ψ, that are square integrable,

$$\int_{-\infty}^{\infty} dx\, |\psi(x)|^2 < \infty\,.$$

Ω is a *nuclear* space, made up of all functions, ϕ, that satisfy

$$\int_{-\infty}^{\infty} dx\, |\phi(x)|^2 \left(1 + |x|\right)^n < \infty\,,$$

for all finite $n = 0, 1, 2, \ldots$. Clearly Ω is a smaller space than $\mathcal{H}$; in contrast its complement, the extended space Ω^x, is larger than $\mathcal{H}$. It contains all functions, χ, for which

$$\int_{-\infty}^{\infty} dx\, \chi^*(x)\phi(x)$$

is finite for all $\phi \in \Omega$. This extended space contains the eigenfunctions of q and p; and there exists a generalized spectral theorem, which states that if an operator is self-adjoint in $\mathcal{H}$, then a complete set of eigenvectors exists in Ω^x.

Gel'fand and Vilenkin realized mathematically the insight of Dirac, namely that observables should correspond to Hermitian operators, having real eigenvalues and a complete set of eigenvectors; but this was not the first attempt to put quantum mechanics on a firm mathematical footing. In the 1930s, von Neumann avoided delta functions with distaste, and with good reason, for they had not yet been made mathematically respectable by the distribution theory of Schwartz; and moreover he kept within the formalism of Hilbert space, thus excluding eigenvectors like e^{ipx}. By using projection operators on Hilbert space, and with a generalized notion of the integral, he made quantum mechanics mathematically well-defined, but in a manner that departed somewhat from Dirac's

formalism. The method of the rigged Hilbert space may well be regarded as the preferable option, since it is also mathematically sound, and it adheres closely to Dirac's notation, which is almost universally employed by working physicists.

One aspect of quantum mechanics that requires generalization is the concept of a state. A pure state may be represented by a vector in Hilbert space; but a more general state is represented by a normalized, self-adjoint, positive operator (i.e., one with non-negative eigenvalues). Such a state operator, or density matrix, ρ, can always be written

$$\rho = \sum_n \rho_n |\phi_n\rangle\langle\phi_n| ,$$

where $\{\phi_n\}$ is a complete set of states. The expectation value of an observable that is represented by a self-adjoint operator, Ω, in a state represented by a state operator, ρ, is defined to be $\mathrm{Tr}\{\rho\Omega\}$. In this operator language, a pure state is represented by the simple expression

$$\rho = |\psi\rangle\langle\psi| ,$$

for some vector $|\psi\rangle$, and so the expectation value of the observable is

$$\mathrm{Tr}\{\rho\Omega\} = \mathrm{Tr}\{|\psi\rangle\langle\psi|\Omega\} = \langle\psi|\Omega|\psi\rangle ,$$

and this agrees with the elementary form of an expectation value in a state represented by one state vector.

In addition to the above matters, there are two other sources of infinities when quantum mechanics is applied to field theories. One is that divergent integrals are encountered in terms in the perturbation expansion of physical quantities like scattering cross-sections. These integrals are rendered finite by a process called regularization: the dimensional method of regularization is expounded in Chapter 8. After this regularization has been performed, physical quantities like masses and coupling constants are *renormalized*, after which the regularization is removed. In this introductory treatment, the process of renormalization will not be explained in detail. Lastly, the renormalized perturbation series is expected to diverge, so that on adding more and more terms, the perturbation sum explodes. The reason that the perturbation series is thought to diverge in QED is that the term which describes the interaction is more singular in the ultraviolet than the free Hamiltonian that is being perturbed. The situation is reminiscent of the anharmonic oscillator, in which the harmonic Hamiltonian, $\frac{1}{2}(p^2 + q^2)$, is perturbed by the anharmonic term, λq^4. All is not lost, however, for the series may be asymptotic, and it may be possible to resum it, for example by the use of Padé approximants (see Problem 7.10 in Volume 1). A more thoroughgoing solution is to replace the perturbation series by the Dyson-Schwinger equations,

which is done in Chapter 9. These equations can be used to describe bound states, and such other nonperturbative phenomena as the creation of masses through chiral symmetry breaking (Sec. **9.4**). Although the last two sources of infinities are dealt with in a somewhat *ad hoc* manner that lacks adequate mathematical justification, the methods do seem to work, as judged by the empirical success of QED.

In Chapter 4 we shall introduce quantum fields, for example the quantized version of the electromagnetic field, $A^\mu(x)$. Although it appears to be defined at one point, x^ν, this appearance is deceptive, for the field should rather be understood as a distribution, defined on a suitable space of test functions, $g(x)$. Thus $A^\mu(x)$ does not have a definite value, whereas

$$A^\mu[g] = \int d^4x\, g(x) A^\mu(x)$$

is well-defined, but as a quartet of self-adjoint operators rather than simply as numbers. Matrix elements of these operators, $\langle A^\mu[g] \rangle$, are bona fide numbers that are functions of g, rather than of x. The test function takes the place of the space-time point, and as such it does justice to the empirical fact that fields are never measured at precisely one point, but always as averages over a small volume in space and a small interval in time. In the above sense, $A^\mu(x)$ is said to be an operator-valued distribution.

After these introductory remarks on quantum mechanics and quantum field theory, we shall devote the rest of this chapter to a résumé of special relativity, and its application to the problem of setting up relativistic equations of motion for particles that move with speeds which are comparable to that of light.

1.3 Special Theory of Relativity

Consider two inertial coordinate systems, $S = (t, x, y, z)$ and $S' = (t', x', y', z')$, such that the space axes of the two systems are coincident when both t and t' are zero, and parallel thereafter, and such that S' has velocity v, along the x-direction, with respect to S. The transformation between the two systems must have the form

$$x' = \gamma(x - vt) \qquad y' = y \qquad z' = z\,, \tag{1.3}$$

for this just expresses the fact that the origin of S' corresponds to the point $x = vt$, $y = 0$, $z = 0$ in S, i.e., S' does indeed have velocity v in the x-direction, as seen from S. The classical assumption (Galileo, Newton) was that $\gamma = 1$, but if we only require that the origin of S' have velocity v with respect to

S, the above more general transformation is possible. What is γ? From the uniformity of space, we see that γ may not depend on the coordinates (in a non-uniform gravitational field, for example near a star, the properties of space are not uniform, i.e., translation-invariant, but that is the domain of general, not of special relativity). In the absence of gravity, γ is independent of t, x, y, z, but it may depend on v. Space is also invariant under rotations: a rotation by 180 degrees about the y-axis should have no measurable effect; but it changes the sign of the x and the z coordinates, and also of the relative velocity of the primed system. Rotational invariance therefore means that γ must be independent of the sign of v, i.e., it is a function only of v^2. The unprimed system has velocity $-v$, with respect to the primed system, along the x-direction, so the inverse relation between the two systems is

$$x = \gamma(x' + vt') \qquad y = y' \qquad z = z' \ , \tag{1.4}$$

where γ is the same as in Eq.(1.3), since it depends on v^2 only.

Note that we have not assumed that the time coordinate, t', in S' is the same as t, the time coordinate in S. Such an assumption was made, quite explicitly, by Newton, in his Principia: "Absolute, true, and mathematical time, of itself, and from its own nature, flows equably without relation to anything external". If we set $t' = t$, with Newton, and do not enquire too closely what, if anything, 'equable flow of *time*' means, then Eq.(1.3) and Eq.(1.4) together imply $\gamma = 1$. Now if light is emitted at the origin of space-time and travels along the x-axis with velocity c, then at time t it will have reached the point $x = ct$. From Eq.(1.3), with $\gamma = 1$ and $t' = t$, we see that $x' = (c - v)t = (c - v)t'$, which means that the speed of light, as measured in S' in the positive direction of x', would be $c - v$.

The experimental fact that the measured speed of light is *independent* of the motion of the source, and of the observer (Michelson-Morley experiment and the tests of 'æther-drag' in the solar-system), forces the conclusions $t' \neq t$ and $\gamma \neq 1$. Since the speed of light in S' is actually c, and not $c - v$, it follows that $x = ct$ in S must correspond to $x' = ct'$ in S'. Putting these values into Eq.(1.3) and Eq.(1.4), we find

$$\begin{aligned} ct' &= \gamma(c - v)t \\ ct &= \gamma(c + v)t' \ ; \end{aligned} \tag{1.5}$$

and by multiplying these two equations together, we obtain

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} \ . \tag{1.6}$$

By eliminating x' between Eq.(1.3) and Eq.(1.4), we see that

$$t' = \gamma\left(t - \frac{v}{c^2}x\right) . \tag{1.7}$$

The mapping $(t, x, y, z) \longrightarrow (t', x', y', z')$, given by Eqs.(1.3), (1.6) and (1.7), is called a Lorentz transformation.

Let us introduce the parametrization $v/c = \tanh u$, so that $\gamma = \cosh u$ and $\gamma v/c = \sinh u$. We shall also write $x^0 = ct$, $x^1 = x$, $x^2 = y$, $x^3 = z$. Then the Lorentz transformation implies

$$x'^0 = x^0 \cosh u - x^1 \sinh u \qquad x'^1 = -x^0 \sinh u + x^1 \cosh u . \tag{1.8}$$

This form looks very much like a rotation: if the space axes are rotated an angle θ about the x^1-axis, we have

$$x'^2 = x^2 \cos\theta + x^3 \sin\theta \qquad x'^3 = -x^2 \sin\theta + x^3 \cos\theta . \tag{1.9}$$

There are some important differences between Eq.(1.8) and Eq.(1.9). In Eq.(1.8), hyperbolic functions replace circular ones, and moreover the signs of the two hyperbolic sines are the same, whereas those of the two circular sines are different. One can regard the Lorentz transformation as a 'hyperbolic rotation' between the x^0-axis and the x^1-axis. Note that

$$(x'^0)^2 - (x'^1)^2 - (x'^2)^2 - (x'^3)^2 = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 ,$$

whether Eq.(1.8) holds (a 'pure' Lorentz transformation), or whether Eq.(1.9) holds (a rotation), or indeed if Eq.(1.8) and Eq.(1.9) are both met.

Contravariant and Covariant Vectors

Consider a linear transformation from coordinates, x^μ, to a new set, x'^μ,

$$x'^\mu = \Lambda^\mu_{\ \nu} x^\nu , \tag{1.10}$$

where the matrix, $\Lambda^\mu_{\ \nu}$, is constant (i.e., independent of x), and is such that

$$g_{\mu\nu} x'^\mu x'^\nu = g_{\mu\nu} x^\mu x^\nu , \tag{1.11}$$

then the mapping is said to be a Lorentz transformation. Here the metric tensor, $g_{\mu\nu}$, is $+1$ if $\mu = 0 = \nu$, or -1 if $\mu = i = \nu$, for $i = 1, 2$, or 3, and to be 0 if $\mu \neq \nu$, and where there is an implicit summation over the repeated indices. This will be a general rule: if a Greek index occurs above (a *contravariant* index), and below (a *covariant* index) in the same term, then a summation over the values $0, 1, 2, 3$ is implied.

Since the matrix, $\Lambda^{\mu}_{\ \nu}$, is independent of x, it follows, by differentiation of Eq.(1.10), that

$$\Lambda^{\mu}_{\ \nu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}}. \tag{1.12}$$

Any quadruple of numbers, a^{μ}, together with the transformation,

$$a'^{\mu} = \Lambda^{\mu}_{\ \nu} a^{\nu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}} a^{\nu}\,, \tag{1.13}$$

defines a contravariant Lorentz four-vector. It is important to understand that Eq.(1.13) applies for all Lorentz transformations, $x^{\mu} \rightarrow x'^{\mu}$, that respect Eq.(1.11). In words, a vector that transforms in the same way as a coordinate is called a contravariant vector.

Suppose now that Φ is a Lorentz invariant function of x (i.e., a function that does not change when the coordinates, x^{μ}, undergo a Lorentz transformation). Then, by the usual chain rule for partial derivatives,

$$\frac{\partial \Phi}{\partial x'^{\mu}} = \frac{\partial x^{\nu}}{\partial x'^{\mu}} \frac{\partial \Phi}{\partial x^{\nu}}. \tag{1.14}$$

Hence the partial differentiation operator does not have the same transformation law as the contravariant vector in Eq.(1.13). Rather, it is an example of a *covariant* vector, b_{μ}, which transforms as follows:

$$b'_{\ \mu} = \frac{\partial x^{\nu}}{\partial x'^{\mu}} b_{\nu}. \tag{1.15}$$

Again, this applies for any Lorentz transformation. In words, a vector that transforms in the same way as a space-time partial derivative is called a covariant vector. The product of any contravariant vector, a^{μ}, and any covariant vector, b_{μ}, which we often write simply as ab, is Lorentz invariant:

$$a'b' = a'^{\mu} b'_{\ \mu} = \frac{\partial x'^{\mu}}{\partial x^{\rho}} \frac{\partial x^{\sigma}}{\partial x'^{\mu}} a^{\rho} b_{\sigma} = a^{\rho} b_{\rho} = ab. \tag{1.16}$$

If a^{ν} is a contravariant vector, and we define

$$a_{\mu} = g_{\mu\nu} a^{\nu}, \tag{1.17}$$

then clearly

$$a'_{\mu} = g_{\mu\nu} a'^{\nu} = g_{\mu\nu} \frac{\partial x'^{\nu}}{\partial x^{\rho}} g^{\rho\sigma} a_{\sigma}, \tag{1.18}$$

where $g^{\rho\sigma}$, the contravariant metric tensor, is equal to $g_{\rho\sigma}$, the covariant metric tensor, element by element. Note that

$$g_{\rho\tau}g^{\rho\sigma} = \delta^{\sigma}_{\tau}, \tag{1.19}$$

where the Kronecker δ is equal to 1 if both indices are equal, and to 0 if they are not. By applying the partial differential operator $\partial_\rho \partial'_\sigma$ to both sides of Eq.(1.11), we obtain

$$g_{\mu\nu}\frac{\partial x'^{\nu}}{\partial x^{\rho}} = g_{\rho\tau}\frac{\partial x^{\tau}}{\partial x'^{\mu}}\,. \tag{1.20}$$

With use of Eq.(1.19), we have

$$g_{\mu\nu}\frac{\partial x'^{\nu}}{\partial x^{\rho}}g^{\rho\sigma} = \frac{\partial x^{\sigma}}{\partial x'^{\mu}}\,, \tag{1.21}$$

and by combining this result with Eq.(1.18), we see that

$$a'_{\mu} = \frac{\partial x^{\sigma}}{\partial x'^{\mu}}a_{\sigma}\,,$$

which means that a_μ indeed transforms as a covariant vector. In a similar way, if b_μ is a covariant vector, then

$$b^{\mu} = g^{\mu\nu}b_{\nu}, \tag{1.22}$$

can be shown to transform as a contravariant vector.

Mechanics of a Free Particle

The relativistically invariant interval, ds, between two infinitesimally separated points is defined by

$$(ds)^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 = g_{\mu\nu}dx^{\mu}dx^{\nu}\,. \tag{1.23}$$

Consider a particle, and let τ be its proper time, i.e., the time coordinate in the particle's rest system. Clearly $ds = c\,d\tau$. In order to guess the correct Lagrangian for a free particle in relativity theory, and hence to analyze relativistic mechanics, we write the Hamilton variational principle, in an inertial frame, in the form (cf., Eq. (1.17) of Volume 1)

$$S = \int_a^b dtL(x_i, \dot{x}_i, t)\,. \tag{1.24}$$

The integration is taken from a proper time point, a, to another one, b. The problem is that we do not know what L should be. It depends asymmetrically on space and time, and certainly is not a relativistic invariant. However, the action only refers to the two proper times, a and b. This is true in all inertial

frames, although of course the time and space coordinates of a and b do depend on the frame that is chosen. The Hamilton variational principle simply says that the physical trajectory between a and b is the one for which $\delta S = 0$. This is a Lorentz invariant specification of the dynamics, and it is consistent with the principle of relativity to require that the action be a Lorentz invariant. However, the only invariants available, out of which the action of a free particle could be made, are the mass, m, and the invariant interval, s, or equivalently the proper time, τ. The action must be translationally invariant, so we cannot include τ itself, but only the invariant measure, $d\tau$. Hence Eq.(1.24) becomes

$$S = \kappa \int_a^b d\tau, \tag{1.25}$$

where κ is a function of the mass of the particle, m, only. Suppose now that we transform from the rest-system to any other inertial frame. From Eq.(1.23),

$$(ds)^2 = (c\,d\tau)^2 = (dt)^2\,(c^2 - \dot{x}_1^2 - \dot{x}_2^2 - \dot{x}_3^2) = (c\,dt)^2(1 - v^2/c^2)\,, \tag{1.26}$$

so we have finally

$$d\tau = dt\sqrt{1 - v^2/c^2}\,. \tag{1.27}$$

From Eq.(1.25) we see that

$$S = \kappa \int_a^b dt\sqrt{1 - v^2/c^2}\,, \tag{1.28}$$

so we can read off the form of the Lagrangian in a general inertial frame:

$$L = \kappa\sqrt{1 - v^2/c^2}. \tag{1.29}$$

The constant, κ, can be identified by expanding this to first order in v^2/c^2:

$$L = \kappa - \tfrac{1}{2}\kappa v^2/c^2 + O(v^4/c^4), \tag{1.30}$$

and it follows that $\kappa = -mc^2$, so that L reduces in the low velocity limit to the correct non-relativistic kinetic energy, $T = \frac{1}{2}mv^2$, aside from $-mc^2$, which, being constant, does not appear in the Euler-Lagrange equation. So Eq.(1.29) becomes

$$L = -mc^2\sqrt{1 - v^2/c^2}\,, \tag{1.31}$$

and we can then calculate the canonical momenta:

$$p^i = \frac{\partial L}{\partial \dot{x}^i} = -mc^2 \frac{\partial}{\partial \dot{x}^i}\left\{1 - \frac{1}{c^2}\sum_{j=1}^{3} \dot{x}^j \dot{x}^j\right\}^{\frac{1}{2}} = \frac{m\dot{x}^i}{\sqrt{1 - v^2/c^2}}\,. \tag{1.32}$$

The Hamiltonian is therefore

$$\begin{aligned} H &= \sum_{j=1}^{3} \dot{x}^j p^j - L \\ &= \sum_{j=1}^{3} \frac{m\dot{x}^j \dot{x}^j}{\sqrt{1 - v^2/c^2}} + mc^2\sqrt{1 - v^2/c^2} \\ &= \frac{mc^2}{\sqrt{1 - v^2/c^2}}. \end{aligned} \tag{1.33}$$

According to the discussion in Sec. **1.2** of Volume 1, if the Lagrangian does not depend explicitly on time (and if the potential energy, if there is one, is conservative, i.e., it does not depend on the velocities, $\dot{x}^j$), then H will be time-independent and equal to the total energy, E. We assume this also to be true relativistically. The energy is identified with the zeroth component of momentum in relativity, and its value is not adjustable. The assumption has far-reaching consequences, for we see that the energy of a free particle in its rest-frame ($v = 0$), is not zero, but is equal rather to mc^2.

The expression Eq.(1.33) is not yet in canonical form, i.e., it is not expressed in terms of the coordinates and momenta. However, from Eq.(1.32),

$$\begin{aligned} \sum_{i=1}^{3} p^i p^i + m^2c^2 &= \sum_{i=1}^{3} \frac{m^2 \dot{x}^i \dot{x}^i}{1 - v^2/c^2} + m^2c^2 \\ &= \frac{m^2c^2}{1 - v^2/c^2}, \end{aligned} \tag{1.34}$$

which, with the help of Eq.(1.33), yields the canonical form:

$$H = c\left\{\sum_{i=1}^{3} p^i p^i + m^2c^2\right\}^{\frac{1}{2}}. \tag{1.35}$$

Note that, for $\vec{p}^{\,2} = \sum_{i=1}^{3} p^i p^i << m^2c^2$,

$$H = mc^2 + \frac{\vec{p}^{\,2}}{2m} - \frac{\vec{p}^{\,4}}{8m^3c^2} + O\left(\frac{\vec{p}^{\,6}}{m^5c^4}\right). \tag{1.36}$$

We recognize the second term as the nonrelativistic kinetic energy. The first term, the rest-mass of the free particle, is the equivalent energy that is locked up in a particle at rest, and which can be liberated on the annihilation of matter and antimatter.

Next we introduce the 4-velocity,

$$u^\mu = c\frac{dx^\mu}{ds} = \frac{dx^\mu}{d\tau}\,, \tag{1.37}$$

which is manifestly a contravariant 4-vector. From Eq.(1.26), we see that

$$u^\mu = \frac{1}{\sqrt{1-v^2/c^2}}\frac{dx^\mu}{dt} \tag{1.38}$$

so that

$$mu^0 = \frac{mc}{\sqrt{1-v^2/c^2}} = E/c \tag{1.39}$$

$$mu^i = \frac{m}{\sqrt{1-v^2/c^2}}\frac{dx^i}{dt} = p^i\,. \tag{1.40}$$

Hence

$$p^\mu \equiv mu^\mu = (E/c, \vec{p}) \tag{1.41}$$

is a contravariant 4-vector. The invariant,

$$p^\mu p_\mu = E^2/c^2 - \vec{p}^{\,2}\,, \tag{1.42}$$

has the same value in any inertial frame. In the rest-frame, $\vec{p} = 0$, it is clearly equal to m^2c^2, so in general

$$E^2 = \vec{p}^{\,2}c^2 + m^2c^4\,. \tag{1.43}$$

As a simple application of this last formula, consider the decay at rest of a π^+ meson, of mass m_π, into a μ^+ lepton, of mass m_μ, and a neutrino, assumed to have mass zero. Since the 3-momentum is conserved, and it is zero before the decay, the momenta of the μ^+ and of the neutrino must be equal and opposite. Suppose that the magnitude of the momentum of the μ^+, which can be measured, is p. The zeroth component of the 4-momentum, the relativistic energy, is also conserved. Before the decay, the energy of the π^+ is just the pion mass times c^2; and after the decay, it is the sum of the the neutrino energy, which is equal to pc [see Eq.(1.43) with $m = 0$], and the muon energy. That is,

$$m_\pi c^2 = pc + \sqrt{p^2c^2 + {m_\mu}^2c^4}\,. \tag{1.44}$$

This equation can be solved to yield

$$p = \frac{{m_\pi}^2 - {m_\mu}^2}{2m_\pi}c\,. \tag{1.45}$$

Evidence has been found that the electron neutrino possesses a very small mass, but in 2002 the Particle Data Group gave only the upper bound $m_\nu < 3$ eV for the electron neutrino's mass. The corrected value for the muon momentum is

$$p = \frac{c}{2m_\pi}\sqrt{(m_\pi^2 - m_\mu^2)^2 - m_\nu^2(2m_\pi^2 + 2m_\mu^2 - m_\nu^2)}\,.$$

1.4 Schrödinger and Klein-Gordon Equations

We recall from Sec. **1.9** of Volume 1, that q_i and p_j, respectively the ith and jth Cartesian components of the position and momentum operators of a particle in 3 dimensions, satisfy the quantization condition

$$[q_i, p_j] = i\hbar\delta_{ij}\,. \tag{1.46}$$

Further the eigenvector $|\vec{x}\rangle = |x_1\rangle|x_2\rangle|x_3\rangle$ of q_i, corresponding to the eigenvalue x_i, satisfies

$$\langle\vec{x}\,|\vec{y}\rangle = \delta(x_1 - y_1)\delta(x_2 - y_2)\delta(x_3 - y_3) \equiv \delta^3(\vec{x} - \vec{y})\,. \tag{1.47}$$

It follows in general that

$$\langle\vec{x}\,|p_i|\vec{y}\rangle = \left[i\hbar\frac{\partial}{\partial y_i} + f_i(\vec{x})\right]\delta^3(\vec{x} - \vec{y})\,, \tag{1.48}$$

where $\vec{f}(\vec{x})$ is a continuous function. For a free particle we use the phase freedom, as in the one-dimensional case, to remove this function, leaving

$$\langle\vec{x}\,|p_i|\vec{y}\rangle = i\hbar\frac{\partial}{\partial y_i}\delta^3(\vec{x} - \vec{y})\,. \tag{1.49}$$

For an arbitrary state vector $|\phi\rangle$ in the Hilbert space spanned by the eigenfunctions of $\vec{q}$, we have

$$|\phi\rangle = \int d^3y|\vec{y}\rangle\langle\vec{y}\,|\phi\rangle\,.$$

Hence

$$\begin{aligned}\langle\vec{x}\,|p_i|\psi(t)\rangle &= \int d^3y\,\langle\vec{x}\,|p_i|\vec{y}\rangle\langle\vec{y}\,|\psi(t)\rangle \\ &= -i\hbar\frac{\partial}{\partial x_i}\langle\vec{x}\,|\psi(t)\rangle \\ &= -i\hbar\frac{\partial}{\partial x_i}\psi(t,\vec{x})\,,\end{aligned} \tag{1.50}$$

where $|\psi(t)\rangle$ is a state vector describing a physical system at time t. Accordingly, we can say that, in the configuration representation, p_i is represented by the operator $-i\hbar\frac{\partial}{\partial x_i}$:

$$\vec{p} \longrightarrow -i\hbar\vec{\nabla} \,. \tag{1.51}$$

In an isolated *classical* system, for which the Lagrangian is not an explicit function of the time, the Hamiltonian is equal to the total energy of the system, which is time-independent. In the nonrelativistic mechanics of a particle of mass m, we have

$$H = \frac{p^2}{2m} + V(\vec{x}\,),$$

where $p^2 = \vec{p}\cdot\vec{p}$, and where $V(\vec{x}\,)$ is the (conservative) potential.

In making the transition to quantum mechanics, we replace the Hamiltonian, the momentum and the position by the corresponding linear operators that have these quantities as their eigenvalues:

$$H = \frac{p^2}{2m} + V(\vec{q}\,). \tag{1.52}$$

Let a physical system be described by a state vector $|\psi(t)\rangle$, which is an eigenvector of the Hamiltonian:

$$i\hbar\frac{d}{dt}|\psi(t)\rangle = H|\psi(t)\rangle = E|\psi(t)\rangle\,, \tag{1.53}$$

where E is the total energy of the system. Hence

$$E\psi(t,\vec{x}\,) = \langle\vec{x}\,|H|\psi(t)\rangle = \langle\vec{x}\,|\frac{p^2}{2m} + V(\vec{q}\,)|\psi(t)\rangle\,. \tag{1.54}$$

Now

$$\langle\vec{x}\,|p_ip_i|\psi(t)\rangle = -\hbar^2\nabla^2\psi(t,\vec{x}\,)\,,$$

where use has been made of Eq.(1.49)-(1.50). Since the eigenvalue of q_i belonging to $\langle\vec{x}\,|$ is x_i, the eigenvalue of $V(\vec{q}\,)$ is $V(\vec{x}\,)$ (at any rate if the function $V(\vec{x}\,)$ has a Fourier transform). Hence from Eq.(1.54) we find

$$i\hbar\frac{\partial}{\partial t}\psi(t,\vec{x}\,) = E\psi(t,\vec{x}\,) = -\frac{\hbar^2}{2m}\nabla^2\psi(t,\vec{x}\,) + V(\vec{x}\,)\psi(t,\vec{x}\,)\,, \tag{1.55}$$

the time-dependent Schrödinger equation.

Since E is time-independent under the conditions stipulated above, we can solve Eq.(1.53):

$$|\psi(t)\rangle = \exp\left(\frac{-iEt}{\hbar}\right)|\psi(0)\rangle\,.$$

Hence

$$\psi(t,\vec{x}\,) \equiv \langle\vec{x}\,|\psi(t)\rangle = \exp\left(\frac{-iEt}{\hbar}\right)\psi(\vec{x}\,)\,,$$

where

$$\psi(\vec{x}\,) = \psi(0,\vec{x}\,)\,. \tag{1.56}$$

Thus the time dependence can be factored out of Eq.(1.55), yielding

$$E\psi(\vec{x}\,) = \left[-\frac{\hbar^2}{2m}\nabla^2 + V(\vec{x}\,)\right]\psi(\vec{x}\,)\,, \tag{1.57}$$

which is Schrödinger's time-independent equation.

When the speed of the particle is not small compared to that of light, the mechanics of special relativity must be used; and instead of Eq.(1.52) we have

$$H^2 = p^2c^2 + m^2c^4\,. \tag{1.58}$$

Here we treat a free particle exclusively, but we will later see how the electromagnetic 4-potential can be incorporated in a relativistically covariant manner. In quantum mechanics, we again replace H and p by the corresponding operators. Sandwiching both sides of Eq.(1.58) between $\langle\vec{x}\,|$ and $|\psi(t)\rangle$ and putting everything on the left, we find

$$\langle\vec{x}\,|H^2 - p^2c^2 - m^2c^4|\psi(t)\rangle = 0.$$

By using Eqs.(1.49), (1.50) and (1.53), we deduce

$$\left(\partial^2 + \frac{m^2c^2}{\hbar^2}\right)\psi(t,\vec{x}\,) = 0\,, \tag{1.59}$$

where the d'Alembertian is defined by

$$\partial^2 = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\,. \tag{1.60}$$

This is the Klein-Gordon equation that generalizes the Schrödinger equation in the relativistic domain.

1.5 Dirac Equation

So far we have discussed the equation of motion of a spinless particle. For the electron let us employ a state vector with two components, $|\psi_i(t)\rangle$, $i = 1, 2$, in order to account for the two possible eigenvalues of the third component of the spin ($\pm\hbar/2$). We require each component of $\psi(t, \vec{x}) = \langle \vec{x} \,|\psi(t)\rangle$ to satisfy the Klein-Gordon equation, so that one of the requirements of relativity will be met. However, we might expect the components to be connected in some way, and we follow Dirac in looking for an equation of the form

$$\left[i\gamma^\mu \partial_\mu - \frac{mc}{\hbar}\right] \psi(t, \vec{x}) = 0\,. \tag{1.61}$$

Here γ^μ, $\mu = 0, 1, 2, 3$ are four liearly independent constant matrices, the *Dirac matrices*, that we will determine; and ∂_μ is the relativistic, covariant derivative operator, with components

$$\partial_0 = \frac{1}{c}\frac{\partial}{\partial t} \qquad \partial_1 = \frac{\partial}{\partial x_1} \qquad \partial_2 = \frac{\partial}{\partial x_2} \qquad \partial_3 = \frac{\partial}{\partial x_3}\,. \tag{1.62}$$

In Eq.(1.61) summation over the repeated index, μ, is implicit, and $\psi(t, \vec{x})$ is a 'matrix' with one column that contains its two spin components. The Dirac matrices multiply this column in the standard matrix fashion.

Dirac was motivated to contemplate an equation of the form of Eq.(1.61) by the consideration that the classical Hamilton equations are *linear* in time derivatives, and he wanted to incorporate this feature also into the quantum theory. However, relativistic covariance should demand linearity also in the spatial derivatives. Dirac's motivation was not sound, since the Klein-Gordon equation, in which the time derivative appears quadratically, does describe spinless particles. Moreover, the equations of motion for the components of the quantized electromagnetic field are also quadratic in the time derivative. The best that can be said at this stage is that Eq.(1.61) is a shot in the dark*. The ultimate justification for the Dirac equation is experimental and *a posteriori*. A theoretical framework arises in the group-theoretical treatment of the Lorentz group: its covering group, $SL(2, C)$, has spinor representations (Sec. **5.2**).

The great game now is to choose the Dirac matrices in such a way that each component of $\psi(t, \vec{x})$ satisfies the Klein-Gordon equation, which we rewrite in the form

$$\left(g^{\mu\nu}\partial_\mu \partial_\nu + \frac{m^2 c^2}{\hbar^2}\right) \psi(t, \vec{x}) = 0\,, \tag{1.63}$$

*See A. Koestler, 'The Sleepwalkers', Hutchinson (1959), for earlier examples in science of genial shots in the dark!

with summation over μ and ν, where $g^{\mu\nu}$ is the metric tensor, defined by

$$g^{00} = 1 \qquad g^{11} = -1 \qquad g^{22} = -1 \qquad g^{33} = -1$$

with all the nondiagonal terms vanishing. Multiply Eq.(1.61) from the left by $[i\gamma^\nu \partial_\nu + mc/\hbar]$ and write out the terms:

$$\left[-\gamma^\nu\gamma^\mu\partial_\nu\partial_\mu - i\frac{mc}{\hbar}\gamma^\nu\partial_\nu + i\frac{mc}{\hbar}\gamma^\mu\partial_\mu - \frac{m^2c^2}{\hbar^2}\right]\psi(t,\vec{x}) = 0\,. \tag{1.64}$$

If the Dirac matrices satisfy

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}\,, \tag{1.65}$$

where it is understood that the right side multiplies a unit matrix, then the following identity between differential operators holds:

$$\begin{aligned}\gamma^\nu\gamma^\mu\partial_\nu\partial_\mu &= \tfrac{1}{2}\left[\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu\right]\partial_\mu\partial_\nu \\ &= g^{\mu\nu}\partial_\mu\partial_\nu\,.\end{aligned} \tag{1.66}$$

Hence, if the Dirac matrices satisfy the anticommutation relations Eq.(1.65), the individual Dirac spinor wave-functions each satisfy the Klein-Gordon equation.

What have we gained by playing with mathematical equations? First of all, it is impossible to satisfy Eq.(1.65) with four 2×2 matrices!! The Pauli matrices,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

satisfy

$$\sigma_i\sigma_j + \sigma_j\sigma_i = 2\delta_{ij}\,,$$

for $i, j = 1, 2, 3$, but there is no fourth matrix that anticommutes with the Pauli matrices. A lesser sleepwalker than Dirac would have given up; but after much worrying Dirac realized that his anticommutation relations could be realized by 4×4 matrices! This means that $\psi(t,\vec{x})$ must be considered to be a column with not two but four components. We expected two components to account for the spin degree of freedom. What is the meaning of the extra two components? Although this is far from obvious, they in fact correspond to the spin states of the *positron*, the antiparticle of the electron, a particle with the same mass and spin as the electron, but with a charge of the same magnitude but *opposite sign.* Playing about with equations, Dirac opened up the world of antimatter!

The standard (Dirac) representation of the γ-matrices is

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\gamma^1 = \begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix} \equiv \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma^2 = \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix} \equiv \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma^3 = \begin{pmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{pmatrix} \equiv \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} .$$

We can write this compactly as follows:

$$\begin{aligned} \gamma^0 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ \vec{\gamma} &= \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix} . \end{aligned} \tag{1.67}$$

Here 0 and 1 are to be understood respectively as the null and the unit 2×2 matrices.

Note that γ^0 is Hermitian and that its square is the unit matrix. The *spacelike* γ's, i.e., γ^1, γ^2, γ^3, are anti-Hermitian, but since they anticommute with γ^0, it is true that for all four Dirac matrices

$$\gamma^0 \gamma^{\mu\dagger} \gamma^0 = \gamma^\mu . \tag{1.68}$$

The Dirac equation is often written in the super-compact form

$$[i\gamma\partial - m]\,\psi(x) = 0 , \tag{1.69}$$

where $\gamma\partial$ simply means the relativistic invariant $\gamma^\mu \partial_\mu$ and x stands for all four dimensions of space-time. To make the equation more beautiful, units of velocity

and action have been adjusted in such a way that $c = 1$ and $\hbar = 1$, so that these extraneous factors can be omitted. This is common practice in high-energy physics; but in this, and the following two chapters we will retain c and $\hbar$ to avoid potential confusion.

The Hermitian conjugate of Eq.(1.69) is $\psi^\dagger(x)[-i\gamma^\dagger \overleftarrow{\partial} - m] = 0$, where $\psi^\dagger(x)$ has been placed to the left: it is a matrix with one row. Multiply by $-\gamma^0$ from the right and use Eq.(1.68):

$$\overline{\psi}(x)\left[i\gamma \overleftarrow{\partial} + m\right] = 0\,, \tag{1.70}$$

where $\overline{\psi}(x) = \psi^\dagger(x)\gamma^0$. This is called the conjugate Dirac equation.

It should be noted that Eq.(1.67) is only one possible representation of the Dirac matrices. If we replace all gamma matrices γ^μ by $\tilde{\gamma}^\mu$, where

$$\tilde{\gamma}^\mu = M\gamma^\mu M^\dagger\,, \tag{1.71}$$

and the four-component wave-function $\psi(x)$ by $\tilde{\psi}(x) = M\psi(x)$, where M is any unitary 4×4 matrix, then all of the equations of this section retain their form. The Dirac matrices are only defined up to a unitary transformation: the important thing is that Eq.(1.65) must be satisfied.

1.6 Dirac's Views

To round out this discussion of Dirac's notation, and our introduction to the Dirac equation, we cite his own words:

"In setting up this form of the equations, Hamilton was influenced only by conditions of mathematical beauty. He might have said: 'It is very nice to write the equations in this way, but ... you could ... continue to use the equations in the form they were originally given by Newton.' But Hamilton seemed to have some remarkable insight into what was important—one of the most remarkable insights, I suppose, that a mathematician has ever had. He found a form of writing the equations of mechanics whose importance would be realized only after a hundred years.

Now, with quantum mechanics, we cannot exclude transitions from positive energy states to negative energy states, and that means that we cannot exclude the negative energy states from our theory. ... We can get a departure from the vacuum in two ways: one way is to bring an electron into a positive energy state; the other way is to have a 'hole' in the distribution of negative energy states ... the 'holes' appear as a new kind of particle having a positive charge. What is the mass of these new particles? Well, when I first thought of this

idea, it occurred to me that the mass would have to be the same as that of the electron because of the symmetry. But I did not dare to put forward that idea, because it seemed to me that if this new kind of particle (having the same mass as the electron and an opposite charge) existed, it would certainly have been discovered by the experimenters. ... I think that Weyl was the first to make the very definite statement that mathematical symmetry demanded that these 'holes' should be particles with the same mass as the mass of the electron. ... and the question arises: 'Why had experimenters never observed them?' I think the only answer to that question is that they were prejudiced against new particles. It was assumed that there were only two basic particles in Nature: the electron and the proton. ... They had never observed positrons, because they really turned a blind eye to them when they had evidence for them."[Dirac, 1978, pages 6-17]

"People have succeeded in setting up certain rules that enable one to discard the infinities produced by the fluctuations in a self-consistent way and have thus obtained a workable theory from which one can calculate results that can be compared with experiment. Good agreement with experiment has been found, showing that there is some validity in the rules. ... They should therefore not be considered as a satisfactory solution of the difficulties."[Dirac, 1958, page 309]

1.7 Exercises

Problem 1

Let b_μ be a covariant Lorentz vector, so it has the transformation rule

$$b'_\mu = \frac{\partial x^\nu}{\partial x'^\mu} b_\nu \, .$$

In terms of this covariant vector, define $b^\mu = g^{\mu\nu} b_\nu$ where $g^{\mu\nu}$ is the contravariant metric tensor. Show that b^μ is a contravariant Lorentz vector, i.e., it has the following transformation rule:

$$b^\mu = \frac{\partial x'^\mu}{\partial x^\nu} b^\nu \, .$$

Problem 2

If $\mathcal{L}$ is a Lorentz scalar, show that $\frac{\partial \mathcal{L}}{\partial A_\nu}$ is a contravariant Lorentz vector, and $\frac{\partial \mathcal{L}}{\partial \partial_\mu A_\nu}$ is a contravariant Lorentz tensor.

Problem 3

Suppose that the speed of light from A to B is different from its speed from B to A, but that the round-trip average speed is constant (i.e., the average speed is the same for all points A and B, and is independent of which inertial system of coordinates is used). Deduce the generalization of the Lorentz transformation and discuss interesting particular cases.

Problem 4

Show that the four Dirac matrices (in the Dirac representation) do indeed anti-commute with one another. Define $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$. Demonstrate

$$\gamma_5^\dagger = \gamma_5 \qquad \gamma_5^2 = 1 \qquad \gamma_5\gamma^\mu + \gamma^\mu\gamma_5 = 0\,,\ \mu = 0,1,2,3.$$

Show that γ_5 anticommutes with γ^μ, $\mu = 0,1,2,3$. Work out the explicit form of γ_5 and of the spin matrix, $\vec{S} = \frac{1}{2}\hbar\gamma_5\gamma^0\vec{\gamma}$, both in the usual representation.

Problem 5

Define a scalar product on the space, $\mathcal{L}^2$, of square-integrable functions, $\psi(x)$, $-\infty < x < \infty$, and show that $\mathcal{L}^2$, equipped with this product, is a Hilbert space. Consider the operator $p = -i\frac{\partial}{\partial x}$, restricted to the line $a < x < b$. Is this operator Hermitian on its domain in $\mathcal{L}^2$? Is this domain dense in $\mathcal{L}^2$? Is p self-adjoint? What are its self-adjoint extensions, if any? What are its eigenstates, and what space do they span? Consider the cases

(1) $a = -\infty$, $b = \infty$,
(2) a and b finite,
(3) a finite and $b = \infty$.

Problem 6

The *Majorana representation* of the Dirac matrices is defined as follows:

$$\gamma^\mu_{\text{Mar}} = M\gamma^\mu_{\text{Dir}}M^\dagger\,,$$

with $M = \frac{1}{\sqrt{2}}\gamma^0_{\text{Dir}}(1+\gamma^2_{\text{Dir}})$. Show that M is unitary and Hermitian. Calculate γ^0_{Mar}, γ^1_{Mar}, γ^2_{Mar}, γ^3_{Mar} and γ^5_{Mar}. Which of the gamma matrices are Hermitian, and which are unitary in the Majorana representation?

Problem 7

The chiral representation of the Dirac matrices is defined by

$$\gamma^0_{\text{Chi}} = -\gamma_{5\,\text{Dir}} \qquad \vec{\gamma}_{\text{Chi}} = \vec{\gamma}_{\text{Dir}}\,.$$

What is $\gamma_{5\,\text{Chi}}$? Find a matrix, M, that effects the transformation from the Dirac to the chiral representation.

Problem 8

Show that, under an infinitesimal Lorentz transformation,

$$x'^{\mu} = x^{\mu} + \omega^{\mu\nu} x_{\nu}$$

a solution of the free Dirac equation becomes

$$\psi'(x) = \left[1 - \frac{i}{2}(L_{\mu\nu} + S_{\mu\nu})\omega^{\mu\nu}\right] \psi(x)\,,$$

where $L_{\mu\nu}$ and $S_{\mu\nu}$ are the orbital and spin operators. Give explicit expressions for these operators.

Problem 9

Let $q_n(t)$, $n = 1, 2, 3, \ldots$ be a complete set of canonical coordinates of a given dynamical quantum system. Let $f_n(\vec{x})$, $n = 1, 2, 3, \ldots$ be an orthonormal basis of a Hilbert space, and define the quantum field

$$\phi(x) = \sum_{n=1}^{\infty} q_n(t) f_n(\vec{x})\,.$$

From the Euler-Lagrange equation for the discrete variables $q_n(t)$, deduce the Euler-Lagrange equation for the field $\phi(x)$, regarded as a continuous canonical quantum coordinate. What is the momentum conjugate to $\phi(x)$? Starting from the canonical commutation relations for the discrete variables, deduce those for the field variables.

Problem 10

> "In fact, while all other quantities (especially those x, y, z closely connected with t by the Lorentz transformation) are represented by operators, there corresponds to the time an ordinary number-parameter t, just as in classical mechanics." [Neumann 1932]

Is it true that space points are represented by operators in quantum mechanics, while time is not quantized? Distinguish carefully between the position of a point particle and (x, y, z), treated as a spatial coordinate, and similarly between the reading of a clock and t, treated as a temporal coordinate, and thereby show that there is in fact no special difficulty in treating space and time on an equal footing in nonrelativistic quantum mechanics. What can one say about the matter in relativistic quantum mechanics?

Chapter 2

Charged Particle in Electromagnetic Field

2.1 Maxwell's Equations

The Maxwell equations, in the presence of a charge-density, $\rho(x)$, and a current-density, $\vec{j}(x)$, are

$$\vec{\nabla} \cdot \vec{E} = \rho\,, \tag{2.1}$$

$$\vec{\nabla} \cdot \vec{B} = 0\,, \tag{2.2}$$

$$\vec{\nabla} \wedge \vec{B} - \frac{1}{c}\frac{\partial \vec{E}}{\partial t} = \frac{\vec{j}}{c}\,, \tag{2.3}$$

$$\vec{\nabla} \wedge \vec{E} + \frac{1}{c}\frac{\partial \vec{B}}{\partial t} = 0\,. \tag{2.4}$$

Note that no polarization or magnetization has been included: these are the equations in vacuo, except that charge distributions are taken into account. In a polarizable and magnetizable medium, one distinguishes between $\vec{D}$ and $\vec{E}$, and between $\vec{B}$ and $\vec{H}$. When, however, one adopts the more fundamental view that all the charges should be explicitly taken into account, this distinction need no longer be made.

Note that no factor of 4π appears on the right-hand sides of Eq.(2.1) and Eq.(2.3), as it does in the Gaussian form of Maxwell's equations. The factor is removed by redefining the unit of electric charge, and Eqs.(2.1)-(2.4) constitute the electromagnetic equations in Heaviside-Lorentz units. These units turn out to be convenient in quantum field theory, as we shall see. However, a price has to be paid for getting rid of the factor 4π in the Maxwell equations, for it reappears in the Coulomb law for the electric field of a point charge, e:

$$\vec{E} = \frac{e_{\mathsf{HL}}}{4\pi r^2}\frac{\vec{r}}{r} = \frac{e_{\mathsf{Gauss}}}{r^2}\frac{\vec{r}}{r}\,. \tag{2.5}$$

Since $\vec{B}$ is divergence-free (there do not seem to be any magnetic monopoles), it follows that there is a vector field, $\vec{A}$, whose curl it is:

$$\vec{B} = \vec{\nabla} \wedge \vec{A}. \tag{2.6}$$

Define the auxiliary vector field

$$\vec{C} = -\vec{E} - \frac{1}{c}\frac{\partial \vec{A}}{\partial t},$$

so that $\vec{\nabla} \wedge \vec{C} = 0$. This implies the existence of a scalar field, Φ, such that

$$-\vec{E} - \frac{1}{c}\frac{\partial \vec{A}}{\partial t} = \vec{C} = \vec{\nabla}\Phi\,,$$

which can be rewritten

$$\vec{E} = -\vec{\nabla}\Phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}\,. \tag{2.7}$$

Substituting Eqs.(2.6)-(2.7) into Eqs.(2.1)-(2.3), we find

$$\begin{aligned} \frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} - \nabla^2\Phi - \frac{1}{c}\frac{\partial}{\partial t}\left(\frac{1}{c}\frac{\partial \Phi}{\partial t} + \vec{\nabla}\cdot\vec{A}\right) &= \rho\,, \\ \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} - \nabla^2\vec{A} + \vec{\nabla}\left(\frac{1}{c}\frac{\partial \Phi}{\partial t} + \vec{\nabla}\cdot\vec{A}\right) &= \vec{\jmath}\,. \end{aligned} \tag{2.8}$$

The above equations can be cast into a more compact form by combining the scalar and the vector potentials into one 4-potential:

$$A^\mu = (\Phi, \vec{A})\,. \tag{2.9}$$

We shall show that this 4-potential transforms as a contravariant Lorentz vector. In terms of the covariant derivative (1.62) and the d'Alembertian (1.60), we can write Eq.(2.8) in the form

$$\begin{aligned} \partial^2 A^0 - \partial_0\left(\partial_\mu A^\mu\right) &= \rho\,, \\ \partial^2 \vec{A} + \vec{\nabla}\left(\partial_\mu A^\mu\right) &= \vec{\jmath}/c\,. \end{aligned} \tag{2.10}$$

These equations cry out to be combined into one covariant form, do they not? However, there is an awkward sign difference in front of the second term on the left. Electric charge is taken to be relativistically invariant, so that the charge, e, of an electron is the same in any inertial system. Thus charge density is not invariant, but the product, $e = \rho(x)d^3x$, is indeed Lorentz invariant. Electric current is caused by the flow of electrons: it is given by the sum of the electric

charges multiplied by their velocities. The current density is accordingly the product of the charge density and the velocity,

$$\vec{j} = \rho\vec{v}, \tag{2.11}$$

so that if we define the 4-current density by

$$j^\mu = \frac{\rho}{c}\frac{dx^\mu}{dt} \equiv (\rho, \vec{j}/c) , \tag{2.12}$$

then we find

$$edx^\mu = \rho\, dx^\mu d^3x = \frac{\rho}{c}\frac{\partial x^\mu}{\partial t} dx^0 d^3x = j^\mu d^4x . \tag{2.13}$$

Now since e and d^4x are Lorentz invariants, and dx^μ is a contravariant 4-vector, it follows that j^μ must also be a contravariant vector. We can rewrite Eq.(2.10) in component form as follows:

$$\begin{aligned} \partial^2 A^0 - \partial_0[\partial_\nu A^\nu] &= j^0 \\ \partial^2 A^k + \partial_k[\partial_\nu A^\nu] &= j^k . \end{aligned} \tag{2.14}$$

Now we can finally understand the apparently awkward sign difference, for the derivative operator is covariant, and rewriting it in the unnatural, contravariant form, $[\partial^0 = \partial_0 \ , \ \partial^k = -\partial_k]$, we pick up a minus sign! Hence Eq.(2.14) can be thrown into the elegant form

$$\partial^2 A^\mu - \partial^\mu[\partial_\nu A^\nu] = j^\mu \tag{2.15}$$

Since j^μ is a contravariant vector, it follows that A^μ must also be a contravariant vector. In fact, Eq.(2.15), which is merely a rewriting of Eq.(2.1)-(2.4), is in relativistically covariant form. The equations *knew* more than did their creator, Maxwell, when he invented them! To do Maxwell and Lorentz justice, they were worried that the electromagnetic equations are not consistent with Galilean covariance, and they did their best to understand this fact.

The 4-potential changes under a Lorentz transformation as follows:

$$A^{\mu\prime}(x') = \frac{\partial x'^\mu}{\partial x^\rho} A^\rho(x) ; \tag{2.16}$$

that is, the transformed field, at the transformed point, is equal to the old field, at the old point, multiplied by the Lorentz-transformation matrix. A covariant version of the 4-potential can be defined:

$$A_\mu(x) = g_{\mu\nu} A^\nu(x) ; \tag{2.17}$$

and the transformation law for this is

$$A_\mu{}'(x') = \frac{\partial x^\rho}{\partial x'^\mu} A_\rho(x) \ . \tag{2.18}$$

It is convenient to introduce the second-order tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \ , \tag{2.19}$$

with the transformation law

$$F_{\mu\nu}{}'(x') = \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} F_{\rho\sigma}(x) \ . \tag{2.20}$$

After these book-keeping preliminaries, we can write the Maxwell equations (2.15) in the still more compact form

$$\partial_\mu F^{\mu\nu} = j^\nu \ . \tag{2.21}$$

The field tensor, which is manifestly antisymmetric, can be expressed directly in terms of the electric field and the magnetic induction, for if i, j, k are restricted to the values $1, 2, 3$, then

$$F_{0k} = \partial_0 A_k - \partial_k A_0 = -\partial_0 A^k - \partial_k A^0 = E_k \ , \tag{2.22}$$

and

$$F_{jk} = \partial_j A_k - \partial_k A_j = \epsilon_{jkl} B_l \ . \tag{2.23}$$

Thus the field tensor can be expressed wholly in terms of $\vec{E}$ and $\vec{B}$, and vice-versa.

Despite the fact that the 4-potential, A^μ, is not uniquely determined by the field tensor, it is an extremely useful quantity. If it is subjected to a *gauge transformation*, i.e.

$$A^\mu \longrightarrow A'^\mu = A^\mu + \partial^\mu G \ , \tag{2.24}$$

where G is any Lorentz scalar field, then clearly the field tensor is unchanged. Such a gauge transformation has no physical consequences: any interactions involving the electromagnetic 4-potential must respect this gauge invariance. The restriction turns out to be very important, with ramifications far outside the field of electromagnetism.

The Lorenz* condition is often imposed on the four-potential:

$$\partial_\mu A^\mu = 0 \ . \tag{2.25}$$

*The Danish physicist L. Lorenz introduced this condition in 1867, although it is almost universally ascribed to H.A. Lorentz.

By means of a gauge transformation, it is always possible to achieve the Lorenz condition, without changing the physics. For under the gauge transformation (2.24),

$$\partial_\mu A'^\mu = \partial_\mu A^\mu + \partial^2 G \,, \tag{2.26}$$

and the right side can be made to vanish by choosing G such that $\partial^2 G = -\partial_\mu A^\mu$. When the Lorenz condition, (2.25), is satisfied, the Maxwell equations, (2.21), become even simpler:

$$\partial^2 A^\nu = j^\nu \,. \tag{2.27}$$

Let us examine the free electromagnetic field. We shall see how the Maxwell equation, (2.21), can be derived from a variational principle. In order to do this, we regard the field, $A^\nu(t, \vec{r})$, as a continuous set of generalized coordinates. For a given time, t, the canonical variables are labeled by ν, and the continuous variable, $\vec{r}$. Since the expression for the Lagrangian will inevitably involve a summation over all space, it is convenient to introduce a Lagrangian density:

$$L = \int d^3x \mathcal{L}(x). \tag{2.28}$$

The action can accordingly be written

$$S = \int_{t_a}^{t_b} dt L = \int_{t_a}^{t_b} dt \int d^3x \mathcal{L}(x) = \int_a^b d^4x \mathcal{L}(x). \tag{2.29}$$

Since the action, S, is a Lorentz invariant, and d^4x is an invariant measure, it follows that the Lagrangian density, $\mathcal{L}$, is Lorentz-invariant.

Consider now a variation in the fields, A^μ, such that the values stay fixed at the spacelike hypersurfaces a and b. The resultant change in the action is

$$\begin{aligned} \delta S &= \int_a^b d^4x [\frac{\partial \mathcal{L}}{\partial A_\nu} \delta A_\nu + \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} \delta(\partial_\mu A_\nu)] \\ &= \int_a^b d^4x [\frac{\partial \mathcal{L}}{\partial A_\nu} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)}] \delta A_\nu. \end{aligned} \tag{2.30}$$

Since δA_ν is arbitrary in the interior, the Hamilton variational principle, $\delta S = 0$, implies

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} - \frac{\partial \mathcal{L}}{\partial A_\nu} = 0 \tag{2.31}$$

This expression is the Euler-Lagrange equation for the electromagnetic potential.

We shall now show that the Lagrangian density,

$$\mathcal{L} = -\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} - A^{\mu}j_{\mu} = -\tfrac{1}{2}F^{\mu\nu}\partial_{\mu}A_{\nu} - A^{\mu}j_{\mu}, \tag{2.32}$$

when inserted into Eq.(2.31), yields the Maxwell equation, (2.21). To do this, we regard $\mathcal{L}$ as a function of the 4 variables A^{ν}, and the 16 variables $\partial^{\mu}A^{\nu}$:

$$\begin{aligned}
\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}A_{\nu})} &= -\tfrac{1}{2}\left\{F^{\rho\sigma}\frac{\partial F_{\rho\sigma}}{\partial(\partial_{\mu}A_{\nu})}\right\} \\
&= -\tfrac{1}{2}F^{\rho\sigma}[\delta^{\mu}_{\rho}\delta^{\nu}_{\sigma} - \delta^{\mu}_{\sigma}\delta^{\nu}_{\rho}] = -F^{\mu\nu} \\
\frac{\partial\mathcal{L}}{\partial A_{\nu}} &= -j^{\nu}\,.
\end{aligned}$$

Thus the Euler-Lagrange field equations do indeed yield the Maxwell equations.

The *Lorentz force* on a particle of charge q and velocity $\vec{v}$ in an electromagnetic field is

$$\vec{F} = q\vec{E} + \frac{q}{c}\vec{v}\wedge\vec{B}\,, \tag{2.33}$$

where $\vec{E}$ and $\vec{B}$ are the electric field strength and the magnetic induction, respectively. This force law can be derived from the following Lagrangian:

$$L = -mc^2\sqrt{1 - \frac{\dot{x}^i\dot{x}^i}{c^2}} - qA^0 + \frac{q}{c}\dot{x}^iA^i\,, \tag{2.34}$$

where m is the mass of the particle. The canonical momentum and Hamiltonian are defined as follows:

$$\begin{aligned}
p_i = \frac{\partial L}{\partial\dot{x}^i} &= \frac{m\dot{x}^i}{\sqrt{1 - v^2/c^2}} + \frac{q}{c}A^i \\
H = p_i\dot{x}^i - L &= \frac{mc^2}{\sqrt{1 - v^2/c^2}} + qA^0\,.
\end{aligned} \tag{2.35}$$

Evidently the influence of the electromagnetic interaction is summarized in the following additions to the field-free quantities:

$$\begin{aligned}
p_i &\longrightarrow p_i + \frac{q}{c}A^i \\
H &\longrightarrow H + qA^0\,.
\end{aligned} \tag{2.36}$$

2.2 Covariant Derivative

In order to find out what the configuration representation of Dirac's Hamiltonian is, we multiply Eq.(1.61) by $\hbar c\gamma^0$ from the left, remembering that $(\gamma^0)^2 = 1$:

$$i\hbar\frac{\partial\psi}{\partial t} = i\hbar c\partial_0\psi = -i\hbar c\gamma^0\vec{\gamma}\cdot\vec{\nabla}\,\psi + mc^2\gamma^0\psi\,.$$

Now from the general definition, Eq.(1.53), we see that the Hamiltonian is

$$H = c\gamma^0\vec{\gamma}\cdot\vec{p} + mc^2\gamma^0 \tag{2.37}$$

where

$$\vec{p} = -i\hbar\vec{\nabla} \tag{2.38}$$

is the configuration-space representation of the momentum operator.

To make the transition from the free Dirac equation to one describing the electromagnetic coupling of an electron, with charge, $q = -e$, we replace H by $H - eA^0$, and $\vec{p}$ by $\vec{p} - e\vec{A}/c$ [cf., Eq.(2.36)]. We suppose that this rule remains good in quantum mechanics, and we replace Eq.(2.37) by

$$H - eA^0 = c\gamma^0\vec{\gamma}\cdot(\vec{p} - e\vec{A}/c) + mc^2\gamma^0\,. \tag{2.39}$$

Allowing the operators to act on ψ, we find

$$\left\{i\hbar c\partial_0 - eA^0\right\}\psi = \left\{-c\gamma^0\vec{\gamma}\cdot\left[i\hbar\vec{\nabla} + \frac{e}{c}\vec{A}\right] + mc^2\gamma^0\right\}\psi\,. \tag{2.40}$$

At first sight, there appears to be a sign mistake; but when we remember that $\partial_\mu = (\partial_0, \vec{\nabla})$ is a *covariant* vector but $A^\mu = (A^0, \vec{A})$ is a *contravariant* one, we see that we can write Eq.(2.40) in the form

$$\left(i\gamma^\mu\mathcal{D}_\mu - \frac{mc}{\hbar}\right)\psi = 0\,, \tag{2.41}$$

where the *covariant derivative* is defined by

$$\mathcal{D}_\mu = \partial_\mu + \frac{ie}{\hbar c}A_\mu\,. \tag{2.42}$$

Since $A_\mu = g_{\mu\nu}A^\nu = (A^0, -\vec{A})$, it is clear that Eq.(2.41) is indeed equivalent to Eq.(2.40). In super-compact form,

$$(i\gamma\mathcal{D} - m)\psi = 0\,, \tag{2.43}$$

where once more c and $\hbar$ have been set equal to unity.

2.3 Pauli Equation

The Dirac equation can be written

$$\left\{ i\hbar\frac{\partial}{\partial t} - eA^0 - mc^2\gamma^0 \right\} \psi = c\gamma^0\vec{\gamma}\cdot\vec{\pi}\,\psi\,, \tag{2.44}$$

where the modified momentum is

$$\vec{\pi} = -i\hbar\vec{\nabla} - \frac{e}{c}\vec{A}\,.$$

We shall write the four-spinor ψ in the form

$$\psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix}, \tag{2.45}$$

where ϕ and χ are two-spinors; and we replace $i\hbar\frac{\partial\psi}{\partial t} = H\psi$ by $E\psi$, and the gamma matrices by their explicit representations:

$$\left\{ E - eA^0 - mc^2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\} \begin{pmatrix} \phi \\ \chi \end{pmatrix} = c \begin{pmatrix} 0 & \vec{\sigma}\cdot\vec{\pi} \\ \vec{\sigma}\cdot\vec{\pi} & 0 \end{pmatrix} \begin{pmatrix} \phi \\ \chi \end{pmatrix}. \tag{2.46}$$

We split off the rest-energy of the electron, writing $E = mc^2 + \overline{E}$, so that Eq.(2.46) takes on the form

$$(\overline{E} - eA^0)\phi = c\vec{\sigma}\cdot\vec{\pi}\,\chi \tag{2.47}$$

$$(\overline{E} - eA^0 + 2mc^2)\chi = c\vec{\sigma}\cdot\vec{\pi}\,\phi \tag{2.48}$$

which is still an exact consequence of the Dirac equation. The nonrelativistic approximation consists in assuming that the energies $\overline{E}$ and eA^0 are very small compared with the rest-energy mc^2. Then Eq.(2.48) is approximated by

$$\chi = \frac{1}{2mc}\vec{\sigma}\cdot\vec{\pi}\,\phi\,, \tag{2.49}$$

from which one sees that χ is small compared with ϕ at nonrelativistic energies. On substituting Eq.(2.49) into Eq.(2.47), we find

$$\begin{aligned} (\overline{E} - eA^0)\phi &= \frac{1}{2m}(\vec{\sigma}\cdot\vec{\pi})^2\,\phi \\ &= \frac{1}{2m}\left[\pi^2 + i\vec{\sigma}\cdot(\vec{\pi}\wedge\vec{\pi})\right]\phi\,. \end{aligned} \tag{2.50}$$

If you have difficulty getting the second from the first line, see the generalized proof in the next section. Now

$$\vec{\pi}\wedge\vec{\pi} = \frac{ie\hbar}{c}\left(\vec{\nabla}\wedge\vec{A} + \vec{A}\wedge\vec{\nabla}\right),$$

and

$$\begin{aligned}(\vec{\nabla}\wedge\vec{A}+\vec{A}\wedge\vec{\nabla})_i\phi &= \epsilon_{ijk}(\partial_j(A^k\phi)+A^j\partial_k\phi)\\ &= \epsilon_{ijk}[(\partial_j A^k)+A^k\partial_j+A^j\partial_k)]\phi\\ &= \left\{(\vec{\nabla}\wedge\vec{A})_i\right\}\phi\,, \end{aligned} \tag{2.51}$$

where the parentheses indicate that the differential operator $\vec{\nabla}$ works only on $\vec{A}$ and not on ϕ. We have then

$$\vec{\pi}\wedge\vec{\pi} = \frac{ie\hbar}{c}\vec{\nabla}\wedge\vec{A} = \frac{ie\hbar}{c}\vec{B}\,. \tag{2.52}$$

Gathering these results together, we express Eq.(2.50) in the form

$$\overline{E}\phi = \left\{-\frac{\hbar^2}{2m}\left(\vec{\nabla}-\frac{ie}{\hbar c}\vec{A}\right)^2 + eA^0 - \frac{e\hbar}{2mc}\vec{\sigma}\cdot\vec{B}\right\}\phi\,. \tag{2.53}$$

This nonrelativistic equation, including an effect of spin, is called the Pauli equation. It was invented a few years before the appearance of the Dirac equation.

With the gauge choice $\vec{\nabla}\cdot\vec{A}=0$, we can write

$$\left(\vec{\nabla}-\frac{ie}{\hbar c}\vec{A}\right)^2\phi = \left(\nabla^2 - \frac{2ie}{\hbar c}\vec{A}\cdot\vec{\nabla} - \frac{e^2}{\hbar^2c^2}A^2\right)\phi\,. \tag{2.54}$$

Consider now the case in which the magnetic field is uniform (i.e., independent of $\vec{r}$). A possible choice for the vector potential is

$$\vec{A} = \tfrac{1}{2}\vec{B}\wedge\vec{r}\,. \tag{2.55}$$

Proof that Eq.(2.55) implies $\vec{B}=\vec{\nabla}\wedge\vec{A}$ and $\vec{\nabla}\cdot\vec{A}=0$ if $\vec{B}$ is uniform:

$$\begin{aligned}\left(\vec{\nabla}\wedge\vec{A}\right)_i &= \tfrac{1}{2}\epsilon_{ijk}\epsilon_{kmn}\partial_j B_m x^n = \tfrac{1}{2}(\delta_{im}\delta_{jn}-\delta_{in}\delta_{jm})B_m\delta_{jn}\\ &= \tfrac{1}{2}B_m(\delta_{im}\delta_{jj}-\delta_{ij}\delta_{jm}) = \tfrac{1}{2}B_m(3\delta_{im}-\delta_{im}) = B_i\\ \vec{\nabla}\cdot\vec{A} &= \tfrac{1}{2}\epsilon_{ijk}\partial_i B_j x^k = \tfrac{1}{2}B_j\epsilon_{ijk}\delta_{ik} = \tfrac{1}{2}B_j\epsilon_{iji} = 0\,.\end{aligned}$$

END OF PROOF

In terms of the *orbital* angular momentum, which has the following form (in the configuration representation),

$$\vec{L} = -i\hbar\vec{r}\wedge\vec{\nabla}\,,$$

we have (remembering that $\vec{B}$ is independent of $\vec{r}$),

$$\vec{A}\cdot\vec{\nabla} = \tfrac{1}{2}(\vec{B}\wedge\vec{r})\cdot\vec{\nabla} = \tfrac{1}{2}(\vec{r}\wedge\vec{\nabla})\cdot\vec{B} = \frac{i}{2\hbar}\vec{L}\cdot\vec{B}\,, \tag{2.56}$$

which can be substituted into Eq.(2.54).

From Eq.(2.53), we now find

$$\overline{E}\phi = \left\{-\frac{\hbar^2}{2m}\nabla^2 + eA^0 - \frac{e}{2mc}(\vec{L}+\hbar\vec{\sigma})\cdot\vec{B} + \frac{e^2}{8mc^2}(\vec{B}\wedge\vec{r})^2\right\}\phi\,. \tag{2.57}$$

This is what the Pauli equation (and thus the Dirac equation), gives for the Schrödinger equation of a nonrelativistic electron, in the presence of a scalar potential and a constant magnetic induction.

The magnetic term has the form $-\vec{\mu}\cdot\vec{B}$, where $\vec{\mu} = e(\vec{L}+g\vec{S})/(2mc)$ is a magnetic moment that couples to the induction. Here $\vec{S} = \frac{1}{2}\hbar\vec{\sigma}$ is the spin matrix of the electron. As we see from Eq.(2.57), the Dirac equation predicts $g = 2$, which turns out to be very close to the value experimentally found for this factor. An attempt to measure the g-factor was performed in 1915 by the Dutchman W.J. de Haas, son-in-law of Lorentz, in wartime Berlin at the Kaiser-Wilhelm-Institut für Physik, under the active stimulation of its newly-appointed director, Professor Einstein! The so-called Einstein-de Haas experiment consists in suspending a vertical iron rod on a fine glass thread, along the axis of a solenoid. The rod twists back and forth, exchanging its angular energy with torsional potential energy in the thread. When the rod is magnetized, some of the electron spins line up parallel to the rotational axis of the rod and hence the angular momentum changes. The change is NS, where N is the number of electrons in the rod whose spin is so aligned, and S is the magnitude of the spin of one electron (namely $\frac{1}{2}\hbar$). Let μ be the magnetic moment of one electron. Then the magnetic moment of the rod is $N\mu$. The *gyromagnetic ratio*, i.e. the ratio of the magnetic moment and the angular momentum, is thus the same for the rod as for one electron. By estimating this quantity for the rod, which can be done by macroscopic measurements, de Haas and Einstein found

$$\mu = g\frac{e}{2mc}\frac{\hbar}{2} \tag{2.58}$$

with $g = 1.02$ and $g = 1.45$ in two separate runs. Since the classical expectation was $g = 1$, Einstein and de Haas concluded 'These preliminary results seemed satisfactory to us, and one can easily understand that we were led to consider the value 1.02 as the better one ...' † Later, experimentalists, whom history has

†W. de Haas in *Proceedings of the Third Solvay Conference,* April 1921, page 206, Gautier-Villars, Paris, 1923.

forgotten — we still speak of the Einstein-de Haas effect — measured $g = 2$, to within experimental errors! The modern value is used as a test of the limits of validity of quantum electrodynamics (see Chapter 8).

In the next subsection, we will consider the largest relativistic corrections; but before we do that, let us consider the Paschen-Back effect in hydrogen (i.e., the normal Zeeman effect), namely the splitting of spectral lines in a magnetic field, as implied by Eq.(2.57). Choose the z-axis in the direction of the external, constant magnetic field. We treat

$$H_1 = -\frac{e}{2mc}(L_3 + 2S_3)B \tag{2.59}$$

as an addition to the usual nonrelativistic Hamiltonian. This is a reasonable approximation if the external field is strong compared with the spin-orbit coupling (the latter will be discussed in the next section). If $|\psi_{nm_\ell m_s}\rangle$ is an eigenstate of the nonrelativistic Hamiltonian, and of L^2, L_3 and S_3, then the energy shift is

$$\begin{aligned} \Delta E_{m_\ell m_s} &= -\frac{e}{2mc}\langle\psi_{nm_\ell m_s}|(L_3 + 2S_3)B|\psi_{nm_\ell m_s}\rangle \\ &= -\frac{e\hbar}{2mc}\,(m_\ell + 2m_s)\,B\,. \end{aligned} \tag{2.60}$$

Note that nondiagonal matrix elements of H_1 vanish with these states. Since $2m_s = \pm 1$ it follows that a degenerate level is split by the magnetic field into a number of equally spaced levels, the spacing being proportional to the applied field. The ground state of the hydrogen atom is twofold degenerate, taking the spin degree of freedom into account, and that is split into two levels. The $n = 2$ level is eightfold degenerate: $\ell = 0$, $m_\ell = 0$ and $\ell = 1$, $m_\ell = -1$, 0, 1, and these four states are doubled because of spin. The magnetic field splits this level into five distinct levels, with equal spacing. The highest and lowest energy levels are simple, while the others are doubly degenerate: for example, the unperturbed level corresponds to a superposition of $m_\ell = 1\,, m_s = -\frac{1}{2}$ and $m_\ell = -1\,, m_s = \frac{1}{2}$ states. Note that this normal Zeeman effect occurs when the relativistic fine-structure can be neglected and the external field is large, but not so large as to invalidate the approximation of working to leading order in the external fields.

2.4 Spin-Orbit Coupling

Let us return to Eq.(2.46), which we write

$$\left(i\hbar\frac{\partial}{\partial t} - eA^0 - mc^2\right)\phi = c\vec{\sigma}\cdot\vec{\pi}\,\chi \tag{2.61}$$

$$\left(i\hbar\frac{\partial}{\partial t} - eA^0 + mc^2\right)\chi = c\vec{\sigma}\cdot\vec{\pi}\,\phi\,. \tag{2.62}$$

To improve on the approximation (2.49) we write

$$\chi = \frac{1}{2mc}\vec{\sigma}\cdot\vec{\pi}\,\phi - \frac{1}{2mc^2}\left(i\hbar\frac{\partial}{\partial t} - eA^0 - mc^2\right)\chi\,, \tag{2.63}$$

which is exact, and then we insert the approximation (2.49) into the right side of Eq.(2.63):

$$\chi \approx \frac{1}{2mc}\vec{\sigma}\cdot\vec{\pi}\,\phi - \frac{1}{4m^2c^3}\left(i\hbar\frac{\partial}{\partial t} - eA^0 - mc^2\right)\vec{\sigma}\cdot\vec{\pi}\,\phi\,. \tag{2.64}$$

When this approximate expression for χ is used in Eq.(2.61), we obtain

$$\begin{aligned}\left(i\hbar\frac{\partial}{\partial t} - eA^0 - mc^2\right)\phi \;&=\; \frac{1}{2m}(\vec{\sigma}\cdot\vec{\pi})^2\phi - \\ &\quad \frac{1}{4m^2c^2}\vec{\sigma}\cdot\vec{\pi}\left(i\hbar\frac{\partial}{\partial t} - eA^0 - mc^2\right)\vec{\sigma}\cdot\vec{\pi}\,\phi\,.\end{aligned} \tag{2.65}$$

Let us write

$$\left(i\hbar\frac{\partial}{\partial t} - mc^2\right)\phi = \overline{E}\phi = eA^0\phi + X + Y + Z\,. \tag{2.66}$$

Here

$$\begin{aligned}X \;&=\; \frac{1}{2m}(\vec{\sigma}\cdot\vec{\pi})^2\phi \\ &=\; -\left\{\frac{\hbar^2}{2m}\nabla^2 + \frac{e}{2mc}\left(\vec{L} + \hbar\vec{\sigma}\right)\cdot\vec{B}\right\}\phi\,,\end{aligned}$$

in the presence of a weak constant magnetic field. This is the nonrelativistic contribution that we studied in the previous section. In Eq.(2.66) we set

$$Y = -\frac{1}{4m^2c^2}(\vec{\sigma}\cdot\vec{\pi})^2\left(i\hbar\frac{\partial}{\partial t} - eA^0 - mc^2\right)\phi\,,$$

and

$$Z = \frac{1}{4m^2c^2}\vec{\sigma}\cdot\vec{\pi}\left[\vec{\sigma}\cdot\vec{\pi}, \left(i\hbar\frac{\partial}{\partial t} - eA^0\right)\right]\phi\,.$$

Now use Eq.(2.61) and replace χ by the lowest approximation, (2.49):

$$Y = -\frac{1}{8m^3c^2}(\vec{\sigma}\cdot\vec{\pi})^4\phi\,. \tag{2.67}$$

The leading contribution in this correction term is obtained by the replacement

$$(\vec{\sigma}\cdot\vec{\pi})^2 \longrightarrow \pi^2 \longrightarrow p^2\,,$$

so that we find

$$Y \approx -\frac{\hbar^4}{8m^3c^2}\nabla^4\phi\,.$$

This may be called the Einstein term, a kinematical relativistic correction to the nonrelativistic kinetic energy.

To evaluate Z, we observe that

$$\begin{aligned}\left[\vec{\pi}\,,\left(i\hbar\frac{\partial}{\partial t}-eA^0\right)\right]\phi &= \left[\left(-i\hbar\vec{\nabla}-\frac{e}{c}\vec{A}\right),\left(i\hbar\frac{\partial}{\partial t}-eA^0\right)\right]\phi \\ &= ie\hbar\left\{\vec{\nabla}A^0+\frac{1}{c}\frac{\partial\vec{A}}{\partial t}\right\}\phi \\ &= -ie\hbar\vec{E}\,\phi\,. \end{aligned} \tag{2.68}$$

Hence

$$Z=-\frac{ie\hbar}{4m^2c^2}(\vec{\sigma}\cdot\vec{\pi})(\vec{\sigma}\cdot\vec{E})\phi\,.$$

To first order in the fields, we can replace $\vec{\pi}$ here by $\vec{p}$, and since

$$(\vec{\sigma}\cdot\vec{p})(\vec{\sigma}\cdot\vec{E}) = (\vec{p}\cdot\vec{E}) + i\vec{\sigma}\cdot(\vec{p}\wedge\vec{E})\,, \tag{2.69}$$

we have

$$Z=-\frac{e\hbar^2}{4m^2c^2}\left\{\vec{\nabla}\cdot\vec{E}+i\vec{\sigma}\cdot(\vec{\nabla}\wedge\vec{E})\right\}\phi\,.$$

The first item in the parentheses is called the *Darwin term*, while the second is the important *Spin-Orbit term*. To see why the latter has this name, consider the case that the electric field is static and central (as in the H-atom), so that

$$\vec{E} = -\vec{\nabla}A^0 = -\frac{\vec{r}}{r}\frac{\partial A^0}{\partial r}\,. \tag{2.70}$$

Then

$$\begin{aligned} i\hbar\vec{\nabla}\wedge\vec{E} &= \vec{p}\wedge\vec{r}\,\frac{1}{r}\frac{\partial A^0}{\partial r} \\ &= -\frac{1}{r}\frac{\partial A^0}{\partial r}\vec{L}\,, \end{aligned} \tag{2.71}$$

where we have used the fact that $\vec{L}$ and r commute. The term takes on the form

$$\begin{aligned} \text{Spin-Orbit Term} &= \frac{e\hbar}{4m^2c^2}\frac{1}{r}\frac{\partial A^0}{\partial r}\vec{\sigma}\cdot\vec{L}\phi \\ &= \frac{e}{2m^2c^2}\frac{1}{r}\frac{\partial A^0}{\partial r}\vec{S}\cdot\vec{L}\phi, \end{aligned} \tag{2.72}$$

from which the nomenclature is evident.

Proof of Eq.(2.69)
Since

$$\{\sigma_i,\sigma_j\} = 2\delta_{ij} \qquad \text{and} \qquad [\sigma_i,\sigma_j] = 2i\epsilon_{ijk}\sigma_k\,,$$

it follows that

$$\sigma_i\sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k\,.$$

Therefore

$$\begin{aligned} (\vec{\sigma}\cdot\vec{p})(\vec{\sigma}\cdot\vec{E}) &= \sigma_i\sigma_j p_i E_j \\ &= p_iE_i + i\epsilon_{ijk}p_iE_j\sigma_k \\ &= (\vec{p}\cdot\vec{E}) + i\vec{\sigma}\cdot(\vec{p}\wedge\vec{E})\,. \end{aligned}$$

END OF PROOF

The complete equation, with the extra terms, can now be written:

$$\begin{aligned} \overline{E}\phi \;=\; & -\frac{\hbar^2}{2m}\nabla^2\phi + eA^0\phi && \text{[Schrödinger]} \\ & -\frac{e}{2mc}\left(\vec{L}+2\vec{S}\right)\cdot\vec{B}\;\phi && \text{[Pauli]} \\ & -\frac{\hbar^4}{8m^3c^2}\nabla^4\phi && \text{[Einstein]} \\ & +\frac{e}{2m^2c^2}\frac{1}{r}\frac{\partial A^0}{\partial r}\vec{S}\cdot\vec{L}\,\phi && \text{[Spin-Orbit]} \\ & -\frac{e\hbar^2}{4m^2c^2}\vec{\nabla}\cdot\vec{E}\,\phi && \text{[Darwin]} \end{aligned} \tag{2.73}$$

where we have written $\vec{S} = \frac{1}{2}\hbar\vec{\sigma}$. The effect of the spin-orbit term is to remove some of the degeneracy of the energy levels. Since

$$J^2 = \left(\vec{L}+\tfrac{1}{2}\hbar\vec{\sigma}\right)^2 = L^2 + \hbar\vec{\sigma}\cdot\vec{L} + \tfrac{3}{4}\hbar^2\,, \tag{2.74}$$

it follows that the eigenvalue of the operator $\vec{\sigma} \cdot \vec{L}$ (in the approximation in which spin and orbital angular momentum are separately conserved), is simply $[j(j+1)-\ell(\ell+1)-\frac{3}{4}]\hbar$. For an $S^{\frac{1}{2}}$ level, this is zero, but for $P^{\frac{1}{2}}$ it is $-2\hbar$ and for $P^{\frac{3}{2}}$ it is $\hbar$. Hence the spin-orbit perturbation alone would lift the ℓ-degeneracy of the $n = 2$ level of the hydrogen atom. However, the combined effect of the spin-orbit, the Einstein and the Darwin terms together is that finally the two $j = \frac{1}{2}$ states, $S^{\frac{1}{2}}$ and $P^{\frac{1}{2}}$, are degenerate, but the $P^{\frac{3}{2}}$ state is not (of course, there is also the azimuthal degeneracy associated with the quantum number m, which is only split by the magnetic Pauli term). We shall prove in the next chapter that the *exact* energy levels of the Dirac hydrogen atom depend on n and j only, so that states with the same n and j but different ℓ, like $S^{\frac{1}{2}}$ and $P^{\frac{1}{2}}$, are indeed degenerate.

Let us now consider the so-called *anomalous Zeeman effect*, in which the splitting of the energy levels depends on ℓ (as well as m). We rewrite the Pauli term in the form (cf., Eq.(2.59))

$$H_1 = -\frac{e}{2mc}(J_3 + S_3)B\,.$$

For magnetic fields that are not strong compared with those existing in the hydrogen atom itself, we consider a first-order perturbation

$$\begin{aligned} \Delta E_{\ell m_j} &= -\frac{e}{2mc}\langle\psi_{nj\ell m_j}|(J_3 + S_3)B|\psi_{nj\ell m_j}\rangle \\ &= -\frac{e\hbar}{2mc}\,(m_j + \delta_s)\,B\,, \end{aligned} \tag{2.75}$$

where

$$\delta_s = \tfrac{1}{2}\langle\psi_{nj\ell m_j}|\sigma_3|\psi_{nj\ell m_j}\rangle\,.$$

To calculate this matrix element, we use the Clebsch-Gordan expansion. In the case $j = \ell + \frac{1}{2}$, this reads

$$|\psi_{nj\ell m_j}\rangle = aY_{\ell,m_j-\frac{1}{2}}\chi_+ + bY_{\ell,m_j+\frac{1}{2}}\chi_-\,, \tag{2.76}$$

where

$$a = \sqrt{\frac{\ell + m_j + \frac{1}{2}}{2\ell+1}} \qquad b = \sqrt{\frac{\ell - m_j + \frac{1}{2}}{2\ell+1}}\,. \tag{2.77}$$

Here Y is a spherical harmonic, being an orbital angular momentum eigenfunction, in configuration representation, while χ is a spin factor, an eigenvector of

the relevant Pauli matrix. In fact $\sigma_3\chi_\pm = \pm\chi_\pm$, so

$$\delta_s = \tfrac{1}{2}\left(a^2 - b^2\right) = \frac{m_j}{2\ell+1}\,.$$

The other possibility is $j = \ell - \frac{1}{2}$, and then

$$|\psi_{nj\ell m_j}\rangle = bY_{\ell,m_j-\frac{1}{2}}\chi_+ - aY_{\ell,m_j+\frac{1}{2}}\chi_-\,, \tag{2.78}$$

and so in this case

$$\delta_s = \tfrac{1}{2}\left(b^2 - a^2\right) = -\frac{m_j}{2\ell+1}\,.$$

Hence for $j = \ell \pm \frac{1}{2}$, we find

$$\Delta E_{\ell,m_j} = -\frac{em_j\hbar}{2mc}\left(1 \pm \frac{1}{2\ell+1}\right)B\,. \tag{2.79}$$

The order of magnitude of the spin-orbit coupling energy is 10^{-4} eV for the hydrogen atom, which we must multiply by Z^2 for an atom with atomic number Z. On the other hand, the Pauli term contributes about $0.6 \times 10^{-4}\,B$ eV, with B expressed in Tesla (1 Tesla is 10^4 Gauss, which is a large, but experimentally attainable field strength). When B is much less than a Tesla, the spin-orbit coupling is the more important and we have to do with the anomalous Zeeman formula Eq.(2.79). For a field strength of a Tesla or more, the spin-orbit term can be treated as a perturbation of the nonrelativistic energy, in which the orbital angular momentum is regarded in first approximation as a good quantum number. This leads to the 'normal Zeeman effect' of Eq.(2.60). For even stronger fields, one has the quadratic Zeeman effect that arises from the A^2-term in Eq.(2.54). In practice the complicated splitting that occurs gives rise to an effective line-broadening that can be calculated.

2.5 Hyperfine Structure

The Einstein relativistic correction and the spin-orbit term together produce what is called the *fine splitting* of the hydrogen energy levels. As we will see explicitly in the next chapter, the excited levels, $n = 2, 3, \ldots$, all exhibit this fine structure, where the split levels can be labeled by j, the total angular momentum quantum number.

In addition to the fine splitting, there is an even smaller *hyperfine splitting*, caused by the coupling of the the magnetic moment of the electron to that of the nucleus of the hydrogen atom, a proton. From Sec. **3.3** of Volume 1, we

recall that the part of the magnetic moment that arises from the orbital motion of the electron is

$$\vec{\mu}_L = \frac{e}{2mc}\vec{L}\,.$$

The part of the magnetic moment that arises from the spin of the electron is

$$\vec{\mu}_S = \frac{eg}{2mc}\vec{S}\,,$$

where, as we have seen, the Dirac equation gives 2 as the g-factor of the electron. Like the electron, the proton has spin $\frac{1}{2}$, and so it also has a magnetic moment,

$$\vec{\mu}_p = \frac{eg_p}{2m_pc}\vec{S}_p\,,$$

where m_p is the mass of the proton, about 2000 times that of the electron, $\vec{S}_p$ being its spin. The g-factor of the proton is g_p, and it is larger than 2, a consequence of the fact that the proton is not elementary, but is a bound state of three quarks. The experimentally measured value is $g_p = 5.56$.

The source of the hyperfine splitting is the coupling of the magnetic moment of the electron to the magnetic field created by the magnetic moment of the proton. This magnetic field is proportional to the proton's magnetic moment, hence to its spin. In general the magnetic moment of the electron has an orbital component, but we shall restrict our attention to S-states, i.e., $\ell = 0$, for which the electron's magnetic moment comes purely from the spin, being proportional to that spin. The energy of the hyperfine interaction is proportional to $\vec{S}\cdot\vec{S}_p$, so we can write the part of the interaction Hamiltonian that is responsible for the hyperfine splitting as

$$H_{\text{Hpfn}} = \mathcal{F}\,\vec{S}\cdot\vec{S}_p\,,$$

with a coefficient, $\mathcal{F}$, that is independent of the electron and the proton spins. The hyperfine shift is, in first order of perturbation theory, the expectation value of this, in a definite S-wave state of the hydrogen atom, $|n,\ell=0,m=0,s,s_p\rangle$:

$$\langle n00ss_p|\mathcal{F}\,\vec{S}\cdot\vec{S}_p|n00ss_p\rangle = \langle n00|\mathcal{F}|n00\rangle\langle ss_p|\vec{S}\cdot\vec{S}_p|ss_p\rangle\,.$$

Let us write the total angular momentum of the electron and the proton as

$$\vec{\mathcal{S}} = \vec{S} + \vec{S}_p\,.$$

This combines into a singlet, of angular momentum $\mathcal{J}=0$, and a triplet, of angular momentum $\mathcal{J}=1$. Using $\vec{\mathcal{S}}\cdot\vec{\mathcal{S}} = \vec{S}\cdot\vec{S} + 2\vec{S}\cdot\vec{S}_p + \vec{S}_p\cdot\vec{S}_p$, we find

$$2\langle ss_p|\vec{S}\cdot\vec{S}_p|ss_p\rangle = \left[\mathcal{J}(\mathcal{J}+1) - \tfrac{1}{2}(\tfrac{1}{2}+1) - \tfrac{1}{2}(\tfrac{1}{2}+1)\right]\hbar^2 = \left[\mathcal{J}(\mathcal{J}+1) - \tfrac{3}{2}\right]\hbar^2\,.$$

The hyperfine shift is $\frac{1}{4}\hbar^2\langle n00\rangle|\mathcal{F}|n00\rangle$ for the triplet and $-\frac{3}{4}\hbar^2\langle n00\rangle|\mathcal{F}|n00\rangle$ for the singlet. The matrix element of the coefficient, $\mathcal{F}$, can be calculated in terms of the modulus of the S-state electron wave-function at the the origin, i.e., at the location of the proton (Problem 2.7).

Hydrogen in our galaxy, far from any star, is in the ground state so far as the electronic configuration is concerned, but is partially excited by collision, at the ambient temperature of 3.5 degrees Kelvin, into the excited triplet state of the spin-spin combination, from which it decays into the singlet ground state, emitting a photon of frequency 1420 MegaHertz, or equivalently of wavelength 21.1 centimeters. This characteristic 21 cm. line in the spectrum of interstellar hydrogen was discovered in 1951; and it is exceedingly sharp, because of the long lifetime for spontaneous decay from the $\mathcal{S} = 1$ to the $\mathcal{S} = 0$ states, namely about 11 million years. This decay is in fact called 'forbidden', because it does not involve a change of orbital angular momentum, since $\ell = 0$ in both states. Because of the sharpness of the spectral line, small Doppler shifts, due to motion of the emitting hydrogen, can be readily measured. It was in this way that it was established that our galaxy is a rotating spiral of stars. The 21 cm. line is also used in the hydrogen maser, yielding an atomic clock that is accurate to better than one part in 10^{14}.

2.6 Exercises

Problem 1

Show the following:

(1) If $\vec{\nabla} \cdot \vec{B} = 0$, then there exists a vector field, $\vec{A}$, such that $\vec{B} = \vec{\nabla} \wedge \vec{A}$.
(2) If $\vec{\nabla} \wedge \vec{C} = 0$, then there exists a scalar field, Φ, such that $\vec{C} = \vec{\nabla}\Phi$.

Problem 2

The classical Lagrangian for a point charge in an electromagnetic field is

$$L = -mc^2\sqrt{1 - \dot{x}^i\dot{x}^i/c^2} - qA^0 + q\dot{x}^iA^i/c\,,$$

where m is the mass of the particle, q is its electric charge, and A^0 and $\vec{A}$ are respectively the scalar and vector potentials. Show that the Euler-Lagrange equation leads to the Lorentz force

$$\vec{F} = q\vec{E} + \frac{q}{c}\vec{v} \wedge \vec{B}\,.$$

How can one demonstrate the Lorentz invariance of electric charge?

Problem 3

Show that any solution of the Dirac equation with minimal coupling,

$$\left(i\gamma^\mu\partial_\mu - \frac{e}{\hbar c}\gamma^\mu A_\mu - \frac{mc}{\hbar}\right)\psi = 0\,,$$

also satisfies

$$\left\{\left(i\partial - \frac{e}{\hbar c}A\right)^2 - \frac{e}{2\hbar c}\sigma^{\mu\nu}F_{\mu\nu} - \frac{m^2c^2}{\hbar^2}\right\}\psi = 0\,,$$

where $\sigma^{\mu\nu} = \frac{1}{2}i[\gamma^\mu, \gamma^\nu]$. Show further that this can be rewritten

$$\left\{\left(i\partial - \frac{e}{\hbar c}A\right)^2 - \frac{ie}{\hbar c}\gamma^0\vec{\gamma}\cdot\vec{E} + \frac{2e}{\hbar^2 c}\vec{S}\cdot\vec{B} - \frac{m^2c^2}{\hbar^2}\right\}\psi = 0\,,$$

where $\vec{S} = \frac{1}{2}\hbar\gamma_5\gamma^0\vec{\gamma}$ is the spin matrix.

Problem 4

The Dirac equation with abnormal coupling is

$$\left(i\gamma^\mu\hat{\mathcal{D}}_\mu - \frac{mc}{\hbar}\right)\psi = 0\,,$$

with the abnormal covariant derivative

$$\hat{\mathcal{D}}_\mu = \partial_\mu + \frac{ie}{\hbar c}A_\mu - \frac{e}{4mc^2}K\gamma^\nu F_{\mu\nu}\,,$$

where $F_{\mu\nu}$ is the electromagnetic field tensor. Obtain the Pauli equation with abnormal coupling and calculate the electric and magnetic moments.

Problem 5

The spin-orbit contribution to the relativistic hydrogen atom Hamiltonian is

$$H_{LS} = \frac{e}{2m^2c^2}\frac{1}{r}\frac{\partial A^0}{\partial r}\vec{S}\cdot\vec{L}\,,$$

where $A^0 = -e/r$ is the Coulomb scalar potential (the Gaussian normalization has been used). Show that the expectation value of this contribution in the $\{n, j, \ell\}$ state of the hydrogen atom is given by

$$\langle H_{LS}\rangle_{nj\ell} = \frac{\alpha^4 mc^2}{2n^3}\frac{j-\ell}{(j+\frac{1}{2})(\ell+\frac{1}{2})}\,,$$

where $\alpha = e^2/(\hbar c)$ is the fine-structure constant.

Problem 6

Calculate the energy shift of the $\{n, \ell, m_\ell, s\}$ level of the nonrelativistic hydrogen atom, to first order in perturbation theory, due to the Einstein correction term ($\propto p^4$). Calculate the shift due to the sum of the Einstein and the $L \cdot S$ coupling term. Why does this sum depend only on n and j, and not on ℓ and s separately?

Problem 7

Calculate the expectation value of the coefficient, $\mathcal{F}$, in the hyperfine interaction,

$$H_{\text{Hpfn}} = \mathcal{F}\, \vec{S} \cdot \vec{S}_P \,,$$

in the ground state of hydrogen.

Problem 8

The totally antisymmetric tensor, $\varepsilon^{\mu\nu\rho\sigma}$, is defined to take the values ± 1 if $\{\mu\nu\rho\sigma\}$ is a cyclic/anticyclic permutation of $\{0, 1, 2, 3\}$, and zero if two or more of the indices are equal. Show that this quantity indeed transforms as a fourth-order Lorentz tensor. Show that the metric, $g^{\mu\nu}$, transforms as a second-order Lorentz tensor. Compute the following scalar functions of the electromagnetic field tensor: $g_{\mu\rho} g_{\nu\sigma} F^{\mu\nu} F^{\rho\sigma}$ and $\varepsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma}$. Show thereby that $\vec{E} \cdot \vec{B}$ and $\vec{E}^{\,2} - \vec{B}^{\,2}$ are Lorentz invariant.

Problem 9

The velocity operator for a free Dirac particle in the Heisenberg picture is given by $\vec{v}\,(t) = i[H, \vec{x}\,]/\hbar$.

(1) Show that the only eigenvalues of $v_j(t)$ are $\pm c$ for all t. How can this result be justified by the Heisenberg uncertainty principle?

(2) Solve the Heisenberg equations of motion to obtain the position of the particle in the form $\vec{x}\,(t) = \vec{x}\,(0) + c^2 \vec{p}\, t H^{-1} + \Xi\, e^{2iHt/\hbar}$, and obtain an expression for the operator Ξ. What is the role of contributions from negative energy states to the last term? What is the average value of $\vec{v}\,(t)$ over a small interval of time?

(3) How do the Dirac γ-matrices depend on time? Show that, at equal times, their anticommutation relations remain valid.

Problem 10

Consider the relativistic corrections to the Schrödinger energy levels of the electron-positron bound states. Compare the relative magnitudes of corrections for positronium, as compared with those for the hydrogen atom. Examine the changes due to the reduced mass, the Einstein term, the spin-orbit interaction, and the hyperfine splitting.

Chapter 3

Dirac Hydrogen Atom

3.1 Scalar Central Potential

The nonrelativistic Schrödinger Hamiltonian, with a central potential, can be written in operator form as follows:

$$H = \frac{p^2}{2m} + V(q^2)\,.$$

The orbital angular momentum,

$$\vec{L} = \vec{q} \wedge \vec{p}\,,$$

commutes with this Hamiltonian.

Proof that $[\vec{L}\,, H] = 0$**:**

$$\begin{aligned} [L_1, p^2] &= [q_2 p_3 - q_3 p_2, p_2^2 + p_3^2] \\ &= [q_2, p_2^2] p_3 - [q_3, p_3^2] p_2 \\ &= 2i\hbar p_2 p_3 - 2i\hbar p_3 p_2 = 0\,, \end{aligned}$$

and similarly for the other components. A similar argument shows that the angular momentum commutes with q^2. Induction can be used in the standard way to show that the angular momentum therefore commutes with any positive integral power of q^2. It commutes also with $\cos(tq^2)$, since the cosine can be expanded in a convergent power series. Therefore it commutes with any Fourier transformable $V(q^2)$:

$$V(q^2) = \int_0^\infty dt\, v(t) \cos(tq^2)\,.$$

If $v(t) \propto 1/\sqrt{t}$ then $V(q^2) \propto 1/\sqrt{q^2}$, i.e. V is the Coulomb potential.

A consequence of the fact that the orbital angular momentum commutes with the Hamiltonian is that this angular momentum is constant in time (in this theory). Moreover, if $|\psi\rangle$ is an eigenvector of the Hamiltonian belonging to the energy eigenvalue E, then

$$H\vec{L}\,|\psi\rangle = \vec{L}\,H|\psi\rangle = E\vec{L}\,|\psi\rangle\,.$$

Hence, if $|\psi\rangle$ is also an eigenvector of L^2, then $L_\pm|\psi\rangle$ is an eigenvector of the Hamiltonian belonging to the same energy E (with the same quantum number, ℓ, but with a different azimuthal quantum number, m_ℓ, not to be confused with the mass, m). This means that the energy eigenvalues in the nonrelativistic hydrogen atom cannot depend on m_ℓ. According to the Schrödinger equation for the hydrogen atom, the energy levels may be written

$$E(n) = -\frac{\alpha^2}{2n^2}mc^2 \tag{3.1}$$

where the *fine structure constant* is defined by

$$\alpha = \frac{e^2}{\hbar c}\,,$$

(see Chapters 5 and 9 of Volume 1). This formula is based on the use of the Coulomb potential, $V(r) = -e^2/r$, which means that Gaussian, rather than Heaviside-Lorentz units were employed. We will retain this usage in the present chapter, since it shortens many of the equations by removing a factor of 4π in denominators; but in the next chapter, and for the rest of the book, we propose to revert to the Heaviside-Lorentz convention. For a given value of the principal quantum number n, the energy is indeed independent of the azimuthal quantum number m_ℓ. In fact, it is independent of ℓ also, but this is an 'accident' of the Coulomb potential. For a modified Coulomb potential, for example

$$\frac{\theta(r-r_0)}{r} \qquad \text{or} \qquad \frac{\theta(r_1-r)}{r}\,,$$

the energy eigenvalues have ℓ-dependence, but no m_ℓ-dependence. The essential point is that all the states with the same angular momentum, i.e., the same value of ℓ, must have the same energy if the potential is spherically symmetric. This is physically reasonable: there is no preferred direction in space, so the choice of the quantization axis around which the azimuthal degree of freedom operates cannot have physical consequences; in particular it cannot affect the energy.

One reason for this lengthy review of the nonrelativistic case is to contrast it with the Dirac equation in a central potential like that of the hydrogen atom.

When there is no vector potential, and the scalar potential is central, the Dirac Hamiltonian Eq.(2.39) reduces to

$$H = c\gamma^0\vec{\gamma}\cdot\vec{p} + mc^2\gamma^0 + eA^0(q^2)\,. \tag{3.2}$$

What is the commutator of the orbital angular momentum with *this* Hamiltonian? The only term in Eq.(3.2) that fails to commute is the first one, and

$$\begin{aligned}\gamma^i[L_1,p_i] &= \gamma^i\{[q_2,p_i]p_3 - [q_3,p_i]p_2\}\\ &= i\hbar(\gamma^2p_3 - \gamma^3p_2)\\ &= i\hbar(\vec{\gamma}\wedge\vec{p})_1\,,\end{aligned} \tag{3.3}$$

and similarly for the other components. Hence

$$[\vec{L},H] = i\hbar c\gamma^0\vec{\gamma}\wedge\vec{p} \tag{3.4}$$

so that in the Dirac theory the orbital angular momentum and the Hamiltonian cannot be simultaneously diagonalized. Consequently, in an eigenstate of the Hamiltonian, a component of the orbital angular momentum does not, in general, have a well-defined value.

Consider now the following quantity:

$$\vec{S} = \tfrac{1}{2}\hbar\gamma_5\gamma^0\vec{\gamma}\,,$$

where

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}.$$

Explicitly,

$$\vec{S} = \tfrac{1}{2}\hbar\begin{pmatrix}\vec{\sigma} & 0\\ 0 & \vec{\sigma}\end{pmatrix}. \tag{3.5}$$

Clearly $\vec{S}$ is the spin matrix that was introduced in the discussion of the Pauli equation. The Pauli matrices have simply been replicated along the diagonal to produce 4×4 matrices.

We are interested in the commutator of $\vec{S}$ with the Hamiltonian Eq.(3.2). Clearly $\vec{S}$ commutes with γ^0, since that is diagonal, and with the scalar potential, since that is a multiple of the unit matrix. The only term in Eq.(3.2) that fails to commute is the first one, and since

$$\gamma^0\vec{\gamma} = \begin{pmatrix}0 & \vec{\sigma}\\ \vec{\sigma} & 0\end{pmatrix}, \tag{3.6}$$

we see that

$$
\begin{aligned}
[S_i, H] &= \tfrac{1}{2}\hbar c \left[\begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix}, \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix} \right] p_j \\
&= \tfrac{1}{2}\hbar c p_j \begin{pmatrix} 0 & [\sigma_i, \sigma_j] \\ [\sigma_i, \sigma_j] & 0 \end{pmatrix} \\
&= i\hbar c p_j \epsilon_{ijk} \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix} \\
&= i\hbar c p_j \epsilon_{ijk} \gamma^0 \gamma^k ; \qquad (3.7)
\end{aligned}
$$

so we obtain

$$
[\vec{S}, H] = -i\hbar c \gamma^0 \vec{\gamma} \wedge \vec{p} \,. \qquad (3.8)
$$

By a remarkable coincidence, this just cancels the right side of Eq.(3.4), and so the operator

$$
\vec{J} = \vec{L} + \vec{S} \qquad (3.9)
$$

does commute with the Hamiltonian Eq.(3.2). We interpret $\vec{J}$ as the total angular momentum operator. This is conserved, whereas the spin and orbital angular momenta are separately not conserved—except in the nonrelativistic limit.

Although we have used an explicit representation to prove Eq.(3.8) and other relations involving γ-matrices, by multiplying both sides from the left by a 4×4 unitary matrix, M, and from the right by its inverse, $M^\dagger$, we have actually effected a proof for an arbitrary representation $\tilde{\gamma}^\mu$, cf. Eq.(1.71). An alternative to the *Dirac representation* of Eq.(1.67) is the *chiral representation:*

$$
\begin{aligned}
\gamma^0_{\text{Chi}} &= \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \\
\vec{\gamma}_{\text{Chi}} &= \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix} . \qquad (3.10)
\end{aligned}
$$

3.2 Solution of Dirac Equation

The Dirac equation for the electron in a hydrogen atom is

$$
H\psi(x) = i\hbar \frac{\partial}{\partial t}\psi(x) = \left\{ -ic\hbar\gamma^0 \vec{\gamma} \cdot \vec{\nabla} + mc^2\gamma^0 - \frac{e^2}{r} \right\} \psi(x) \,. \qquad (3.11)
$$

We shall solve for the eigenstates and eigenvalues without approximation. The Hamiltonian appearing in Eq.(3.11) can be rewritten

$$H = c\gamma_5\sigma_r\left(p_r + \frac{ih}{r}\gamma^0\omega\right) + mc^2\gamma^0 - \frac{e^2}{r}, \tag{3.12}$$

where

$$p_r = -\frac{i\hbar}{r}(\vec{r}\cdot\vec{\nabla}+1) = -i\hbar\left(\frac{\partial}{\partial r}+\frac{1}{r}\right) \tag{3.13}$$

$$\sigma_r = \frac{2}{\hbar}\frac{\vec{S}\cdot\vec{r}}{r} = \begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{r}}{r} & 0 \\ 0 & \frac{\vec{\sigma}\cdot\vec{r}}{r} \end{pmatrix} \tag{3.14}$$

$$\omega = \gamma^0\left[1+\frac{2}{\hbar^2}\vec{S}\cdot\vec{L}\right]. \tag{3.15}$$

Proof of Eq.(3.12)

We know that

$$\gamma^0\vec{\gamma}\cdot\vec{\nabla} = \begin{pmatrix} 0 & \vec{\sigma}\cdot\vec{\nabla} \\ \vec{\sigma}\cdot\vec{\nabla} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \vec{\sigma}\cdot\vec{\nabla} & 0 \\ 0 & \vec{\sigma}\cdot\vec{\nabla} \end{pmatrix}. \tag{3.16}$$

Moreover

$$\begin{aligned}\vec{\sigma}\cdot\vec{\nabla} &= \frac{1}{r^2}(\vec{\sigma}\cdot\vec{r})(\vec{\sigma}\cdot\vec{r})(\vec{\sigma}\cdot\vec{\nabla}) \\ &= \frac{1}{r^2}(\vec{\sigma}\cdot\vec{r})\left\{(\vec{r}\cdot\vec{\nabla})+i\vec{\sigma}\cdot(\vec{r}\wedge\vec{\nabla})\right\} \\ &= \frac{1}{r^2}(\vec{\sigma}\cdot\vec{r})\left\{(\vec{r}\cdot\vec{\nabla})-\frac{1}{\hbar}\vec{\sigma}\cdot\vec{L}\right\}.\end{aligned} \tag{3.17}$$

Thus, referring to the definitions (3.13)-(3.15)

$$\begin{aligned}-i\hbar\gamma^0\vec{\gamma}\cdot\vec{\nabla} &= -\frac{2i}{r^2}\gamma_5(\vec{S}\cdot\vec{r})\left\{\vec{r}\cdot\vec{\nabla}-\frac{2}{\hbar^2}\vec{S}\cdot\vec{L}\right\} \\ &= \gamma_5\sigma_r\left\{-\frac{i\hbar}{r}\left(\vec{r}\cdot\vec{\nabla}+1\right)+\frac{i\hbar}{r}\gamma^0\gamma^0\left(1+\frac{2}{\hbar^2}\vec{S}\cdot\vec{L}\right)\right\}. \\ &= \gamma_5\sigma_r\left(p_r+\frac{ih}{r}\gamma^0\omega\right).\end{aligned}$$

END OF PROOF

We shall next prove that ω commutes with the Hamiltonian, Eq.(3.12), and so is a constant of motion.

Proof that ω commutes with H

We first establish two helpful lemmata:

Lemma 1

$$[p_i, L_j] = i\hbar\epsilon_{ijk}p_k \tag{3.18}$$

Proof

$$\begin{aligned} [p_i, L_j] &= \epsilon_{jnk}(p_i q_n p_k - q_n p_k p_i) \\ &= \epsilon_{jnk}[p_i, q_n]p_k \\ &= -i\hbar\epsilon_{jik}p_k = i\hbar\epsilon_{ijk}p_k\,. \end{aligned} \tag{3.19}$$

Lemma 2

$$S_i = \tfrac{1}{4}i\hbar\epsilon_{ijk}\gamma^j\gamma^k \tag{3.20}$$

Proof

$$\begin{aligned} \gamma^1\gamma^2 &= \begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix} \\ &= \begin{pmatrix} -\sigma_1\sigma_2 & 0 \\ 0 & -\sigma_1\sigma_2 \end{pmatrix} \\ &= -i\begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}, \end{aligned} \tag{3.21}$$

and similarly for cyclic permutations of {1,2,3}, so we have

$$\gamma^j\gamma^k = -\frac{2i}{\hbar}\epsilon_{ijk}S_i - \delta_{jk}\,, \tag{3.22}$$

where we have introduced the spin matrix

$$\vec{S} = \tfrac{1}{2}\hbar\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}. \tag{3.23}$$

From Eq.(3.22) we obtain

$$\begin{aligned} \epsilon_{ijk}\,\gamma^j\gamma^k &= -\frac{2i}{\hbar}\epsilon_{ijk}\,\epsilon_{jkl}\,\gamma^j\gamma^l S_l \\ &= -\frac{2i}{\hbar}(\delta_{jj}\delta_{il} - \delta_{jl}\delta_{ij})S_l \\ &= -\frac{4i}{\hbar}S_i\,. \end{aligned} \tag{3.24}$$

END OF LEMMATA

Theorem

$$[H,\omega]=0\,. \tag{3.25}$$

Proof

$$[H,\omega]=[H,\gamma^0]+\frac{2}{\hbar^2}[H,\gamma^0\vec{S}\cdot\vec{L}\,]\,. \tag{3.26}$$

Now

$$\begin{aligned}[H,\gamma^0] &= c\gamma^0\vec{\gamma}\cdot\vec{p}\gamma^0-c\vec{\gamma}\cdot\vec{p}\\ &= -2c\vec{\gamma}\cdot\vec{p}\,,\end{aligned}$$

while

$$[H,\gamma^0\vec{S}\cdot\vec{L}\,]=c[\gamma^0\vec{\gamma}\cdot\vec{p},\gamma^0\vec{S}\cdot\vec{L}\,], \tag{3.27}$$

since γ^0 commutes with $\vec{S}$ and $\vec{L}$ commutes with $A^0(q^2)$. Though γ^0 commutes with $\vec{S}$ it anticommutes with $\vec{\gamma}$, so

$$\begin{aligned}[H,\gamma^0\vec{S}\cdot\vec{L}\,] &= -c\left\{\vec{\gamma}\cdot\vec{p}\vec{S}\cdot\vec{L}+\vec{S}\cdot\vec{L}\vec{\gamma}\cdot\vec{p}\right\}\\ &= -\tfrac{1}{2}c\hbar\gamma_5\gamma^0(\gamma^i\gamma^jp_iL_j+\gamma^j\gamma^iL_jp_i)\\ &= -\tfrac{1}{2}c\hbar\gamma_5\gamma^0\left\{(\gamma^j\gamma^i+\gamma^i\gamma^j)L_jp_i+\gamma^i\gamma^j[p_i,L_j]\right\}\\ &= -\tfrac{1}{2}c\hbar\gamma_5\gamma^0\left\{2g^{ij}L_jp_i+i\hbar\epsilon_{ijk}\gamma^i\gamma^jp_k\right\}\,,\end{aligned} \tag{3.28}$$

where Lemma 1 has been used to get the last line. Now

$$L_ip_i=\epsilon_{ijk}q_jp_kp_i=0\,; \tag{3.29}$$

and so by Lemma 2,

$$\begin{aligned}[H,\gamma^0\vec{S}\cdot\vec{L}\,] &= -2c\hbar\gamma_5\gamma^0S_kp_k\\ &= -c\hbar^2\gamma_5\gamma^0\gamma_5\gamma^0\gamma^kp_k\\ &= c\hbar^2\vec{\gamma}\cdot\vec{p}\,.\end{aligned} \tag{3.30}$$

Hence

$$[H,\omega]=-2c\vec{\gamma}\cdot\vec{p}+\frac{2}{\hbar^2}c\hbar^2\vec{\gamma}\cdot\vec{p}=0\,. \tag{3.31}$$

END OF PROOF

Since H and ω commute, we can diagonalize them simultaneously, obtaining from the Hamiltonian (3.22) the equation

$$E\psi = \left\{ c\gamma_5\sigma_r \left(p_r + \frac{ih}{r}\gamma^0 w \right) + mc^2\gamma^0 - \frac{e^2}{r} \right\} \psi \,, \tag{3.32}$$

where H and ω have been replaced by their eigenvalues, respectively E and w. Using the fact that $\gamma_5^2 = 1$, we rewrite Eq.(3.32) as follows:

$$\gamma_5 (E + \frac{e^2}{r} - mc^2\gamma^0)\psi = -i\hbar c\sigma_r \left(\frac{\partial}{\partial r} + \frac{1}{r} - \frac{w}{r}\gamma^0 \right) \psi \,. \tag{3.33}$$

In terms of two-spinors,

$$\psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix} \,, \tag{3.34}$$

this equation can be written

$$\begin{aligned} &\left\{ (E + \frac{e^2}{r}) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - mc^2 \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\} \begin{pmatrix} \phi \\ \chi \end{pmatrix} \\ &\qquad = -i\hbar c\sigma_r \left\{ \frac{\partial}{\partial r} + \frac{1}{r} - \frac{w}{r} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\} \begin{pmatrix} \phi \\ \chi \end{pmatrix} . \end{aligned} \tag{3.35}$$

Now

$$\frac{\vec{\sigma} \cdot \vec{r}}{r} = \begin{pmatrix} \cos\theta & \sin\theta\, \mathrm{e}^{-i\phi} \\ \sin\theta\, \mathrm{e}^{i\phi} & -\cos\theta \end{pmatrix} ; \tag{3.36}$$

and so σ_r depends only on the angles and thus commutes with the differential operator $\partial/\partial r$. Moreover its square is the unit matrix. Hence

$$\begin{aligned} (E + \frac{e^2}{r} + mc^2)\frac{\vec{\sigma} \cdot \vec{r}}{r}\chi &= -i\hbar c \left(\frac{\partial}{\partial r} + \frac{1-w}{r} \right) \phi \\ (E + \frac{e^2}{r} - mc^2)\phi &= -i\hbar c \left(\frac{\partial}{\partial r} + \frac{1+w}{r} \right) \frac{\vec{\sigma} \cdot \vec{r}}{r}\chi \,. \end{aligned} \tag{3.37}$$

Define

$$F = r\phi \qquad\qquad G = -i\vec{\sigma} \cdot \vec{r}\chi \,. \tag{3.38}$$

The equations reduce to the following:

$$\begin{aligned} (E + \frac{e^2}{r} + mc^2)G &= -\hbar c \left(\frac{\partial}{\partial r} - \frac{w}{r} \right) F \\ (E + \frac{e^2}{r} - mc^2)F &= \hbar c \left(\frac{\partial}{\partial r} + \frac{w}{r} \right) G \,. \end{aligned} \tag{3.39}$$

We introduce the scaled radial variable,

$$\rho = \frac{2r}{\hbar c}\sqrt{m^2c^4 - E^2}\,, \tag{3.40}$$

in terms of which we find

$$\begin{aligned}\left(\frac{\partial}{\partial\rho} - \frac{w}{\rho}\right)F + \left(\lambda_+ + \frac{\alpha}{\rho}\right)G = 0\\ \left(\frac{\partial}{\partial\rho} + \frac{w}{\rho}\right)G + \left(\lambda_- - \frac{\alpha}{\rho}\right)F = 0\,,\end{aligned} \tag{3.41}$$

where

$$\lambda_\pm = \frac{1}{2}\sqrt{\frac{mc^2 \pm E}{mc^2 \mp E}}\,, \tag{3.42}$$

and where α is the fine-structure constant,

$$\alpha = \frac{e^2}{\hbar c} \approx \frac{1}{137}\,. \tag{3.43}$$

Insert the following expansions in Eq.(3.41):

$$\begin{aligned}F &= \exp(-\frac{\rho}{2})\rho^s \sum_{\nu=0}^{\infty} a_\nu \rho^\nu\\ G &= \exp(-\frac{\rho}{2})\rho^s \sum_{\nu=0}^{\infty} b_\nu \rho^\nu\,.\end{aligned} \tag{3.44}$$

By equating coefficients of the lowest power, we find

$$\begin{aligned}sa_0 - wa_0 + \alpha b_0 = 0\\ sb_0 + wb_0 - \alpha a_0 = 0\,,\end{aligned} \tag{3.45}$$

which implies

$$\frac{a_0}{b_0} = -\frac{\alpha}{s-w} = \frac{s+w}{\alpha}\,, \tag{3.46}$$

so that

$$s = \sqrt{w^2 - \alpha^2}\,. \tag{3.47}$$

For $\nu \geq 0$, we find the recursion relations

$$(\nu + s + 1)a_{\nu+1} - \tfrac{1}{2}a_\nu - wa_{\nu+1} + \lambda_+ b_\nu + \alpha b_{\nu+1} = 0 \tag{3.48}$$

$$(\nu + s + 1)b_{\nu+1} - \tfrac{1}{2}b_\nu + wb_{\nu+1} + \lambda_- a_\nu - \alpha a_{\nu+1} = 0\,. \tag{3.49}$$

We can use Eq.(3.48)-(3.49) to determine a_n and b_n, to each order n, in terms of arbitrary a_0 and b_0, and since these zeroth order coefficients are related by Eq.(3.46), it follows that only one arbitrary (normalization) constant remains. The recursion relations may be decoupled by multiplying Eq.(3.49) by $2\lambda_+$ and using the fact that $\lambda_+\lambda_- = \frac{1}{4}$:

$$2\lambda_+(\nu + s + w + 1)b_{\nu+1} - \lambda_+ b_\nu + \tfrac{1}{2}a_\nu - 2\alpha\lambda_+ a_{\nu+1} = 0\,. \tag{3.50}$$

Add this to Eq.(3.48):

$$(\nu + s - w + 1)a_{\nu+1} + 2\lambda_+(\nu + s + w + 1)b_{\nu+1} - \alpha(2\lambda_+ a_{\nu+1} - b_{\nu+1}) = 0\,. \tag{3.51}$$

Replace $\nu + 1$ by ν, obtaining

$$a_\nu = -\frac{1}{2\lambda_-}\frac{\nu + s + w + 2\alpha\lambda_-}{\nu + s - w - 2\alpha\lambda_+}\, b_\nu\,, \tag{3.52}$$

and inject this into Eq.(3.49):

$$2\left[\nu + 1 + s + w + \frac{\alpha}{2\lambda_-}\frac{\nu + s + w + 2\alpha\lambda_-}{\nu + s - w - 2\alpha\lambda_+}\right] b_{\nu+1} \quad = \quad \left[\frac{\nu + s + w + 2\alpha\lambda_-}{\nu + s - w - 2\alpha\lambda_+} + 1\right] b_\nu\,. \tag{3.53}$$

Note that, for $\nu = 0$, Eq.(3.52) is consistent with Eq.(3.46). Relations (3.52)-(3.53) constitute the decoupled recursion formulas. As in the nonrelativistic case, one can show that, if the series does not terminate, it behaves like e^ρ, which means that F and G behave like $e^{\rho/2}$, up to powers of ρ, which is incompatible with the probability interpretation. Hence the series must terminate.

There must therefore be an integer, ν, positive or zero, such that

$$\frac{\nu + s + w + 2\alpha\lambda_-}{\nu + s - w - 2\alpha\lambda_+} + 1 = 0\,,$$

for then $b_{\nu+1}$ and all the higher terms vanish, and, because of Eq.(3.52), the same is true for the a_ν. This condition reduces to

$$\frac{\nu + s}{\alpha} = \lambda_+ - \lambda_- = \frac{E}{\sqrt{m^2c^4 - E^2}}\,. \tag{3.54}$$

Square this, solve for E^2, and then take the square root of the answer:

$$E = mc^2\left\{1 + \frac{\alpha^2}{\left(\nu + \sqrt{w^2 - \alpha^2}\right)^2}\right\}^{-\frac{1}{2}}\,, \tag{3.55}$$

where Eq.(3.47) has been used to eliminate s.

In order to relate the numbers ν and w to known quantities, we expand Eq.(3.55) to lowest order in the small quantity α^2:

$$E - mc^2 = -\frac{mc^2\alpha^2}{2(\nu + |w|)^2} + O(\alpha^4)\,. \tag{3.56}$$

We recognize this as the nonrelativistic formula (3.1), on condition that

$$n = \nu + |w| \tag{3.57}$$

is the principal quantum number. In the nonrelativistic theory, ℓ is a good quantum number that has one of the values $0, 1, 2, ..., n-1$. Since $j = \ell \pm \frac{1}{2}$, unless $\ell = 0$, in which case $j = \frac{1}{2}$, it follows that j can take on the values $\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, ..., n - \frac{1}{2}$. Since $\vec{J}$ commutes with the Hamiltonian in the Dirac theory, but $\vec{L}$ does not, it is in terms of n and j that we should seek to write the energy. In fact, from the definition (3.15),

$$\omega^2 = 1 + \frac{4}{\hbar^2}\vec{S}\cdot\vec{L} + \frac{4}{\hbar^4}\left(\vec{S}\cdot\vec{L}\right)^2\,. \tag{3.58}$$

Using an identity between the Pauli matrices,

$$\begin{aligned}(\vec{\sigma}\cdot\vec{L})(\vec{\sigma}\cdot\vec{L}) &= \vec{L}\cdot\vec{L} + i\vec{\sigma}\cdot(\vec{L}\wedge\vec{L}) \\ &= \vec{L}\cdot\vec{L} + i\vec{\sigma}\cdot(i\hbar\vec{L}) \\ &= L^2 - \hbar\vec{\sigma}\cdot\vec{L}\,,\end{aligned} \tag{3.59}$$

we find that

$$\frac{4}{\hbar^2}\left(\vec{S}\cdot\vec{L}\right)^2 = L^2 - 2\vec{S}\cdot\vec{L}\,. \tag{3.60}$$

We have

$$\begin{aligned}\omega^2 &= 1 + \frac{4}{\hbar^2}\vec{S}\cdot\vec{L} + \frac{1}{\hbar^2}\left(L^2 - 2\vec{S}\cdot\vec{L}\right) \\ &= \frac{1}{\hbar^2}\left(\vec{L}+\vec{S}\right)^2 + 1 - \frac{1}{\hbar^2}S^2 = \frac{1}{\hbar^2}J^2 + \frac{1}{4}\,.\end{aligned} \tag{3.61}$$

The eigenvalues of ω are therefore given by

$$|w| = \sqrt{j(j+1) + \tfrac{1}{4}} = j + \tfrac{1}{2}\,. \tag{3.62}$$

Together with Eq.(3.57), this shows that

$$\nu = n - j - \tfrac{1}{2}\,. \tag{3.63}$$

With the identifications (3.62) and (3.63) of $|w|$ and ν, the solution of the bound-state problem is complete.

3.3 Bound State Formula

Label the energy level expression (3.55) with the quantum numbers n and j:

$$E(n,j) = mc^2 \left\{ 1 + \frac{\alpha^2}{\left[n - j - \frac{1}{2} + \sqrt{(j+\frac{1}{2})^2 - \alpha^2} \right]^2} \right\}^{-\frac{1}{2}}, \tag{3.64}$$

where the identifications (3.62) and (3.63) have been used. The principal quantum number, n, is a positive integer, and j can take on the values $\frac{1}{2}, \frac{3}{2}, ..., n - \frac{1}{2}$. Since the fine-structure constant is so small,

$$\alpha = \frac{e^2}{\hbar c} \approx \frac{1}{137}, \qquad \alpha^2 \approx 5.328 \times 10^{-5},$$

it makes sense to expand this exact result in a power series in α. The result is

$$\frac{E(n,j)}{mc^2} = 1 - \frac{\alpha^2}{2n^2} - \frac{\alpha^4}{2n^4}\left(p - \tfrac{3}{4}\right) - \frac{\alpha^6}{8n^6}\left(p^3 + 3p^2 - 6p + \tfrac{5}{2}\right) + \ldots,$$

where $p = n/(j + \frac{1}{2})$. To order α^4, the first three energy levels of the hydrogen atom, according to the Dirac equation, are as follows:

$$\begin{aligned} \frac{E(1,\frac{1}{2})}{mc^2} &= 1 - \frac{\alpha^2}{2} - \frac{\alpha^4}{8} \\ \frac{E(2,\frac{1}{2})}{mc^2} &= 1 - \frac{\alpha^2}{8} - \frac{5\alpha^4}{128} \\ \frac{E(2,\frac{3}{2})}{mc^2} &= 1 - \frac{\alpha^2}{8} - \frac{\alpha^4}{128}. \end{aligned} \tag{3.65}$$

The fine-structure splitting between the second and third of these levels is

$$\frac{E(2,\frac{3}{2}) - E(2,\frac{1}{2})}{mc^2} = \frac{\alpha^4}{32},$$

which agrees with experiment, to this level of accuracy.

When $j = \frac{1}{2}$, the exact Dirac level can be written

$$\frac{E(n,\frac{1}{2})}{mc^2} = \frac{n - 1 + \sqrt{1-\alpha^2}}{\sqrt{n^2 - 2n + 2 + 2(n-1)\sqrt{1-\alpha^2}}}, \tag{3.66}$$

and the ground-state energy, corresponding to $n = 1$ and $j = \frac{1}{2}$ is

$$\frac{E(1,\frac{1}{2})}{mc^2} = \sqrt{1-\alpha^2}. \tag{3.67}$$

Clearly something strange would happen if the fine-structure constant were greater than unity, for then the above $j = \frac{1}{2}$ energies would become complex! This difficulty is not purely hypothetical, for in a hydrogen-like atom of atomic number Z (i.e. there are Z protons in the nucleus but only one electron outside), the Dirac formula would be expected to apply, but with α replaced by $Z\alpha$. So if $Z > 137$, there would be problems. Admittedly this is still not very interesting, for in the first place it would be difficult to ionize an atom of atomic number greater than 137 down to just one electron, and in the second place the nucleus of such an atom is wildly unstable by fission. Nevertheless the tantalizing question arises: what if? Does the Dirac equation simply cease to be valid when the effective fine-structure constant is strong (i.e. larger than unity)?

It is believed that the system protects itself from this disastrous 'collapse into the center' by a phase-change: as one increases the value of α, the ground-state energy decreases to zero at $\alpha = 1$ (see Eq.(3.67)). This may be related to what happens in the strong interactions of quarks, leading to their confinement.

The fact that the $j = \frac{1}{2}$ S and P waves (i.e. $\ell = 0$ and $\ell = 1$) are still degenerate in the Dirac theory is a consequence of the fact that the Dirac Hamiltonian commutes with the total angular momentum operator. Experimentally there is actually a tiny splitting of these levels: the famous Lamb shift. This is explained by quantum electrodynamics (QED): the theory of interacting electrons and photons, both expressed as *quantum fields.* The Lamb shift energy has been measured, in practice by detecting the radio photon that is emitted when the electron shifts from the S to the slightly lower P state. The comparison with the results of QED calculations of this frequency for hydrogen, deuterium and singly ionized helium is as follows:

	Experiment	**QED**
H	1057.8 ± 0.1	1058.0 ± 0.2
D	1059.0 ± 0.1	1059.4 ± 0.2
He^+	14040 ± 5	14057 ± 3

Lamb shift frequency $2^2S_{\frac{1}{2}} - 2^2P_{\frac{1}{2}}$ in MHz

3.4 Dirac Spinors

As we remarked in Sec. **1.5**, the free Dirac equation can be written in the compact form

$$[i\gamma\partial - m]\,\psi(x) = 0\,, \tag{3.68}$$

in which $\hbar = 1 = c$, and the Lorentz indices, as well as the spinor multiplication, are left implicit. We shall now show that this equation is already in a covariant form, on condition that ψ transforms suitably. Under a Lorentz transformation, $x \to x'$, it is supposed that $\psi(x) \to \psi'(x')$, and we wish the Dirac equation (3.68) to become

$$[i\gamma\partial' - m]\,\psi'(x') = 0\,. \tag{3.69}$$

Note that we assume the γ matrices do not change, although in principle we could allow $\gamma \to \gamma'$, so long as the γ' satisfy the same anticommutation relations. However, we do not need to use this freedom: it is possible to choose a particular representation for the γ matrices and to keep this unchanged from one inertial frame to another. However, the Dirac wave function, ψ, cannot be expected to be a scalar. In general, we must write

$$\psi'(x') = S\psi(x)\,, \tag{3.70}$$

where S is a 4×4 matrix that depends on the Lorentz transformation parameters, and which has the effect of mixing up the four components of ψ. Multiply Eq.(3.69) from the left by S^{-1} and use Eq.(3.70):

$$\left(iS^{-1}\gamma^\mu S\frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial}{\partial x^\nu} - m\right)\psi(x) = 0\,. \tag{3.71}$$

For this to be the same as Eq.(3.68), we need

$$S^{-1}\gamma^\mu S\frac{\partial x^\nu}{\partial x'^\mu} = \gamma^\nu\,,$$

or equivalently

$$S^{-1}\gamma^\mu S = \frac{\partial x'^\mu}{\partial x^\nu}\gamma^\nu = \Lambda^\mu{}_\nu\gamma^\nu\,. \tag{3.72}$$

We propose to show now, by construction, that S exists, such that Eq.(3.72) holds. This will define the correct way to transform a Dirac wave function under a Lorentz transformation.

From the invariance of the speed of light, we have $x^2 = x'^2$, i.e.

$$g_{\rho\sigma}x^\rho x^\sigma = g_{\mu\nu}x'^\mu x'^\nu = g_{\mu\nu}\Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma x^\rho x^\sigma\,,$$

and hence

$$g_{\mu\nu}\Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma = g_{\rho\sigma}\,. \tag{3.73}$$

Consider an infinitesimal Lorentz transformation of the form

$$\Lambda^\mu{}_\nu = \delta^\mu_\nu + \omega^\mu{}_\nu\,,$$

where $\omega^{\mu}{}_{\nu}$ is infinitesimal. Substitute this into Eq.(3.73):

$$g_{\rho\sigma} + g_{\mu\nu}\left[\delta^{\mu}_{\sigma}\omega^{\nu}{}_{\rho} + \delta^{\nu}_{\rho}\omega^{\mu}{}_{\sigma}\right] = g_{\rho\sigma}\,,$$

to first order in ω, or

$$\omega_{\rho\sigma} = -\omega_{\sigma\rho}\,,$$

i.e. $\omega_{\rho\sigma}$ is antisymmetric. Let us write

$$S = 1 - \frac{i}{4}\sigma^{\mu\nu}\omega_{\mu\nu}\,.$$

We have to evaluate the 4×4 matrices $\sigma^{\mu\nu}$. To first order in ω, the inverse of S is

$$S^{-1} = 1 + \frac{i}{4}\sigma^{\mu\nu}\omega_{\mu\nu}\,,$$

so

$$\frac{i}{4}\left[\sigma^{\mu\nu}, \gamma_{\rho}\right]\omega_{\mu\nu} = \omega_{\rho\sigma}\gamma^{\sigma}\,.$$

Since this must hold for arbitrary antisymmetric $\omega_{\rho\sigma}$, it follows that

$$\left[\sigma^{\mu\nu}, \gamma^{\rho}\right] = 2i\left[\gamma^{\mu}g^{\nu\rho} - \gamma^{\nu}g^{\mu\rho}\right]\,.$$

This equation is solved by

$$\sigma^{\mu\nu} = \frac{i}{2}\left[\gamma^{\mu}, \gamma^{\nu}\right]\,,$$

as may be checked by substitution. We may write

$$S = 1 + \frac{1}{8}[\gamma^{\mu}, \gamma^{\nu}]\omega_{\mu\nu}\,,$$

and the Hermitian conjugate of this is

$$S^{\dagger} = 1 - \frac{1}{8}[\gamma^{\dagger\mu}, \gamma^{\dagger\nu}]\omega_{\mu\nu}\,.$$

We know from Eq.(1.68) that $\gamma^0\gamma^{\mu\dagger}\gamma^0 = \gamma^{\mu}$, and so

$$\gamma^0 S^{\dagger}\gamma^0 = 1 - \frac{1}{8}[\gamma^{\mu}, \gamma^{\nu}]\omega_{\mu\nu} = S^{-1}\,.$$

We can use the above results to show that

$$j^{\mu}(x) = \overline{\psi}(x)\gamma^{\mu}\psi(x) \tag{3.74}$$

is a conserved current density that transforms as a Lorentz contravariant vector:

$$\begin{aligned}\partial_\mu j^\mu(x) &= [\partial_\mu \overline{\psi}(x)]\gamma^\mu\psi(x) + \overline{\psi}(x)\gamma^\mu\partial_\mu\psi(x) \\ &= \overline{\psi}(x)[i\gamma\overleftarrow{\partial} + m]\psi(x) + \overline{\psi}(x)\left[i\gamma\partial - m\right]\psi(x) = 0\,.\end{aligned}$$

Under the infinitesimal Lorentz transformation,

$$\begin{aligned}j'^\mu(x') &= \psi^\dagger(x)S^\dagger\gamma^0\gamma^\mu S\psi(x) \\ &= \psi^\dagger(x)\gamma^0 S^{-1}\gamma^\mu S\psi(x) \\ &= \Lambda^\mu{}_\nu j^\nu(x)\,.\end{aligned}$$

It is left as an exercise to show that these two properties of the current density remain valid when the coupling to the electromagnetic field is included.

We now define standard solutions of the free Dirac equation in momentum space — these will shortly be used in setting up the second quantized fields. There are two types of solution, of the forms $u(\vec{p})\,\mathrm{e}^{-ipx}$ and $v(\vec{p})\,\mathrm{e}^{ipx}$, where u and v are 1×4 column matrices called *spinors,* and where the *mass-shell* condition is implicit, i.e.

$$p^0 = \omega_p = \sqrt{\vec{p}^{\,2} + m^2}\,. \tag{3.75}$$

On substituting these trial solutions into the free Dirac equation, we find that the spinors must satisfy

$$\begin{aligned}(\gamma p - m)u(\vec{p}) &= 0 \\ (\gamma p + m)v(\vec{p}) &= 0\,.\end{aligned} \tag{3.76}$$

The mass-shell condition (3.75) implies $(\gamma p)^2 = p^2 = m^2$, so that solutions of Eq.(3.76) may be written in the form

$$\begin{aligned}u(\vec{p}) &= (\gamma p + m)x \\ v(\vec{p}) &= (\gamma p - m)y\,.\end{aligned} \tag{3.77}$$

where x and y are arbitrary 1×4 column matrices. In the rest frame of the particle, $\vec{p} = 0$, $p^0 = m$, Eqs.(3.76) reduce to

$$\begin{aligned}(\gamma^0 - 1)u(\vec{0}) &= 0 \\ (\gamma^0 + 1)v(\vec{0}) &= 0\,.\end{aligned} \tag{3.78}$$

Now we know that the matrix γ^0 has two independent eigenfunctions belonging to the eigenvalue 1, and two belonging to -1. In the representation (1.67) of

the γ-matrices, we may define the following orthonormal set:

$$u(\vec{0},1) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad u(\vec{0},2) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$v(\vec{0},1) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \qquad v(\vec{0},2) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} . \tag{3.79}$$

We now choose the matrices x and y in Eq.(3.77) to be proportional to these rest-frame quantities:

$$\begin{aligned} u(\vec{p},s) &= \frac{m+\gamma p}{\sqrt{2m(m+\omega_p)}} u(\vec{0},s) \\ v(\vec{p},s) &= \frac{m-\gamma p}{\sqrt{2m(m+\omega_p)}} v(\vec{0},s) , \end{aligned} \tag{3.80}$$

where s, the spin quantum number, can take the values 1 and 2. The constants of proportionality in Eq.(3.79) have been chosen to ensure the normalizations

$$\overline{u}(\vec{p},s)u(\vec{p},t) = \delta_{st} = -\overline{v}(\vec{p},s)v(\vec{p},t) . \tag{3.81}$$

Note the negative sign in front of the v spinor — it arises from the eigenvalue -1 of the matrix γ^0. The particle and antiparticle spinors are orthogonal:

$$\overline{v}(\vec{p},s)u(\vec{p},t) = 0 = \overline{u}(\vec{p},s)v(\vec{p},t) . \tag{3.82}$$

Without the matrix γ^0, the normalization reads

$$u^\dagger(\vec{p},s)u(\vec{p},t) = \frac{\omega_p}{m}\delta_{st} = v^\dagger(\vec{p},s)v(\vec{p},t) . \tag{3.83}$$

3.5 Particle and Spin Projection Operators

The particle and antiparticle projection operators may be defined by

$$\Lambda_+(\vec{p}) = \sum_{s=1}^{2} u(\vec{p},s)\overline{u}(\vec{p},s) \qquad \Lambda_-(\vec{p}) = -\sum_{s=1}^{2} v(\vec{p},s)\overline{v}(\vec{p},s) .$$

The particle projection operators thus have the form

$$\Lambda_+(\vec{p}) = \frac{m+\gamma p}{2m(m+\omega_p)} \sum_{s=1}^{2} u(\vec{0},s)\overline{u}(\vec{0},s)(m+\gamma p) , \tag{3.84}$$

and the sum over the rest-frame spinors is

$$\sum_{s=1}^{2} u(\vec{0},s)\overline{u}(\vec{0},s) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix}$$
$$= \mathrm{diag}\begin{pmatrix} 1 & 1 & 0 & 0 \end{pmatrix}$$

i.e., a diagonal matrix with the given elements on the diagonal. Compactly,

$$\sum_{s=1}^{2} u(\vec{0},s)\overline{u}(\vec{0},s) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

where 1 and 0 stand for the unit and null matrices in two dimensions. Thus

$$(m+\gamma p)\sum_{s=1}^{2} u(\vec{0},s)\overline{u}(\vec{0},s)(m+\gamma p)$$
$$= \begin{pmatrix} m+\omega_p & \vec{\sigma}\cdot\vec{p} \\ -\vec{\sigma}\cdot\vec{p} & m-\omega_p \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} m+\omega_p & \vec{\sigma}\cdot\vec{p} \\ -\vec{\sigma}\cdot\vec{p} & m-\omega_p \end{pmatrix}$$
$$= \begin{pmatrix} (m+\omega_p)^2 & (m+\omega_p)\vec{\sigma}\cdot\vec{p} \\ -(m+\omega_p)\vec{\sigma}\cdot\vec{p} & -(\vec{p})^2 \end{pmatrix}.$$

On mass shell, $(\vec{p})^2 = \omega_p^2 - m^2$, and so the above matrix has precisely the form $(m+\omega_p)(m+\gamma p)$. We see then from Eq.(3.84) that

$$\Lambda_+(\vec{p}) = \frac{m+\gamma p}{2m}. \tag{3.85}$$

In a similar way,

$$\Lambda_-(\vec{p}) = -\frac{m-\gamma p}{2m(m+\omega_p)}\sum_{s=1}^{2} v(\vec{0},s)\overline{v}(\vec{0},s)(m-\gamma p). \tag{3.86}$$

$$\sum_{s=1}^{2} v(\vec{0},s)\overline{v}(\vec{0},s) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & -1 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & -1 & 0 \end{pmatrix}$$
$$= \mathrm{diag}\begin{pmatrix} 0 & 0 & -1 & -1 \end{pmatrix},$$

which we may write in the compact form

$$\sum_{s=1}^{2} v(\vec{0},s)\overline{v}(\vec{0},s) = -\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

We find in this case

$$
\begin{aligned}
&-(m-\gamma p)\sum_{s=1}^{2} v(\vec{0},s)\overline{v}(\vec{0},s)(m-\gamma p) \\
&\quad = \begin{pmatrix} m-\omega_p & -\vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & m+\omega_p \end{pmatrix}\begin{pmatrix} m-\omega_p & -\vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & m+\omega_p \end{pmatrix} \\
&\quad = \begin{pmatrix} -(\vec{p})^2 & -(m+\omega_p)\vec{\sigma}\cdot\vec{p} \\ (m+\omega_p)\vec{\sigma}\cdot\vec{p} & (m+\omega_p)^2 \end{pmatrix}.
\end{aligned}
$$

On mass shell, $(\vec{p})^2 = \omega_p^2 - m^2$, and so the above matrix has precisely the form $(m+\omega_p)(m-\gamma p)$. It follows from Eq.(3.86) that

$$\Lambda_-(\vec{p}) = \frac{m-\gamma p}{2m}. \tag{3.87}$$

Now $(\gamma p)^2 = p^2$, and this is equal to m^2 on mass-shell, so

$$\Lambda_\pm(\vec{p})^2 = \tfrac{1}{4}(1 \pm 2\gamma p/m + 1) = \Lambda_\pm(\vec{p}). \tag{3.88}$$

Moreover

$$
\begin{aligned}
\Lambda_\pm(\vec{p})\Lambda_\mp(\vec{p}) &= 0 \\
\Lambda_+(\vec{p}) + \Lambda_-(\vec{p}) &= 1.
\end{aligned}
$$

Thus $\Lambda_\pm(\vec{p})$ are projection operators. The most general spinor may be written

$$w(\vec{p}) = \beta_1 u(\vec{p},1) + \beta_2 u(\vec{p},2) + \delta_1 v(\vec{p},1) + \delta_2 v(\vec{p},2),$$

and clearly

$$\Lambda_+(\vec{p})w(\vec{p}) = \beta_1 u(\vec{p},1) + \beta_2 u(\vec{p},2) \qquad \Lambda_-(\vec{p})w(\vec{p}) = \delta_1 v(\vec{p},1) + \delta_2 v(\vec{p},2).$$

In other words, Λ_+ projects onto the particle spinor content, $u(\vec{p},1)$ and $u(\vec{p},2)$, while Λ_- projects onto the antiparticle spinor content, $v(\vec{p},1)$ and $v(\vec{p},2)$.

We shall now construct the spin projection operator, $\Sigma_s(\vec{p})$, which does not distinguish between particle and antiparticle, and which satisfies

$$
\begin{aligned}
\Sigma_s(\vec{p})u(\vec{p},s') &= \delta_{ss'}u(\vec{p},s) \\
\Sigma_s(\vec{p})v(\vec{p},s') &= \delta_{ss'}v(\vec{p},s).
\end{aligned}
$$

In view of Eqs.(3.81)-(3.82), this operator can be written

$$\Sigma_s(\vec{p}) = u(\vec{p},s)\overline{u}(\vec{p},s) - v(\vec{p},s)\overline{v}(\vec{p},s),$$

and in the rest-frame we have

$$\begin{aligned}\Sigma_1(\vec{0}) &= \begin{pmatrix}1\\0\\0\\0\end{pmatrix}\begin{pmatrix}1&0&0&0\end{pmatrix} - \begin{pmatrix}0\\0\\0\\1\end{pmatrix}\begin{pmatrix}0&0&0&-1\end{pmatrix} \\ &= \operatorname{diag}\begin{pmatrix}1&0&0&1\end{pmatrix} \end{aligned} \tag{3.89}$$

$$\begin{aligned}\Sigma_2(\vec{0}) &= \begin{pmatrix}0\\1\\0\\0\end{pmatrix}\begin{pmatrix}0&1&0&0\end{pmatrix} - \begin{pmatrix}0\\0\\1\\0\end{pmatrix}\begin{pmatrix}0&0&-1&0\end{pmatrix} \\ &= \operatorname{diag}\begin{pmatrix}0&1&1&0\end{pmatrix} \end{aligned} \tag{3.90}$$

These matrices can be subsumed in one formula:

$$\Sigma_s(\vec{0}) = \tfrac{1}{2}\left[1 + (-1)^s \gamma_5 \gamma^3\right]. \tag{3.91}$$

This is clear, since

$$\gamma_5\gamma^3 = \begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}0&\sigma_3\\-\sigma_3&0\end{pmatrix} = \begin{pmatrix}-\sigma_3&0\\0&\sigma_3\end{pmatrix}. \tag{3.92}$$

To generalize this formula to an arbitrary Lorentz frame, and to an arbitrary direction, $\vec{n}$, for the quantization of the spin, we define

$$\Sigma_s(\vec{p},\vec{n}) = \tfrac{1}{2}\left[1 - (-1)^s \gamma_5 \gamma^\mu n_\mu)\right]. \tag{3.93}$$

If n^μ is chosen to be a spacelike four-vector that reduces to

$$n^\mu = (0,0,0,1)$$

in the rest-frame of the particle, then

$$\gamma^\mu n_\mu = \gamma^3 n_3 = -\gamma^3 n^3 = -\gamma^3,$$

and so $\Sigma_s(\vec{p},\vec{n})$ reduces to $\Sigma_s(\vec{p})$ in the rest-frame. By generalizing suitably the choice of n^μ, we obtain the spin projection operator for any direction of the spin quantization axis.

The above analysis applies to massive Dirac particles, like the electron. A different treatment is necessary for a massless particle (actually, it seems that the electron neutrino has a nonvanishing, very small mass, but for most purposes it can be treated as though it were strictly massless). For a particle solution of the

massless Dirac equation, $u(\vec{p})\,\mathrm{e}^{-ipx}$, and for an antiparticle solution, $v(\vec{p})\,\mathrm{e}^{ipx}$, we find

$$\gamma^\mu p_\mu u(\vec{p}) = 0 = \gamma^\mu p_\mu v(\vec{p})\,. \tag{3.94}$$

Choose the z-axis as the direction of propagation of the particle or antiparticle, and let p be its momentum, so $p^\mu = (p,0,0,p)$, since the mass is zero. Hence

$$p(\gamma^0 - \gamma^3)u(\vec{p}) = 0 = p(\gamma^0 - \gamma^3)v(\vec{p})\,,$$

and so

$$u(\vec{p}) = \gamma^0\gamma^3 u(\vec{p}) \qquad\qquad v(\vec{p}) = \gamma^0\gamma^3 v(\vec{p})\,.$$

Since $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, it follows that

$$\gamma_5 u(\vec{p}) = i\gamma^1\gamma^2 u(\vec{p}) \qquad\qquad \gamma_5 v(\vec{p}) = i\gamma^1\gamma^2 v(\vec{p})\,.$$

Now in the standard representation of the Dirac matrices,

$$\begin{aligned}
\gamma^1\gamma^2 &= \begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix} \\
&= \begin{pmatrix} -\sigma_1\sigma_2 & 0 \\ 0 & -\sigma_1\sigma_2 \end{pmatrix} \\
&= -i\begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix} = -\frac{2i}{\hbar}S_3\,,
\end{aligned} \tag{3.95}$$

where the four-dimensional spin matrix was defined in Eq.(3.5). It follows that

$$\tfrac{1}{2}\hbar\gamma_5 u(\vec{p}) = S_3 u(\vec{p}) \qquad\qquad \tfrac{1}{2}\hbar\gamma_5 v(\vec{p}) = S_3 v(\vec{p})\,.$$

Thus $\frac{1}{2}\hbar\gamma_5$, which is called the chiral operator, has the same effect on a spinor as does the helicity operator, which is defined to be the projection of the spin operator in the direction of propagation of the particle or antiparticle. If $u(\vec{p},s)$ and $v(\vec{p},s)$ are spinors of helicity $\frac{1}{2}\hbar(-1)^{s+1}$, $s = 1,2$, then clearly

$$\begin{aligned}
\gamma_5 u(\vec{p},s) &= (-1)^{s+1}u(\vec{p},s) \\
\gamma_5 v_\pm(\vec{p}) &= (-1)^{s+1}v(\vec{p},s)\,.
\end{aligned}$$

Right and left chiral projection operators are defined by

$$P_R = \tfrac{1}{2}(1+\gamma_5) \qquad P_L = \tfrac{1}{2}(1-\gamma_5)\,, \tag{3.96}$$

and we have proved that, for the massless spinors,

$$P_R u(\vec{p},1) = u(\vec{p},1) \qquad P_L u(\vec{p},2) = u(\vec{p},2)$$
$$P_R v(\vec{p},1) = v(\vec{p},1) \qquad P_L v(\vec{p},2) = v(\vec{p},2)\,.$$

Thus positive helicity massless fermions are right-handed, while negative helicity massless fermions are left-handed.

3.6 Exercises

Problem 1

Given the on-mass-shell definitions

$$u(\vec{p},s) = \frac{m+\gamma p}{\sqrt{2m(m+\omega_p)}}\,u(\vec{0},s)$$
$$v(\vec{p},s) = \frac{m-\gamma p}{\sqrt{2m(m+\omega_p)}}\,v(\vec{0},s)\,,$$

demonstrate the following:

(1) $(\gamma p - m)u(\vec{p},s) = 0 = \overline{v}(\vec{p},s)(\gamma p + m)$
(2) $\overline{u}(\vec{p},s)u(\vec{p},t) = \delta_{st}$
(3) $\overline{v}(\vec{p},s)v(\vec{p},t) = -\delta_{st}$
(4) $\overline{v}(\vec{p},s)u(\vec{p},t) = 0$
(5) $u^\dagger(\vec{p},s)v(-\vec{p},t) = 0$
(6) $u^\dagger(\vec{p},s)u(\vec{p},t) = \frac{\omega_p}{m}\delta_{st} = v^\dagger(\vec{p},s)v(\vec{p},t)$

Problem 2

Let $u(\vec{p},s)$ be a spinor satisfying

$$(\gamma p - m)u(\vec{p},s) = 0\,.$$

Investigate the the Lorentz transformation properties of the following quantities:

$$\bar{u}(\vec{p},s)u(\vec{p},s) \qquad \bar{u}(\vec{p},s)\gamma_5 u(\vec{p},s)$$
$$\bar{u}(\vec{p},s)\gamma^\mu u(\vec{p},s) \qquad \bar{u}(\vec{p},s)\gamma^\mu\gamma_5 u(\vec{p},s)$$
$$\bar{u}(\vec{p},s)\gamma^\mu\gamma^\nu u(\vec{p},s)$$

Problem 3

Prove the Gordon identity, namely

$$2m\bar{u}(\vec{p}\,',s')\gamma^\mu u(\vec{p},s) = \bar{u}(\vec{p}\,',s')\left[P^\mu + i\sigma^{\mu\nu}k_\nu\right]u(\vec{p},s)$$

where $P^\mu = p'^\mu + p^\mu$ and $k^\mu = p'^\mu - p^\mu$.

Problem 4

Solve the Dirac equation for an electron in the presence of a constant, uniform magnetic induction. Obtain the general energy eigenstates, and show that the energy eigenvalues are given by

$$E = \sqrt{m^2c^4 + c^2p_3^2 + 2ne\hbar cB}$$

where $n = 0, 1, 2, \cdots$. Determine all the constants of motion.

Problem 5

Let F be the column vector

$$F = \begin{pmatrix} s \\ v \\ t \\ a \\ p \end{pmatrix} \equiv \begin{pmatrix} \bar{u}(\vec{p}_4, s_4)u(\vec{p}_1, s_1)\bar{u}(\vec{p}_3, s_3)u(\vec{p}_2, s_2) \\ \bar{u}(\vec{p}_4, s_4)\gamma^\mu u(\vec{p}_1, s_1)\bar{u}(\vec{p}_3, s_3)\gamma_\mu u(\vec{p}_2, s_2) \\ \frac{1}{2}\bar{u}(\vec{p}_4, s_4)\sigma^{\mu\nu}u(\vec{p}_1, s_1)\bar{u}(\vec{p}_3, s_3)\sigma_{\mu\nu}u(\vec{p}_2, s_2) \\ \bar{u}(\vec{p}_4, s_4)\gamma_5\gamma^\mu u(\vec{p}_1, s_1)\bar{u}(\vec{p}_3, s_3)\gamma_\mu\gamma_5 u(\vec{p}_2, s_2) \\ \bar{u}(\vec{p}_4, s_4)\gamma_5 u(\vec{p}_1, s_1)\bar{u}(\vec{p}_3, s_3)\gamma_5 u(\vec{p}_2, s_2) \end{pmatrix},$$

and let F' be the vector obtained from F by means of the Fierz interchange $(\vec{p}_1, s_1) \longleftrightarrow (\vec{p}_2, s_2)$. Compute the Fierz matrix, M, that satisfies

$$F' = MF\,.$$

What is the transformation property of the combination $v - a$?

Problem 6

Calculate the Foldy-Wouthuysen unitary operator, U, such that

$$H_{\rm FW} = UH_{\rm D}U^\dagger = \gamma^0\left(\vec{p}^{\,2}c^2 + m^2c^4\right)^{\frac{1}{2}},$$

where $H_{\rm D} = \gamma^0\left(\vec{\gamma}\cdot\vec{p} + mc^2\right)$ is the free Dirac Hamiltonian. Show that the large and the small components of the Dirac wave function decouple. What can be said when there is electromagnetic coupling?

Problem 7

Discuss how the number of nodes of the radial wave functions, $F(r)$ and $G(r)$, in the solution of the Dirac equation in the presence of a Coulomb potential, are related to quantum numbers (n, j, ℓ). For what values of α, the fine-structure constant, is the Hamiltonian self-adjoint? Discuss the cases

(1) $\alpha^2 < \frac{3}{4}$
(2) $\frac{3}{4} < \alpha^2 < 1$
(3) $\alpha^2 > 1$.

Problem 8

Consider a Dirac particle subject to a (three-dimensional) spherical well potential

$$V(r) = \begin{bmatrix} V_0 < 0 & \text{for} & r < r_0 \\ \\ 0 & \text{for} & r > r_0 \end{bmatrix}$$

(1) Obtain the exact 4-component energy eigenfunctions for $j = \frac{1}{2}$ bound states, where the 2 upper components have even orbital parity.
(2) Obtain an equation for the energy eigenvalues.
(3) What happens if the strength of the potential is increased so that V_0 becomes comparable to or larger than $2mc^2$?

Problem 9

Determine the bound state energies for a spinless particle, of mass m and charge $-e$, that obeys the Klein-Gordon equation in the presence of the Coulomb field a fixed particle of charge Ze. Determine the fine structure of the energy levels, and consider the limit $Ze^2/(\hbar c) \to \frac{1}{2}$.

Problem 10

Show that the Dirac equation for a particle of mass m in the presence of the scalar oscillator potential $V(r) = \frac{1}{2}m\omega^2 r^2$ does not have energy eigenstates, because one cannot keep all components of the wave function bounded as $r \to \infty$. However, show that for the potential

$$V(r) = -im\omega\,(\vec{\gamma} \cdot \vec{r}\,),$$

the Dirac equation has energy eigenstates, and determine them.

Chapter 4

Quantum Field Theory

In quantum field theory, a field, $\phi(x) = \phi(t, \vec{x})$, is a generalized coordinate. A particular way to achieve this is to imagine all of space, at a given time, t, to be divided into a countably infinite set of cells. To each cell we assign a single coordinate, namely the average value of $\phi(t, \vec{x})$ over that cell at the given time. A more sophisticated approach is to suppose that $\phi(x)$ belongs to a separable Hilbert space with an orthonormal basis, $\{f_n(x)\}$, so that one can write

$$\phi(t, \vec{x}) = \sum_n q_n(t) f_n(\vec{x}) \,. \tag{4.1}$$

The time-dependent coefficients, $q_n(t)$, are regarded as constituting a discrete set of canonical quantum coordinates. Given a Lagrangian for the system, one writes the Euler-Lagrange equation, defines the canonical momenta, $p_n(t)$, calculates the Hamiltonian and writes down the standard commutation relations between the coordinates and the momenta. The next step is to consider Eq.(4.1) as a canonical transformation, with inverse

$$q_n(t) = \int d^3x \, \phi(t, \vec{x}) f_n(\vec{x}) \,.$$

The Lagrangian is written

$$L(t) = \int d^3x \, \mathcal{L}(t, \vec{x}) \,,$$

where $\mathcal{L}(t, \vec{x}) = \mathcal{L}(x)$ is the Lagrangian density. It is a straightforward matter to show that the Euler-Lagrange equation, written in terms of the $q_n(t)$, leads to the continuum version,

$$\partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right] - \frac{\partial \mathcal{L}}{\partial \phi} = 0 \,, \tag{4.2}$$

where the Lagrangian density is considered, as far as the partial differentiations are concerned, to be a function of ϕ and its partial derivatives. The momentum conjugate to $\phi(t,\vec{x})$ is

$$\pi(t,\vec{x}) = \dot{\phi}(t,\vec{x}),$$

and the standard commutation relations between the discrete variables q_n and p_n lead to the continuum analog (with $\hbar = 1$)

$$\begin{aligned}
[\phi(t,\vec{x}),\pi(t,\vec{y})] &= i\delta^3(\vec{x}-\vec{y}) \\
[\phi(t,\vec{x}),\phi(t,\vec{y})] &= 0 \\
[\pi(t,\vec{x}),\phi(t,\vec{y})] &= 0,
\end{aligned}$$

which are called the equal-time commutators (Problem 1.9).

4.1 Scalar Field

Consider a spinless particle of mass m, the state of which is described by a vector $|\phi(t)\rangle$. The configuration space wave function is

$$\phi(t,\vec{x}) = \langle\vec{x}\,|\phi(t)\rangle\,.$$

In the absence of interactions, this satisfies the Klein-Gordon equation,

$$(\partial^2 + m^2)\phi = 0\,. \tag{4.3}$$

The procedure that leads to this equation is sometimes called *first quantization* (Sec. **1.4**). It turns out that certain difficulties arise in treating it as a one-particle equation — difficulties that are avoided by considering it in terms of a many-particle field theory. This we do by regarding $\phi(x)$ not as a c-number quantity, but rather as a q-number (to use the language of Dirac), i.e., as an operator on Fock space, a superposition of a creation and an annihilation part: $\phi(x) = a(x) + a^\dagger(x)$. This reinterpretation of ϕ is called *second quantization* by some people. The Lagrangian density

$$\mathcal{L} = \tfrac{1}{2} : (\partial_\mu\phi)(\partial^\mu\phi) - m^2\phi^2 :$$

gives rise to the Klein-Gordon equation (4.3) as the Euler-Lagrange equation. Between the pair of colons in this equation (and elsewhere), all creation operators must be placed to the left of *all* annihilation operators — a procedure that is called *normal ordering*. The canonical momentum is

$$\pi(x) = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \dot{\phi}\,,$$

and the canonical equal-time commutation relations are

$$\begin{aligned}
[\phi(t,\vec{x}),\phi(t,\vec{y})] &= 0 \\
[\dot{\phi}(t,\vec{x}),\dot{\phi}(t,\vec{y})] &= 0 \\
[\phi(t,\vec{x}),\dot{\phi}(t,\vec{y})] &= i\delta^3(\vec{x}-\vec{y})\,. \qquad (4.4)
\end{aligned}$$

Consider next the 3-dimensional Fourier transform

$$\phi(x) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int \frac{d^3k}{\sqrt{2\omega_k}}\left[a(\vec{k})\,\mathrm{e}^{-ikx} + a^\dagger(\vec{k})\,\mathrm{e}^{ikx}\right]_{\text{mass-shell}}\,.$$

where 'mass-shell' means that $k^0 = \omega_k = \sqrt{\vec{k}\cdot\vec{k}+m^2}$. The equal-time commutation relations Eq.(4.4) can be shown to lead to

$$\begin{aligned}
[a(\vec{k}),a^\dagger(\vec{p})] &= \delta^3(\vec{k}-\vec{p}) \\
[a(\vec{k}),a(\vec{p})] &= 0 \\
[a^\dagger(\vec{k}),a^\dagger(\vec{p})] &= 0\,. \qquad (4.5)
\end{aligned}$$

These equations are precisely the continuum analogs of the commutation relations between the ladder operators for a quantum mechanical simple harmonic oscillator. We shall exploit this equivalence by interpreting $a(\vec{k})$ not merely as an operator that effects a step down the ladder of energy eigenstates, but as a fully fledged annihilation operator that destroys one quantum of momentum $\vec{k}$. Its Hermitian conjugate, $a^\dagger(\vec{k})$, is a creation operator that calls the quantum into existence. The Hamiltonian density is defined by

$$\begin{aligned}
\mathcal{H}(x) &= :\pi(x)\dot{\phi}(x): - \mathcal{L}(x) \\
&= \tfrac{1}{2}:\dot{\phi}^2 + (\vec{\nabla}\phi)^2 + m^2\phi^2: \qquad (4.6)
\end{aligned}$$

and from this one may evaluate the Hamiltonian itself,

$$H = \int d^3x\,\mathcal{H}(x) = \int d^3k\,\omega_k a^\dagger(\vec{k})a(\vec{k})\,. \qquad (4.7)$$

This formula is readily intelligible: $a^\dagger(\vec{k})a(\vec{k})$ is the number density operator in momentum space, and ω_k is the energy associated with momentum $\vec{k}$ (recall that $\hbar = 1$). This Hamiltonian is a positive definite operator. To show that it is, suppose that $|\Phi\rangle$ is an arbitrary ket vector. Then

$$\langle\Phi|H|\Phi\rangle = \int d^3k\,\omega_k\langle\Psi(k)|\Psi(k)\rangle \geq 0\,,$$

where $|\Psi(k)\rangle = a(\vec{k})|\Phi\rangle$.

As we will discover later, it is important to calculate the time-ordered product of two fields, defined by

$$T[\phi(x),\phi(y)] = \phi(x)\phi(y)\theta(x^0-y^0) + \phi(y)\phi(x)\theta(y^0-x^0)\,.$$

Differentiating once with respect to x^0, we obtain

$$\frac{\partial}{\partial x^0}T[\phi(x),\phi(y)] = [\phi(x),\phi(y)]\delta(x^0-y^0) + T[\dot{\phi}(x),\phi(y)]\,. \tag{4.8}$$

Now from Eq.(4.4), we have

$$[\phi(x),\phi(y)]\delta(x^0-y^0) = [\phi(x^0,\vec{x}),\phi(x^0,\vec{y})]\delta(x^0-y^0) = 0\,,$$

i.e., the first term in Eq.(4.8) vanishes. On differentiating that equation once more with respect to x^0, we find

$$\left(\frac{\partial}{\partial x^0}\right)^2 T[\phi(x),\phi(y)] = [\dot{\phi}(x),\phi(y)]\delta(x^0-y^0) + T[\ddot{\phi}(x),\phi(y)]\,.$$

From Eq.(4.4), we have

$$\begin{aligned}[\dot{\phi}(x),\phi(y)]\delta(x^0-y^0) &= [\dot{\phi}(x^0,\vec{x}),\phi(x^0,\vec{y})]\delta(x^0-y^0)\\ &= -i\delta^3(\vec{x}-\vec{y})\delta(x^0-y^0) = -i\delta^4(x-y)\,.\end{aligned}$$

Since spatial derivatives are indifferent to the time ordering, finally we have

$$\begin{aligned}(\partial_x^2+m^2)T[\phi(x),\phi(y)] &= -i\delta^4(x-y) + T[(\partial_x^2+m^2)\phi(x),\phi(y)]\\ &= -i\delta^4(x-y)\,.\end{aligned} \tag{4.9}$$

Taking vacuum expectation values of both sides of this equation, we obtain

$$(\partial_x^2+m^2)\langle 0|T[\phi(x),\phi(y)]|0\rangle = -i\delta^4(x-y)\,,$$

for which a solution is given in terms of the following distribution:

$$\langle 0|T[\phi(x),\phi(y)]|0\rangle = i\Delta_F(x-y)\,, \tag{4.10}$$

where the Feynman Green's function is

$$\Delta_F(x-y) = \frac{1}{(2\pi)^4}\int d^4k \frac{\mathrm{e}^{-ik(x-y)}}{k^2-m^2+i\epsilon}. \tag{4.11}$$

The commutation relations Eq.(4.5) lead to the specification that the limit $\epsilon \to 0$ should be taken through positive values (Problem 4.2).

4.2 Electromagnetic Field

The electromagnetic Lagrangian density is

$$\mathcal{L}_{\text{photons}} = -\tfrac{1}{4} : F^{\mu\nu} F_{\mu\nu} : \tag{4.12}$$

where the electromagnetic tensor is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \,.$$

This gives rise to the Maxwell equation in vacuo as the Euler-Lagrange equation. There are difficulties with this form in quantum theory, however, which have to do with maintaining the gauge invariance of the Lagrangian. It is customary to add a *gauge fixing term:*

$$\mathcal{L}_{\text{gfix}} = -\tfrac{1}{2} : \partial_\mu A^\mu \partial_\nu A^\nu : \; \equiv -\tfrac{1}{2} : \left(\partial_\mu A^\mu\right)^2 :$$

The complete Lagrangian of the free electromagnetic field, in the Gupta-Bleuler method, is written as

$$\mathcal{L}_{\text{em}} = \mathcal{L}_{\text{photons}} + \mathcal{L}_{\text{gfix}} \,. \tag{4.13}$$

We find for the field derivatives

$$\frac{\partial \mathcal{L}_{\text{em}}}{\partial A_\nu} = 0 \qquad\qquad \frac{\partial \mathcal{L}_{\text{em}}}{\partial \partial_\mu A_\nu} = -F^{\mu\nu} - g^{\mu\nu} \partial_\rho A^\rho \,.$$

The equation of motion, i.e., the Euler-Lagrange equation, is

$$\partial_\mu F^{\mu\nu} + \partial^\nu \partial_\rho A^\rho = 0 \,, \tag{4.14}$$

which can be rewritten

$$\partial^2 A^\nu = 0 \,.$$

The momenta conjugate to the A_ν are

$$\pi^\nu = \frac{\partial \mathcal{L}_{\text{em}}}{\partial \partial_0 A_\nu} = -F^{0\nu} - g^{0\nu} \partial_\rho A^\rho \;;$$

and the equal-time commutation relations are

$$\begin{aligned} [A_\mu(x), \pi^\nu(y)]_{\text{et}} &= i\delta_\mu^\nu \delta^3(\vec{x} - \vec{y}) \\ [A_\mu(x), A^\nu(y)]_{\text{et}} &= 0 \\ [\pi_\mu(x), \pi^\nu(y)]_{\text{et}} &= 0 \,. \end{aligned} \tag{4.15}$$

Note that the *contravariant* vector π^μ is the canonical momentum corresponding to the *covariant* A_μ.

The timelike canonical momentum is

$$\pi^0 = -\partial_\mu A^\mu = -\left[\dot{A}^0 + \vec{\nabla}\cdot\vec{A}\right],$$

while the spacelike momenta are

$$\vec{\pi} = -\dot{\vec{A}} - \vec{\nabla} A^0,$$

and hence the equal-time commutation relations between the fields and their time derivatives are

$$\begin{aligned} [A_0(x), \dot{A}_0(y)]_{\text{et}} &= -i\delta^3(\vec{x}-\vec{y}) \\ [A_i(x), \dot{A}_j(y)]_{\text{et}} &= i\delta_{ij}\delta^3(\vec{x}-\vec{y}), \end{aligned} \tag{4.16}$$

with all other equal time commutators vanishing. Just as in the scalar case, let us make a momentum space decomposition:

$$A^\mu(x) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int \frac{d^3k}{2|\vec{k}|}\left[a^\mu(\vec{k})\,\mathrm{e}^{-ikx} + a^{\mu\dagger}(\vec{k})\,\mathrm{e}^{ikx}\right]_{\text{mass-shell}}. \tag{4.17}$$

The commutation relations between the creation and annihilation operators are

$$\begin{aligned} [a_0(\vec{k}), a_0^\dagger(\vec{p})] &= -\delta^3(\vec{k}-\vec{p}) \\ [a_i(\vec{k}), a_j^\dagger(\vec{p})] &= \delta_{ij}\,\delta^3(\vec{k}-\vec{p}), \end{aligned} \tag{4.18}$$

with other commutators vanishing. These commutation relations are equivalent to Eq.(4.16), as for the scalar case. It should be noted that the right-hand sides of the timelike relations (4.16), (4.18) have an unexpected minus sign. This is potentially troublesome, for the Hamiltonian becomes

$$\begin{aligned} H &= \int d^3x\,\mathcal{H}(x) \\ &= \int d^3k\,\omega_k\,[\vec{a}^{\,\dagger}(\vec{k})\cdot\vec{a}\,(\vec{k}) - a_0^\dagger(\vec{k})a_0(\vec{k})], \end{aligned} \tag{4.19}$$

which is not positive definite.

The trick of Gupta and Bleuler is to define a *physical subspace* of Fock space. It is spanned by all the ket vectors, $|\Phi\rangle_{\text{phys}}$, that satisfy

$$k^\mu a_\mu(\vec{k})|\Phi\rangle_{\text{phys}} = 0. \tag{4.20}$$

From Eq.(4.17), we may write

$$A^\mu(x) = a^\mu(x) + a^{\mu\dagger}(x),$$

where

$$a^{\mu}(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int \frac{d^3k}{2|\vec{k}|} a^{\mu}(\vec{k})\, \mathrm{e}^{-ikx} \Big|_{\text{mass-shell}}, \tag{4.21}$$

so that

$$\partial_{\mu} a^{\mu}(x) |\Phi\rangle_{\text{phys}} = 0\,.$$

The matrix element ${}_{\text{phys}}\langle\Phi_1|\partial_{\mu}A^{\mu}(x)|\Phi_2\rangle_{\text{phys}}$ is always zero, since it may be rewritten

$${}_{\text{phys}}\langle\Phi_1|\partial_{\mu}a^{\mu}(x)|\Phi_2\rangle_{\text{phys}} + {}_{\text{phys}}\langle\Phi_2|\partial_{\mu}a^{\mu}(x)|\Phi_1\rangle^{*}_{\text{phys}} = 0\,,$$

i.e., the Lorentz condition (2.25) is valid on the physical subspace. Note that

$${}_{\text{phys}}\langle\Phi_1|\partial_{\mu}F^{\mu\nu}(x)|\Phi_2\rangle_{\text{phys}} = {}_{\text{phys}}\langle\Phi_1|\partial^2 A^{\nu}(x)|\Phi_2\rangle_{\text{phys}} = 0\,,$$

so that the modified equation of motion Eq.(4.14) is equivalent to the correct free Maxwell equation, $\partial_{\mu}F^{\mu\nu}(x) = 0$, when the latter is restricted to the physical space. To understand how the introduction of the physical space resolves the problem that the Hamiltonian is not positive definite, it is convenient to introduce polarization vectors. These are defined in momentum space. For every 4-momentum, k^{μ}, on mass shell, choose coordinate axes such that $\vec{k}$ lies in the positive z direction:

$$k^{\mu} = (|\vec{k}|, 0, 0, |\vec{k}|)\,.$$

Introduce four orthonormal polarization vectors, $\varepsilon_{\mu}(\vec{k}, \lambda)$, as follows:

$$\begin{aligned}
\varepsilon^{\mu}(\vec{k}, 0) &= (1,0,0,0) \\
\varepsilon^{\mu}(\vec{k}, 1) &= (0,1,0,0) \\
\varepsilon^{\mu}(\vec{k}, 2) &= (0,0,1,0) \\
\varepsilon^{\mu}(\vec{k}, 3) &= (0,0,0,1)\,.
\end{aligned} \tag{4.22}$$

Define new annihilation operators in momentum space, $a(\vec{k}, \lambda)$, by

$$a^{\mu}(\vec{k}) = \sum_{\lambda=0}^{3} \varepsilon^{\mu}(\vec{k}, \lambda) a(\vec{k}, \lambda)\,. \tag{4.23}$$

Notice that $\varepsilon^{\mu}(\vec{k}, 1)$ and $\varepsilon^{\mu}(\vec{k}, 2)$ are orthogonal to k^{μ}, and that

$$k^{\mu} a_{\mu}(\vec{k}) = |\vec{k}| \left[a(\vec{k}, 0) - a(\vec{k}, 3)\right]\,.$$

Hence Eq.(4.20) implies that, for all nonvanishing k^μ,

$$a(\vec{k},0)|\Phi\rangle_{\text{phys}} = a(\vec{k},3)|\Phi\rangle_{\text{phys}}\,, \tag{4.24}$$

so that

$${}_{\text{phys}}\langle\Phi_1|a^\dagger(\vec{k},0)a(\vec{k},0)|\Phi_2\rangle_{\text{phys}} = {}_{\text{phys}}\langle\Phi_1|a^\dagger(\vec{k},3)a(\vec{k},3)|\Phi_2\rangle_{\text{phys}}\,.$$

Thus the matrix elements of the Hamiltonian, taken between physical states, involve only the transverse polarization degrees of freedom:

$${}_{\text{phys}}\langle\Phi_1|H|\Phi_2\rangle_{\text{phys}} = \int d^3k\,\omega_k\ {}_{\text{phys}}\langle\Phi_1|[a^\dagger(\vec{k},1)a(\vec{k},1) + a^\dagger(\vec{k},2)a(\vec{k},2)]|\Phi_2\rangle_{\text{phys}}\,,$$

i.e., the longitudinal and timelike (ghost) degrees of freedom cancel each other, leaving only the physically expected transverse polarizations. In the restriction to the physical subspace, the Hamiltonian is positive definite.

The derivation of the vacuum expectation value of the time-ordered product of two fields proceeds in analogy to the scalar case, yielding

$$\langle 0|T[A^\mu(x)A^\nu(y)]|0\rangle = -i\frac{g^{\mu\nu}}{(2\pi)^4}\int d^4k\frac{\mathrm{e}^{-ik(x-y)}}{k^2+i\epsilon}\,.$$

It is sometimes useful to generalize this expression by introducing what is called a *gauge parameter*, a, which involves changing the gauge fixing term to

$$\mathcal{L}_{\text{gfix}} = -\frac{1}{2a} : \partial_\mu A^\mu \partial_\nu A^\nu :$$

In this case one finds

$$\langle 0|T[A^\mu(x)A^\nu(y)]|0\rangle = \frac{i}{(2\pi)^4}\int d^4k\, D_F^{\mu\nu}(k)\,\mathrm{e}^{-ik(x-y)}\,,$$

where the momentum-space representation of the Feynman propagator, in a *general gauge* a, is

$$D_F^{\mu\nu}(k) = \frac{-g^{\mu\nu} + (1-a)\frac{k^\mu k^\nu}{k^2}}{k^2+i\epsilon}\,.$$

The choice $a = 1$ is called the Feynman gauge, while $a = 0$ specifies the Landau gauge, which is convenient for some purposes, since the photon propagator is transverse to the momentum in the Landau gauge. That is,

$$k_\mu D_F^{\mu\nu}(k) = 0\,,$$

if $a = 0$, but not otherwise.

4.3 Spinor Field

We may start with the Lagrangian density

$$\mathcal{L} =: \overline{\psi}(i\gamma\partial - m)\psi :$$

for which

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \psi} &= -m\overline{\psi} \\ \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} &= i\overline{\psi}\gamma^\mu . \end{aligned}$$

This gives, as the Euler-Lagrange equation,

$$\overline{\psi}(i\gamma \overleftarrow{\partial} + m) = 0 ,$$

which is the adjoint of the Dirac equation,

$$(i\gamma\partial - m)\psi = 0 .$$

The momentum conjugate to ψ is

$$\pi = \frac{\partial \mathcal{L}}{\partial(\partial_0 \psi)} = i\overline{\psi}\gamma^0 = i\psi^\dagger . \tag{4.25}$$

The momentum-space decomposition of the Dirac field is

$$\psi(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \sum_{s=1}^{2} \int d^3p \sqrt{\frac{m}{\omega_p}} \left[b(\vec{p}, s) u(\vec{p}, s)\, \mathrm{e}^{-ipx} + d^\dagger(\vec{p}, s) v(\vec{p}, s)\, \mathrm{e}^{ipx} \right] , \tag{4.26}$$

where b is a particle annihilation operator and $d^\dagger$ an antiparticle creation operator. The adjoint of this equation is

$$\psi^\dagger(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \sum_{s=1}^{2} \int d^3p \sqrt{\frac{m}{\omega_p}} \left[b^\dagger(\vec{p}, s) u^\dagger(\vec{p}, s)\, \mathrm{e}^{ipx} + d(\vec{p}, s) v^\dagger(\vec{p}, s)\, \mathrm{e}^{-ipx} \right] . \tag{4.27}$$

Since particles satisfying the Dirac equation (such as the electron) are fermions, we require their creation and annihilation operators to satisfy anticommutation relations — instead of commutation relations — to guarantee the antisymmetry of two particle wave functions. Thus

$$\begin{aligned} \left\{ b(\vec{p}, s), b^\dagger(\vec{q}, t) \right\} &= \delta_{st} \delta^3(\vec{p} - \vec{q}) \\ \left\{ d(\vec{p}, s), d^\dagger(\vec{q}, t) \right\} &= \delta_{st} \delta^3(\vec{p} - \vec{q}) , \end{aligned}$$

with all other anticommutators vanishing. These anticommutation relations are equivalent to the following equal-time anticommutation relations between the fields and their canonical momenta:

$$\{\psi_\alpha(x), \pi_\beta(y)\}_{\text{et}} = i\delta_{\alpha\beta}\delta^3(\vec{x} - \vec{y})\,.$$

From Eq.(4.25) we see that this implies

$$\{\psi_\alpha(x), \psi^\dagger_\beta(y)\}_{\text{et}} = \delta_{\alpha\beta}\delta^3(\vec{x} - \vec{y})\,;$$

and all other equal-time anticommutators vanish; for example

$$\{\psi_\alpha(x), \psi_\beta(y)\}_{\text{et}} = 0\,.$$

The Hamiltonian density is

$$\mathcal{H} =: \pi\partial_0\psi : -\mathcal{L} = - : \overline{\psi}(i\vec{\gamma}.\vec{\nabla} - m)\psi := i : \psi^\dagger\partial_0\psi :$$

the last formula following from the Dirac equation. The Hamiltonian itself may be expressed as a momentum space integral:

$$\begin{aligned} H = \int d^3x\,\mathcal{H}(x) &= \sum_{s=1}^{2}\int d^3p\,\omega_p : b^\dagger(\vec{p},s)b(\vec{p},s) - d(\vec{p},s)d^\dagger(\vec{p},s) : \\ &= \sum_{s=1}^{2}\int d^3p\,\omega_p\{b^\dagger(\vec{p},s)b(\vec{p},s) + d^\dagger(\vec{p},s)d(\vec{p},s)\} \quad (4.28) \end{aligned}$$

To obtain the last line, we introduced the rule that the normal ordering of fermion operators brings in a minus sign:

$$: d(\vec{p},s)d^\dagger(\vec{p},s) : \; = - : d^\dagger(\vec{p},s)d(\vec{p},s) : \; = -d^\dagger(\vec{p},s)d(\vec{p},s)\,.$$

The motivation for this minus sign is that the spinor operators satisfy anticommutation — rather than commutation — relations. Normal ordering, with this minus sign, therefore amounts to an infinite shift in the energies. Because of this minus sign, the Hamiltonian is a positive definite operator; without it, that would not be so. This is the essential reason that anticommutators must be used for fields that satisfy the Dirac equation.

Consider next the current density Eq.(3.74), with normal ordering:

$$j^\mu(x) = \; : \overline{\psi}(x)\gamma^\mu\psi(x) : \qquad (4.29)$$

This current can be readily shown to be conserved, that is,

$$\partial_\mu A^\mu = 0\,.$$

The conserved charge is

$$\begin{aligned} Q &= \int d^3x : \psi^\dagger(x)\psi(x) : \\ &= \sum_{s=1}^{2} \int d^3p \{ : b^\dagger(\vec{p},s)b(\vec{p},s) + d(\vec{p},s)d^\dagger(\vec{p},s) : \} \\ &= \sum_{s=1}^{2} \int d^3p \{ b^\dagger(\vec{p},s)b(\vec{p},s) - d^\dagger(\vec{p},s)d(\vec{p},s) \} . \end{aligned} \tag{4.30}$$

Whereas the Hamiltonian (4.28) is the sum of the number density operators $b^\dagger(\vec{p},s)b(\vec{p},s)$ and $d^\dagger(\vec{p},s)d(\vec{p},s)$, weighted with the energy ω_p, the charge is their difference, with energy-independent weight. This shows that the charge of particle and antiparticle are equal in magnitude and opposite in sign. The magnitude is here normalized to unity; but by multiplying the current density Eq.(4.29) by $-e$, that can be adjusted.

The time-ordered product of two spinor fields is defined with an extra minus sign for every interchange of order of two fields, much as in the case of normal-ordered products. Thus, by definition,

$$T[\psi_\alpha(x), \overline{\psi}_\beta(y)] = \psi_\alpha(x)\overline{\psi}_\beta(y)\theta(x^0 - y^0) - \overline{\psi}_\beta(y)\psi_\alpha(x)\theta(y^0 - x^0) .$$

We find, for the vacuum expectation value of this time-ordered product,

$$\langle 0|T[\psi_\alpha(x)\overline{\psi}_\beta(y)]|0\rangle = iS_{F\alpha\beta}(x-y) = \frac{i}{(2\pi)^4} \int d^4p S_{F\,\alpha\beta}(p)\, \mathrm{e}^{ip(x-y)} ,$$

where the Feynman propagator in momentum representation is

$$S_F(p) = \left[\gamma p - m + i\epsilon\right]^{-1} = \frac{\gamma p + m}{p^2 - m^2 + i\epsilon} .$$

4.4 Exercises

Problem 1

Show that the equal-time commutation relations of the scalar quantum field and its conjugate momentum imply, and are implied by the commutation relations between the creation and annihilation operators. Use arguments of Lorentz invariance to show that the commutator of two fields at different points vanishes if the separation between the points is spacelike. What can you say about the relation of microcausality to macroscopic, or ordinary causality?

Problem 2

Show that the equal-time commutation relations for the scalar field lead to the following expression for the Feynman propagator:

$$\langle 0|T[\phi(x),\phi(y)]|0\rangle = i\Delta_F(x-y) = \frac{i}{(2\pi)^4}\int d^4k \frac{e^{-ik(x-y)}}{k^2-m^2+i\epsilon}\,.$$

Problem 3

The Lagrangian density of two independent Hermitian fields, ϕ_1 and ϕ_2, is

$$\mathcal{L}(x) = \tfrac{1}{2}\sum_{j=1}^{2} : \partial_\mu\phi_j(x)\partial^\mu\phi_j(x) - m^2\phi_j^2(x) :$$

with the canonical momenta $\pi_j(x) = \dot{\phi}_j(x)$, $j = 1,2$. Define the *nonHermitian* field $\phi(x) = \frac{1}{\sqrt{2}}[\phi_1(x) + i\phi_2(x)]$, which has a Fourier decomposition

$$\phi(x) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int \frac{d^3k}{\sqrt{2\omega(k)}}\left[a(\vec{k})\,e^{-ikx} + b^\dagger(\vec{k})\,e^{ikx}\right]_{\text{mass-shell}}\,.$$

(1) Starting with the standard commutation relations between the creation and annihilation operators $a_i^\dagger(\vec{k})$ and $a_j(\vec{p})$, deduce the commutation relations between $a^\dagger(\vec{k})$, $a(\vec{p})$, $b^\dagger(\vec{q})$ and $b(\vec{r})$. Next calculate the commutation relations between $\phi(x)$ and $\phi(y)$, for arbitrary times, and thence the equal-time commutation relations between $\phi(x)$ and its canonical conjugate momentum.

(2) Calculate the Hamiltonian in terms of $\phi(x)$, and then in terms of the a and b number operators.

(3) Define the current

$$J^\mu = i : (\partial^\mu\phi^\dagger)\phi - (\partial^\mu\phi)\phi^\dagger :$$

Show that the charge, namely the space integral of $J^0(x)$, is constant in time, and express it in terms of the number operators.

Problem 4

Consider the Lagrangian density for a massive vector field,

$$\mathcal{L} = -\frac{1}{4} : F_{\mu\nu}F^{\mu\nu} : + \frac{m^2}{2} : A_\mu A^\mu : - \frac{1}{2a} : \left(\partial_\mu A^\mu\right)^2 :$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Calculate the Feynman propagator and consider the massless limit, $m \to 0$.

Problem 5

The Becchi-Rouet-Stora (BRS) Lagrangian for QED is

$$\mathcal{L} = -\tfrac{1}{4} : F^{\mu\nu} F_{\mu\nu} : + : \delta G :,$$

where δ signifies an infinitesimal BRS transformation, and

$$G = \bar{c}\{\partial_\mu A^\mu + \tfrac{1}{2} a B\},$$

B being an auxiliary scalar field and $\bar{c}$ a ghost field. The BRS transformation of the fields is specified by

$$\begin{aligned} \delta A_\mu &= i[Q^{\mathrm{BRS}}, A_\mu] = \partial_\mu c \\ \delta B &= i[Q^{\mathrm{BRS}}, B] = 0 \\ \delta c &= i\{Q^{\mathrm{BRS}}, c\} = 0 \\ \delta \bar{c} &= i\{Q^{\mathrm{BRS}}, \bar{c}\} = B, \end{aligned}$$

where c is another ghost field, and where Q^{BRS} is called the generator of the BRS transformation. Show that δ is nilpotent, i.e., δ^2, working on any field, yields zero. Work out the equations of motion for the various fields. Obtain an integral representation for the generator, Q^{BRS}. Show that the Maxwell equations are valid in a subspace of Fock space defined by the requirement

$$Q^{\mathrm{BRS}} |\Phi\rangle_{\mathrm{phys}} = 0.$$

Show that Q^{BRS} is nilpotent and that it commutes with the Hamiltonian.

Problem 6

Show that the equal-time anticommutation relations of the spinor quantum field and its conjugate momentum imply, and are implied by the anticommutation relations between the creation and annihilation operators. Use arguments of Lorentz covariance to show that the anticommutator of two fields at different points vanishes if the separation between the points is spacelike.

Problem 7

Calculate the Hamiltonian in configuration and in momentum space for

(1) A neutral scalar field,
(2) The electromagnetic field with a general gauge parameter, a,
(3) A spinor field.

Problem 8

Show that the free Dirac Hamiltonian,

$$H = \gamma^0(\vec{\gamma} \cdot \vec{p} + m)\,,$$

is equivalent to the free Dirac Hamiltonian in second quantization, in the sense that both satisfy $H\psi = i\frac{d}{dt}\psi$, where ψ is to be understood as a Dirac wave function in the first case, and as a spinor field in the second case.

Problem 9

Consider the alternative Hermitian Lagrangian for the free spinor field,

$$\mathcal{L} =: \overline{\psi}(\tfrac{1}{2}i\gamma \overleftrightarrow{\partial} - m)\psi :$$

where $\overleftrightarrow{\partial} = \overrightarrow{\partial} - \overleftarrow{\partial}$.

(1) Deduce the Euler-Lagrange equations for ψ and for $\overline{\psi}$.

(2) Does this Lagrangian lead to any new physical features, as compared to the standard non-Hermitian Lagrangian?

Problem 10

Two photons fly apart from one another, and are in oppositely oriented circularly polarized states. One strikes a polaroid film with axis parallel to the unit vector $\vec{a}$, the other a polaroid with axis parallel to the unit vector $\vec{b}$. Let $P_{++}(\vec{a}\,,\vec{b}\,)$ be the joint probability that both photons are transmitted through their respective polaroids. Similarly $P_{--}(\vec{a}\,,\vec{b}\,)$ is the probability that both photons are absorbed by the polaroids, $P_{+-}(\vec{a}\,,\vec{b}\,)$ is the probability that the photon at the $\vec{a}$ polaroid is transmitted, while the other is absorbed, and finally $P_{-+}(\vec{a}\,,\vec{b}\,)$ is the probability that the photon at the $\vec{a}$ polaroid is absorbed, while the other is transmitted. The correlation coefficient is defined by

$$C(\vec{a}\,,\vec{b}\,) = P_{++}(\vec{a}\,,\vec{b}\,) + P_{--}(\vec{a}\,,\vec{b}\,) - P_{+-}(\vec{a}\,,\vec{b}\,) - P_{-+}(\vec{a}\,,\vec{b}\,)\,.$$

Show that quantum electrodynamics predicts

$$C(\vec{a}\,,\vec{b}\,) = \cos 2(\theta_a - \theta_b)\,.$$

Chapter 5

Group Theory and the Noether Theorem

We propose to consider symmetries of a quantum field theory that form a group. A set of such operations, $\{g_i\}$, together with an operation of composition, $\times$, is called a group, $\mathcal{G}$, if the following conditions are satisfied:

(1) The composition of two symmetry operations in $\mathcal{G}$, g_i and g_j, written $g_i \times g_j$, is an element of $\mathcal{G}$.
(2) This composition is associative, i.e., $g_i \times [g_j \times g_k] = [g_i \times g_j] \times g_k$.
(3) There is an element, e, in $\mathcal{G}$, such that $g \times e = g$, for every element g.
(4) Every element, g, has an inverse, g^{-1}, in $\mathcal{G}$, such that $g \times g^{-1} = e$.

The composition rule need not be commutative, but if $g_i \times g_j = g_j \times g_i$ holds for all elements, the group is called Abelian, otherwise it is non-Abelian. As an example of a simple symmetry, the complex scalar field Lagrangian,

$$\mathcal{L} =: (\partial_\mu \phi^\dagger)(\partial^\mu \phi) - m^2 \phi^\dagger \phi :$$

is invariant under multiplication of the field ϕ by a phase factor,

$$\phi \to e^{-i\alpha} \phi .$$

The symmetry group is $U(1)$, the Abelian group of unitary one-dimensional matrices. A gauged version of this $U(1)$ symmetry is none other than quantum electrodynamics with minimal coupling. The operation of 'gauging' a global symmetry, in which a parameter, for example α here, is no longer held fixed, but is allowed to depend upon space-time, has proved very fruitful in high-energy physics. The symmetry can only be maintained by postulating the existence of a gauge field, in this case the photon, with a specified symmetry transformation.

5.1 Rotation Group and $SU(2)$

Consider the set of all rotations in two dimensions, described by orthogonal matrices of the form,

$$U(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} . \tag{5.1}$$

These matrices obey the (matrix multiplication) composition law

$$U(\theta_1) \times U(\theta_2) = U(\theta_1 + \theta_2) = U(\theta_2) \times U(\theta_1) ,$$

and thus form an Abelian group, $SO(2)$. We can use the infinitesimal rotation

$$\delta U(\delta\theta) = 1 - iL\delta\theta \quad \text{where} \quad L = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \sigma_2 , \tag{5.2}$$

to construct the finite rotation,

$$U(\theta) = \lim_{N\to\infty} [\delta U(\theta/N)]^N = \lim_{N\to\infty} [1 - iL\theta/N]^N = \exp[-iL\theta] . \tag{5.3}$$

From the properties $L^{2N} = 1$, $L^{2N+1} = L$, for $N = 1, 2, \cdots$, we find

$$U(\theta) = \exp[-iL\theta] = \cos\theta - iL\sin\theta$$

The matrix L is called the generator of two-dimensional rotations.

In examining the symmetry of physical theories, we are primarily interested in finding and categorizing all irreducible representations of the symmetries. A representation of a group is a set of matrices that has the same multiplication table as the group elements themselves; and an irreducible representation is one that cannot be reduced by a transformation, MUM^{-1}, into block diagonal form. Note that the same M must be used for each each U. The defining representation, Eq.(5.1), is two dimensional, but $SO(2)$ has only one-dimensional irreducible representations. Indeed, consider

$$M = \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} \quad \text{and} \quad M^{-1} = \tfrac{1}{2}\begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix} ,$$

for which the matrix $MU(\theta)M^{-1}$ has the reduced form:

$$MU(\theta)M^{-1} = D(\theta) = \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix} .$$

Thus the matrices of the irreducible representations are $e^{i\theta}$ and $e^{-i\theta}$. The irreducible representations of all Abelian groups are one dimensional.

The group $SO(3)$ of rotations in three dimensions is not Abelian. Let $R(\hat{n},\theta)$ be a rotation by an angle θ about an axis $\hat{n}$. Note that $R(\hat{n},\theta) = R(-\hat{n},-\theta)$. Let us first consider a rotation in the x_1–x_2 plane (that is, about the x_3 axis), and proceed in analogy with Eq.(5.2) to define the generator L_3:

$$\delta U(\hat{3},\delta\theta_3) \;=\; 1 - iL_3\,\delta\theta_3 \quad \text{where} \quad L_3 = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} .$$

Similarly, we may define rotations about the x_1 and x_2 axes:

$$\delta U(\hat{1},\delta\theta_1) \;=\; 1 - iL_1\,\delta\theta_1 \quad \text{where} \quad L_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix}$$

$$\delta U(\hat{2},\delta\theta_2) \;=\; 1 - iL_2\,\delta\theta_2 \quad \text{where} \quad L_2 = \begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{bmatrix} .$$

These matrices obey the familiar commutation relations

$$[L_i, L_j] = i\epsilon_{ijk}L_k \; . \tag{5.4}$$

We construct the infinitesimal rotation matrix for an arbitrary axis, specified by a unit vector, $\hat{n}$:

$$\delta U(\hat{n},\delta\theta) = 1 - i(\vec{L}\cdot\hat{n})\,\delta\theta \; ,$$

and then compute the finite rotational matrix

$$U(\hat{n},\theta) = \exp[-i(\vec{L}\cdot\hat{n})\,\theta] \, . \tag{5.5}$$

One may establish the relations

$$\begin{aligned} (\vec{L}\cdot\hat{n})^{2N+1} &= \vec{L}\cdot\hat{n} \\ (\vec{L}\cdot\hat{n})^{2N+2} &= (\vec{L}\cdot\hat{n})^2 \, , \end{aligned}$$

for $N = 1, 2, 3, \cdots$, and thereby express this rotational matrix in the form

$$U(\hat{n},\theta) = \exp[-i(\vec{L}\cdot\hat{n})\,\theta] = 1 - i(\vec{L}\cdot\hat{n})\,\sin\theta + (\vec{L}\cdot\hat{n})^2\,(\cos\theta - 1) \; .$$

The three matrices L_i constitute the defining representation of the generators of $SO(3)$. They generate, via Eq.(5.5), all elements of the group; and they satisfy the algebra (5.4). The matrices L_i are unitarily equivalent to the matrices discussed in Chapter 5 of Volume 1 (see Problem 5.9 of Volume 1, as well as Problem 5.1 of the present volume). Any representation of $SO(3)$, which may contain matrices of dimension other than 3, has the property that it is isomorphic

to the defining representation, i.e., the matrices have the same multiplication table as in that representation.

All representations of $SO(3)$ lead to representations of the algebra (5.4), but not all representations of the algebra generate representations of $SO(3)$. To see this, we recall from Chapter 3 of Volume 1 that if the J_i satisfy the same algebra (5.4) as the L_i, then the only possible eigenvalues of J^2 are $j(j+1)$, with $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \cdots$. Moreover, for a given j, the possible eigenvalues of J_3 are $m = -j, -j+1, \cdots j$ (in Volume 1 we had not set $\hbar = 1$, as we have done here). The elements, $|jm\rangle$, of a multiplet of vectors satisfy

$$\begin{aligned} J_\pm|jm\rangle &= \sqrt{(j \mp m)(j \pm m + 1)}\, |j, m \pm 1\rangle\,, \\ J_3|jm\rangle &= m\,|jm\rangle\,, \end{aligned}$$

(see Eq.(3.14) of Volume 1). The spin-j representation of the rotation group is defined by

$$D^j_{m'm}(\hat{n}, \theta) = \langle jm'|\exp[-i(\vec{J}\cdot\hat{n})\,\theta]|jm\rangle \tag{5.6}$$

For $j = 0$, $D^0(\hat{n}, \theta) = 1$ for every rotation (every group possesses this "trivial representation"). For $j = \frac{1}{2}$ we use the Pauli spin matrices:

$$J_i = \tfrac{1}{2}\sigma_i\,.$$

It follows from the relation $(\vec{\sigma}\cdot\hat{n})^2 = 1$ that

$$\begin{aligned} D^{\frac{1}{2}} &= \cos\theta/2 - i\vec{\sigma}\cdot\hat{n}\,\sin\theta/2 \\ &= \begin{bmatrix} \cos\theta/2 - in_3\,\sin\theta/2 & (-in_1 - n_2)\,\sin\theta/2 \\ (-in_1 + n_2)\,\sin\theta/2 & \cos\theta/2 + in_3\,\sin\theta/2 \end{bmatrix} \end{aligned} \tag{5.7}$$

The matrices $D^{\frac{1}{2}}(\hat{n}, \theta)$ are unitary, and they have determinant $+1$. Indeed, the set of all such matrices, for $\theta \in [0, 4\pi)$ and all real unit vectors, $\hat{n}$, constitutes the group of special (determinant 1) unitary matrices in two dimensions, written $SU(2)$ for short. Since the elements of $SU(2)$ corresponding to θ and $\theta + 2\pi$, for a given $\hat{n}$, are not identical (they differ by a sign), the defining representation of $SU(2)$ is not a representation of $SO(3)$. On the other hand, the defining representation of $SO(3)$ is a representation of $SU(2)$, although it is not a faithful one. A faithful representation is one in which there is a $1:1$ correspondence: here the correspondence is $1:2$. $SU(2)$ is said to be the covering group of $SO(3)$; all representations of the algebra (5.4) generate representations of $SU(2)$, but not all such representations generate those of $SO(3)$.

5.2 Poincaré Group and $SL(2,C)$

Consider the homogeneous Lorentz group, i.e., all linear transformations of a four-vector field $V^\mu \equiv (V^0, \vec{V})$ that preserve the invariant length:

$$V^2 = (V^0)^2 - \vec{V} \cdot \vec{V} \ . \tag{5.8}$$

For a given vector field we define the 2-dimensional Hermitian matrix

$$v = V^\mu \sigma_\mu = \begin{bmatrix} V^0 + V^3 & V^1 - iV^2 \\ V^1 + iV^2 & V^0 - V^3 \end{bmatrix} ,$$

where $\sigma_\mu \equiv (1, \vec{\sigma})$. If we make the transformation

$$v \to v' = AvA^\dagger \ , \tag{5.9}$$

where A is a 2-dimensional matrix with determinant $+1$, then the matrix v' is also Hermitian, and

$$\det v = (V^0)^2 - \vec{V} \cdot \vec{V} = \det v' \ .$$

Thus every transform Eq.(5.9) preserves the invariant length Eq.(5.8), and so it is a homogeneous Lorentz transform of the vector field V^μ:

$$AV^\mu \sigma_\mu A^\dagger = [\Lambda^\mu_\nu(A) V^\nu] \, \sigma_\mu \ .$$

Furthermore, the product of two such matrices, A and B, of determinant $+1$, is also a matrix of determinant $+1$, and

$$ABV^\mu \sigma_\mu (AB)^\dagger = A[\Lambda^\mu_\nu(B) V^\nu \, \sigma_\mu] A^\dagger = \Lambda^\mu_\rho(A) \Lambda^\rho_\nu(B) V^\nu \, \sigma_\mu \ .$$

Thus the homogeneous Lorentz group $\Lambda(A)$ is locally isomorphic to the group $SL(2,C)$, the elements of which are all the 2-dimensional complex matrices A of determinant $+1$.

Using the Polar Decomposition Theorem (Problem 5.4), we shall express any element of $SL(2,C)$ uniquely in the form

$$A = UP,$$

where U is a unitary matrix of determinant $+1$, and P is a positive definite (Hermitian) matrix of determinant $+1$. In terms of Pauli matrices, we thus write

$$A(\hat{n}, \theta; \hat{m}, r) = \exp[-i(\vec{\sigma} \cdot \hat{n}) \, \theta/2] \, \exp[(-\vec{\sigma} \cdot \hat{m}) \, r/2] \tag{5.10}$$

where the 6 parameters $(\hat{n}, \theta; \hat{m}, r)$ may be chosen independently.

Consider the case that A is unitary, when the Lorentz transformation reduces to a rotation. For the particular choice

$$A(\hat{n},\theta;\hat{m},r) = \begin{bmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{bmatrix},$$

the $SL(2,C)$ matrix is

$$v' = \begin{bmatrix} V^0+V^3 & e^{-i\theta}(V^1-iV^2) \\ e^{i\theta}(V^1+iV^2) & V^0-V^3 \end{bmatrix},$$

which gives the transformation

$$\Lambda(A) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

a rotation by an angle θ about the x_3 axis.

If, on the other hand, the matrix A is Hermitian, the transformation is a true Lorentz transformation, often called a Lorentz boost. For the choice

$$A = \begin{bmatrix} e^{-r/2} & 0 \\ 0 & e^{r/2} \end{bmatrix},$$

the $SL(2,C)$ matrix is

$$v' = \begin{bmatrix} e^{-r}(V^0+V^3) & V^1-iV^2 \\ V^1+iV^2 & e^{r}(V^0-V^3) \end{bmatrix},$$

which gives the transformation

$$\Lambda(A) = \begin{bmatrix} \cosh r & 0 & 0 & -\sinh r \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh r & 0 & 0 & \cosh r \end{bmatrix},$$

a Lorentz boost with velocity $v = \tanh r$ in the x_3-direction.

A Lorentz transformation can be expressed as the composition of a rotation of angle θ about axis $\hat{n}$, and a boost of rapidity r in the direction $\hat{m}$, corresponding to the general $SL(2,C)$ matrix (5.10). Since the matrices $(A,-A)$ correspond to the same homogeneous Lorentz transformation, this correspondence is $2:1$. All representations of the Lorentz group are representations of $SL(2,C)$, but not the other way around. $SL(2,C)$ is the covering group of the Lorentz group, much as $SU(2)$ is the covering group of the rotation group in three dimensions.

For a general infinitesimal transform we have

$$A(\hat{n},\delta\theta;\hat{m},\delta r) = 1 - i(\vec{\sigma}\cdot\hat{n})\delta\theta/2 + (\vec{\sigma}\cdot\hat{m})\delta r/2\,.$$

$$\Lambda(\hat{n},\delta\theta;\hat{m},\delta r) \;=\; 1 - \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -n_3 & n_2 \\ 0 & n_3 & 0 & -n_1 \\ 0 & -n_2 & n_1 & 0 \end{bmatrix}\delta\theta + \begin{bmatrix} 0 & m_1 & m_2 & m_3 \\ m_1 & 0 & 0 & 0 \\ m_2 & 0 & 0 & 0 \\ m_3 & 0 & 0 & 0 \end{bmatrix}\delta r\,.$$

We can read off the generators of the Lorentz group,

$$J_1 = i\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad K_1 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$J_2 = i\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \qquad K_2 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$J_3 = i\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad K_3 = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}.$$

The commutator algebra is

$$[J_i, J_j] = i\epsilon_{ijk}J_k \qquad [J_i, K_j] = i\epsilon_{ijk}K_k$$
$$[K_i, K_j] = -i\epsilon_{ijk}J_k\,. \tag{5.11}$$

We may transform these generators to a new basis by defining

$$M_i = J_i + iK_i \qquad N_i = J_i - iK_i\,.$$

The commutator algebra in this new basis is

$$[M_i, M_j] = i\epsilon_{ijk}M_k \qquad [N_i, N_j] = i\epsilon_{ijk}N_k$$
$$[M_i, N_j] = 0\,. \tag{5.12}$$

Thus the Lorentz group is locally isomorphic to the group $SU(2)_M \otimes SU(2)_N$, with independent generators $(\vec{M}\,,\vec{N}\,)$. There are two Casimir operators (i.e., operators that commute with all the generators), namely $M^2 = M_1^2 + M_2^2 + M_3^2$ and $N^2 = N_1^2 + N_2^2 + N_3^2$.

The representations of the Lorentz group may be labelled by (j_M, j_N), quantum numbers that take on integer or half-integer values. The simplest nontrivial representations, $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$, are 2-dimensional. The Weyl representation $(\frac{1}{2},0)$ is given by Eq.(5.10) for the Lorentz transformation $(\hat{n},\theta;\hat{m},r)$, and the (inequivalent!) representation $(0,\frac{1}{2})$ is given by making the replacement $r \to -r$ in that relation. All finite dimensional representations of the Lorentz group can be generated by taking direct products of the fundamental representations, $(0,\frac{1}{2})$ and $(\frac{1}{2},0)$. For example, four-vector transformation formulae, involving Λ^μ_ν, correspond to the $(\frac{1}{2},\frac{1}{2})$ representation (see Problem 5.6).

Next we consider the Poincaré group, namely Lorentz transformations extended by translations in space-time,

$$V'^\mu = a^\mu + \Lambda^\mu_\nu V^\nu \,,$$

the composition rule for transformations (a,Λ) and (b,Σ) being

$$(a,\Lambda)\times(b,\Sigma) = (a+\Lambda b, \Lambda\Sigma)\,.$$

The generators of the Poincaré transformation are the four-momentum, P^μ, and the generators $\vec{J}$ and $\vec{K}$ of the Lorentz transformation. The commutator algebra of the Poincaré group consists of Eq.(5.11), as well as

$$\begin{aligned}
[P^\mu, P^\nu] &= 0 \\
[P^0, J_i] &= 0 \\
[P_i, J_j] &= i\epsilon_{ijk}P_k \\
[P_i, K_j] &= i\delta_{ij}P^0 \\
[P^0, K_i] &= iP_i\,.
\end{aligned}$$

An infinitesimal Poincaré transformation may be written

$$U(\hat{n},\delta\theta;\hat{m},\delta r;\delta\vec{x},\delta t) = 1 - i(\vec{J}\cdot\hat{n})\delta\theta - (\vec{K}\cdot\hat{m})\delta r - i\vec{P}\cdot\delta\vec{x} - iP_0\delta t\,. \quad (5.13)$$

In terms of the covariant, antisymmetric tensor,

$$J_{\mu\nu} = \begin{bmatrix} 0 & K_1 & K_2 & K_3 \\ -K_1 & 0 & -J_3 & J_2 \\ -K_2 & J_3 & 0 & -J_1 \\ -K_3 & -J_2 & J_1 & 0 \end{bmatrix},$$

we may express the commutation relations in covariant form:

$$\begin{aligned}
[J_{\mu\nu}, J_{\rho\sigma}] &= i(g_{\mu\rho}J_{\sigma\nu} - g_{\mu\rho}J_{\sigma\mu} + g_{\nu\rho}J_{\mu\sigma} - g_{\nu\sigma}J_{\mu\rho}) \\
[P_\mu, J_{\rho\sigma}] &= i(g_{\mu\rho}P_\sigma - g_{\mu\nu}P_\rho) \qquad [P_\mu, P_\nu] = 0\,.
\end{aligned}$$

The full Poincaré Group has two Casimir operators, $P^2 = (P^0)^2 - \vec{P} \cdot \vec{P}$, and the square of the "covariant spin operator" or Pauli-Lubanski vector,

$$W^\mu = \tfrac{1}{2}\epsilon^{\mu\nu\rho\sigma} J_{\nu\rho} P_\sigma \,,$$

the Poincaré invariant being

$$W^2 = W^\mu W_\mu \,.$$

One may show by direct computation that

$$W^\mu = (\vec{J} \cdot \vec{P}\,,\, \vec{K} \times \vec{P} - P^0 \vec{J}) \,,$$

and thus one can calculate W^2 in the rest frame of a particle of mass m, obtaining $-m^2 J^2$. The spin, as well as the mass, are thus Poincaré invariant. Massless particles may be described by their three-momentum, $\vec{P}$, and the component of spin along the momentum vector, their "helicity" (Problem 5.7).

5.3 Noether Theorem

For every continuous symmetry of the Lagrangian density, there is a conserved physical quantity. We shall illustrate this theorem by considering the invariance of the free electromagnetic Lagrangian density, $\mathcal{L}$, under time translations, space translations, and Lorentz transformations. These invariances lead respectively to the conservation of energy, momentum, and angular momentum.

Under a transformation of the space-time points $x \to x'$, and of the fields, $A^\mu(x) \to A'^\mu(x')$, the Lagrangian density is unchanged, i.e.,

$$\mathcal{L}(\partial'^\mu A'^\nu(x')) = \mathcal{L}(\partial^\mu A^\nu(x)) \,. \tag{5.14}$$

Define the infinitesimal quantities

$$\delta A^\mu(x) = A'^\mu(x) - A^\mu(x) \,,$$

and

$$\delta\mathcal{L}(x) = \mathcal{L}(\partial^\mu A'^\nu(x)) - \mathcal{L}(\partial^\mu A^\nu(x)) \,. \tag{5.15}$$

Note that $A'^\mu(x)$ and $\partial^\mu A'^\nu(x)$ occur in these definitions, and not $A'^\mu(x')$ and $\partial'^\mu A'^\nu(x')$. We find therefore that

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial A^\mu}\delta(A^\mu) + \frac{\partial\mathcal{L}}{\partial(\partial^\mu A^\nu)}\delta(\partial^\mu A^\nu) \,, \tag{5.16}$$

where the space-time argument, x, has been suppressed. Now

$$\begin{aligned}\partial^\mu[\frac{\partial\mathcal{L}}{\partial(\partial^\mu A^\nu)}\delta A^\nu] &= \partial^\mu[\frac{\partial\mathcal{L}}{\partial(\partial^\mu A^\nu)}]\delta A^\nu + \frac{\partial\mathcal{L}}{\partial(\partial^\mu A^\nu)}\partial^\mu(\delta A^\nu), \\ &= \frac{\partial\mathcal{L}}{\partial A^\mu}\delta A^\mu + \frac{\partial\mathcal{L}}{\partial(\partial^\mu A^\nu)}\delta(\partial^\mu A^\nu)\,,\end{aligned}$$

where the Euler-Lagrange equation has been used to obtain the last relation. On comparing this with Eq.(5.16), we see that

$$\delta\mathcal{L} = \partial^\mu[\frac{\partial\mathcal{L}}{\partial(\partial^\mu A^\nu)}\delta A^\nu]\,, \tag{5.17}$$

which is the general form of the Noether equation.

To evaluate $\delta\mathcal{L}(x)$ in detail, we replace x' in Eq.(5.14) by x, which means that we must replace x by $\overline{x}$, where the transformation $x \to \overline{x}$ is the inverse of $x \to x'$. So in place of Eq.(5.14), we have

$$\mathcal{L}(\partial^\mu A'^\nu(x)) = \mathcal{L}(\overline{\partial}^\mu A^\nu(\overline{x})).$$

Thus Eq.(5.15) can be written

$$\begin{aligned}\delta\mathcal{L}(x) &= \mathcal{L}(\overline{\partial}^\mu A^\nu(\overline{x})) - \mathcal{L}(\partial^\mu A^\nu(x)) \\ &= \mathcal{L}(\overline{x}) - \mathcal{L}(x) \\ &= (\overline{x}-x)^\rho\partial_\rho\mathcal{L}(x) + O((\overline{x}-x)^2)\,. \end{aligned}\tag{5.18}$$

Under space-time translations, $x^\mu \to x'^\mu = x^\mu + a^\mu$, the transformed field at the transformed point is just the original field at the original point, that is to say, $A'^\mu(x') = A^\mu(x)$, so that $A'^\mu(x) = A^\mu(\overline{x})$, where $\overline{x}^\mu = x^\mu - a^\mu$. Thus

$$A'^\mu(x) = A^\mu(x-a) = A^\mu(x) - a^\nu\partial_\nu A^\mu(x) + O(a^2). \tag{5.19}$$

Hence, for infinitesimal a^μ,

$$\delta A^\mu(x) = -a^\nu\partial_\nu A^\mu(x),$$

$$\delta\mathcal{L}(x) = -a^\nu\partial_\nu\mathcal{L}(x).$$

The Noether equation, Eq.(5.17), takes the form

$$a^\nu\partial_\nu\mathcal{L} = a^\nu\partial_\mu[\frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\rho)}\partial_\nu A_\rho].$$

Since this equation must hold for arbitrary a^ν , it follows that

$$\partial_\mu T^{\mu\nu} = 0, \tag{5.20}$$

where the energy-momentum tensor is defined to be

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\rho)} \partial^\nu A_\rho - g^{\mu\nu}\mathcal{L}.$$

From the explicit form (4.12), we obtain

$$T^{\mu\nu} = -F^{\mu\rho}\partial^\nu A_\rho + \tfrac{1}{4} g^{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} ,$$

(the normal ordering being implicit). The four-momentum of the electromagnetic field is defined by

$$P^\nu = \int d^3x\, T^{0\nu},$$

so that, by using Eq.(5.20), we find

$$\dot{P}^\nu = \int d^3x\, \partial_0 T^{0\nu} = -\int d^3x \partial_i T^{i\nu} = 0,$$

on condition that the fields vanish at spatial infinity. The four quantities, P^ν, are the conserved quantities of the Noether theorem.

The zeroth component is just the Hamiltonian, since

$$P^0 = \int d^3x\, T^{00} = \int d^3x\, [-F^{0\rho}\partial^0 A_\rho + \tfrac{1}{4} F_{\rho\sigma} F^{\rho\sigma}] = \int d^3x\, \mathcal{H} = H.$$

The space components of the field four-momentum are

$$\begin{aligned} P^i = \int d^3x T^{0i} &= -\int d^3x F^{0\rho}\partial^i A_\rho \\ &= -\int d^3x [\partial^0 A^j - \partial^j A^0]\partial^i A_j. \end{aligned}$$

However, the vector product of the electric field and the magnetic induction is

$$\begin{aligned} [\vec{E} \wedge \vec{B}]_i &= [\vec{E} \wedge (\vec{\nabla} \wedge \vec{A})]_i = E_j[\partial_i A^j - \partial_j A^i] \\ &= [\partial_0 A_j - \partial_j A_0][\partial_i A^j - \partial^j A_i] \\ &= -[\partial^0 A^j - \partial^j A^0]\partial^i A_j - \partial^j \{[\partial_0 A_j - \partial_j A_0] A_i\} . \end{aligned} \tag{5.21}$$

where we have used the free Maxwell equation to obtain the last line. Hence

$$\vec{P} = \int d^3x \vec{E} \wedge \vec{B}, \tag{5.22}$$

where $\vec{P} = (P^1, P^2, P^3)$. Note that the integral of the last term in Eq.(5.21), a perfect differential, has been omitted, since it vanishes when sampled by test functions of compact support. The integrand of the above equation, the field momentum density, is the Poynting vector.

The Lagrangian density is invariant under under a Lorentz transformation; but the four potential is not, since it is a four vector. If we write

$$x'^{\mu} = \Lambda^{\mu}_{\ \nu} x^{\nu} \ ,$$

then

$$A'^{\mu}(x') = \Lambda^{\mu}_{\ \nu} A^{\nu}(x) \ .$$

In order to obtain δA, we need to calculate $A'^{\mu}(x)$ rather than $A'^{\mu}(x')$. This is done by replacing x' by x, which means that x must be replaced by $\overline{x} = \Lambda^{-1} x$:

$$A'^{\mu}(x) = \Lambda^{\mu}_{\ \nu} A^{\nu}(\overline{x}).$$

We consider an infinitesimal Lorentz transformation, and its inverse,

$$\Lambda^{\mu}_{\ \nu} = \delta^{\mu}_{\nu} + \varepsilon^{\mu}_{\ \nu} \ , \qquad (\Lambda^{-1})^{\rho}_{\ \sigma} = \delta^{\rho}_{\sigma} - \varepsilon^{\rho}_{\ \sigma}.$$

Hence

$$A'^{\mu}(x) = [\delta^{\mu}_{\nu} + \varepsilon^{\mu}_{\ \nu}][1 - \varepsilon^{\rho}_{\ \sigma} x^{\sigma} \partial_{\rho}] A^{\nu}(x),$$

so that

$$\delta A^{\mu} = \varepsilon^{\mu}_{\ \nu} A^{\nu} - \varepsilon^{\rho}_{\ \sigma} x^{\sigma} \partial_{\rho} A^{\mu}. \tag{5.23}$$

Since the Lagrangian density is Lorentz invariant, we find from Eq.(5.18) that

$$\delta \mathcal{L}(x) = -\varepsilon^{\rho}_{\ \sigma} x^{\sigma} \partial_{\rho} \mathcal{L}(x) \ . \tag{5.24}$$

By inserting Eq.(5.23) and Eq.(5.24) into Eq.(5.17), we obtain

$$\varepsilon_{\rho\sigma} x^{\sigma} \partial^{\rho} \mathcal{L} = \varepsilon_{\rho\sigma} \partial_{\mu} \left[\frac{\partial \mathcal{L}}{\partial(\partial_{\mu} A_{\sigma})} A^{\rho} + \frac{\partial \mathcal{L}}{\partial(\partial_{\mu} A_{\nu})} x^{\sigma} \partial^{\rho} A_{\nu} \right] \ .$$

Since $\varepsilon_{\sigma\rho}$ is antisymmetric, but otherwise arbitrary, it follows that the odd part of its coefficient in the above equation must vanish. Thus

$$\partial_{\mu} J^{\mu\rho\sigma} = 0, \tag{5.25}$$

where

$$\begin{aligned} J^{\mu\rho\sigma} &= \frac{\partial \mathcal{L}}{\partial(\partial_{\mu} A_{\sigma})} A^{\rho} + \frac{\partial \mathcal{L}}{\partial(\partial_{\mu} A_{\nu})} x^{\sigma} \partial^{\rho} A_{\nu} - g^{\mu\rho} x^{\sigma} \mathcal{L} - [\rho \leftrightarrow \sigma] \\ &= L^{\mu\rho\sigma} + S^{\mu\rho\sigma}, \end{aligned}$$

in which the extrinsic (or orbital) angular momentum density is

$$\begin{aligned} L^{\mu\rho\sigma} &= x^\sigma \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} \partial^\rho A_\nu - g^{\mu\rho} \mathcal{L} \right] - [\rho \leftrightarrow \sigma] \\ &= T^{\mu\rho} x^\sigma - T^{\mu\sigma} x^\rho, \end{aligned}$$

and the intrinsic (or spin) angular momentum density is

$$\begin{aligned} S^{\mu\rho\sigma} &= \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\sigma)} A^\rho - (\rho \leftrightarrow \sigma) \\ &= F^{\mu\rho} A^\sigma - F^{\mu\sigma} A^\rho. \end{aligned}$$

We define the angular momentum tensor by

$$J^{\rho\sigma} = \int d^3x J^{0\rho\sigma}.$$

In view of Eq.(5.25),

$$j^{\rho\sigma} = \int d^3x \, \partial_0 J^{0\rho\sigma} = -\int d^3x \, \partial_i J^{i\rho\sigma} = 0,$$

so the components of the angular momentum tensor are time independent (they are the Noether currents that correspond to the invariance of the Lagrangian density under Lorentz transformation). Consider

$$L^{21} = \int d^3x L^{021} = \int d^3x \left[x^1 T^{02} - x^2 T^{01} \right].$$

Since T^{0i} are the momentum densities, we see that L^{21} is the third component of the orbital angular momentum, which we often write L^3. Then $S^3 \equiv S^{21}$ is the third component of the intrinsic, or spin angular momentum. We write $J^{21} \equiv J^3 = L^3 + S^3$. In a similar way, by permuting $1, 2, 3$ cyclically, we define the other components: $J^1 = L^1 + S^1$ and $J^2 = L^2 + S^2$.

5.4 Parity, Time Reversal and Charge Conjugation

So far in this chapter, we have looked at symmetries of the Lagrangian that depend continuously on a parameter; but we turn now to discrete symmetries. Consider first the inversion of the position vector, $\vec{r} \to -\vec{r}$, applied to the infinitesimal Poincaré transformation (5.13). Under this operation the energy, P^0, and the angular momentum, $\vec{J}$, remain unchanged, but the momentum, $\vec{P}$, and the Lorentz boost, $\vec{K}$, change sign. The angular momentum should indeed generate spatial rotations in a direction that is unchanged under space

inversion, but the direction of an infinitesimal boost should be reversed, so the parity operator on Fock space must satisfy

$$\begin{aligned} \mathcal{P}P^0\mathcal{P}^{-1} &= P^0 \\ \mathcal{P}\vec{J}\,\mathcal{P}^{-1} &= \vec{J} \\ \mathcal{P}\vec{P}\,\mathcal{P}^{-1} &= -\vec{P} \\ \mathcal{P}\vec{K}\,\mathcal{P}^{-1} &= -\vec{K}\,. \end{aligned} \tag{5.26}$$

The four-momentum of one-particle states obeys $(p^0, \vec{p}) \to (p^0, -\vec{p})$. To explore the effect of space inversion on internal degrees of freedom, consider an irreducible basis set of states $|j_M, j_{M3}; j_M, j'_{M3}\rangle$, that transform according to the $(\, j_M, j_M \,)$ irreducible representation of the Lorentz group. It follows from the Lorentz group algebra and (5.26) that

$$\mathcal{P}|j_M, j_{M3}; j_M, j'_{M3}\rangle = \eta_P |j_M, j_{M3}; j_M, j'_{M3}\rangle\,,$$

where $\eta_P = \pm 1$ is the intrinsic parity for the basis set. Dirac spinors, which transform according to the $(\frac{1}{2}, \frac{1}{2})$ representation of the Lorentz group, fall into this category. However, an irreducible basis set for the representation (j_M, j_N), with $M \neq N$, is mapped into a different irreducible representation of the Lorentz group, in fact

$$\mathcal{P}|j_M, j_{M3}; j_N, j_{N3}\rangle \equiv |j_N, j_{N3}; j_M, j_{M3}\rangle$$

These states are therefore not eigenstates of the parity operator. Left- and right-handed Weyl spinors, which transform according to $(0, \frac{1}{2})$ and $(\frac{1}{2}, 0)$, respectively, are of this type (Problem 5.6).

The effect of the parity transformation on the second-quantized free Dirac field, $\psi(t, \vec{x}\,)$, is realized by

$$\mathcal{P}\psi(t, \vec{x}\,)\mathcal{P}^{-1} = \gamma^0 \psi(t, -\vec{x}\,)\,.$$

For spinors, we see from the explicit representation (3.80) that

$$\gamma^0 u(\vec{p}, s) = u(-\vec{p}, s) \qquad \gamma^0 v(\vec{p}, s) = -v(-\vec{p}, s).$$

From Eq.(4.26) we find therefore that

$$\begin{aligned} \mathcal{P}b(\vec{p}, s)\,\mathcal{P}^{-1} &= b(-\vec{p}, s) \\ \mathcal{P}d(\vec{p}, s)\,\mathcal{P}^{-1} &= -d(-\vec{p}, s)\,, \end{aligned}$$

so the annihilation (and creation) operators for spin-half particles and antiparticles have opposite intrinsic parity.

Time Reversal

Next we consider the operation of time reversal, $t \to t' = -t$. Unlike space inversion, time reversal cannot properly be described by a unitary operator, $\mathcal{T}$. The reason is that, if $\mathcal{T}$ were a linear, unitary operator on Fock space, then the Hamilton-Heisenberg equation for a scalar field,

$$\frac{d}{dt}\phi(t,\vec{x}) = i[H, \phi(t,\vec{x})]\,,$$

would give the desirable Hamilton-Heisenberg equation

$$\frac{d}{dt'}\phi(t',\vec{x}) = i[H, \phi(t',\vec{x})]\,,$$

only on condition that $\mathcal{T}^\dagger H \mathcal{T} = -H$, which would lead to negative energies in a time-reversed state. To avoid this undesirable consequence, we make $\mathcal{T}$ anti-linear, i.e., it is required to satisfy

$$\begin{aligned} \mathcal{T}\big[|\phi_1\rangle + |\phi_2\rangle\big] &= \mathcal{T}|\phi_1\rangle + \mathcal{T}|\phi_2\rangle \\ \text{but} \qquad \mathcal{T}[c|\phi\rangle] &= c^*\,\mathcal{T}|\phi\rangle\,, \end{aligned}$$

where c is a complex number. In particular, $\mathcal{T}\big[i|\phi\rangle\big] = -i\mathcal{T}|\phi\rangle$. Instead of $\mathcal{T}^\dagger H \mathcal{T} = -H$, we have $\mathcal{T}^\dagger i H \mathcal{T} = -i\mathcal{T}^\dagger H \mathcal{T}$ and so $\mathcal{T}^\dagger H \mathcal{T} = H$. The operator $\mathcal{T}$ is also antiunitary, meaning that

$$\langle \mathcal{T}\psi | \mathcal{T}\phi\rangle = \langle \psi|\phi\rangle^*\,.$$

To see why this is necessary, let $|\phi\rangle$ be a one-particle state of definite momentum, p, so that the configuration wave function is

$$\langle x|\phi\rangle = \exp[i(\vec{p}\cdot\vec{r} - \omega_p t)]\,.$$

If $\mathcal{T}$ were unitary, we would obtain the following time-reversed wave function:

$$\langle x|\mathcal{T}\phi\rangle = \langle \mathcal{T}x|\phi\rangle = \exp[i(\vec{p}\cdot\vec{r} + \omega_p t)]\,,$$

again with unacceptable negative energy. With an *antiunitary* $\mathcal{T}$, we find

$$\langle x|\mathcal{T}\phi\rangle = \langle \mathcal{T}\mathcal{T}x|\mathcal{T}\phi\rangle = \langle \mathcal{T}x|\phi\rangle^* = \exp[-i(\vec{p}\cdot\vec{r} + \omega_p t)]\,,$$

which has the same energy as the original state and a momentum that is reversed, as should be the case.

Like unitary operators, antiunitary operators also satisfy the requirement

$$|\langle \mathcal{T}\psi|\mathcal{T}\phi\rangle|^2 = |\langle\psi|\phi\rangle|^2\,,$$

and are suitable for describing symmetry operations. Because of the factor i in the terms involving $P^0, \vec{P}$ and $\vec{J}$ in the infinitesimal Poincaré transformation

(5.13), the transformations of these generators acquire an extra minus sign, so we obtain the relations

$$\begin{aligned} \mathcal{T}P^0\mathcal{T}^{-1} &= P^0 \\ \mathcal{T}\vec{J}\mathcal{T}^{-1} &= -\vec{J} \\ \mathcal{T}\vec{P}\mathcal{T}^{-1} &= -\vec{P} \\ \mathcal{T}\vec{K}\mathcal{T}^{-1} &= \vec{K}\,. \end{aligned} \tag{5.27}$$

Consider the spin multiplets of a particle of mass m and spin s at rest, $|\vec{0},\, s, m_s\rangle$. According to these relations, the operator $\mathcal{T}$ has the effect of reversing the three-component of spin:

$$\begin{aligned} \mathcal{T}J^2\,|\vec{0},\, s,\, m_s\rangle &= s(s+1)\mathcal{T}\,|\vec{0},\, s,\, m_s\rangle \\ \mathcal{T}J_3\,|\vec{0},\, s,\, m_s\rangle &= -m_s\mathcal{T}\,|\vec{0},\, s,\, m_s\rangle \end{aligned}$$

In other words, the effect of time reversal on these states is equivalent to a rotation by $R(\hat{2},\pi)$. Since

$$R(\hat{2},\pi)|\vec{0},\, s,\, m_s\rangle = (-1)^{s+m_s}\,|\vec{0},\, s,\, -m_s\rangle\,,$$

we take

$$\mathcal{T}\,|\vec{0},\, s,\, m_s\rangle = (-1)^{s+m_s}\eta_T\,|\vec{0},\, s,\, -m_s\rangle$$

where $\eta_T = \pm 1$ is the intrinsic time-parity. Let us apply a Lorentz boost $\Lambda(\vec{p})$ to this state to obtain

$$|\vec{p}, s,\, m_s\rangle = U[\Lambda(\vec{p})]\,|\vec{0}, s,\, m_s\rangle\,.$$

Note that $\mathcal{T}\,U[\Lambda(\vec{p})]\,\mathcal{T} = U[\Lambda(-\vec{p})]$, so that

$$\mathcal{T}\,|\vec{p}, s,\, m_s\rangle = (-1)^{s+m_s}\eta_T|-\vec{p}, s,\, -m_s\rangle\,.$$

For massless particles we can specify only the momentum and helicity (the component of spin along the momentum) (Problem 5.7). For the eigenstate of three-momentum $\vec{p}$ and helicity σ, $|\vec{p},\sigma\rangle$, the operator $\mathcal{T}$ changes the direction of momentum $\vec{p}$ and spin $\vec{J}$, but not of the helicity σ. In other words, we may take

$$\mathcal{T}\,|\vec{p},\sigma\rangle = e^{i\pi\sigma}\,|-\vec{p},\sigma\rangle\,.$$

For a free Dirac state, with or without mass, $\psi(t,\vec{x}\,)$, the time-reversal operator can be realized by

$$\mathcal{T}\psi(t,\vec{x}\,)\mathcal{T}^{-1} = \gamma^3\gamma^1\,\psi(-t,\vec{x}\,)\,.$$

Since $\left[\gamma^3\gamma^1\right]^{-1} = -\gamma^3\gamma^1 = \gamma^1\gamma^3$, and further

$$\gamma^3\gamma^1\gamma^0\gamma^1\gamma^3 = \gamma^0 \qquad \gamma^3\gamma^1\gamma^1\gamma^1\gamma^3 = -\gamma^1$$
$$\gamma^3\gamma^1\gamma^2\gamma^1\gamma^3 = \gamma^2 \qquad \gamma_3\gamma^1\gamma^3\gamma^1\gamma^3 = -\gamma^3\,,$$

it follows that

$$\begin{aligned}
\gamma^3\gamma^1 u^*(\vec{p},1) &= u(-\vec{p},2)\\
\gamma^3\gamma^1 u^*(\vec{p},2) &= -u(-\vec{p},1)\\
\gamma^3\gamma^1 v^*(\vec{p},1) &= -v(-\vec{p},2)\\
\gamma^3\gamma^1 v^*(\vec{p},2) &= v(-\vec{p},1)\,.
\end{aligned}$$

Correspondingly, for the creation and annihilation operators,

$$\mathcal{T}b(\vec{p},1)\mathcal{T}^{-1} = b(-\vec{p},2) \qquad \mathcal{T}b(\vec{p},2)\mathcal{T}^{-1} = -b(-\vec{p},1)$$
$$\mathcal{T}d(\vec{p},1)\mathcal{T}^{-1} = -d(-\vec{p},2) \qquad \mathcal{T}d(\vec{p},2)\mathcal{T}^{-1} = d(-\vec{p},1)\,.$$

Hence the time-reversal operator reverses the three momentum of particles and antiparticles, and it flips the spin, i.e., the spin direction is also reversed, as should be the case. Note that particles and antiparticles are *not* interchanged by the time-reversal operator alone. To do that, one needs the charge-conjugation operator.

Charge Conjugation

The charge-conjugation operator, $\mathcal{C}$, converts particle states into antiparticle states, and we require

$$\begin{aligned}
\mathcal{C}b(\vec{p},s)\,\mathcal{C}^{-1} &= (-1)^{s+1}d(\vec{p},s)\\
\mathcal{C}d(\vec{p},s)\,\mathcal{C}^{-1} &= (-1)^{s+1}b(\vec{p},s)\,,
\end{aligned}$$

in words, the annihilation (and creation) operators for spin-half particles are transformed into those for the antiparticles, and vice versa, without change of momentum or spin.

We see from the explicit representation (3.80) that

$$i\gamma^2 u(\vec{p},s) = (-1)^{s+1}v^*(\vec{p},s) \qquad i\gamma^2 v(\vec{p},s) = (-1)^{s+1}u^*(\vec{p},s)\,,$$

and this leads to

$$\mathcal{C}\psi(x)\mathcal{C}^{-1} = -i\left[\psi^\dagger(x)\gamma^2\right]^T = -i\left[\overline{\psi}(x)\gamma^0\gamma^2\right]^T\,,$$

whre the superscript, T, here indicates the transpose of the spinor matrices in question. Since the transpose of $\psi^\dagger$ is a spinor field, rather than the transpose

of a spinor field, we see that $\mathcal{C}$, unlike $\mathcal{T}$, is a linear operator. The free Dirac Hamiltonian is invariant under charge conjugation, whereas the charge operator (4.30) changes sign.

Finally, consider the following types of factors involving Dirac fields, which may be present in a theory

$$\begin{aligned} M(x) &= :\psi^\dagger(x) m \psi(x): && \text{mass} \\ S(x) &= :\overline{\psi}(x)\psi(x): && \text{scalar} \\ V^\mu(x) &= :\overline{\psi}(x)\gamma^\mu\psi(x): && \text{vector} \\ T^{\mu\nu}(x) &= :\overline{\psi}(x)\sigma^{\mu\nu}\psi(x): && \text{tensor} \\ A^\mu(x) &= :\overline{\psi}(x)\gamma^5\gamma^\mu\psi(x): && \text{pseudovector} \\ P(x) &= :i\overline{\psi}(x)\gamma^5\psi(x): && \text{pseudoscalar} \end{aligned}$$

Here is a table indicating the transformation of these factors under the discrete symmetries $\mathcal{P}$, $\mathcal{C}$, $\mathcal{T}$, and $\mathcal{PCT}$; note that $\tilde{x} \equiv (x^0, -\vec{x})$:

	$M(x)$	$S(x)$	$V^\mu(x)$	$T^{\mu\nu}(x)$	$A^\mu(x)$	$P(x)$
$\mathcal{P}$	$M(\tilde{x})$	$S(\tilde{x})$	$V^\mu(\tilde{x})$	$T^{\mu\nu}(\tilde{x})$	$-A^\mu(\tilde{x})$	$-P(\tilde{x})$
$\mathcal{C}$	$M(x)$	$S(x)$	$-V^\mu(x)$	$-T^{\mu\nu}(x)$	$A^\mu(x)$	$P(x)$
$\mathcal{T}$	$M(-\tilde{x})$	$S(-\tilde{x})$	$V^\mu(-\tilde{x})$	$-T^{\mu\nu}(-\tilde{x})$	$A^\mu(-\tilde{x})$	$-P(-\tilde{x})$
$\mathcal{PCT}$	$M(-x)$	$S(-x)$	$-V^\mu(-x)$	$T^{\mu\nu}(-x)$	$-A^\mu(-x)$	$P(-x)$

The operator $\mathcal{PCT}$ leaves the Lagrangian invariant for any Poincaré invariant local quantum field theory involving scalar, vector, and spinor fields. This result, known as the Pauli-Villars theorem, remains valid when appropriate forms of

higher spin fields are introduced. The effect of $\mathcal{PCT}$ on the Lagrangian density $\mathcal{L}(x)$ amounts to the following:

(1) On all space-time arguments, make the replacement $x^\mu \to -x^\mu$.
(2) For consistency with (1), all derivatives must change sign: $\partial^\mu \to -\partial^\mu$.
(3) A vector field transforms like the gradient of a scalar field: $V^\mu \to -V^\mu$.
(4) A bilinear form with $\overline{\psi}\cdots\psi$ changes sign by $(-1)^s$, where s is the number of γ-matrices that lie between the spinor fields.
(5) All constants are complex-conjugated: $g \to g^*$.

The theorem is a consequence of this algorithm. However, one should be aware that the relevance of the $\mathcal{PCT}$ theorem is predicated upon the assumption that physics is described by a Poincaré invariant, local quantum field theory.

5.5 Exercises

Problem 1

Show that the defining representation, $\tilde{R}$, of $SO(3)$,

$$U(\hat{n},\theta) = \exp[-i(\vec{L}\cdot\hat{n})\,\theta] = 1 - i(\vec{L}\cdot\hat{n})\,\sin\theta + (\vec{L}\cdot\hat{n})^2\,(\cos\theta - 1)\,,$$

$$L_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix} \quad L_2 = \begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{bmatrix} \quad L_3 = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

is equivalent to the representation, R, obtained by the replacement $L_i \to J_i$, where

$$J_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad J_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix} \quad J_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

That is, find a unitary matrix, M, independent of θ and $\hat{n}$, that transforms one form into the other:

$$M\,\tilde{U}(\hat{n},\theta)\,M^\dagger = U(\hat{n},\theta)\,.$$

Use these results to show that

$$\exp[-i(\vec{L}\cdot\hat{n})\,\theta] = 1 - i(\vec{L}\cdot\hat{n})\,\sin\theta + (\vec{L}\cdot\hat{n})^2\,(\cos\theta - 1)\,.$$

Problem 2

Show that the direct product $D^{\frac{1}{2}}(R) \otimes D^{\frac{1}{2}}(R)$ is unitarily equivalent to the direct sum $D^0(R) \oplus D^1(R)$. That is, find the (rotation-independent) Clebsch-Gordan matrix M such that

$$M\, D^{\frac{1}{2}}(R) \otimes D^{\frac{1}{2}}(R) = D^0(R) \oplus D^1(R)\, M\ .$$

Remark: The standard form for these irreducible representations [Condon-Shortley phase convention] is

$$D^j_{m'm}(\hat{n}, \theta) = \langle jm' | \exp[-i(\vec{J} \cdot \hat{n})\, \theta] | jm \rangle$$

In particular, with the Condon-Shortley phase convention,

$$D^{\frac{1}{2}} = \cos\theta/2 - i(\vec{\sigma} \cdot \hat{n}) \sin\theta/2 = \begin{bmatrix} \cos\theta/2 - in_3 \sin\theta/2 & -in_- \sin\theta/2 \\ -in_+ \sin\theta/2 & \cos\theta/2 + in_3 \sin\theta/2 \end{bmatrix}$$

Problem 3

Show that the composition of two rotations $R(\hat{m}_1, \phi_1) \times R(\hat{m}_2, \phi_2)$ yields a rotation $R(\hat{n}, \theta)$, using the spin-$\frac{1}{2}$ representations $D^{\frac{1}{2}}$ to show that

$$\begin{aligned} \cos\theta/2 &= \cos\phi_1/2 \cos\phi_2/2 - \hat{m}_1 \cdot \hat{m}_2 \sin\phi_1/2 \sin\phi_2/2 \\ \hat{n} \sin\theta/2 &= -\hat{m}_1 \sin\phi_1/2 \cos\phi_2/2 - \hat{m}_2 \sin\phi_2/2 \cos\phi_1/2 \\ &\qquad + (\hat{m}_1 \wedge \hat{m}_2) \sin\phi_1/2 \sin\phi_2/2 \end{aligned}$$

Note that the angle of rotation θ is symmetric under the interchange of order of rotations, whereas the axis $\hat{n}$ is not, in general.

Problem 4

Show that any finite-dimensional, nonsingular square matrix, A, may be expressed uniquely as the product of a unitary matrix U and a positive definite matrix P,

$$A = UP\,.$$

Problem 5

Consider the composition of Lorentz boosts in different directions:

$$L(\vec{0}, 0; \hat{m}_1, r_1) \times L(\vec{0}, 0; \hat{m}_2, r_2) = L(\hat{n}, \theta; \hat{m}, r)\ .$$

Using the Weyl representation,

$$A(\hat{n}, \theta; \hat{m}, r) = \exp[-i(\vec{\sigma} \cdot \hat{n})\, \theta/2]\ \exp[-(\vec{\sigma} \cdot \hat{m})\, r/2]$$

express the composite transformation in terms of the parameters $(\hat{n}, \theta; \hat{m}, r)$.

Problem 6

The representation $D^{(\frac{1}{2},0)}(\hat{n},\theta;\hat{m},r)$, of the Lorentz group is

$$\exp[-i(\vec{\sigma}\cdot\hat{n})\,\theta/2]\;\exp[(-\vec{\sigma}\cdot\hat{m})\,r/2]\;;$$

whereas the representation $D^{(0,\frac{1}{2})}(\hat{n},\theta;\hat{m},r)$, is obtained from that expression by making the replacement $r\to -r$. Show that the direct product representation

$$\hat{D} = D^{(\frac{1}{2},0)}\otimes D^{(0,\frac{1}{2})}$$

is isomorphic to the transformation matrix Λ for a four-vector:

$$V'^{\mu} = \Lambda^{\mu}_{\nu}(\hat{n},\theta;\hat{m},r)V^{\nu}\,.$$

Problem 7

The Pauli-Lubanski vector is defined as

$$W^{\mu} = \tfrac{1}{2}\epsilon^{\mu\nu\rho\sigma}J_{\nu\rho}P_{\sigma} = (W^0,\vec{W}\,)\,.$$

Show that $W^2 = W^{\mu}W_{\mu}$ and $P^2 = P^{\mu}P_{\mu}$ commute with all the generators of the Poincaré group.

(1) Show that, in the rest frame of a particle of mass m, $W^2 = -m^2J^2$, so that the mass and spin are good quantum numbers.

(2) For a massless particle, restrict consideration to eigenstates of $P_{\mu} = (p_0,0,0,p_0)$. With this restriction show that the following relations are satisfied.

$$\begin{aligned} W^0 &= W^3 \\ [W^1,W^2] &= 0 \\ [W^3,W^1] &= ip_0W^2 \\ [W^3,W^2] &= -ip_0W^1\,. \end{aligned}$$

Show that $\vec{W}/p_0$ satisfies the same algebra as does the two-dimensional Euclidean group that has the generators P_1,P_2,J_3. For this Euclidean group, there is a set of eigenstates $|0,s\rangle$, for which

$$\begin{aligned} P_1\,|0,s\rangle &= 0 \\ P_2\,|0,s\rangle &= 0 \\ J_3\,|0,s\rangle &= s\,|0,s\rangle\,. \end{aligned}$$

Show that

$$W^{\mu}\,|p_0,0,s\rangle = sP^{\mu}\,|p_0,0,s\rangle\,.$$

Problem 8

(1) Show that the Lagrangian density

$$\mathcal{L} =: \bar{\psi}(x)(i\gamma\partial - m)\psi(x) :$$

is invariant under the global $U(1)$ transformation

$$\psi'(x) = \mathrm{e}^{-ie\theta}\psi(x)\,.$$

Show that the conserved Noether current is $j^\mu(x) = e : \bar{\psi}(x)\gamma^\mu\psi(x) :$

(2) Compute the total charge and write it in terms of creation and annihilation operators.

(3) In the case that θ does depend on space and time, the transformation is said to be a local, or *gauge* transformation. Show how the minimal substitution restores the invariance of the Lagrangian density in this case.

(4) What is now the Noether current, and is it conserved?

Problem 9

Suppose that an electromagnetic radiation field is specified by

$$A^1 = \Delta\cos k(x^0 - x^3)\,, \qquad A^2 = \Delta\sin k(x^0 - x^3)\,, \qquad A^0 = 0 = A^3\,,$$

and that this radiation falls upon and is absorbed by a surface orthogonal to the x^3 axis. Calculate the transfer to the surface, per unit area per unit time, of energy, momentum, orbital angular momentum and intrinsic angular momentum (spin). What can you conclude about the properties of the photon?

Problem 10

Assume these transformation properties of the electron and positron annihilation operators under parity inversion, charge conjugation and time reversal:

$$\begin{aligned}
\mathcal{P}b(\vec{p},s)\,\mathcal{P}^{-1} &= b(-\vec{p},s)\\
\mathcal{P}d(\vec{p},s)\,\mathcal{P}^{-1} &= -d(-\vec{p},s)\\
\mathcal{C}b(\vec{p},1)\,\mathcal{C}^{-1} &= d(\vec{p},1)\\
\mathcal{C}b(\vec{p},2)\,\mathcal{C}^{-1} &= -d(\vec{p},2)\\
\mathcal{T}b(\vec{p},1)\,\mathcal{T}^{-1} &= b(-\vec{p},2)\\
\mathcal{T}d(\vec{p},1)\,\mathcal{T}^{-1} &= -d(-\vec{p},2)\,.
\end{aligned}$$

Calculate the transformed fields, $\mathcal{P}\psi(x)\,\mathcal{P}^{-1}$, $\mathcal{C}\psi(x)\,\mathcal{C}^{-1}$ and $\mathcal{T}\psi(x)\,\mathcal{T}^{-1}$. How do the Hamiltonian and the charge operators transform? Show that applying time reversal twice to $\psi(x)$ produces $-\psi(x)$.

Chapter 6

Scattering Theory and Feynman Graphs

A theory of noninteracting electrons and photons is described by the sum of the Dirac and the electromagnetic Lagrangians. An interacting theory for the two is obtained through the minimal coupling prescription,

$$\mathcal{L} =: \overline{\psi}(i\gamma^\mu \mathcal{D}_\mu - m)\psi - \tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} :$$

where the *covariant derivative* is defined as

$$\mathcal{D}_\mu = \partial_\mu + ieA_\mu \,, \tag{6.1}$$

see Sec. **2.2**, but recall that here $c = 1 = \hbar$. This Lagrangian density is invariant under the gauge transformation

$$A_\mu \longrightarrow A_\mu + \partial_\mu G \qquad\qquad \psi \longrightarrow \mathrm{e}^{-ieG}\psi \,,$$

where $G(x)$ is an arbitrary scalar field. The idea of inducing interactions from the requirement of gauge invariance has proved to be very fruitful — this is the simplest (Abelian) example of the principle.

To implement the quantization of the electromagnetic field, we introduced also a gauge-breaking term, which does not respect the gauge invariance of the rest of the Lagrangian, yielding

$$\mathcal{L} =: \overline{\psi}(i\gamma^\mu \mathcal{D}_\mu - m)\psi - \tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2a}(\partial_\mu A^\mu)^2 :$$

Gauge invariance is not valid on the whole of Fock space; but it does hold on the physical subspace.

We may write the Lagrangian as the sum of a part that is quadratic in the fields, say $\mathcal{L}_f$, which leads to a linear (free) contribution to the equations of motion, and the remainder, say $\mathcal{L}_i$, which is of higher order in the fields, leading

to a nonlinear (interaction) contribution. Thus

$$\mathcal{L}_f =: \overline{\psi}(i\gamma^\mu \partial_\mu - m)\psi - \tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2a}(\partial_\mu A^\mu)^2 :$$

and

$$\mathcal{L}_i = -e : \overline{\psi}\gamma^\mu A_\mu \psi : \tag{6.2}$$

In the Feynman gauge, $a = 1$, the corresponding Euler-Lagrange equations are

$$[i\gamma\partial - m]\,\psi = e\gamma^\mu A_\mu \psi\,,$$

$$\partial^2 A^\mu = e\overline{\psi}\gamma^\mu \psi\,.$$

This distinction between the quadratic part of the Lagrangian and the remainder is of paramount importance for the development of perturbation theory. In the modern theories with non-Abelian gauge structure for the weak, electromagnetic and strong interactions, this basic distinction between a quadratic part, which can be treated exactly, and a cubic and quartic part, which has to be treated perturbatively, remains operative. For the rest of this chapter, we shall treat a simpler, scalar interacting theory. This will serve as an introduction to the more interesting gauge theories like quantum electrodynamics.

6.1 Asymptotic Fields

For simplicity we will introduce the general formalism of scattering for a neutral scalar theory. An example is that of neutral π-mesons, or pions, interacting with one another by a contact interaction (this is a so-called effective theory, approximately valid only at low energies — at a deeper level, it is replaced by QCD, in which the pion is a bound state of a quark and an antiquark, interacting by exchange of gluons).

The effective theory can be written

$$\mathcal{L} = \mathcal{L}_f + \mathcal{L}_i \tag{6.3}$$

with

$$\mathcal{L}_f = \tfrac{1}{2} : \partial_\mu\phi\,\partial^\mu\phi - m_0^2\phi^2 :$$

and

$$\mathcal{L}_i = -\lambda : \phi^4 :$$

which describes a pion-pion self-interaction term. Here m_0 is called the bare mass — as we shall see, it is not equal to the physically measured pion mass.

A scattering process is ideally described as follows: at time $t = -\infty$, there were only stable particles, far apart from one another, so that their mutual interactions can be ignored. As time goes on, the particles can come together and interact; but at $t = \infty$, all unstable particles have decayed, and there will be only stable particles far apart again, with negligible mutual interaction. The "in" states describe the situation at $t = -\infty$, before the experiment, and the "out" states describe the situation after the scattering, at $t = \infty$. We want to calculate the probability amplitudes between the in and out states (the so-called scattering, or S-matrix), in order to be able to predict observables like the cross-section for scattering.

The supposed state of the system at $t = -\infty$ involves no interactions between different particles, because they are far apart from one another. However, these particles of the in state are not the particles of the free theory that would be described by the Lagrangian $\mathcal{L}_f$ alone, for they have self-interactions caused by $\mathcal{L}_i$. An isolated physical particle can emit and reabsorb virtual particles, as we shall see later in detail. The effect of this cloud of virtual particles is one of inertia: the effective mass of the physical particles is not the bare mass, m_0; rather there is a shift to, say, $m^2 = m_0^2 + \delta m^2$, the square of the physically measurable mass.

At $t = -\infty$, we define a free field, ϕ_{in}, that satisfies a free Klein-Gordon equation

$$(\partial^2 + m^2)\phi_{\mathsf{in}} = 0\,.$$

This can be Fourier-analyzed in the usual way, in terms of annihilation and creation operators, $a_{\mathsf{in}}(\vec{k})$ and $a_{\mathsf{in}}^\dagger(\vec{k})$. In a similar way, at $t = \infty$, we define a free field, ϕ_{out}, that satisfies

$$(\partial^2 + m^2)\phi_{\mathsf{out}} = 0\,.$$

A Fock space can be built up from the vacuum state, the one-particle states obtained by allowing the operator $a_{\mathsf{in}}^\dagger(\vec{k})$ to act on the vacuum, for any value of the momentum $\vec{k}$, the corresponding two-particle states, three-particle states, and so on indefinitely. Similarly, a Fock space can be built up by using the out instead of the in operators: it is a fundamental assumption that these two Fock spaces are identical. It is assumed further that the vacuum and the one-particle states are stable, in the sense that

$$|0;\mathsf{in}\rangle = |0;\mathsf{out}\rangle = |0\rangle \qquad |\vec{k};\mathsf{in}\rangle = |\vec{k};\mathsf{out}\rangle = |\vec{k}\rangle\,. \tag{6.4}$$

However,

$$|\vec{k}_1, \vec{k}_2; \mathsf{in}\rangle \neq |\vec{k}_1, \vec{k}_2; \mathsf{out}\rangle\,,$$

because of the possibility of scattering. Indeed, the two-particle sector of the S-matrix is defined by

$$|\vec{k}_1, \vec{k}_2; \mathsf{in}\rangle = S|\vec{k}_1, \vec{k}_2; \mathsf{out}\rangle\,, \tag{6.5}$$

and more generally, the n-particle sector yields

$$|\vec{k}_1, ...\vec{k}_n; \mathsf{in}\rangle = S|\vec{k}_1, ...\vec{k}_n; \mathsf{out}\rangle\,. \tag{6.6}$$

Thus S maps any out vector on to the corresponding in vector, with the same momenta. More succinctly, we may write

$$|\alpha; \mathsf{in}\rangle = S|\alpha; \mathsf{out}\rangle\,, \tag{6.7}$$

where α represents any state.

We take the in and out states to be orthonormal,

$$\langle\alpha; \mathsf{in}|\beta; \mathsf{in}\rangle = \delta_{\alpha\beta} = \langle\alpha; \mathsf{out}|\beta; \mathsf{out}\rangle\,,$$

which of course is shorthand for a continuum normalization in terms of Dirac distributions. It is assumed that the in and out Fock spaces are identical, being spanned both by the in and by the out state vectors:

$$\sum_\alpha |\alpha; \mathsf{in}\rangle\langle\alpha; \mathsf{in}| = 1 = \sum_\alpha |\alpha; \mathsf{out}\rangle\langle\alpha; \mathsf{out}|\,.$$

From this it follows that S is unitary:

$$\begin{aligned} S^\dagger S &= \sum_{\alpha,\beta} |\alpha; \mathsf{out}\rangle\langle\alpha; \mathsf{out}|S^\dagger S|\beta; \mathsf{out}\rangle\langle\beta; \mathsf{out}| \\ &= \sum_{\alpha,\beta} |\alpha; \mathsf{out}\rangle\langle\alpha; \mathsf{in}|\beta; \mathsf{in}\rangle\langle\beta; \mathsf{out}| \\ &= \sum_\alpha |\alpha; \mathsf{out}\rangle\langle\alpha; \mathsf{out}| = 1\,. \end{aligned}$$

Thus

$$\begin{aligned} S^\dagger|\alpha; \mathsf{in}\rangle = S^\dagger S|\alpha; \mathsf{out}\rangle &= |\alpha; \mathsf{out}\rangle \\ \langle\alpha; \mathsf{in}|S &= \langle\alpha; \mathsf{out}| \end{aligned} \tag{6.8}$$

and hence

$$\begin{aligned} SS^\dagger &= \sum_{\alpha,\beta} |\alpha;\mathsf{in}\rangle\langle\alpha;\mathsf{in}|SS^\dagger|\beta;\mathsf{in}\rangle\langle\beta;\mathsf{in}| \\ &= \sum_{\alpha,\beta} |\alpha;\mathsf{in}\rangle\langle\alpha;\mathsf{out}|\beta;\mathsf{out}\rangle\langle\beta;\mathsf{in}| \\ &= \sum_{\alpha} |\alpha;\mathsf{in}\rangle\langle\alpha;\mathsf{in}| = 1\,. \end{aligned}$$

Clearly S must also relate ϕ_{in} and ϕ_{out}. Since

$$\langle\alpha;\mathsf{out}|\phi_{\mathsf{in}}S|\beta;\mathsf{out}\rangle = \langle\alpha;\mathsf{out}|\phi_{\mathsf{in}}|\beta;\mathsf{in}\rangle\,,$$

and $\phi_{\mathsf{in}}|\beta;\mathsf{in}\rangle$ is an in state that is equal to S operating on the corresponding out state, i.e.

$$\phi_{\mathsf{in}}|\beta;\mathsf{in}\rangle = S\phi_{\mathsf{out}}|\beta;\mathsf{out}\rangle\,,$$

it follows that

$$\langle\alpha;\mathsf{out}|\phi_{\mathsf{in}}S|\beta;\mathsf{out}\rangle = \langle\alpha;\mathsf{out}|S\phi_{\mathsf{out}}|\beta;\mathsf{out}\rangle\,.$$

From the completeness of the out states, we obtain

$$\begin{aligned} \phi_{\mathsf{in}}S &= S\phi_{\mathsf{out}} \\ \phi_{\mathsf{in}} &= S\phi_{\mathsf{out}}S^\dagger \\ \phi_{\mathsf{out}} &= S^\dagger\phi_{\mathsf{in}}S\,. \end{aligned}$$

The aim of scattering theory is to calculate cross-sections and other observables. The probability amplitude for a state α at $t = -\infty$ scattering into a state β at $t = \infty$ is

$$\langle\beta;\mathsf{out}|\alpha;\mathsf{in}\rangle = \langle\beta;\mathsf{in}|S|\alpha;\mathsf{in}\rangle\,,$$

[cf., Eq.(6.8)]. We need then to calculate the S-matrix elements with respect to the in representation.

While ϕ_{in} and ϕ_{out} satisfy linear equations of motion, the general field ϕ satisfies the nonlinear equation

$$(\partial^2 + m_0^2)\phi = -4\lambda : \phi^3 : \tag{6.9}$$

Naively, we might hope that a unitary evolution operator, $U(t)$, exists, such that

$$\begin{aligned} \phi(t,\vec{x}) &= U^\dagger(t)\phi_{\mathsf{in}}(t,\vec{x})U(t) \\ \pi(t,\vec{x}) &= U^\dagger(t)\pi_{\mathsf{in}}(t,\vec{x})U(t)\,. \end{aligned} \tag{6.10}$$

We might expect that $U(-\infty) = 1$, so that $\phi(t, \vec{x})$ goes over into $\phi_{\text{in}}(t, \vec{x})$ at $t = -\infty$, and $U(\infty) = S$, but this turns out to be untrue. It is only in a weak sense that the limit exists, and then only if we introduce a regularization procedure. The difficulty arises from the highly singular nature of the product of different fields at the same space-time point. We proceed nevertheless formally; we will later learn how to handle these divergent integrals.

In order to calculate the evolution operator, we need to apply the dynamics of the field. In quantum theory, the field version of the Hamilton equations is

$$\begin{aligned} \frac{\partial \phi}{\partial t} &= i[H, \phi]\,, \\ \frac{\partial \pi}{\partial t} &= i[H, \pi]\,. \end{aligned} \tag{6.11}$$

In classical mechanics, any quantity that has vanishing Poisson brackets with all the coordinates and with all the canonical momenta is not a dynamical variable. The quantum mechanical version of this statement is that any quantity that commutes with all the coordinates and with all the canonical momenta is also not a dynamical variable: it is a c-number and not an operator. Here it is crucial that we have to do with a *complete* set of dynamical variables and their canonical momenta; in our case they are respectively $\phi(t, \vec{x})$ and $\pi(t, \vec{x})$ for all $\vec{x}$ at a given t. Our strategy in determining the evolution operator will be to prove that a certain function of it indeed commutes with ϕ and π, and therefore that it must be a c-number. We cannot compute this c-number; but fortunately it cancels out of the final expression for the S-matrix.

The Hamiltonian corresponding to the Lagrangian density Eq.(6.3) is

$$H(\phi, \pi) = \int d^3x \left[\mathcal{H}_f(\phi, \pi) + \mathcal{H}_i(\phi)\right], \tag{6.12}$$

where

$$\begin{aligned} \mathcal{H}_f(\phi, \pi) &= \tfrac{1}{2} : \pi^2 + \vec{\nabla}\,\phi . \vec{\nabla}\,\phi + m_0^2 \phi^2 : \\ \mathcal{H}_i(\phi) &= \lambda : \phi^4 : \end{aligned}$$

The field ϕ_{in} and its momentum π_{in} are however governed by the Hamiltonian

$$H_{\text{in}}(\phi_{\text{in}}, \pi_{\text{in}}) = \int d^3x \mathcal{H}_{\text{in}}(\phi_{\text{in}}, \pi_{\text{in}}) \tag{6.13}$$

where

$$\mathcal{H}_{\text{in}}(\phi_{\text{in}}, \pi_{\text{in}}) = \tfrac{1}{2} : \pi_{\text{in}}^2 + \vec{\nabla}\,\phi_{\text{in}} . \vec{\nabla}\,\phi_{\text{in}} + m^2 \phi_{\text{in}}^2 :$$

Note that $\mathcal{H}_{\text{in}}$ differs from $\mathcal{H}_f$ in two respects: it is a function of ϕ_{in} and π_{in}, instead of ϕ and π, and it contains the physical mass, m, in place of the bare mass, m_0. Recall that $m^2 = m_0^2 + \delta m^2$.

It is convenient to rewrite Eq.(6.12) as follows:

$$\begin{aligned} H &= \int d^3x : \left\{ \tfrac{1}{2}\left[\pi^2 + \vec{\nabla}\phi \cdot \vec{\nabla}\phi + m^2\phi^2 - \delta m^2 \phi^2\right] + \lambda\phi^4 \right\} : \\ &= H_{\text{in}}(\phi,\pi) + H_I(\phi)\,, \end{aligned}$$

where

$$H_I(\phi) = \int d^3x : \left\{ \lambda\phi^4 - \tfrac{1}{2}\delta m^2 \phi^2 \right\} : \tag{6.14}$$

The dynamical equations (6.11) can be written

$$\begin{aligned} \frac{\partial\phi}{\partial t} &= i[H_{\text{in}}(\phi,\pi) + H_I(\phi), \phi]\,, \\ \frac{\partial\pi}{\partial t} &= i[H_{\text{in}}(\phi,\pi) + H_I(\phi), \pi]\,. \end{aligned} \tag{6.15}$$

The analogous equations for the in fields are

$$\begin{aligned} \frac{\partial\phi_{\text{in}}}{\partial t} &= i[H_{\text{in}}(\phi_{\text{in}},\pi_{\text{in}}), \phi_{\text{in}}]\,, \\ \frac{\partial\pi_{\text{in}}}{\partial t} &= i[H_{\text{in}}(\phi_{\text{in}},\pi_{\text{in}}), \pi_{\text{in}}]\,, \end{aligned} \tag{6.16}$$

where the dependence of H_{in} on ϕ_{in} and π_{in} has been indicated explicitly.

We wish to compute now the evolution operator, $U(t)$. Since $U^\dagger U = 1$, it follows that $\dot{U}^\dagger U + U^\dagger \dot{U} = 0$, or $\dot{U}^\dagger = -U^\dagger \dot{U} U^\dagger$. Now $\phi_{\text{in}} = U\phi U^\dagger$, so we have

$$\begin{aligned} \frac{\partial\phi_{\text{in}}}{\partial t} &= \dot{U}\phi U^\dagger + U\frac{\partial\phi}{\partial t}U^\dagger + U\phi\dot{U}^\dagger \\ &= \dot{U}U^\dagger U\phi U^\dagger + iU[H_{\text{in}}(\phi,\pi) + H_I(\phi), \phi]U^\dagger - U\phi U^\dagger \dot{U}U^\dagger \\ &= [\dot{U}U^\dagger, \phi_{\text{in}}] + i[H_{\text{in}}(\phi_{\text{in}},\pi_{\text{in}}), \phi_{\text{in}}] + i[H_I(\phi_{\text{in}}), \phi_{\text{in}}]\,. \end{aligned}$$

From Eq.(6.16), we see that the commutator involving H_{in} cancels the left-hand side, and we are left with

$$[\dot{U}U^\dagger + iH_I(\phi_{\text{in}}), \phi_{\text{in}}] = 0\,.$$

The same technique suffices to show that also

$$[\dot{U}U^\dagger + iH_I(\phi_{\text{in}}), \pi_{\text{in}}] = 0\,.$$

At a given time, t, the quantity $\dot{U}U^\dagger + iH_I(\phi_{\text{in}})$ commutes with $\phi_{\text{in}}(t,\vec{x})$ and $\pi_{\text{in}}(t,\vec{x})$ for every value of $\vec{x}$. It must then be a c-number, say

$$\dot{U}U^\dagger + iH_I(\phi_{\text{in}}) = -iE(t)\,,$$

where $E(t)$ may depend on time, but commutes with all operators of the theory.

From the preceding equation, we have

$$\dot{U}(t) = -iH_I'(t)U(t)\,,$$

where we have set

$$H_I'(t) = H_I(t;\phi_{\text{in}}) + E(t)\,.$$

We want to solve this differential equation, subject to the condition $U(-\infty) = 1$. This is done by iteration (Problem 6.1):

$$\begin{aligned}
U(t) &= 1 - i\int_{-\infty}^{t} dt_1 H_I'(t_1)U(t_1) \\
&= 1 + \ldots (-i)^n \int_{-\infty}^{t} dt_1 \int_{-\infty}^{t_1} dt_2 \ldots \int_{-\infty}^{t_{n-1}} dt_n H_I'(t_1)H_I'(t_2)\ldots H_I'(t_n) + \ldots \\
&= \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{t} dt_1 \int_{-\infty}^{t} dt_2 \ldots \int_{-\infty}^{t} dt_n T[H_I'(t_1)H_I'(t_2)\ldots H_I'(t_n)] \\
&= T\sum_{n=0}^{\infty} \frac{1}{n!}\Big[-i\int_{-\infty}^{t} dt_1 H_I'(t_1)\Big]^n = T\exp\Big[-i\int_{-\infty}^{t} dt_1 H_I'(t_1)\Big]\,.
\end{aligned}$$

In the third line, a time ordering has been introduced, and simultaneously all integrations have been taken up to t, and this has been compensated by $1/n!$, a weight factor. One can show that the third line follows from the second by an inductive argument. However, it is easier to show directly that the expression in the third line does indeed satisfy the differential equation. The exponential is purely formal: the meaning of the time-ordered exponential is precisely the time-ordered power series expansion that precedes it.

The S-operator can now be identified with $U(\infty)$ (in the weak sense):

$$\begin{aligned}
S &= U(\infty) = T\exp\left[-i\int_{-\infty}^{\infty} dt_1 H_I'(t_1)\right] \\
&= \exp\left[-i\int_{-\infty}^{\infty} dt_1 E(t_1)\right]\; T\exp\left[-i\int_{-\infty}^{\infty} dt_1 H_I(t_1)\right]\,. \qquad (6.17)
\end{aligned}$$

The phase factor involving E has been removed from the time ordering, since it commutes with everything.

6.2 LSZ Reduction

Let us consider the scattering of two spinless particles, with momenta p_1 and p_2 before, and momenta p_3 and p_4 after the scattering process. The required S-matrix element is

$$\langle p_3, p_4; \mathsf{out}|p_1, p_2; \mathsf{in}\rangle = \langle p_3, p_4; \mathsf{in}|S|p_1, p_2; \mathsf{in}\rangle\,. \tag{6.18}$$

We shall extract the particle of momentum p_1 from the initial state, obtaining

$$\langle p_3, p_4; \mathsf{out}|a^\dagger_{\mathsf{in}}(p_1)|p_2\rangle\,. \tag{6.19}$$

The label in has been dropped for the one-particle state $|p_2\rangle$ [see Eq.(6.4)].

The creation and annihilation operators are the Fourier transforms of ϕ_{in}:

$$\phi_{\mathsf{in}}(x_1) = \int d^3p_1 \left[a_{\mathsf{in}}(p_1) f_{p_1}(x_1) + a^\dagger_{\mathsf{in}}(p_1) f^*_{p_1}(x_1)\right],$$

where

$$f_{p_1}(x_1) = (2\pi)^{-\frac{3}{2}} (2p_1^0)^{-\frac{1}{2}}\, \mathrm{e}^{-ip_1x_1}\Big|_{p_1^0=\omega_1}$$

with $\omega_1 = \left(\vec{p}_1^{\,2} + m^2\right)^{\frac{1}{2}}$. The inverse is

$$\begin{aligned} a_{\mathsf{in}}(p_1) &= i\int d^3x_1 f^*_{p_1}(x_1) \overset{\leftrightarrow}{\partial}_{10} \phi_{\mathsf{in}}(x_1) \\ a^\dagger_{\mathsf{in}}(p_1) &= -i\int d^3x_1 f_{p_1}(x_1) \overset{\leftrightarrow}{\partial}_{10} \phi_{\mathsf{in}}(x_1)\,, \end{aligned} \tag{6.20}$$

where $a \overset{\leftrightarrow}{\partial}_{10} b = a\partial b/\partial x_1^0 - \partial a/\partial x_1^0\, b$.

At first sight, one might think that the creation and annihilation operators could depend on the time, x_1^0. However, this is not so, because

$$\begin{aligned} \frac{\partial a_{\mathsf{in}}(p_1)}{\partial x_1^0} &= i\int d^3x_1 f^*_{p_1}(x_1)\Big[\,\overrightarrow{\partial}^2_{10} - \overleftarrow{\partial}^2_{10}\,\Big]\phi_{\mathsf{in}}(x_1) \\ &= i\int d^3x_1 f^*_{p_1}(x_1)\Big[\,\overrightarrow{\nabla}^2_1 - \overleftarrow{\nabla}^2_1\,\Big]\phi_{\mathsf{in}}(x_1) \\ &= i\int d^3x_1\, \overrightarrow{\nabla}_1 \cdot\Big[f^*_{p_1}(x_1) \overset{\leftrightarrow}{\nabla}_1 \phi_{\mathsf{in}}(x_1)\Big] = 0\,. \end{aligned}$$

In going from the first line to the second, we have used the fact that both $\phi_{\mathsf{in}}(x_1)$ and $f^*_{p_1}(x_1)$ are annihilated by the Klein-Gordon operator $(\partial^2 + m^2)$. The last result follows because the space integral of a spatial derivative yields fields at spatial infinity, and these, evaluated on a space of test functions of compact support, vanish.

Since $a_{\text{in}}(p_1)$ is independent of x_1^0, we can substitute Eq.(6.20) into Eq.(6.19) and freely take the limit $x_1^0 \longrightarrow -\infty$, obtaining

$$-i \lim_{x_1^0 \to -\infty} \langle p_3, p_4; \text{out} | \int d^3x_1 f_{p_1}(x_1) \overset{\leftrightarrow}{\partial}_{10} \phi_{\text{in}}(x_1) | p_2 \rangle \,. \tag{6.21}$$

Now as $x_1^0 \longrightarrow -\infty$, $\phi_{\text{in}}(x_1)$ describes a free field of mass m, and hence its matrix elements, for spatially localized states, are equal to those of $\phi(x_1)$ itself, at least up to a normalization. Thus we replace Eq.(6.21) by

$$-iZ^{-\frac{1}{2}} \lim_{x_1^0 \to -\infty} \langle p_3, p_4; \text{out} | \int d^3x_1 f_{p_1}(x_1) \overset{\leftrightarrow}{\partial}_{10} \phi(x_1) | p_2 \rangle \,, \tag{6.22}$$

where Z is called a renormalization constant. In general $Z \neq 1$.

Let us return to Eq.(6.18), writing the matrix element artificially as

$$\langle p_3, p_4; \text{in} | S | p_1, p_2; \text{in} \rangle = \langle p_3, p_4; \text{out} | p_1, p_2; \text{out} \rangle + \langle p_3, p_4; \text{out} | a_{\text{in}}^\dagger(p_1) - a_{\text{out}}^\dagger(p_1) | p_2 \rangle \,.$$

The first term on the right is trivial, consisting of 4-momentum delta functions, corresponding to particles that go through without scattering. In the second term, we repeat the manipulations that we applied to $a_{\text{in}}^\dagger(p_1)$, but now with $a_{\text{out}}^\dagger(p_1)$, except that we take the limit $x_1^0 \longrightarrow \infty$ instead of $x_1^0 \longrightarrow -\infty$, assuming that matrix elements of ϕ_{out} agree with those of ϕ in this limit, up to the same renormalization. Thus

$$\begin{aligned} &\langle p_3, p_4; \text{in} | S | p_1, p_2; \text{in} \rangle - \langle p_3, p_4; \text{out} | p_1, p_2; \text{out} \rangle \\ &= iZ^{-\frac{1}{2}} \Big\{ \lim_{x_1^0 \to \infty} - \lim_{x_1^0 \to -\infty} \Big\} \langle p_3, p_4; \text{out} | \int d^3x_1 f_{p_1}(x_1) \overset{\leftrightarrow}{\partial}_{10} \phi(x_1) | p_2 \rangle \,. \end{aligned} \tag{6.23}$$

Since any differentiable function satisfies the identity

$$\Big\{ \lim_{x_1^0 \to \infty} - \lim_{x_1^0 \to -\infty} \Big\} F(x_1) = \int_{-\infty}^{\infty} dx_1^0 \frac{\partial}{\partial x_1^0} F(x_1) \,,$$

we can write the term on the right of Eq.(6.23) in the form

$$\begin{aligned} & iZ^{-\frac{1}{2}} \langle p_3, p_4; \text{out} | \int d^4x_1 \partial_{10} \Big\{ f_{p_1}(x_1) \overset{\leftrightarrow}{\partial}_{10} \phi(x_1) \Big\} | p_2 \rangle \\ = \; & iZ^{-\frac{1}{2}} \langle p_3, p_4; \text{out} | \int d^4x_1 \Big\{ f_{p_1}(x_1) \overset{\leftrightarrow}{\partial}{}^2_{10} \phi(x_1) \Big\} | p_2 \rangle \\ = \; & iZ^{-\frac{1}{2}} \langle p_3, p_4; \text{out} | \int d^4x_1 \Big\{ f_{p_1}(x_1) \partial_{10}^2 \phi(x_1) - \big[(\nabla_1^2 - m^2) f_{p_1}(x_1) \big] \phi(x_1) \Big\} | p_2 \rangle \\ = \; & iZ^{-\frac{1}{2}} \langle p_3, p_4; \text{out} | \int d^4x_1 f_{p_1}(x_1) \big(\partial_1^2 + m^2 \big) \phi(x_1) | p_2 \rangle \,. \end{aligned}$$

In going from the second to the third lines, we have used the fact that $f_{p_1}(x_1)$ is annihilated by the differential operator $\partial_1^2 + m^2$, while the last line was obtained by two partial integrations, which transfer the action of ∇_1^2 from $f_{p_1}(x_1)$ to $\phi(x_1)$. Note that $\phi(x_1)$ is not annihilated by $\partial_1^2 + m^2$, for we can rewrite Eq.(6.9) in the form

$$(\partial_1^2 + m^2)\phi(x_1) = -4\lambda : \phi^3(x_1) : +\delta m^2 \phi(x_1) .$$

Next we shall extract the particle of momentum p_3 in the final state:

$$\begin{aligned} &\langle p_3, p_4; \mathsf{in}|S|p_1, p_2; \mathsf{in}\rangle - \langle p_3, p_4; \mathsf{out}|p_1, p_2; \mathsf{out}\rangle \\ &\quad = iZ^{-\frac{1}{2}} \int d^4x_1 f_{p_1}(x_1)\left[\partial_1^2 + m^2\right]\langle p_4|a_{\mathsf{out}}(p_3)\phi(x_1)|p_2\rangle . \end{aligned} \tag{6.24}$$

Now it can be shown that

$$\int d^4x_1 f_{p_1}(x_1)\left[\partial_1^2 + m^2\right]\langle p_4|\phi(x_1)a_{\mathsf{in}}(p_3)|p_2\rangle = 0 ,$$

(Problem 6.3), so that

$$\begin{aligned} &\langle p_3, p_4; \mathsf{in}|S|p_1, p_2; \mathsf{in}\rangle - \langle p_3, p_4; \mathsf{out}|p_1, p_2; \mathsf{out}\rangle \\ &\qquad = iZ^{-\frac{1}{2}} \int d^4x_1 f_{p_1}(x_1)(\partial_1^2 + m^2)\sigma(x_1) , \end{aligned} \tag{6.25}$$

where

$$\begin{aligned} \sigma(x_1) &= \langle p_4|a_{\mathsf{out}}(p_3)\phi(x_1) - \phi(x_1)a_{\mathsf{in}}(p_3)|p_2\rangle \\ &= i\langle p_4|\int d^3x_3 f^*_{p_3}(x_3) \overset{\leftrightarrow}{\partial}_{30} \left[\phi_{\mathsf{out}}(x_3)\phi(x_1) - \phi(x_1)\phi_{\mathsf{in}}(x_3)\right] \\ &= iZ^{-\frac{1}{2}}\langle p_4|\Big\{ \lim_{x_{30}\to\infty} \int d^3x_3 f^*_{p_3}(x_3) \overset{\leftrightarrow}{\partial}_{30} \phi(x_3)\phi(x_1) \\ &\qquad\qquad - \lim_{x_{30}\to-\infty} \int d^3x_3 f^*_{p_3}(x_3) \overset{\leftrightarrow}{\partial}_{30} \phi(x_1)\phi(x_3)\Big\}|p_2\rangle \\ &= iZ^{-\frac{1}{2}}\langle p_4|\int d^4x_3 f^*_{p_3}(x_3)(\partial_3^2 + m^2)T\left[\phi(x_1)\phi(x_3)\right]|p_2\rangle . \end{aligned}$$

(Note the time ordering.) We then have

$$\begin{aligned} &\langle p_3, p_4; \mathsf{in}|S|p_1, p_2; \mathsf{in}\rangle - \langle p_3, p_4; \mathsf{out}|p_1, p_2; \mathsf{out}\rangle = \\ &\qquad -Z^{-1}\int\int d^4x_1 d^4x_3 f_{p_1}(x_1) f^*_{p_3}(x_3) \\ &\qquad\qquad (\partial_1^2 + m^2)(\partial_3^2 + m^2)\langle p_4|T\left[\phi(x_1)\phi(x_3)\right]|p_2\rangle . \end{aligned} \tag{6.26}$$

The extraction of the remaining two particles in an analogous manner is left as an exercise (Problem 6.4). The final result is

$$\langle p_3, p_4; \mathsf{in}|S|p_1, p_2; \mathsf{in}\rangle - \langle p_3, p_4; \mathsf{out}|p_1, p_2; \mathsf{out}\rangle =$$
$$Z^{-2}\int\int\int\int d^4x_1 d^4x_2 d^4x_3 d^4x_4 f_{p_1}(x_1) f_{p_2}(x_2) f^*_{p_3}(x_3) f^*_{p_4}(x_4) \qquad (6.27)$$
$$(\partial_1^2 + m^2)(\partial_2^2 + m^2)(\partial_3^2 + m^2)(\partial_4^2 + m^2)\tau(x_1, x_2, x_3, x_4)\,,$$

where the τ-function is

$$\tau(x_1, x_2, x_3, x_4) = \langle 0|T\big[\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\big]|0\rangle\,. \qquad (6.28)$$

Similar expressions are obtained with any number of particles in the initial or final states. Thus the problem of calculating the scattering matrix elements has been reduced to that of computing the vacuum expectation values of time-ordered products of fields. These τ-functions are also called the Green's functions of the theory.

6.3 Wick's Theorem

Now we insert the evolution operator into Eq.(6.28), using Eq.(6.10), to obtain the following expression for $\tau(x_1, x_2, x_3, x_4)$:

$$\langle 0|T\big[U^\dagger(x_1^0)\phi_{\mathsf{in}}(x_1)U(x_1^0)U^\dagger(x_2^0)\phi_{\mathsf{in}}(x_2)\cdots U(x_3^0)U^\dagger(x_4^0)\phi_{\mathsf{in}}(x_4)U(x_4^0)\big]|0\rangle\,.$$

We introduce a reference time, t, and use the fact that

$$U^\dagger(t)U(t) = 1 = U^\dagger(-t)U(-t)$$

to obtain

$$\langle 0|T\big[U^\dagger(t)U(t, x_1^0)\phi_{\mathsf{in}}(x_1)U(x_1^0, x_2^0)\phi_{\mathsf{in}}(x_2)\cdots U(x_3^0, x_4^0)\phi_{\mathsf{in}}(x_4)U(x_4^0, -t)U(-t)\big]|0\rangle$$

where we have defined

$$U(t_1, t_2) = U(t_1)U^\dagger(t_2) = T\exp\left[-i\int_{t_2}^{t_1} dy^0 H_I'(y^0)\right]\,.$$

If we choose t such that $t > \max\{x_1, x_2, x_3, x_4\}$ and $-t < \min\{x_1, x_2, x_3, x_4\}$, then we can pull the factor $U^\dagger(t)$ out of the time ordering to the left, and $U(-t)$ to the right. The vacuum is stable, so $U(-t)|0\rangle$ is equal to the vacuum state, perhaps up to a renormalization constant, and similarly for $\langle 0|U^\dagger(t)$. Hence

$$\tau(x_1, x_2, x_3, x_4) = c(t)\langle 0|T\big[U(t, -t)\phi_{\mathsf{in}}(x_1)\phi_{\mathsf{in}}(x_2)\phi_{\mathsf{in}}(x_3)\phi_{\mathsf{in}}(x_4)\big]|0\rangle\,,$$

where

$$\begin{aligned} U(t,-t) &= U(t,x_1^0)U(x_1^0,x_2^0)U(x_2^0,x_3^0)U(x_3^0,x_4^0)U(x_4^0,-t) \\ &= T\exp\left[-i\int_{-t}^{t} dy^0 H_I'(y^0)\right]. \end{aligned}$$

In the above, $c(t)$ is a possibly time-dependent renormalization constant; it can be evaluated by repeating the above manipulations for a Green's function involving *no* in fields:

$$1 = \langle 0|0\rangle = c(t)\langle 0|T\left[U(t,-t)\right]|0\rangle .$$

Inserting the resultant expression for $c(t)$ and letting the reference time, t, tend to infinity, we obtain

$$\tau(x_1,x_2,x_3,x_4) = \frac{\langle 0|T\left[U(\infty,-\infty)\,\phi_{\text{in}}(x_1)\phi_{\text{in}}(x_2)\phi_{\text{in}}(x_3)\phi_{\text{in}}(x_4)\right]|0\rangle}{\langle 0|T\left[U(\infty,-\infty)\right]|0\rangle} . \tag{6.29}$$

In the weak sense, $U(\infty,-\infty) = U(\infty) = S$, which was calculated in Eq.(6.17):

$$U(\infty,-\infty) = \exp\left[-i\int_{-\infty}^{\infty} dtE(t)\right]\; T\exp\left[-i\int d^4y\mathcal{H}_I(\phi_{\text{in}};y)\right] , \tag{6.30}$$

with [cf., Eq.(6.14)]

$$\mathcal{H}_I(\phi_{\text{in}};y) \;=\; \lambda : \phi_{\text{in}}^4(y) : -\tfrac{1}{2}\delta m^2 : \phi_{\text{in}}^2(y) : \tag{6.31}$$

Finally, we substitute Eq.(6.30) into Eq.(6.29), noting that the phase factor involving $E(t)$ cancels between the numerator and the denominator:

$$\tau(x_1,x_2,x_3,x_4) = \frac{\langle 0|T\left\{\exp\left[-i\int d^4y\mathcal{H}_I(\phi_{\text{in}};y)\right]\,\phi_{\text{in}}(x_1)\phi_{\text{in}}(x_2)\phi_{\text{in}}(x_3)\phi_{\text{in}}(x_4)\right\}|0\rangle}{\langle 0|T\left\{\exp\left[-i\int d^4y\mathcal{H}_I(\phi_{\text{in}};y)\right]\right\}|0\rangle} \tag{6.32}$$

The Green's function, τ, is hereby expressed as the quotient of two vacuum expectation values of time-ordered products of fields in the in representation. To proceed further, we expand the exponentials in numerator and denominator of Eq.(6.32). The numerator becomes

$$\sum_{p=0}^{\infty}\frac{(-i)^p}{p!}\int d^4y_1\ldots\int d^4y_p\langle 0|T\left[\phi_{\text{in}}(x_1)\ldots\phi_{\text{in}}(x_4)\mathcal{H}_I(y_1)\ldots\mathcal{H}_I(y_p)\right]|0\rangle . \tag{6.33}$$

On insertion of Eq.(6.31) into this expression, we reduce the problem to the evaluation of the vacuum expectation values of time-ordered products of in fields.

Two incoming fields:
Consider first the vacuum expectation value of the time-ordered product of two in fields. We define the positive and negative frequency parts of the in fields:

$$\phi^+_{\rm in}(x_1) = \int d^3p_1 a_{\rm in}(p_1) f_{p_1}(x_1)\,,$$

which is an annihilation operator, and

$$\phi^-_{\rm in}(x_1) = \int d^3p_1 a^\dagger_{\rm in}(p_1) f^*_{p_1}(x_1)$$

which is a creation operator, so that

$$\phi_{\rm in}(x_1) = \phi^+_{\rm in}(x_1) + \phi^-_{\rm in}(x_1)\,.$$

Since the normal-ordered product of two fields is

$$:\phi_{\rm in}(x_1)\phi_{\rm in}(x_2): = \phi^+_{\rm in}(x_1)\phi^+_{\rm in}(x_2) + \phi^-_{\rm in}(x_1)\phi^+_{\rm in}(x_2) + \phi^-_{\rm in}(x_2)\phi^+_{\rm in}(x_1) + \phi^-_{\rm in}(x_1)\phi^-_{\rm in}(x_2)\,,$$

and this differs only by the third term from the ordinary product of the two fields, we find

$$\phi_{\rm in}(x_1)\phi_{\rm in}(x_2) - :\phi_{\rm in}(x_1)\phi_{\rm in}(x_2): = \left[\phi^+_{\rm in}(x_1), \phi^-_{\rm in}(x_2)\right].$$

This commutator can be readily calculated:

$$\begin{aligned}\left[\phi^+_{\rm in}(x_1), \phi^-_{\rm in}(x_2)\right] &= \int\int d^3p_1\, d^3p_2\, f_{p_1}(x_1) f^*_{p_2}(x_2)\left[a_{\rm in}(p_1), a^\dagger_{\rm in}(p_2)\right] \\ &= \frac{1}{(2\pi)^3}\int \frac{d^3p_1}{2p^0_1}\, {\rm e}^{-ip_1(x_1-x_2)}\Big|_{p^0_1=\omega_1}\,.\end{aligned}$$

Consider now the difference between the time-ordered and the normal-ordered products of the two fields:

$$\begin{aligned}& T\left[\phi_{\rm in}(x_1)\phi_{\rm in}(x_2)\right] - :\phi_{\rm in}(x_1)\phi_{\rm in}(x_2): \\ =\;& \theta(x^0_1 - x^0_2)\left[\phi^+_{\rm in}(x_1), \phi^-_{\rm in}(x_2)\right] + \theta(x^0_2 - x^0_1)\left[\phi^+_{\rm in}(x_2), \phi^-_{\rm in}(x_1)\right] \\ =\;& \frac{1}{(2\pi)^3}\int \frac{d^3p}{2p^0}\left[{\rm e}^{-ip(x_1-x_2)}\theta(x^0_1 - x^0_2) + {\rm e}^{ip(x_1-x_2)}\theta(x^0_2 - x^0_1)\right]_{p^0=\omega} \\ =\;& i\Delta_F(x_1 - x_2)\,, \qquad (6.34)\end{aligned}$$

where the Feynman propagator is defined in configuration space by

$$\Delta_F(x) = \frac{1}{(2\pi)^4}\int d^4p \frac{{\rm e}^{-ipx}}{p^2 - m^2 + i\epsilon}\,. \qquad (6.35)$$

The last step in Eq.(6.34) is justified by treating the p^0-integration as a contour integral in the complex p^0-plane and using Cauchy's theorem (Problem 4.2). Since the normal-ordered product contains all annihilation operators written to the right, it is clear that

$$\langle 0| : \phi_{\text{in}}(x_1)\phi_{\text{in}}(x_2) : |0\rangle = 0 .$$

Hence the required vacuum expectation value of the time-ordered product of two in fields has the form

$$\langle 0|T\big[\phi_{\text{in}}(x_1)\phi_{\text{in}}(x_2)\big]|0\rangle = i\Delta_F(x_1 - x_2) . \tag{6.36}$$

Three incoming fields:
For brevity we set

$$\phi_k = \phi_{\text{in}}(x_k) \qquad \Gamma_{jk} = \big[\phi^+_{\text{in}}(x_j), \phi^-_{\text{in}}(x_k)\big] .$$

Note that Γ_{jk} is not an operator, but a c-number distribution. Now

$$\phi_1\phi_2\phi_3 - : \phi_1\phi_2\phi_3 : \;=\; \phi_1\Big(\phi_2\phi_3 - : \phi_2\phi_3 :\Big) + \Big[\phi_1^+, : \phi_2\phi_3 :\Big] ,$$

and we have already seen that

$$\phi_2\phi_3 - : \phi_2\phi_3 := \Gamma_{23} ,$$

and we may write

$$\begin{aligned}\Big[\phi_1^+, : \phi_2\phi_3 :\Big] &= \Big[\phi_1^+, \phi_2^-\phi_3^-\Big] + \Big[\phi_1^+, \phi_2^-\Big]\phi_3^+ + \Big[\phi_1^+, \phi_3^-\Big]\phi_2^+ \\ &= \Gamma_{12}\phi_3^- + \Gamma_{13}\phi_2^- + \Gamma_{12}\phi_3^+ + \Gamma_{13}\phi_2^+ \\ &= \Gamma_{13}\phi_2 + \Gamma_{12}\phi_3 .\end{aligned}$$

Hence the product of three fields can be written

$$\phi_1\phi_2\phi_3 = \; : \phi_1\phi_2\phi_3 : + \Gamma_{23}\phi_1 + \Gamma_{13}\phi_2 + \Gamma_{12}\phi_3 .$$

Now we are really interested in a time-ordered product of the three fields, $T[\phi_1\phi_2\phi_3]$, which means that, for example, Γ_{12} should be replaced by

$$\theta(x_1^0 - x_2^0)\Gamma_{12} + \theta(x_2^0 - x_1^0)\Gamma_{21} = i\Delta_F(x_1 - x_2) ,$$

and similarly for Γ_{13} and Γ_{23}. We have then finally

$$T\Big[\phi_1\phi_2\phi_3\Big] =: \phi_1\phi_2\phi_3 : + i\big\{\Delta_F(x_2 - x_3)\phi_1 + \Delta_F(x_3 - x_1)\phi_2 + \Delta_F(x_1 - x_2)\phi_3\big\} .$$

Since $\langle 0|\phi_1|0\rangle = 0$, etc., it follows that $\langle 0|T[\phi_1\phi_2\phi_3]|0\rangle = 0$.

Four incoming fields:
It is easy to see that

$$\phi_1\phi_2\phi_3\phi_4 - :\phi_1\phi_2\phi_3\phi_4: = \phi_1\big(\phi_2\phi_3\phi_4 - :\phi_2\phi_3\phi_4:\big) + \big[\phi_1^+, :\phi_2\phi_3\phi_4:\big]\,,$$

and from this one can check that

$$\begin{aligned}\phi_1\phi_2\phi_3\phi_4 \;=\; & :\phi_1\phi_2\phi_3\phi_4: \\ & +\Gamma_{12}:\phi_3\phi_4: +\Gamma_{13}:\phi_2\phi_4: +\Gamma_{14}:\phi_2\phi_3: +\Gamma_{23}:\phi_1\phi_4: \\ & +\Gamma_{12}\Gamma_{34}+\Gamma_{13}\Gamma_{24}+\Gamma_{14}\Gamma_{23}\,.\end{aligned}$$

The time-ordered form may now be written down by inspection, as in the case of three fields:

$$\begin{aligned}T\big[\phi_1\phi_2\phi_3\phi_4\big] \;=\; & :\phi_1\phi_2\phi_3\phi_4: \\ & +i\big\{\Delta_F(x_1-x_2):\phi_3\phi_4: +\text{permutations}\big\} \\ & -\big\{\Delta_F(x_1-x_2)\Delta_F(x_3-x_4)+\text{permutations}\big\}\,.\end{aligned}$$

All terms involving the vacuum expectation value of normal-ordered products of fields vanish, so the vacuum expectation value of the time-ordered product of four fields can be written

$$\langle 0|T\big[\phi_1\phi_2\phi_3\phi_4\big]|0\rangle = -\big\{\Delta_F(x_1-x_2)\Delta_F(x_3-x_4)+\text{permutations}\big\}\,.$$

The vacuum expectation of the time-ordered product of three fields vanishes, as does that of any odd number of fields. The vacuum expectation of the time-ordered product of $2n$ fields can be written, by an extension of the procedure, in the following form, due to Wick (Problem 6.10):

$$\begin{aligned}\langle 0|T\big[\phi_1\phi_2\ldots\phi_{2n}\big]|0\rangle \;=\; & i^n\big\{\Delta_F(x_1-x_2)\Delta_F(x_3-x_4)\ldots\Delta_F(x_{2n-1}-x_{2n}) \\ & +\text{permutations}\big\}\,. \qquad (6.37)\end{aligned}$$

Thus we can in principle calculate any term in the expansion (the perturbation series) of the numerator or the denominator of Eq.(6.32), since it involves nothing but the vacuum expectation value of the time-ordered product of in fields.

6.4 Diagrams in Momentum Space

For the sake of illustration, we shall consider the contribution of the term $p = 3$ in the sum (6.33) to the numerator of Eq.(6.32):

$$\frac{(-i)^3}{3!}\langle 0|T\Big\{\phi_{\rm in}(x_1)\cdots\phi_{\rm in}(x_4)\int\int\int d^4y_1 d^4y_2 d^4y_3 \mathcal{H}_I(y_1)\mathcal{H}_I(y_2)\mathcal{H}_I(y_3)\Big\}|0\rangle\,, \tag{6.38}$$

where the argument $\phi_{\rm in}$ in $\mathcal{H}_I$ has been suppressed. Because there are two terms in Eq.(6.31), there will be eight terms in the above third-order contribution. For definiteness, let us look at the contribution that comes from the λ term in $\mathcal{H}_I(y_1)$ and $\mathcal{H}_I(y_2)$, but the δm^2 term in $\mathcal{H}_I(y_3)$, namely

$$-\frac{(-i)^3}{3!}\frac{\lambda^2\delta m^2}{2}\int\int\int d^4y_1 d^4y_2 d^4y_3 \tag{6.39}$$
$$\langle 0|T\Big\{\phi_{\rm in}(x_1)\phi_{\rm in}(x_2)\phi_{\rm in}(x_3)\phi_{\rm in}(x_4) : \phi^4_{\rm in}(y_1) :: \phi^4_{\rm in}(y_2) :: \phi^2_{\rm in}(y_3) : \Big\}|0\rangle\,.$$

We consider this term in isolation purely for the sake of clarity: eventually we must include the other seven terms too. For the moment, we ignore the normal ordering inside the above time-ordered product; but we shall come back to this matter. With this simplification, the vacuum expectation value in Eq.(6.39) can be written, thanks to Wick's formula (6.37), as

$$i^7\Big\{\Delta_F(x_1-y_1)\Delta_F(x_2-y_1)\big[\Delta_F(y_1-y_2)\big]^2\Delta_F(y_2-y_3)\Delta_F(y_3-x_3)\Delta_F(y_2-x_4)$$
$$+\ \text{permutations}\Big\}\,.$$

The permutations include every possible combination of the fourteen arguments, taken two at a time (y_1 and y_2 are each counted four times, y_3 twice). We introduce a convenient diagrammatic way of representing the term that has been written explicitly, namely

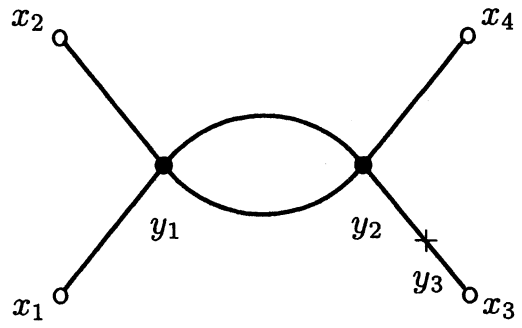


Figure 6.1
Configuration space diagram

This is called a Feynman diagram in configuration space. The open points in the

diagram represent the four external variables on which the τ-function depends. The internal variables, y_1, y_2 and y_3, are to be integrated out in Eq.(6.39). The black points represent the action of the λ coupling term, while the mass counter term is represented by the cross. The lines in the diagram represent i times the Feynman propagator, $i\,\Delta_F$. Some of the other terms included in the 'permutations' are depicted as follows:

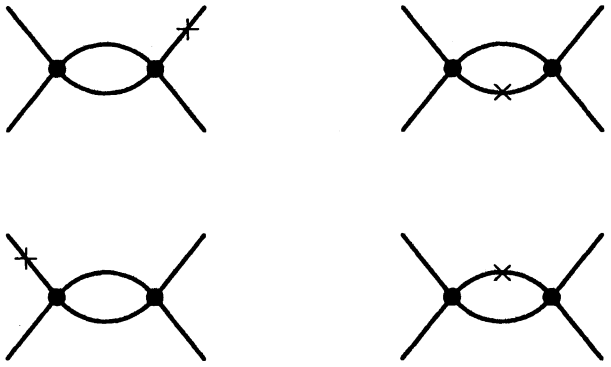


Figure 6.2
Other third-order terms

Here y_3 is always represented by a cross. There are precisely as many terms as there are ways of connecting the x and the y, such that

(1) one line is connected to each x,
(2) four lines are connected to y_1 and four to y_2,
(3) two lines are connected to y_3.

The diagrammatic method is a useful book-keeping device.

If there were no normal ordering, $:\phi^4_{\text{in}}:$ and $:\phi^2_{\text{in}}:$ in Eq.(6.39), we would have to include diagrams of the sort

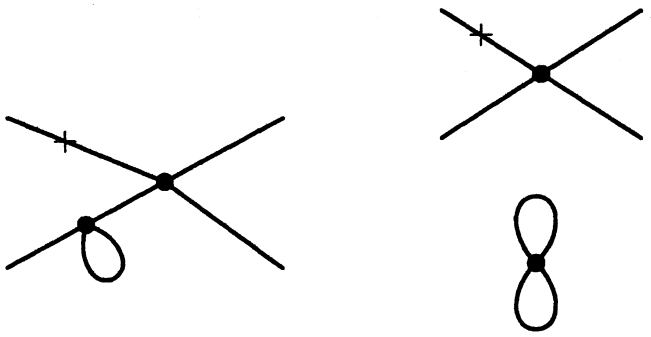


Figure 6.3
Tadpoles and butterflies

In terms involving $\phi_{\text{in}}(y_1)$ with $\phi_{\text{in}}(z)$, where z is some other argument, the normal ordering in $:\phi^4_{\text{in}}:$ has no effect, since the time ordering specifies the order in which the creation and annihilation operators are to be written. However, in

a term involving $\phi_{\text{in}}(y_1)$ twice, we do not obtain $i\,\Delta_F(y_1 - y_1)$, but rather

$$\langle 0|T\Big\{ : \phi_{\text{in}}(y_1)\phi_{\text{in}}(y_1) : \Big\}|0\rangle\,.$$

Here it is the time ordering that has no effect, and the normal ordering ensures that the vacuum expectation value is zero. Hence we do not need to draw any diagram that has a line which doubles back upon itself, that is, a line that starts and ends at the same space-time point.

There are many different diagrams, and therefore terms in the expansion, that give precisely the same contribution to the τ-function. For example, there is a term represented by the diagram given in Figure 6.1, but with y_1 and y_2 interchanged. Considering all eight terms in Eq.(6.38), corresponding to the two terms in Eq.(6.31), there will be in all 3! equivalent diagrams, differing only in the labeling of y_1, y_2 and y_3. More generally, in the contribution of the pth term in Eq.(6.33), for any diagram there are $p!$ equivalent diagrams that have the same topology and differ only in the permutation of the internal labels, $y_1 \cdots y_p$. We agree to cancel the denominator $p!$ in Eq.(6.33), and *not* to permute the y_i in future. We only consider one labeling of the internal points for each topologically distinct diagram.

There remains a less obvious source of equivalence. Even if we agree not to permute the y_i, there is still the possibility of permuting the lines leading to the same y_i. To make the point clear, suppose for a moment that we assign four different colors to the four fields, so that the interaction term in the Hamiltonian density is $\lambda : \phi^{\text{red}}_{\text{in}}(y_1)\phi^{\text{yellow}}_{\text{in}}(y_1)\phi^{\text{green}}_{\text{in}}(y_1)\phi^{\text{blue}}_{\text{in}}(y_1) :$ By Wick's theorem, we would obtain a different term for each of the 4! possible ways of assigning colors to the four lines emanating from y_1. The interaction is color blind, and so, for every diagram that involves the λ interaction vertex, we should include a factor 4! Similarly, every mass counter term should be accompanied by a factor 2!

The denominator in the expression (6.32) can be expanded into a series, leading to terms that can be represented by Feynman diagrams, just as was done for the numerator. The only difference is that that there are no external fields, $\phi_{\text{in}}(x)$. A third-order contribution to the denominator, corresponding to the action of the λ term at y_1 and y_2, and the mass counter term δm^2 at y_3, is depicted in Figure 6.4.

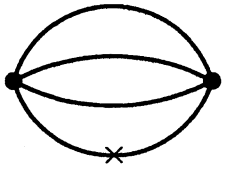
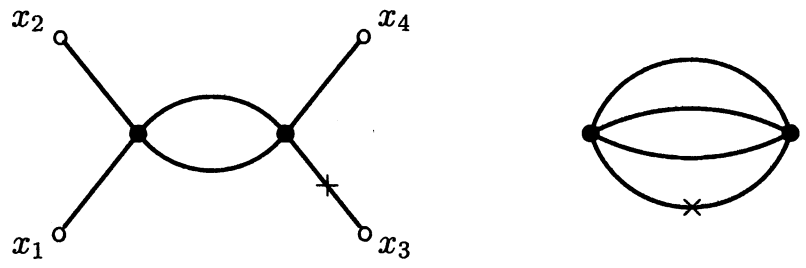


Figure 6.4
Vacuum part

It is a so-called vacuum part, i.e., it represents virtual fluctuations of the vacuum. Although we have eliminated lines that start and end at the same space-time point, the full third-order contribution to the numerator includes the vacuum part depicted in Figure 6.4, with external lines joining x_1 and x_2 directly to x_3 and x_4. In higher orders, one can have less trivial diagrams of this disconnected sort. For example, at sixth order, the numerator contains the term shown in Figure 6.5,

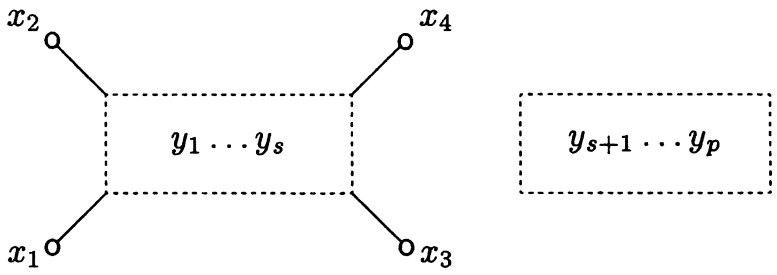
Figure 6.5
Disconnected diagram



which is precisely the diagram of Figure 6.1 and the vacuum part of Figure 6.4. In general, we may divide any diagram corresponding to a term in the numerator into a connected part, containing all the x_i, and a vacuum part that contains the remaining y_j that are wholly disconnected from the x_i.

We propose to show now that the vacuum part of the numerator precisely cancels the denominator in Eq.(6.32). The numerator may be written as in Eq.(6.33), where the factor $p!$ had not yet been cancelled, so all permutations of the y_j must be included. One set of diagrams at order p will have the form shown in Figure 6.6

Figure 6.6
Diagrams of order p



The internal points, $y_1 \cdots y_s$ are connected, directly or indirectly, to the x_i, and the remainder of the y_j are disconnected. To a typical diagram of this kind would correspond a term

$$i^A \{\Delta_F(x_1 - y_a)\Delta_F(x_2 - y_b) \cdots \Delta_F(y_m - y_n)\}\{\Delta_F(y_\alpha - y_\beta) \cdots \Delta_F(y_\mu - y_\nu)\}\,.$$

The terms in the first parentheses belong to the connected part, those in the second parentheses to the vacuum part. For any given term in the connected part, for example the one written above, there will be diagrams corresponding to any possible ordering and interconnection of the $y_{s+1}\cdots y_p$ in the second parenthesis. Such a sum of terms (with the appropriate power of i), is precisely

$$\langle 0|T\left[\mathcal{H}_I(y_{s+1})\ldots\mathcal{H}_I(y_p)\right]|0\rangle$$

which therefore multiplies each term of the connected part. Hence, for a given separation of $y_1\cdots y_p$ into $y_1\cdots y_s$ connected and $y_{s+1}\cdots y_p$ disconnected internal vertices, the vacuum expectation value in Eq.(6.33) can be rewritten

$$\langle 0|T\left[\phi_{\text{in}}(x_1)\ldots\phi_{\text{in}}(x_4)\mathcal{H}_I(y_1)\ldots\mathcal{H}_I(y_s)\right]|0\rangle_c\langle 0|T\left[\mathcal{H}_I(y_{s+1})\ldots\mathcal{H}_I(y_p)\right]|0\rangle\,,$$

where $\langle\,\rangle_c$ indicates that only connected parts are included. We may have any set containing s of the y_j in the connected part and the remaining $p-s$ in the disconnected part, and s may be anything from 0 to p, we should multiply by the the appropriate combinatorial factor and sum over s, so (6.33) becomes

$$\sum_{p=0}^{\infty}\frac{(-i)^p}{p!}\int d^4y_1\ldots\int d^4y_p\sum_{s=0}^{p}\frac{p!}{s!(p-s)!}$$
$$\langle 0|T\left[\phi_{\text{in}}(x_1)\ldots\phi_{\text{in}}(x_4)\mathcal{H}_I(y_1)\ldots\mathcal{H}_I(y_s)\right]|0\rangle_c\langle 0|T\left[\mathcal{H}_I(y_{s+1})\ldots\mathcal{H}_I(y_p)\right]|0\rangle\,.$$

We interchange the order of the summations, at the same time setting $r=p-s$ and introducing the relabeling $z_1=y_{s+1}$, $z_2=y_{s+2},\cdots,z_r=y_p$. This yields

$$\sum_{s=0}^{\infty}\frac{(-i)^s}{s!}\int d^4y_1\ldots\int d^4y_s\langle 0|T\left[\phi_{\text{in}}(x_1)\ldots\phi_{\text{in}}(x_4)\mathcal{H}_I(y_1)\ldots\mathcal{H}_I(y_s)\right]|0\rangle_c$$
$$\sum_{r=0}^{\infty}\frac{(-i)^r}{r!}\int d^4y_1\ldots\int d^4y_r\langle 0|T\left[\mathcal{H}_I(z_1)\ldots\mathcal{H}_I(z_r)\right]|0\rangle\,.$$

The second term here is

$$\langle 0|T\Big\{\exp\Big[-i\int d^4z\mathcal{H}_I(z)\Big]\Big\}|0\rangle\,,$$

which cancels the denominator in Eq.(6.32). The τ-function is therefore

$$\langle 0|T\Big\{\exp\Big[-i\int d^4y\mathcal{H}_I(\phi_{\text{in}};y)\Big]\;\phi_{\text{in}}(x_1)\phi_{\text{in}}(x_2)\phi_{\text{in}}(x_3)\phi_{\text{in}}(x_4)\Big\}|0\rangle_c\,,\qquad(6.40)$$

so we need henceforth only consider connected parts.

We shall now calculate a particular graph in detail, and for simplicity we take the second-order graph shown in Figure 6.7

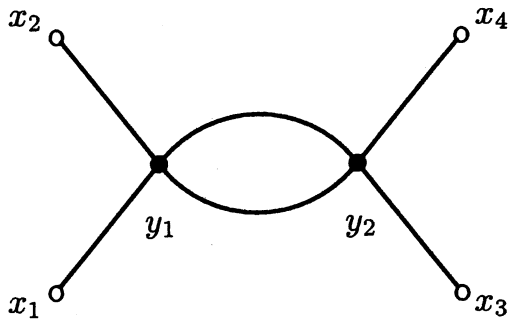


Figure 6.7
Second-order graph

which has a λ interaction at y_1 and y_2 and no δm^2 interaction. The contribution to the τ-function is

$$i^6\lambda^2 \int\int d^4y_1 d^4y_2 \Delta_F(y_1-x_1)\Delta_F(y_1-x_2)\left[\Delta_F(y_2-y_1)\right]^2\Delta_F(x_3-y_2)\Delta_F(x_4-y_2)\,.$$

This is sometimes called an amputated diagram, because the external legs are missing. The legs are grafted on by the action of the Klein-Gordon operators in the LSZ expression for the S-matrix element [Eq.(6.27)]. From the expression (6.35) for the Feynman propagator, we see that

$$(\partial^2 + m^2)\Delta_F(x) = -\frac{1}{(2\pi)^4}\int d^4p\, \mathrm{e}^{-ipx} = -\delta^4(x)\,. \tag{6.41}$$

The contribution of the above graph is thus

$$-\left[\frac{\lambda}{Z}\right]^2 \int\int d^4y_1 d^4y_2 \int\cdots\int d^4x_1\cdots d^4x_4 f_{p_1}(x_1) f_{p_2}(x_2) f^*_{p_3}(x_3) f^*_{p_4}(x_4)$$
$$\delta^4(y_1-x_1)\delta^4(y_1-x_2)\left[\Delta_F(y_2-y_1)\right]^2\delta^4(x_3-y_2)\delta^4(x_4-y_2)\,, \tag{6.42}$$

where for brevity the unscattered wave has been omitted. The x-integrations can be performed trivially, and with use of Eq.(6.35) for the remaining Feynman propagators, we obtain

$$-\left[\frac{\lambda}{Z}\right]^2 (2\pi)^{-6}(16\,\omega_1\omega_2\omega_3\omega_4)^{-\frac{1}{2}} \tag{6.43}$$
$$\int\int d^4y_1 d^4y_2 \exp[-ip_1y_1 - ip_2y_1 + ip_3y_2 + ip_4y_2]\big|_{p_i^0=\omega_i}$$
$$(2\pi)^{-8}\int d^4k_1 \frac{\exp[-ik_1(y_2-y_1)]}{k_1^2-m^2+i\epsilon}\int d^4k_2 \frac{\exp[-ik_2(y_2-y_1)]}{k_2^2-m^2+i\epsilon}\,,$$

where the exponential functions, $f_p(x)$, have been written out explicitly. Since y_1 and y_2 occur only in exponential functions, these integrals can be performed trivially, and we obtain a factor $\delta^4(p_1+p_2-k_1-k_2)\,\delta^4(p_3+p_4-k_1-k_2)$,

which can be replaced by $\delta^4(p_1+p_2-k_1-k_2)\,\delta^4(p_1+p_2-p_3-p_4)$. The first δ-distribution factors out of the integral and simply expresses the conservation of four-momentum. This momentum conserving δ-distribution is a common feature of any Feynman diagram, and it is usual to factor it out from the start by defining the T-matrix element in terms of the S-matrix element:

$$\langle p_3,p_4;\mathsf{in}|S|p_1,p_2;\mathsf{in}\rangle = \langle p_3,p_4;\mathsf{out}|p_1,p_2;\mathsf{out}\rangle + i\delta^4(p_1+p_2-p_3-p_4)\langle p_3,p_4;\mathsf{in}|T|p_1,p_2;\mathsf{in}\rangle\,. \quad (6.44)$$

The contribution of the second-order diagram of Figure 6.7 to the T-matrix element is

$$\frac{i}{4}\left[\frac{\lambda}{Z}\right]^2(2\pi)^{-6}(\omega_1\omega_2\omega_3\omega_4)^{-\frac{1}{2}}\int\frac{d^4k_1}{[k_1^2-m^2+i\epsilon][(p_1+p_2-k_1)^2-m^2+i\epsilon]}\,,$$

where the factor $\delta^4(p_1+p_2-k_1-k_2)$ has been used to get rid of the k_2-integral. The calculation of this term has thus been reduced to that of computing one four-dimensional integral. A similar transformation to momentum space may be made for any Feynman graph.

Normally one draws Feynman graphs directly in momentum space, including the external legs. The above expression would then be drawn as in Figure 6.8.

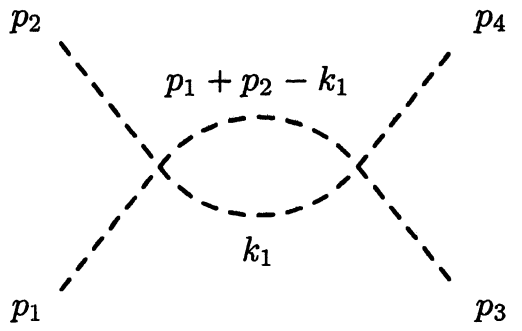


Figure 6.8
Momentum space graph

The four-momenta of the initial, intermediate and final particles are shown. Because of the δ-distributions, four-momentum is conserved at each vertex of a Feynman graph in momentum space. On the other hand, since the intermediate energy, k_1^0, is integrated over all values from $-\infty$ to ∞, *the intermediate particles are not on mass shell.* The external particles, being physical, are on mass-shell.

A set of rules for writing down the mathematical expression that corresponds to any Feynman diagram in the scalar theory can be formulated:

(1) Insert all momenta, with conservation at each vertex,
(2) a factor $(2\omega Z)^{-\frac{1}{2}}$ for each external particle,
(3) a factor $i[k^2 - m^2 + i\epsilon]^{-1}$ for each propagator,
(4) a factor λ for each vertex at which four propagators meet,
(5) a factor $-\frac{1}{2}\delta m^2$ for each vertex at which two propagators meet,
(6) and for each internal loop momentum, an integration, $(2\pi)^{-4}\int d^4k$.

In the center of momentum frame, each particle, incoming or outgoing, has the same energy, say E_{cm}, and the differential cross-section in this frame is

$$\frac{d\sigma}{d\Omega} = \frac{1}{256\pi^2 E_{\mathsf{cm}}^2}|\mathcal{M}|^2\,, \tag{6.45}$$

where $\mathcal{M} = \langle p_3, p_4; \mathsf{in}|T|p_1, p_2; \mathsf{in}\rangle$. The relation (6.45) is the relativistic analog of the nonrelativistic expression to be found in Eq.(8.19) of Volume 1 (the derivation has been relegated to Problem 6.7).

6.5 Exercises

Problem 1

Show that

$$\int_{-\infty}^{t} dt_1 \int_{-\infty}^{t_1} dt_2 \ldots \int_{-\infty}^{t_{n-1}} dt_n \phi(t_1)\phi(t_2)\ldots\phi(t_n)$$
$$= \frac{1}{n!}\int_{-\infty}^{t} dt_1 \int_{-\infty}^{t} dt_2 \ldots \int_{-\infty}^{t} dt_n T[\phi(t_1)\phi(t_2)\ldots\phi(t_n)]$$

Problem 2

What is wrong with the following argument?

$$\begin{aligned}
\langle 0|T\left[\phi(x_1)\phi(x_2)\right]|0\rangle &= \langle 0|T\left[U^\dagger(x_1^0)\phi_{\mathsf{in}}(x_1)U(x_1^0)U^\dagger(x_2^0)\phi_{\mathsf{in}}(x_2)U(x_2^0)\right]|0\rangle \\
&= \langle 0|T\left[U^\dagger(x_1^0)U(x_1^0)U^\dagger(x_2^0)U(x_2^0)\phi_{\mathsf{in}}(x_1)\phi_{\mathsf{in}}(x_2)\right]|0\rangle \\
&= \langle 0|T\left[\phi_{\mathsf{in}}(x_1)\phi_{\mathsf{in}}(x_2)\right]|0\rangle
\end{aligned}$$

Problem 3

Show that, in the neutral scalar quantum field theory with self-interaction,

$$\int d^4x_1 f_{p_1}(x_1)\left[\partial_1^2 + m^2\right]\langle p_4|\phi(x_1)a_{\mathsf{in}}(p_3)|p_2\rangle = 0\,.$$

Problem 4

Complete the LSZ reduction by showing that

$$\begin{aligned}\langle p_4|T\left[\phi(x_1)\phi(x_3)\right]|p_2\rangle \;=\; & -Z^{-1}\int\int d^4x_2 d^4x_4 f_{p_2}(x_2) f^*_{p_4}(x_4) \\ & \left(\partial_2^2 + m^2\right)\left(\partial_4^2 + m^2\right)\langle 0|T\left[\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\right]|0\rangle\end{aligned}$$

Problem 5

Consider two scalar fields, A and B, with the interaction Hamiltonian density

$$\mathcal{H}_i(x) = \lambda : A^2(x)B^2(x) :$$

(a) Deduce the LSZ reduction formula appropriate to the scattering matrix for $A + B \longrightarrow A + B$.
(b) Give the contribution to order λ in the above formula.
(c) Reduce this to an expression involving only vacuum expectation values of time-ordered products of pairs of quantum fields.

Problem 6

For scalar field theory, in the in representation, Fock space is spanned by the vacuum state, $|0\rangle$, one-particle states, $|p;\mathsf{in}\rangle = a^\dagger_{\mathsf{in}}(p)|0\rangle$, two-particle states, $|p_1,p_2;\mathsf{in}\rangle = a^\dagger_{\mathsf{in}}(p_1)a^\dagger_{\mathsf{in}}(p_2)|0\rangle$, and so on. Show that

$$\Omega \equiv \;:\exp\left\{-\int d^3k\, a^\dagger_{\mathsf{in}}(k)a_{\mathsf{in}}(k)\right\}: \;= |0\rangle\langle 0|\,.$$

Problem 7

If a scattering matrix element is written

$$\langle\alpha;\mathsf{in}|S|\beta;\mathsf{in}\rangle = \delta_{\alpha\beta} + i\langle\alpha;\mathsf{in}|T|\beta;\mathsf{in}\rangle = \delta_{\alpha\beta} + i\delta^4(p_\alpha - p_\beta)\mathcal{M}\,,$$

show that, in the case of the elastic scattering of two scalar particles, the differential cross-section is given in the center of momentum by

$$\frac{d\sigma}{d\Omega} = \frac{1}{256\pi^2 E^2_{\mathsf{cm}}}|\mathcal{M}|^2\,.$$

What is the correct formula for the elastic scattering of spin-half particles?

Problem 8

Deduce an explicit form for the commutator

$$[\phi_{\text{in}}(x), \phi_{\text{in}}(y)] \equiv i\Delta(x-y,m)$$

in configuration space. Show that it vanishes if the separation, $x-y$, is spacelike, the condition of microcausality. Does the Feynman propagator, $\Delta_F(x-y)$, vanish under the same circumstances?

Problem 9

Prove that a Lorentz scalar field satisfies

$$\phi(x) = \mathrm{e}^{iPx}\phi(0)\,\mathrm{e}^{-iPx}\,,$$

where P^μ is the generator of space-time translations, i.e., the inhomogeneous part of the Poincaré transformation. Consider the vacuum expectation value of the commutator of two fields at different space-time points, and insert a complete set of in states between the fields. From the weak asymptotic condition,

$$\lim_{x_0\to-\infty}\langle\phi(x)\rangle = \sqrt{Z}\langle\phi_{\text{in}}(x)\rangle\,,$$

demonstrate the Källen-Lehmann representation,

$$\langle 0|\,[\phi(x),\phi(y)\,]|0\rangle = iZ\Delta(x-y,m) + i\int_{m^2}^{\infty} dm'^2\rho(m'^2)\Delta(x-y,m')\,,$$

and prove that $\rho(m'^2)$ is a positive spectral function. Here $i\Delta(x-y,m')$ is the free commutator corresponding to mass m' (Problem 6.8). Deduce $0<Z<1$ from the equal-time commutation relations.

Problem 10

Prove Wick's theorem in the form

$$A_1\ldots A_n = \sum_{p=0}^{[\frac{n}{2}]}\sum(p \text{ contracted pairs}) * : (n-2p)A\text{'s} :$$

Here $[\frac{n}{2}]$ means the integral part of $\frac{n}{2}$, i.e. $\frac{n}{2}$ for even n and $\frac{n-1}{2}$ for odd n. A contracted pair is by definition the vacuum expectation value of the product of two fields, of the form $\langle 0|A_jA_n|0\rangle$. The unspecified sum indicates that all possible combinations of p contracted pairs are to be taken, each combination being multiplied by the normal ordered product of the remaining A_i, where i runs over all the values that are *not* involved in these contracted pairs.

Chapter 7

Quantum Electrodynamics

7.1 LSZ Formalism for QED

In this chapter, we shall extend the results that we have just obtained, so that we can write down at sight the contribution that any Feynman graph in QED makes to the scattering matrix.

We begin with the momentum-space decomposition of the electromagnetic field operator on mass shell, Eq.(4.17), which we may rewrite with use of the polarization vectors (4.22)-(4.23). In the Gupta-Bleuler approach, the photon field has four degrees of freedom, but we only need the transverse field operator,

$$A^T_{\mathsf{in}\mu}(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int \frac{d^3k}{2|\vec{k}\,|} \sum_{\lambda=1}^{2} \varepsilon_\mu(\vec{k}\,,\lambda) \left[a_{\mathsf{in}}(\vec{k}\,,\lambda)\, \mathrm{e}^{-ikx} + a^\dagger_{\mathsf{in}}(\vec{k}\,,\lambda)\, \mathrm{e}^{ikx} \right] , \quad (7.1)$$

for application to physical in states. We restrict our considerations to the Feynman gauge, for which the free-field operator satisfies the wave equation $\partial^2 A_{\mathsf{in}\mu} = 0$. The inverse of the Fourier transform (7.1) is

$$a_{\mathsf{in}}(\vec{k}\,,\lambda) \;=\; \frac{i}{(2\pi)^{\frac{3}{2}}} \frac{1}{\sqrt{2|\vec{k}\,|}} \int d^3x\; \varepsilon^\mu(\vec{k}\,,\lambda)\, \mathrm{e}^{ikx} \overset{\leftrightarrow}{\partial_0} A^T_{\mathsf{in}\mu}(x) , \quad (7.2)$$

and its Hermitian conjugate. The operator $a(\vec{k}\,,\lambda)$ is time-independent for free fields (Problem 7.2), in particular for those of the in and out representations.

Consider an S-matrix element between an 'in' photon state of momentum $\vec{k}$ and polarization λ, and other incoming particles, labeled generically α, and 'out' states, labeled β,

$$\langle \beta; \mathsf{out}|\, (\vec{k}\,,\lambda)\,, \alpha; \mathsf{in}\rangle = \langle \beta; \mathsf{out}| a^\dagger_{\mathsf{in}}(\vec{k}\,,\lambda) |\alpha; \mathsf{in}\rangle\,.$$

We shall subtract from this

$$\langle\beta;\mathsf{out}|a^\dagger_{\mathsf{out}}(\vec{k}\,,\lambda)|\alpha;\mathsf{in}\rangle\,,$$

which is either zero, in the case that $\langle\beta;\mathsf{out}|$ contains no photon states, or is a disconnected term, involving a delta function, which indicates that a photon goes through without scattering, much as in the scalar case [see the unnumbered equation between Eq.(6.22) and Eq.(6.23)]. We may write then

$$\begin{aligned}\langle\beta;\mathsf{out}|\,(\vec{k}\,,\lambda),\alpha;\mathsf{in}\rangle \quad &- \quad \text{disconnected part}\\ &= -\frac{i}{(2\pi)^{\frac{3}{2}}}\frac{1}{\sqrt{2|\vec{k}\,|}}\int d^3x\langle\beta;\mathsf{out}|\,\varepsilon^\mu(\vec{k}\,,\lambda)\,\mathrm{e}^{-ikx}\overset{\leftrightarrow}{\partial_0}\,[A^T_{\mathsf{in}\mu}(x)-A^T_{\mathsf{out}\mu}(x)]\,|\alpha;\mathsf{in}\rangle\\ &= +\frac{i}{\sqrt{Z_3}}\frac{1}{(2\pi)^{\frac{3}{2}}}\frac{1}{\sqrt{2|\vec{k}\,|}}\int d^4x\langle\beta;\mathsf{out}|\,\partial_0\,[\varepsilon^\mu(\vec{k}\,,\lambda)\,\mathrm{e}^{-ikx}\overset{\leftrightarrow}{\partial_0}\,A^T_\mu(x)]\,|\alpha;\mathsf{in}\rangle.\end{aligned}$$

In the last equation we have inserted the asymptotic limits (weak, and applicable *only* when applied to physical states),

$$A_\mu(x)\to\sqrt{Z_3}\,A^T_{\mathsf{in}\mu}(x)\;\;t\to-\infty\quad;\quad A_\mu(x)\to\sqrt{Z_3}\,A^T_{\mathsf{out}\mu}(x)\;\;t\to+\infty\,,$$

where Z_3 is the photon renormalization constant. We now integrate by parts, using the Feynman gauge wave equation, in analogy with the scalar case:

$$\langle\beta|\,a^\dagger_{\mathsf{in}}(\vec{k}\,,\lambda)|\alpha\rangle\to\frac{i}{\sqrt{Z_3}(2\pi)^{\frac{3}{2}}}\int\frac{d^4x}{\sqrt{2|\vec{k}\,|}}\,\mathrm{e}^{-ikx}\overset{\rightarrow}{\partial^2_x}\,\langle\beta|\varepsilon^\mu(\vec{k}\,,\lambda)\,A_\mu(x)|\alpha\rangle\,,$$

where we have written $|\alpha\rangle$ as a shorthand notation for $|\alpha;\mathsf{in}\rangle$, and $\langle\beta|$ for $\langle\beta;\mathsf{out}|$, a practice that we shall repeat.

For a photon in the final state, we obtain from Eq.(7.2),

$$\langle\beta;\mathsf{out}|a_{\mathsf{out}}(\vec{k}\,,\lambda)\,|\alpha;\mathsf{in}\rangle\to\frac{-i}{\sqrt{Z_3}(2\pi)^{\frac{3}{2}}}\int\frac{d^4x}{\sqrt{2|\vec{k}\,|}}\langle\beta|\varepsilon^\mu(\vec{k}\,,\lambda)\,A_\mu(x)|\alpha\rangle\,\overset{\leftarrow}{\partial^2_x}\,\mathrm{e}^{ikx}.$$

These expressions are used to reduce a given matrix element until all photons are removed from the in and out states. The operators are normal-ordered, i.e., creation operators are placed to the left of annihilation operators; but we need them in time-ordered form. Setting

$$A_{\mathsf{in}\mu}(x)=A^+_{\mathsf{in}\mu}(x)+A^-_{\mathsf{in}\mu}(x)\quad,\quad\{A^+_{\mathsf{in}\mu}(x)\}^\dagger=A^-_{\mathsf{in}\mu}(x)\,,$$

we obtain, in analogy with Eq.(6.34),

$$\begin{aligned}T\big[A_\mu(x_1)A_\mu(x_2)\big]\quad-\quad&:A_{\mathsf{in}\mu}(x_1)A_{\mathsf{in}\mu}(x_2):=\\ &\theta(x^0_1-x^0_2)\Big[A^+_{\mathsf{in}\mu}(x_1),A^-_{\mathsf{in}\mu}(x_2)\Big]+\theta(x^0_2-x^0_1)\Big[A^+_{\mathsf{in}\mu}(x_2),A^-_{\mathsf{in}\mu}(x_1)\Big]\,.\end{aligned}$$

In analogy with the discussion in Chapter 6 for the scalar field, we can calculate the vacuum expectation value of this time-ordered product. In the Feynman gauge, the result is (Problem 7.3)

$$\langle 0|T\left[A_{\mathsf{in}\mu}(x_1)A_{\mathsf{in}\nu}(x_2)\right]|0\rangle = \frac{-ig_{\mu\nu}}{(2\pi)^4}\int \frac{d^4p}{p^2-m^2+i\epsilon}\,\mathrm{e}^{-ip(x_1-x_2)}\,.$$

The treatment of three, four or more photon fields in a given matrix element is a simple extension of that presented in Chapter 6 for the scalar field, involving the Wick theorem, and we need not repeat the details.

Let us next consider the reduction of a Dirac field. We invert the momentum-space decomposition of the Dirac field operator, Eq.(4.26), and its adjoint, Eq.(4.27), to obtain the Dirac annihilation operators

$$\begin{aligned} b_{\mathsf{in}}(\vec{p},s) &= \frac{1}{(2\pi)^{\frac{3}{2}}}\sqrt{\frac{m}{\omega_p}}\int d^3x\; u^\dagger(\vec{p},s)\,\mathrm{e}^{ipx}\;\psi_{\mathsf{in}}(x) \\ d_{\mathsf{in}}(\vec{p},s) &= \frac{1}{(2\pi)^{\frac{3}{2}}}\sqrt{\frac{m}{\omega_p}}\int d^3x\;\psi^\dagger_{\mathsf{in}}(x)\;v(\vec{p},s)\,\mathrm{e}^{ipx}\,. \end{aligned} \tag{7.3}$$

The annihilation and creation operators $b_{\mathsf{in}}, d_{\mathsf{in}}, b^\dagger_{\mathsf{in}}, d^\dagger_{\mathsf{in}}$ are all time independent for free fields (Problem 7.4).

Consider an S-matrix element with, among other things, an in Dirac particle of momentum $\vec{p}$ and spin s. We separate a disconnected term in the usual way, and obtain

$$\begin{aligned} &\langle\beta;\mathsf{out}|b^\dagger_{\mathsf{in}}(\vec{p},s)|\alpha;\mathsf{in}\rangle \quad - \quad \text{disconnected part} \\ &\quad = +\frac{1}{(2\pi)^{\frac{3}{2}}}\sqrt{\frac{m}{\omega_p}}\int d^3x\langle\beta|\,\left[\psi^\dagger_{\mathsf{in}}(x)-\psi^\dagger_{\mathsf{out}}(x)\right]\,|\alpha\rangle u(\vec{p},s)\,\mathrm{e}^{-ipx} \\ &\quad = -\frac{1}{\sqrt{Z_2}}\frac{1}{(2\pi)^{\frac{3}{2}}}\sqrt{\frac{m}{\omega_p}}\int d^4x\langle\beta|\,\partial_0\left[\overline{\psi}(x)\gamma^0 u(\vec{p},s)\,\mathrm{e}^{-ipx}\right]\,|\alpha\rangle\,. \end{aligned} \tag{7.4}$$

In the last equation we employed the (weak) asymptotic limits

$$\psi(x)\to\sqrt{Z_2}\,\psi_{\mathsf{in}}(x)\;(t\to-\infty)\,,\qquad \psi(x)\to\sqrt{Z_2}\,\psi_{\mathsf{out}}(x)\;(t\to\infty)\,,$$

where Z_2 is the renormalization constant for the Dirac field. The function $u(\vec{p},s)\,\mathrm{e}^{-ipx}$ satisfies the free Dirac equation, so

$$\gamma^0\partial_0\;u(\vec{p},s)\;\mathrm{e}^{-ipx} = (-\vec{\gamma}\cdot\vec{\nabla}\;-im)\;u(\vec{p},s)\;\mathrm{e}^{-ipx}\;;$$

and thus we can integrate (7.4) by parts to yield

$$\langle\beta|b^\dagger_{\mathsf{in}}|\alpha\rangle\to\frac{-i}{\sqrt{Z_2}(2\pi)^{\frac{3}{2}}}\int d^4x\langle\beta|\overline{\psi}(x)|\alpha\rangle(-i\gamma\overleftarrow{\partial}-m)\sqrt{\frac{m}{\omega_p}}u(\vec{p},s)\,\mathrm{e}^{-ipx}\,.$$

One may proceed in an analogous fashion to remove a Dirac antiparticle from the in state:

$$\langle\beta|d^\dagger_{\rm in}|\alpha\rangle \to \frac{i}{\sqrt{Z_2}(2\pi)^{\frac{3}{2}}}\int d^4x\sqrt{\frac{m}{\omega_p}}\bar{v}(\vec{p},s)\,{\rm e}^{-ipx}(i\gamma\cdot\overrightarrow{\partial}-m)\langle\beta|\psi(x)|\alpha\rangle\,.$$

Here use has been made of the adjoint Dirac equation,

$$\partial_0\, v^\dagger(\vec{p},s)\ {\rm e}^{-ipx}\ \gamma^0 = v^\dagger(\vec{p},s)\ {\rm e}^{-ipx}\ (-\vec{\gamma}\cdot\overleftarrow{\nabla}+im)\,.$$

To remove a Dirac particle from the out state we have

$$\langle\beta|b_{\rm out}|\alpha\rangle \to \frac{-i}{\sqrt{Z_2}}\frac{1}{(2\pi)^{\frac{3}{2}}}\int d^4x\sqrt{\frac{m}{\omega_p}}\bar{u}(\vec{p},s)\,{\rm e}^{ipx}(i\gamma\cdot\overrightarrow{\partial}-m)\,\langle\beta|\psi(x)|\alpha\rangle\,,$$

and to remove a Dirac antiparticle from the out state,

$$\langle\beta|d_{\rm out}|\alpha\rangle \to \frac{i}{\sqrt{Z_2}(2\pi)^{\frac{3}{2}}}\int d^4x\langle\beta|\overline{\psi}(x)\ |\alpha\rangle\,(-i\gamma\cdot\overleftarrow{\partial}-m)\sqrt{\frac{m}{\omega_p}}v(\vec{p},s)\,{\rm e}^{-ipx}\,.$$

We use these expressions to remove all Dirac particles from the in and out states. Again we must re-express the normal-ordered product as a time-ordered one; to do this we write $:\psi_{\rm in}(x_1)\overline{\psi}_{\rm in}(x_2):$ as

$$\psi^+_{\rm in}(x_1)\overline{\psi}^+_{\rm in}(x_2)+\psi^-_{\rm in}(x_1)\overline{\psi}^+_{\rm in}(x_2)-\overline{\psi}^-_{\rm in}(x_2)\psi^+_{\rm in}(x_1)+\psi^-_{\rm in}(x_1)\overline{\psi}^-_{\rm in}(x_2)\,, \quad (7.5)$$

where $\psi^+_{\rm in}(x_1)$ is an annihilation operator for particles, and $\psi^-_{\rm in}(x_1)$ a creation operator for antiparticles. The adjoint, $\overline{\psi}^-_{\rm in}(x_2) = \psi^{+\dagger}_{\rm in}(x_2)\gamma^0$, is a creation operator for particles, and $\overline{\psi}^+_{\rm in}(x_2) = \psi^{-\dagger}_{\rm in}(x_2)\gamma^0$, is an annihilation operator for antiparticles. The order of factors in the third term of Eq.(7.5), and its sign, have been switched, in keeping with the requirement for normal ordering of anticommuting fermion field operators. When the times for these events satisfy $x_{10} > x_{20}$, the time ordering differs from the normal ordering only in the third term; but for $x_{10} < x_{20}$ the time-ordered product differs from the normal-ordered term only in the second factor. In general,

$$T\,[\psi_{\rm in}(x_1)\overline{\psi}_{\rm in}(x_2)] - :\psi_{\rm in}(x_1)\overline{\psi}_{\rm in}(x_2): =$$
$$\theta(x_{10}-x_{20})\,\{\,\psi^+_{\rm in}(x_1),\overline{\psi}^-_{\rm in}(x_2)\,\}-\theta(x_{20}-x_{10})\,\{\,\psi^+_{\rm in}(x_2),\overline{\psi}^-_{\rm in}(x_1)\,\}$$

Eqs.(4.26)-(4.27) are used to calculate the vacuum expectation value of this time-ordered product (Problem 7.5), yielding

$$\langle 0|\,T[\psi_{\rm in}(x_1)\overline{\psi}_{\rm in}(x_2)]\,|0\rangle = \frac{i}{(2\pi)^4}\int d^4p\frac{\gamma p+m}{p^2-m^2+i\epsilon}\,{\rm e}^{-ip(x_1-x_2)}\,.$$

The application of Wick's theorem to simplify matrix elements follows the lines described in Sect. **6.3** for the scalar field, except that the permutation terms in the fermion analog of Eq.(6.37) each carry a factor $(-1)^m$, where m is the total number of interchanges of fermion fields that must be made, in order to bring the original ordering of these fields into the order corresponding to the term in question. Aside from this point, the reduction of anticommuting Dirac field operators is a direct extension of the method used for scalar and for vector fields.

The interaction Hamiltonian density in QED is $\mathcal{H}_i = e : \overline{\psi}\gamma_\mu A^\mu \psi :$ but we should rather write e_0, the *bare* charge, in place of the physical e, which is related to it through multiplication by a suitable renormalization constant. Since the free part of the Dirac Lagrangian density contains the bare mass, m_0, there will have to be a mass-shift term, as in the scalar case, so that the effective interaction Hamiltonian density, including the mass shift, is

$$\mathcal{H}_I(\psi_{\text{in}}, A^\mu_{\text{in}}; y) = e_0 : \overline{\psi}_{\text{in}}\gamma_\mu \psi_{\text{in}} A^\mu_{\text{in}} : - \delta m : \overline{\psi}_{\text{in}}\psi_{\text{in}} :$$

where $\delta m = m - m_0$. This density plays the same role as did Eq.(6.31) in the scalar theory.

By way of illustration, let us consider how the LSZ formula, Eq.(6.27), is generalized in the case of Compton scattering, which describes the collision of a photon and an electron. Suppose that the electron has momentum p before, and p' after the collision, with spin respectively s and s'. The incident photon has momentum and polarization q and λ, the emergent photon q' and λ'. The disconnected term, corresponding to no scattering, is

$$\delta_{ss'}\delta_{\lambda\lambda'}\delta^3(\vec{p} - \vec{p}\,')\delta^3(\vec{q} - \vec{q}\,') .$$

The four integrals of Eq.(6.27) will be present; but the renormalization factor, Z^{-2}, is replaced by $(Z_2 Z_3)^{-1}$. The exponential functions are $e^{i(p_3+p_4-p_1-p_2)}$, multiplied by the normalization factor

$$(2\pi)^{-6}\frac{m}{2\sqrt{\omega_p\,\omega_{p'}\,|\vec{q}\,||\vec{q}\,'|}} .$$

Instead of the Klein-Gordon operators, we have

$$\overline{u}(\vec{p}\,', s')(i\gamma\overrightarrow{\partial}_3 - m)\epsilon^\nu(q', \lambda')\overrightarrow{\partial}_4^2\,\tau(x_1, x_2, x_3, x_4)\overleftarrow{\partial}_2^2\,\epsilon^\mu(q, \lambda)(i\gamma\overleftarrow{\partial}_1 + m)u(\vec{p}, s) .$$

The cancellation of the vacuum parts occurs as in the scalar case, and we may write the τ-function as the connected part

$$\tau(x_1, x_2, x_3, x_4) = \langle 0|T\left[\psi(x_3)A_\nu(x_4)\overline{\psi}(x_1)A_\mu(x_2)\right]|0\rangle_c .$$

The evolution operator is introduced as before, in order to reduce the τ-function to vacuum expectation values of a time-ordered products of in fields; and then Wick's theorem is invoked to reduce terms in the perturbation series to products of propagators.

Care is needed if there are closed loops of fermion propagators. Consider, for example, a term that arises in the calculation of the vacuum polarization:

$$\begin{aligned}\langle 0|T\big[:\overline{\psi}_{\mathsf{in}}(y_1)\psi_{\mathsf{in}}(y_1)\overline{\psi}_{\mathsf{in}}(y_2)\psi_{\mathsf{in}}(y_2):\big]|0\rangle \\ &= -\langle 0|T\big[:\psi_{\beta,\,\mathsf{in}}(y_2)\overline{\psi}_{\alpha,\,\mathsf{in}}(y_1)\psi_{\alpha,\,\mathsf{in}}(y_1)\overline{\psi}_{\beta,\,\mathsf{in}}(y_2):\big]|0\rangle \\ &= -\mathsf{Tr}\Big\{\langle 0|T\big[:\psi_{\mathsf{in}}(y_2)\overline{\psi}_{\mathsf{in}}(y_1)\psi_{\mathsf{in}}(y_1)\overline{\psi}_{\mathsf{in}}(y_2):\big]|0\rangle\Big\} \\ &= -\mathsf{Tr}\Big\{\langle 0|T[\psi_{\mathsf{in}}(y_2)\overline{\psi}_{\mathsf{in}}(y_1)]|0\rangle\,\langle 0|T[\psi_{\mathsf{in}}(y_1)\overline{\psi}_{\mathsf{in}}(y_2)]|0\rangle\Big\}\,.\end{aligned}$$

Here the minus sign arises because of the odd number of interchanges of fermion operators, and summation is implicit over the spinor indices, α and β, leading to the trace operation. This procedure can immediately be generalized to any closed loop of fermions: the last ψ_{in} operator is moved to the front, passing over an odd number of fermion fields in the process, and this generates a minus sign and a trace of the whole product of fields.

Lastly, transformation to momentum space yields the Feynman rules for quantum electrodynamics:

(1) A factor $\frac{i}{\gamma p - m + i\epsilon} = \frac{i(\gamma p + m)}{p^2 - m^2 + i\epsilon}$ for each electron propagator.
(2) A factor $-\frac{ig_{\mu\nu}}{k^2 + i\epsilon}$ for each photon propagator.
(3) A factor $-ie\gamma_\mu$ for each electron-photon vertex.
(4) For each external photon, multiply by $(2|\vec{k}\,|)^{-\frac{1}{2}}(2\pi)^{-\frac{3}{2}}\varepsilon^\mu(k,\lambda)$, where only the physical values of the polarization, $\lambda = 1, 2$, are allowed.
(5) For each incoming electron multiply by $(m/\omega)^{-\frac{1}{2}}(2\pi)^{-\frac{3}{2}}u(p,s)$, and for each outgoing positron by $(m/\omega)^{-\frac{1}{2}}(2\pi)^{-\frac{3}{2}}v(p,s)$, the spinors being written on the right; for an outgoing electron by $(m/\omega)^{-\frac{1}{2}}(2\pi)^{-\frac{3}{2}}\overline{u}(p,s)$, and for an incoming positron by $(m/\omega)^{-\frac{1}{2}}(2\pi)^{-\frac{3}{2}}\overline{v}(p,s)$, these spinors being written on the left.
(6) For each closed fermion loop, take the trace and include a minus sign.
(7) For each internal loop momentum, perform an integration, $(2\pi)^{-4}\int d^4q$.

7.2 Compton Scattering

The rules suffice for a basic understanding of Feynman graphs. We begin with the Compton effect, which has already been mentioned, namely the process

in which light can be scattered from electrons with, in general, a change of frequency. We shall use the physical m and e, rather than the bare m_0 and e_0, which is conventional in low-order perturbation calculations. It is only when one considers higher-order graphs that one has to worry about the mass shift and the charge renormalization. One of the lowest order diagrams we may draw for photon-electron scattering is as shown.

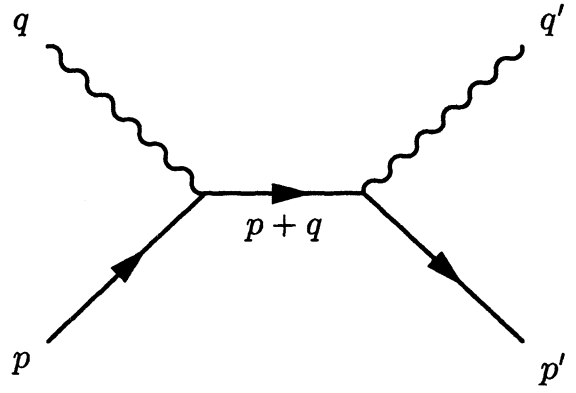


Figure 7.1
Compton scattering

An electron of four-momentum p and spin s absorbs a photon of four-momentum q and polarization λ. The intermediate electron propagator has four-momentum $p+q$, because of momentum conservation, so there is no internal momentum left to be integrated. The final state consists of an electron of momentum p' and spin s' and a photon of momentum q' and polarization λ'. Overall momentum conservation implies that $p+q=p'+q'$.

According to the Feynman rules, the contribution to the scattering matrix will be

$$\frac{1}{(2\pi)^6}\sqrt{\frac{m^2}{4\omega_p\omega_{p'}\omega_q\omega_{q'}}}\,\overline{u}(p',s')[-ie\gamma_\mu\varepsilon^\mu(q',\lambda')]\frac{i}{\gamma p+\gamma q-m+i\epsilon}[-ie\gamma_\nu\varepsilon^\nu(q,\lambda)]u(p,s)\,, \tag{7.6}$$

where $\omega_p=\sqrt{\vec{p}^2+m^2}$, $\omega_q=|\vec{q}|$, and similarly for the primed quantities. This may be written

$$-\frac{ie^2}{(2\pi)^6}\frac{m}{2\sqrt{\omega_p\omega_{p'}\omega_q\omega_{q'}}}\overline{u}(p',s')R\,u(p,s)\,,$$

where

$$R=[\gamma\varepsilon(q',\lambda')][\gamma p+\gamma q+m][\gamma\varepsilon(q,\lambda)]/[2pq]\,.$$

To obtain this last result, use has been made of the mass-shell conditions

$$p^2=m^2=p'^2 \qquad q^2=0=q'^2$$

so that

$$(p+q)^2 - m^2 = 2pq\,.$$

We have dropped the $i\epsilon$ in Eq.(7.6) because

$$pq = m\omega_q\,,$$

in the rest frame of the incoming electron, and is always positive, so the pole does not lie in the physical scattering region.

There is another graph in second order, as shown.

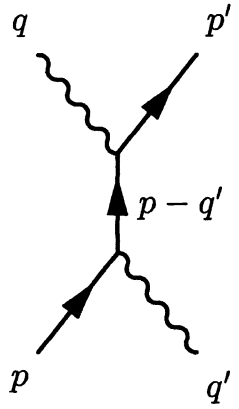


Figure 7.2
Crossed diagram

This contribution may be obtained from the contribution (7.6) by means of the interchanges $q \longleftrightarrow -q'$ and $\lambda \longleftrightarrow \lambda'$. It is important to include both terms to order e^2; and it is only the sum that is invariant under the crossing interchange. The sum of the two graphs leads to very good agreement with the measured cross-section for Compton scattering. If the spin of the initial electron is not measured, one must average over the two possible values of s; and if the spin of the final electron is not measured, one must sum over the two possible values of s'. Similarly, if no polarizations of the photons are measured, one must average over initial polarization λ and sum over final polarization λ'.

We may mention another aspect of crossing. Pair annihilation,

$$e^+ + e^- \longrightarrow \gamma + \gamma\,,$$

or pair creation,

$$\gamma + \gamma \longrightarrow e^+ + e^-\,,$$

is described in second order by the same two graphs, except that p' is now the four momentum of an *incoming* positron, and q that of an *outgoing* photon.

To improve the accuracy of the agreement with experiment we include also the contribution of the fourth order box diagram depicted. There is also a similar

graph obtained by the crossing interchange, and this must be included in the calculation. According to the Feynman rules, the graph gives the contribution

$$\frac{1}{(2\pi)^6}\sqrt{\frac{m^2}{4\omega_p\omega_{p'}\omega_q\omega_{q'}}}\int\frac{d^4k}{(2\pi)^4}\overline{u}(p',s')[-ie\gamma_\mu]\frac{i[\gamma(p'+k)+m]}{(p'+k)^2-m^2+i\epsilon}\cdot$$
$$[-ie\gamma_\nu\varepsilon^\nu(q',\lambda')]\frac{i\gamma(p+q+k)+m}{(p+q+k)^2-m^2+i\epsilon}[-ie\gamma_\rho\varepsilon^\rho(q,\lambda)]\cdot$$
$$\frac{i[\gamma(p+k)+m]}{(p+k)^2-m^2+i\epsilon}[-ie\gamma_\sigma]\frac{-ig^{\mu\sigma}}{k^2+i\epsilon}u(p,s)\,. \tag{7.7}$$

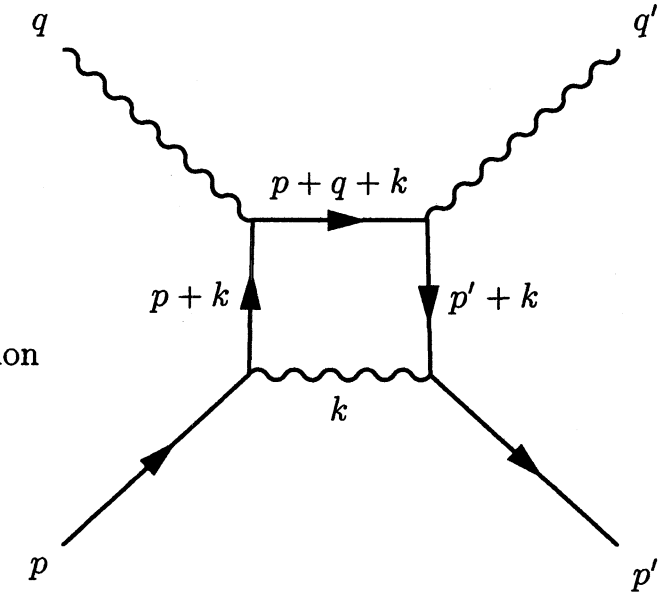


Figure 7.3
Fourth-order contribution

Momentum conservation at the four vertices leaves one four momentum free, which we have taken to be that of the photon, k. This must be integrated, which means that the internal, virtual particles can have any values for their four momenta, in particular off mass shell values. The order of the spinors and the gamma matrices is important: if one writes from left to right, one should begin with an outgoing fermion (or an incoming antifermion), and then follow the fermion line in the direction opposed to the arrow, until one arrives at the incoming fermion (or outgoing antifermion). In this way the order of the spinors, the fermion propagators, and the vertices, is correct. The positions of the polarization vectors for external photon lines, and of photon propagators, is immaterial, but the Lorentz index of a polarization vector must be contracted with that of the gamma matrix that belongs to the vertex to which the photon line is connected in the Feynman graph. Similarly, the Lorentz indices of the metric tensor in the photon propagator must be contracted with those of the gamma matrices that belong to the vertices at the extremities of the propagator.

The expression Eq.(7.7) may be simplified by using the Dirac equation and the mass shell conditions; note that the denominator behaves like k^8 as $k \to \infty$, while the numerator behaves like k^3. Thus the four dimensional integral converges, so that its numerical value can be computed. The crossing symmetric contribution, obtained by means of the interchange $(q, \lambda \leftrightarrow -q', \lambda')$, must then be added, and finally the resultant fourth order expression is added to the second order S-matrix that we obtained above. The agreement with experiment is thereby improved.

7.3 Møller Scattering

We turn now to Møller scattering, which is electron-electron scattering, mediated by photons. The T matrix is defined in general by

$$\langle f|S|i\rangle = \delta_{fi} + i(2\pi)^4 \delta^4(P_f - P_i)\langle f|T|i\rangle$$

where P_i, P_f are the momenta of the initial and final states i, f. For an incident state that consists of two identical Dirac fermions of mass m in states (p_1, s_1) and (p_2, s_2), this expression leads to the differential cross-section

$$d\sigma = \frac{m^2}{\sqrt{(p_1 p_2)^2 - m^4}} \sum_f (2\pi)^4 \, \delta^4(P_f - p_1 - p_2) \, |\langle f|T|p_1, s_1; p_2, s_2\rangle|^2 \,,$$

for an arbitrary final state f (Problem 8). There are two diagrams in second order:

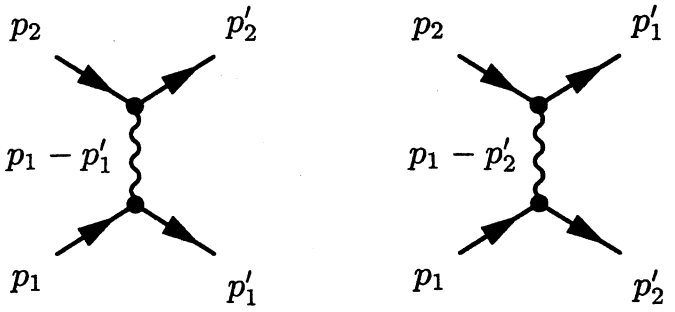


Figure 7.4
Møller scattering

Here the final state consists of the same two Dirac fermions in states (p_1', s_1') and (p_2', s_2'), and the differential cross-section may be written

$$d\sigma = \frac{m^2}{\sqrt{(p_1 p_2)^2 - m^4}} \int \frac{d^3 p_1'}{(2\pi)^3} \frac{m}{\omega_1'} \int \frac{d^3 p_2'}{(2\pi)^3} \frac{m}{\omega_2'} \, (2\pi)^4 \, \delta^4(p_1' + p_2' - p_1 - p_2) \, |\mathcal{M}|^2 \,,$$

where $\mathcal{M} = \langle p_1', s_1'; p_2', s_2' | T | p_1, s_1; p_2, s_2 \rangle$. We may integrate over the two-body phase space in order to determine the differential cross-section per unit solid angle of the outgoing particle of momentum p_1'. In the center of momentum frame, in which the incident particles each have energy E, we obtain

$$\frac{d\sigma}{d\Omega} = \frac{m^4}{4E^2} \frac{1}{(4\pi)^2} |\mathcal{M}|^2 \,. \tag{7.8}$$

The Feynman rules yield, for the first graph (the direct term),

$$\frac{ie^2\mathcal{D}}{(p_1 - p_1')^2} \quad \text{where} \quad \mathcal{D} = [\overline{u}(p_1', s_1')\gamma^\mu u(p_1, s_1)]\,[\overline{u}(p_2', s_2')\gamma_\mu u(p_2, s_2)]\,, \tag{7.9}$$

and for the second graph (the exchange term),

$$\frac{ie^2\mathcal{E}}{(p_1 - p_2')^2} \quad \text{where} \quad \mathcal{E} = [\overline{u}(p_2', s_2')\gamma^\mu u(p_1, s_1)]\,[\overline{u}(p_1', s_1')\gamma_\mu u(p_2, s_2]\,. \tag{7.10}$$

The contributions Eq.(7.9) and Eq.(7.10) must be *subtracted* from one another, since the relative ordering of the fermion operators is odd.

This is the final answer to this order, in the case that the initial and final spins are held fixed. If spins are not measured, as is often the case, then we can simplify the numerators after averaging over the initial spin configurations and summing over the final spins. This involves the insertion of the summation $\frac{1}{4}\sum_{s_1,s_2,s_1',s_2'}$, and we shall show how these sums can be evaluated. Consider

$$\sum_{s_1,s_2,s_1',s_2'} |\mathcal{D}|^2 = \sum_{s_1,s_2,s_1',s_2'} \overline{u}'_{1\alpha}\gamma^\mu_{\alpha\beta}u_{1\beta}\overline{u}'_{2\gamma}\gamma_{\mu\gamma\delta}u_{2\delta}\overline{u}_{1\epsilon}\gamma^\nu_{\epsilon\eta}u'_{1\eta}\overline{u}_{2\sigma}\gamma_{\nu\sigma\omega}u'_{2\omega}\,,$$

where for brevity we have introduced the notation u_1 for $u(p_1, s_1)$, and $\overline{u}_1'$ for $\overline{u}(p_1', s_1')$, and similarly with index 2. Here α, β, $\cdots$ are the explicit spinor indices. This product of numbers may be reordered and then rewritten as a product of traces of matrices:

$$\begin{aligned}\sum_{s_1,s_2,s_1',s_2'} |\mathcal{D}|^2 &= \sum_{s_1,s_2,s_1',s_2'} u'_{1\eta}\overline{u}'_{1\alpha}\gamma^\mu_{\alpha\beta}u_{1\beta}\overline{u}_{1\epsilon}\gamma^\nu_{\epsilon\eta}u'_{2\omega}\overline{u}'_{2\gamma}\gamma_{\mu\gamma\delta}u_{2\delta}\overline{u}_{2\sigma}\gamma_{\nu\sigma\omega} \\ &= \sum_{s_1,s_2,s_1',s_2'} \mathrm{Tr}\,[u_1'\overline{u}_1'\gamma^\mu u_1\overline{u}_1\gamma^\nu]\;\mathrm{Tr}\,[u_2'\overline{u}_2'\gamma_\mu u_2\overline{u}_2\gamma_\nu]\,. \end{aligned} \tag{7.11}$$

Referring to the particle projection operator, Eq.(3.84)-(3.85), we see that

$$\begin{aligned}\sum_{s_1,s_1'} \mathrm{Tr}\,[u_1'\overline{u}_1'\gamma^\mu u_1\overline{u}_1\gamma^\nu] &= \mathrm{Tr}\left[\frac{m + \gamma p_1'}{2m}\gamma^\mu\frac{m + \gamma p_1}{2m}\gamma^\nu\right] \\ &= \frac{1}{4m^2}\mathrm{Tr}\,[\gamma^\rho\gamma^\mu\gamma^\sigma\gamma^\nu p'_{1\rho}p_{1\sigma} + m^2\gamma^\mu\gamma^\nu]\,. \end{aligned}$$

Here we have used the fact that the trace of a product of three gamma matrices vanishes. This may be simply proved by using the properties of γ_5:

$$\mathrm{Tr}\,[\gamma^\rho\gamma^\mu\gamma^\sigma] = \mathrm{Tr}\,[\gamma_5^2\gamma^\rho\gamma^\mu\gamma^\sigma] = \mathrm{Tr}\,[\gamma_5\gamma^\rho\gamma^\mu\gamma^\sigma\gamma_5] = -\mathrm{Tr}\,[\gamma_5\gamma^\rho\gamma^\mu\gamma^\sigma\gamma_5]\,.$$

The third expression follows from the second by cyclicity of the trace, and the fourth expression follows from the second by anticommuting γ_5 through the product of the three gamma matrices. The trace is equal to minus itself and is therefore zero.

To complete the calculation, we need the traces of products of two and of four gamma matrices. Since the trace is cyclic, it follows that

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu} \quad \Longrightarrow \quad \mathrm{Tr}\,[\gamma^\mu\gamma^\nu] = 4g^{\mu\nu}\,, \tag{7.12}$$

because $\mathrm{Tr}\ 1 = 4$. It follows easily from the above that

$$\gamma^\rho\gamma^\mu\gamma^\sigma\gamma^\nu + \gamma^\nu\gamma^\rho\gamma^\mu\gamma^\sigma = 2g^{\sigma\nu}\gamma^\rho\gamma^\mu - 2g^{\mu\nu}\gamma^\rho\gamma^\sigma + 2g^{\nu\rho}\gamma^\mu\gamma^\sigma\,.$$

The traces of the two terms on the left are equal, and so we find, on using the result we have just proved for the trace of two gamma matrices, that

$$\mathrm{Tr}\,[\gamma^\rho\gamma^\mu\gamma^\sigma\gamma^\nu] = 4[g^{\rho\mu}g^{\sigma\nu} - g^{\rho\sigma}g^{\mu\nu} + g^{\rho\nu}g^{\mu\sigma}]\,. \tag{7.13}$$

From the above formulae, we find

$$\sum_{s_1,s_1'} \mathrm{Tr}\,[u_1'\overline{u}_1'\gamma^\mu u_1\overline{u}_1\gamma^\nu] \;=\; \frac{4}{m^2}\{p_1'^\mu p_1^\nu + p_1'^\nu p_1^\mu + g^{\mu\nu}[m^2 - (p_1'p_1)]\}\,.$$

The second factor in Eq.(7.11) can similarly be calculated, with p_2 and p_2' in place of p_1 and p_1'. The product of the two factors yields

$$\delta = \sum_{s_1,s_2,s_1',s_2'} |\mathcal{D}|^2 = \frac{4}{m^4}\Big\{(p_1p_2)^2 + (p_1p_2')^2 + 2m^2[(p_1p_2') - (p_1p_2)]\Big\} \tag{7.14}$$

where we have used the identities

$$(p_1p_2) = (p_1'p_2') \quad (p_1'p_2) = (p_1p_2') \quad (p_1'p_1) = (p_2'p_2)\,,$$

which follow from 4-momentum conservation and the mass-shell conditions, to eliminate p_1'. By similar manipulations, we find

$$\varepsilon = \sum_{s_1,s_2,s_1',s_2'} |\mathcal{E}|^2 = \frac{4}{m^4}\Big\{(p_1p_2)^2 + (p_2p_2')^2 + 2m^2[(p_2p_2') - (p_1p_2)]\Big\} \tag{7.15}$$

and, for the cross-term in Eq.(7.17),

$$\gamma = \sum_{s_1,s_2,s_1',s_2'} \mathcal{D}^*\mathcal{E} + \mathcal{D}\mathcal{E}^* = \frac{8}{m^4}\Big\{(p_1p_2)^2 - 2m^2(p_1p_2)\Big\} \tag{7.16}$$

On insertion of the contributions (7.14)-(7.16) into Eq.(7.8), we finally obtain the differential cross-section for Møller scattering to order e^4,

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E^2}\left[\frac{\delta}{[(p_1-p_1')^2]^2} - \frac{\gamma}{(p_1-p_1')^2\,(p_1-p_2')^2} + \frac{\varepsilon}{[(p_1-p_2')^2]^2}\right], \tag{7.17}$$

where $\alpha = e^2/(4\pi)$, the fine-structure constant. An important feature of Møller scattering is a strong peaking of the differential cross-section in the forward and backward directions. This is a direct consequence of the photon propagators, for we may set, in the center of mass system of the electrons,

$$\begin{aligned} p_1 &= (E,\vec{p}) & p_2 &= (E,-\vec{p}) \\ p_1' &= (E,\vec{p}\,') & p_2' &= (E,-\vec{p}\,') \\ |\vec{p}| &= p = |\vec{p}\,'| & \vec{p}\cdot\vec{p}\,' &= p^2\cos\theta \\ E &= \sqrt{p^2+m^2}\,. \end{aligned} \tag{7.18}$$

The propagator in Eq.(7.9) is

$$\frac{1}{(p_1-p_1')^2} = -\frac{1}{(\vec{p}_1-\vec{p}_1')^2} = -\frac{1}{2p^2(1-\cos\theta)} = -\frac{1}{4p^2\sin^2\frac{\theta}{2}},$$

and this explains the forward peak (at $\theta = 0$). The crossed graph contains

$$\frac{1}{(p_1-p_2')^2} = -\frac{1}{(\vec{p}_1+\vec{p}_1')^2} = -\frac{1}{2p^2(1+\cos\theta)} = -\frac{1}{4p^2\cos^2\frac{\theta}{2}},$$

so this term produces the backward peak (at $\theta = \pi$). Eq.(7.17) becomes

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E^2}\left[\frac{2E^2-m^2}{E^2-m^2}\right]^2\left[\frac{4}{\sin^4\theta} - \frac{3}{\sin^2\theta} + \left[\frac{E^2-m^2}{2E^2-m^2}\right]^2\left(1+\frac{4}{\sin^2\theta}\right)\right] \tag{7.19}$$

In the relativistic limit, $E \to \infty$, we obtain

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E^2}\left[\frac{4}{\sin^4\theta} - \frac{2}{\sin^2\theta} + \frac{1}{4}\right] \tag{7.20}$$

At low energies, $E - m \approx p^2/(2m) \ll m$, and we obtain

$$\frac{d\sigma}{d\Omega} = \frac{m^2\alpha^2}{4p^4}\left[\frac{4}{\sin^4\theta} - \frac{3}{\sin^2\theta}\right]. \tag{7.21}$$

It is of interest to compare these results with the Born approximation to nonrelativistic scattering in a Coulomb potential. On page 136 of Volume 1 the cross-section in this approximation was given, namely

$$\frac{d\sigma}{d\Omega} = \frac{e^4}{16E^2 \sin^4 \frac{\theta}{2}} . \tag{7.22}$$

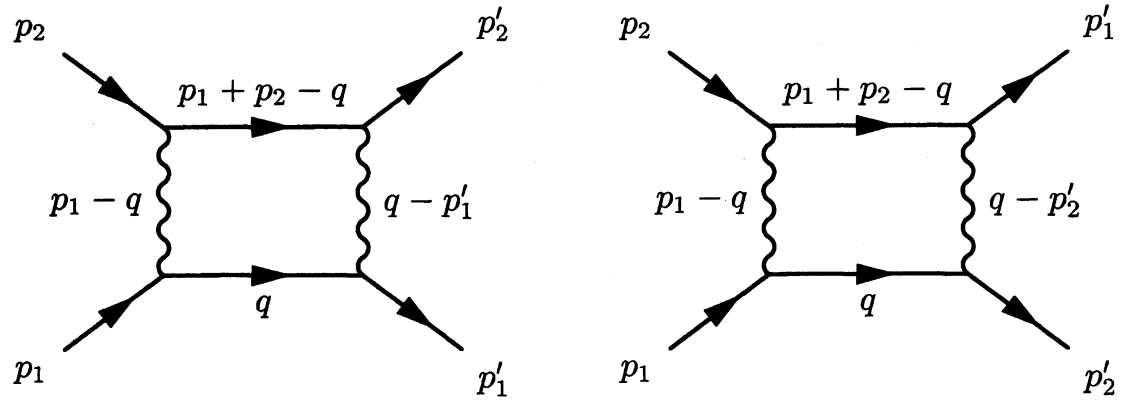


Figure 7.5 Fourth-order diagrams

However, unlike the E in Eq.(7.19), E in Eq.(7.22) is the nonrelativistic energy, which we can write $p^2/(2m_{red})$, p being, as in the relativistic calculation, the center of mass three-momentum of one of the electrons. The reduced mass in a two-electron system is $m_{red} = \frac{1}{2}m$, and we employ units such that $\hbar = 1 = c$. In the nonrelativistic Born approximation calculation, the Coulomb potential was $-e^2/r$, which means that Gaussian, rather than Heaviside-Lorentz units were being used for the charge. With Gaussian units, the fine-structure constant is $\alpha = e^2/(\hbar c) = e^2$, so finally we can write the cross-section in nonrelativistic Born approximation as

$$\frac{d\sigma}{d\Omega} = \frac{m^2\alpha^2}{16p^4 \sin^4 \frac{\theta}{2}} . \tag{7.23}$$

To compare this with the first-order relativistic formula Eq.(7.17), we retain only the direct term involving $|\mathcal{D}|^2$, and we take only the lowest order in small p. In lowest order, the relativistic energy, E, is just m, and $\mathcal{D} = 8$. This yields precisely the same result, Eq.(7.23), if we recall that the Feynman rules where set up with an electromagnetic interaction in which the rationalized, Heaviside-Lorentz units were used for the charge. With these units, the fine-structure constant is $\alpha = e^2/(4\pi)$. The simple nonrelativistic calculation with Coulomb potential and Born approximation did not take cognizance of the exchange term.

Such an exchange term can always be put in by hand in the nonrelativistic calculation, much as we did in the calculation of the exchange effects in the helium atom. The relativistic calculation, it should be noted, automatically takes care of all exchange contributions, thanks to the crossing symmetry of the S matrix.

We may increase the accuracy by adding the contribution from the fourth order graphs shown in Figure 7.5, but we must also add the two extra fourth order graphs of Figure 7.6:

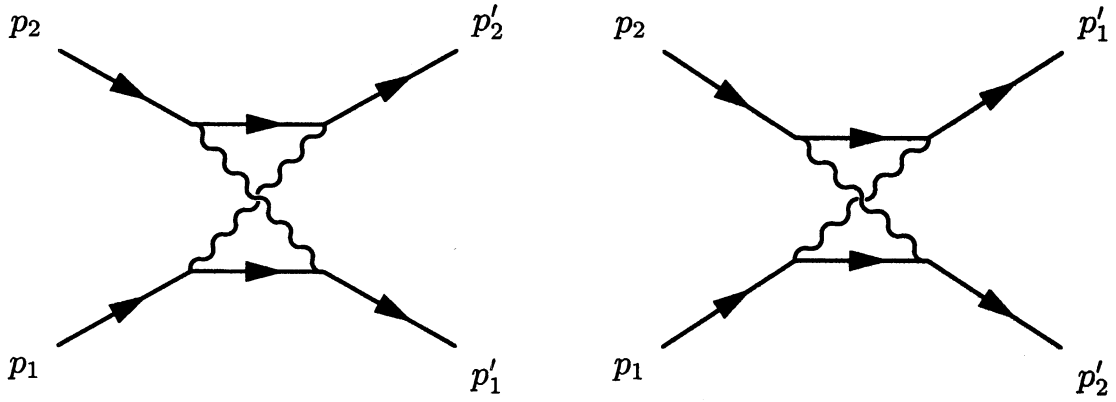


Figure 7.6 Crossed diagrams

7.4 Photon-Photon Scattering

In QED it is possible to describe the scattering of light by light. This amounts to a definite violation of the classical Maxwell theory, in which linearity in the field strengths precludes such an effect. The lowest-order diagrams involve a closed electron box, with a photon attached to each vertex. The fourth-order contribution to the T-matrix can be written

$$\frac{1}{(2\pi)^{10}}\frac{e^4}{4\sqrt{\omega_1\omega_2\omega_3\omega_4}}\epsilon^{\mu}(k_1,\lambda_1)\epsilon^{\nu}(k_2,\lambda_2)\epsilon^{\sigma}(k_3,\lambda_3)\epsilon^{\rho}(k_4,\lambda_4)\mathcal{M}_{\mu\nu\rho\sigma}(k_1,k_2,k_3,k_4)\,,$$

where the tensor $\mathcal{M}_{\mu\nu\rho\sigma}$ is the sum of the contributions of six diagrams, of which we show one in Figure 7.7. The other five are obtained by permuting k_2, k_3 and k_4, and the corresponding Lorentz indices, where for convenience we have defined the four k_i as if they were all incoming. In fact three of the diagrams are mathematically equivalent to the other three, since they only differ by the sense

in which the k_i follow one another around the box, and this is inconsequential.

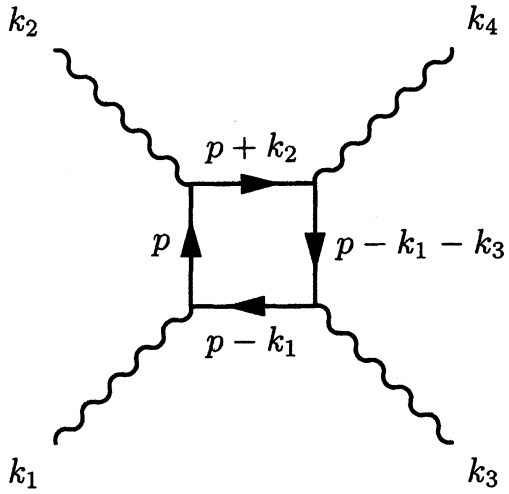


Figure 7.7
Photon-photon scattering

Accordingly, we may write

$$\mathcal{M}_{\mu\nu\rho\sigma}(k_1,k_2,k_3,k_4) \;=\; 2\left[\mathcal{N}_{\mu\nu\rho\sigma}(k_1,k_2,k_3,k_4)+\mathcal{N}_{\mu\nu\sigma\rho}(k_1,k_2,k_4,k_3)+\mathcal{N}_{\mu\rho\nu\sigma}(k_1,k_3,k_2,k_4)\right],$$

where $k_1 + k_2 + k_3 + k_4 = 0$, and

$$\mathcal{N}_{\mu\nu\rho\sigma}(k_1,k_2,k_3,k_4) \;=\; \frac{i}{(2\pi)^4}\mathrm{Tr}\int d^4p\, S_F(p-k_1)\gamma_\mu S_F(p)\gamma_\nu S_F(p+k_2)\gamma_\sigma S_F(p-k_1-k_3)\gamma_\rho\,.$$

Since $S_F(p)$ behaves like p^{-1} for large p, this integral is logarithmically divergent in the ultraviolet, i.e., if we insert a cut-off at $p^2 = \Lambda^2$, after rotation to Euclidean space, the integral diverges like $\log\Lambda$ as $\Lambda \to \infty$. Gauge invariance of the physical process implies that, if the gauge parameter, a, is changed, the tensor $\mathcal{M}_{\mu\nu\rho\sigma}(k_1,k_2,k_3,k_4)$ should be invariant, which is guaranteed if

$$k_1^\mu \mathcal{M}_{\mu\nu\rho\sigma}(k_1,k_2,k_3,k_4) = 0\,,$$

and similarly for the other vertices. Despite the divergence, this condition is satisfied automatically if one uses the method of dimensional regularization, to which we turn in the next chapter (see also Problem 8.8).

7.5 Exercises

Problem 1
Consider the free photon field operator,

$$A_\mu(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int \frac{d^3k}{2|\vec{k}|} \sum_{\lambda=0}^{3} \left[\varepsilon_\mu(\vec{k},\lambda) a(\vec{k},\lambda)\, \mathrm{e}^{-ikx} + \varepsilon_\mu(\vec{k},\lambda) a^\dagger(\vec{k},\lambda)\, \mathrm{e}^{ikx} \right] .$$

Obtain expressions for the creation and annihilation operators, $a^\dagger(\vec{k},\lambda)$ and $a(\vec{k},\lambda)$. Show that they are time-independent.

Problem 2
Show that, in Feynman gauge, the vacuum expectation value of the time-ordered product of two free electromagnetic fields, at different space-time points, is

$$\langle 0|\, T[A^\mu(x), A^\nu(y)]\, |0\rangle = iD_F^{\mu\nu}(x-y) = \frac{-i}{(2\pi)^4} \int d^4k \frac{g^{\mu\nu}}{k^2 + i\epsilon}\, \mathrm{e}^{-ik(x-y)}$$

Evaluate this integral explicitly as a distribution in the variable x^2.

Problem 3
Consider the Dirac free-field operator

$$\psi(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \sum_{s=1}^{2} \int d^3p \sqrt{\frac{m}{\omega_p}} \left[b(\vec{p},s) u(\vec{p},s)\, \mathrm{e}^{-ipx} + d^\dagger(\vec{p},s) v(\vec{p},s)\, \mathrm{e}^{ipx} \right] .$$

Using this expression and its adjoint, obtain expressions for the momentum creation and annihilation operators, $b(\vec{p},s)$, $d(\vec{p},s)$, $b^\dagger(\vec{p},s)$ and $d^\dagger(\vec{p},s)$. Show that they are time-independent.

Problem 4
Show that the vacuum expectation value of the time-ordered product of two spinor fields, at different space-time points, is

$$\langle 0|\, T[\psi(x), \psi(y)]\, |0\rangle = iS_F(x-y) = \frac{i}{(2\pi)^4} \int d^4p \frac{\gamma p + m}{p^2 - m^2 + i\epsilon}\, \mathrm{e}^{-ip(x-y)} .$$

Problem 5
Show that the QED Lagrangian density, projected on to the physical subspace, is invariant under the complete gauge transformation

$$A_\mu \longrightarrow A_\mu + \partial_\mu G \qquad\qquad \psi \longrightarrow \mathrm{e}^{-ieG}\psi ,$$

where $G(x)$ is an arbitrary scalar field.

Problem 6
The Gupta-Bleuler condition does not define a unique physical state, but rather a subspace of states. Construct the mapping from one physical state to another, and show that this mapping is precisely a gauge transformation.

Problem 7
Show that any Feynman diagram containing a closed electron loop, to which is attached an an odd number of photon propagators, is cancelled by another diagram, and hence may always be omitted.

Problem 8
Consider the elastic scattering of a positron and a muon.

(1) Write down the complete contribution to the T matrix to order e^2, in the centre of mass system.
(2) Is the cross-section finite or infinite in the forward and in the backward directions?
(3) Calculate the differential scattering cross-section, after averaging over initial and summing over final spins.

Problem 9
Consider the elastic scattering of polarized electrons and polarized positrons (Bhabha scattering).

(1) Write down the mathematical expression for the T-matrix to order e^2.
(2) Calculate the differential scattering cross-section, after summing over the final spins only.

Problem 10
By first considering the annihilation of an electron and a positron into two photons, estimate the lifetime of parapositronium (in which e^-e^+ is in a singlet spin state). Estimate also the lifetime of orthopositronium, in which e^-e^+ is in a triplet spin state.

Chapter 8

Dimensional Regularization

8.1 Rationale for the Method

As we saw in the previous chapter, some of the integrals involved in calculating Feynman diagrams diverge in the ultraviolet, i.e., toward infinite momentum. To cope with this problem, a method has been invented that consists in three steps: first the integral is *regularized*, which amounts to a change of definition, such that the new integral is finite (an ultraviolet cut-off is a crude example of this technique); second, the integral is written as the sum of a momentum-independent part and a more highly convergent integral that does depend on the external momenta; third, the regularizing prescription is removed, leaving a convergent integral and a formally divergent constant, which in principle has to be absorbed in an experimentally measured constant quantity.

It must be stressed that the method, whose full implementation is called renormalization theory, does not meet the highest standards of mathematical rigor. Dirac himself, although no connoisseur of the more abstract aspects of mathematical quantum theory, unequivocally rejected the renormalization method*, and even Feynman, its foremost inventor, expressed dissatisfaction and disappointment. It must be admitted that the method is unsatisfactory, for some 'divergent constants' are supposed to represent measurable quantities that are finite. For many other measurable quantities, as Dirac admitted, "Good agreement with experiment has been found, showing that there is some validity in the rules." An impressive example is the electron's g-factor [see Eq.(2.58)]. In 2002 the Particle Data Group quoted an experimental value with a standard deviation of only 4×10^{-12}.† The most accurate QED calculation, dating from

*see the last quotation on page 16

†http://pdg.lbl.gov

1981, was done by Kinoshita et al.[‡]. Their result is also quoted with an error, 2×10^{-10}, partly due to uncertainty in the precise value of the fine-structure constant, and partly due to the estimated effect of higher order terms. In the table below the results are displayed, and the agreement, to 1 part in 10^{-9}, is very impressive, and it is believed that the anomaly (the disagreement between the calculation and experiment) is due to the fact that the quantum electrodynamics of the electron-photon interaction is not a complete physical theory. Indeed, effects due to the embedding of QED in the electroweak theory make themselves felt at the the level of the anomaly.

	Value of g	Error
Experiment	2.002 319 304 374	0.000 000 000 008
Theory	2.002 319 304 9	0.000 000 000 4

Electron g-factor

The mathematical reason for the divergence of some integrals in quantum field theory can be traced back to the singular nature of the interaction between the spinor (electron) and the vector (photon) quantum fields. This interaction involves the product of distribution-valued operators at the same space-time point, and it lacks a precise mathematical sense. The adoption of a particular regularization procedure is tantamount to changing the nature of the interaction, and as a result making the momentum-dependent integrals finite. The theory, in effect, is specified both by the Lagrangian *and* the regularization procedure that is adopted. The price we have to pay is that the choice of regularization is painfully *ad hoc*, and moreover some physical constants cannot be calculated (they are the subtraction constants).

To illustrate the above ideas, let us consider part of a Feynman diagram made up of a loop involving two different scalar particles, of masses m_1 and m_2:

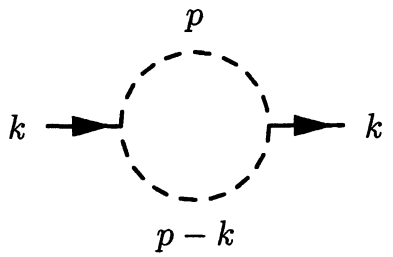


Figure 8.1
Scalar loop

[‡]T.Kinoshita and W.B. Lindquist, Phys. Rev. Lett. **47** (1981) 1573.

The contribution of the loop to the expression for the complete Feynman graph of which this loop is a part is

$$\frac{ig^2}{(2\pi)^4}\int \frac{d^4k}{[(p+k)^2-m_1^2+i\epsilon][k^2-m_2^2+i\epsilon]}\,, \tag{8.1}$$

where g is the coupling constant in the theory. This integral is logarithmically divergent, and the divergence is typical of the ill that is palliated by the method of regularization. We shall explain the prescription of dimensional regularization, as it applies to this loop in a scalar theory, extending the method to QED in the next two sections.

In dimensional regularization the number of space-time dimensions is changed from 4 to n, and n is regarded as being variable. Let us suppress the coupling, g, and write Eq.(8.1) in n dimensions as

$$I(p^2;m_1^2,m_2^2)=\frac{i}{(2\pi)^n}\int \frac{d^nk}{[(p+k)^2-m_1^2+i\epsilon][k^2-m_2^2+i\epsilon]}\,, \tag{8.2}$$

where it is supposed that there is one time dimension and $n-1$ space dimensions. The integral (8.2) is convergent for $n=1,2$ or 3, but not for $n=4$. The first step in the evaluation consists in using the Feynman formula,

$$\frac{1}{AB}=\int_0^1 \frac{dz}{[Az+B(1-z)]^2}\,, \tag{8.3}$$

with the substitutions

$$A=(p-k)^2-m_1^2+i\epsilon\,, \qquad B=k^2-m_2^2+i\epsilon\,,$$

and in changing the integration variable from k^μ to $q^\mu=k^\mu+zp^\mu$. This gives

$$I(p^2;m_1^2,m_2^2)=\frac{i}{(2\pi)^n}\int_0^1 dz\int d^nq\,\frac{1}{[q^2-C+i\epsilon]^2}\,, \tag{8.4}$$

in terms of the q-independent constant

$$C=m_1^2z+m_2^2(1-z)-p^2z(1-z)\,. \tag{8.5}$$

To calculate Eq.(8.4), we think of the q^0-integral domain as a contour along the real axis in the complex q^0-plane, and we rotate this contour anticlockwise from the real to the imaginary axis (this is called the Wick rotation). Because of the $i\epsilon$ prescription in Eq.(8.4), no poles are crossed by the rotating contour; and, for $n\leq 3$, the contributions from the arcs at infinity are zero. If we now substitute $q_0=iq_n$, then q_n is real, and we obtain

$$I(p^2;m_1^2,m_2^2)=-\frac{1}{(2\pi)^n}\int_0^1 dz\int d^nq\,\frac{1}{[q^2+C]^2}\,, \tag{8.6}$$

where the integral and the square, q^2, are now Euclidean. Let us initially limit our attention to $\mathrm{Re}\, C > 0$, and write

$$\frac{1}{[q^2+C]^2} = \int_0^\infty dx\; x\, \exp[-x(q^2+C)]. \tag{8.7}$$

On integrating this with respect to q, *under the x-integral*, we find

$$\begin{aligned}\int d^n q\, \frac{1}{[q^2+C]^2} &= \int_0^\infty dx\; x\, e^{-xC} \prod_{j=1}^{n} \int_{-\infty}^{\infty} dq_j \exp[-xq_j^2] \\ &= \int_0^\infty dx\; x\, e^{-xC} \left(\frac{\pi}{x}\right)^{\frac{n}{2}} \\ &= \pi^{\frac{n}{2}} \Gamma\left(2-\frac{n}{2}\right) C^{\frac{n-4}{2}}, \end{aligned} \tag{8.8}$$

where the so-called Gamma function is defined by

$$\Gamma(a) = \int_0^\infty dz\, z^{a-1}\, \mathrm{e}^{-z},$$

for $\mathrm{Re}\, a > 0$ and by analytic continuation in a for $\mathrm{Re}\, a \leq 0$. Note that, for any complex a, $a\Gamma(a) = \Gamma(a+1)$, and for n a positive integer, $\Gamma(n) = (n-1)!$ (Problem 8.1). The formula (8.8), initially calculated in the half-plane $\mathrm{Re}\, C > 0$, may be continued analytically into the C-plane, with the negative real axis excised. The integral Eq.(8.6) becomes

$$\begin{aligned} &I(p^2; m_1^2, m_2^2) = \\ &\quad -(4\pi)^{-\frac{n}{2}} \Gamma\left(2-\frac{n}{2}\right) \int_0^1 dz \left[m_1^2 z + m_2^2(1-z) - p^2 z(1-z)\right]^{\frac{n-4}{2}}. \end{aligned} \tag{8.9}$$

The expression is well defined if $n = 1, 2$ or 3, and if p^2 is anywhere in the complex p^2-plane, except possibly on the positive real axis, $0 < p^2 < \infty$. In fact, it is well defined also for non-integral, and even for complex values of n. The only points in the complex n-plane where it diverges, because of the occurrence of poles of the Γ-function, are $n = 4, 6, 8, \ldots$. The *ad hoc* nature of the dimensional prescription is very clear in the adoption of the formula (8.9) for any value of n: it amounts in fact to a *definition*. Let us next differentiate Eq.(8.9) with respect to p^2:

$$\begin{aligned}\frac{\partial}{\partial p^2} I(p^2; m_1^2, m_2^2) &= -(4\pi)^{-\frac{n}{2}} \Gamma\left(3-\frac{n}{2}\right) \int_0^1 dz z(1-z) \\ &\qquad \left[m_1^2 z + m_2^2(1-z) - p^2 z(1-z)\right]^{\frac{n-6}{2}}. \end{aligned}$$

Whereas Eq.(8.9) does not have a finite limit $n \to 4$, we see that

$$\lim_{n\to 4} \frac{\partial}{\partial p^2} I(p^2; m_1^2, m_2^2) = -\frac{1}{16\pi^2} \int_0^1 dz \left[\frac{m_1^2}{1-z} + \frac{m_2^2}{z} - p^2 \right]^{-1} .$$

Change the integration variable from z to

$$s = \frac{m_1^2}{1-z} + \frac{m_2^2}{z} , \tag{8.10}$$

which is equivalent to

$$z = z_\pm(s) = \frac{s - m_1^2 + m_2^2 \pm \{[s-(m_1+m_2)^2][s-(m_1-m_2)^2]\}^{\frac{1}{2}}}{2s} , \tag{8.11}$$

the plus sign being correct for z greater than $\frac{m_2}{m_1+m_2}$, the minus sign for z less than this value. Hence

$$\begin{aligned} \lim_{n\to 4} \frac{\partial}{\partial p^2} I(p^2; m_1^2, m_2^2) &= \frac{1}{16\pi^2} \int_{(m_1+m_2)^2}^\infty \frac{ds}{p^2 - s} \left\{ \frac{dz_+(s)}{ds} - \frac{dz_-(s)}{ds} \right\} \\ &= -\frac{1}{16\pi^2} \int_{(m_1+m_2)^2}^\infty \frac{ds}{(p^2-s)^2} \{z_+(s) - z_-(s)\} , \end{aligned}$$

where we have used a partial integration to obtain the second line. Since

$$I(p^2; m_1^2, m_2^2) = I(0; m_1^2, m_2^2) + \int_0^{p^2} dp'^2 \frac{\partial}{\partial p'^2} I_n(p'^2; m_1^2, m_2^2),$$

we may write

$$\begin{aligned} &\lim_{n\to 4} [I(p^2; m_1^2, m_2^2) - I(0; m_1^2, m_2^2)] \\ &\quad = \frac{1}{16\pi^2} \int_{(m_1+m_2)^2}^\infty ds \left[\frac{1}{p^2 - s} + \frac{1}{s} \right] \{z_+(s) - z_-(s)\} . \end{aligned}$$

On substituting for $z_\pm(s)$ from Eq.(8.11), we write the result in the form

$$\begin{aligned} I(p^2; m_1^2, m_2^2) &= I(0; m_1^2, m_2^2) \\ &\quad + \frac{p^2}{16\pi^2} \int_{(m_1+m_2)^2}^\infty \frac{ds}{p^2 - s} \frac{\{[s-(m_1+m_2)^2][s-(m_1-m_2)^2]\}^{\frac{1}{2}}}{s^2} , \end{aligned} \tag{8.12}$$

where the divergence has been absorbed into the subtraction constant, $I(0; m_1^2, m_2^2)$, leaving behind a finite, momentum-independent integral. In the following sections, we will show how this trickery can be extended to QED.

8.2 Fermion Self-Energy

The method of Feynman graphs can be applied just as well to propagators as to complete scattering amplitudes. To order e^2, the full electron propagator, S'_F, can be depicted as follows:

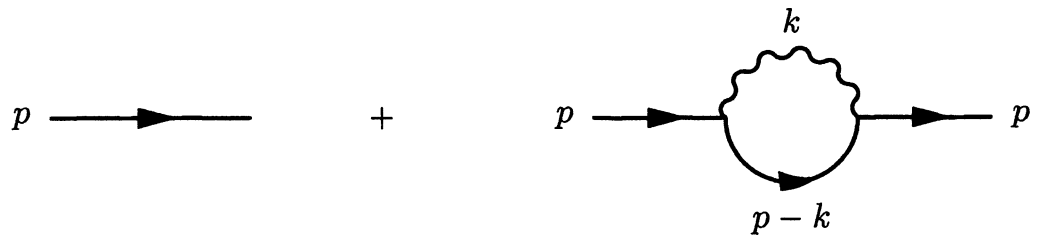


Figure 8.2 Electron propagator

and we can express this in the form

$$S'_F(p) = S_F(p) + S_F(p)\Sigma(p)S_F(p)\,,$$

where $S_F(p) = \big(\gamma p - m + i\epsilon\big)^{-1}$ is the bare electron propagator, and where

$$\Sigma(p) = \frac{ie^2}{(2\pi)^4}\int d^4k\gamma_\mu S_F(p+k)\gamma_\nu\left[-g^{\mu\nu} + (1-a)\frac{k^\mu k^\nu}{k^2}\right]\frac{1}{k^2+i\epsilon} \tag{8.13}$$

is called the electron self-energy. By rationalizing Eq.(8.13), we find

$$\Sigma(p) = \frac{ie^2}{(2\pi)^4}\int d^4k\,\frac{A(p,k)}{[(p+k)^2 - m^2 + i\epsilon][k^2+i\epsilon]}\,, \tag{8.14}$$

where

$$A(p,k) = \gamma_\mu\big[\gamma(p+k)+m\big]\gamma_\nu\left[-g^{\mu\nu} + (1-a)\frac{k^\mu k^\nu}{k^2}\right]. \tag{8.15}$$

Like the scalar loop integral of the previous section, the integral in Eq.(8.14) diverges. We shall adopt the method of dimensional regularization, replacing $(2\pi)^{-4}d^4k$ by $(2\pi)^{-n}d^nk$.

It should be noted that, while the basic anticommutator

$$\{\gamma^\mu,\gamma^\nu\} = 2g^{\mu\nu} \tag{8.16}$$

is retained unchanged in n dimensions, we have

$$g_{\mu\nu}\gamma^\mu\gamma^\nu = \tfrac{1}{2}g_{\mu\nu}\{\gamma^\mu,\gamma^\nu\} = \delta^\mu_\mu = n\,. \tag{8.17}$$

As a consequence of this, we find

$$A(p,k) = (n-3+a)\gamma p + (n-1+a)m - (n-1-a)\gamma k + 2(1-a)\frac{pk}{k^2}\gamma k\,. \quad (8.18)$$

We may write the n-dimensional version of Eq.(8.14) in the form

$$\begin{aligned}\Sigma(p) &= e^2\Big\{\big[(n-3+a)\gamma p - (n-1+a)m\big]M(p^2) \\ &\quad +(n-1-a)\gamma_\mu M^\mu(p) + 2(1-a)p_\mu\gamma_\nu M^{\mu\nu}(p)\Big\}\,, \end{aligned} \quad (8.19)$$

where the scalar, vector and tensor integrals are

$$\begin{aligned} M(p^2) &= \frac{i}{(2\pi)^n}\int d^nk \frac{1}{[(p+k)^2 - m^2 + i\epsilon][k^2+i\epsilon]} \\ M^\mu(p) &= \frac{i}{(2\pi)^n}\int d^nk \frac{k^\mu}{[(p+k)^2 - m^2 + i\epsilon][k^2+i\epsilon]} \\ M^{\mu\nu}(p) &= \frac{i}{(2\pi)^n}\int d^nk \frac{k^\mu k^\nu}{[(p+k)^2 - m^2 + i\epsilon][k^2+i\epsilon]^2}\,. \end{aligned} \quad (8.20)$$

It can be readily seen from Eq.(8.2) that $M(p^2)$ is precisely $I(p^2;m^2,0)$. From Lorentz covariance, $M^\mu(p) = p^\mu N(p^2)$, where $N(p^2)$ is some scalar field, and

$$\begin{aligned} 2p^2N(p^2) &= 2p_\mu M^\mu(p) \\ &= \frac{i}{(2\pi)^n}\int d^nk \frac{[(p+k)^2-m^2] - p^2 + m^2 - k^2}{[(p+k)^2 - m^2 + i\epsilon][k^2+i\epsilon]} \\ &= -(p^2-m^2)M(p^2) - \frac{i}{(2\pi)^n}\int d^nk\left\{\frac{1}{(p+k)^2-m^2+i\epsilon} - \frac{1}{k^2+i\epsilon}\right\} \\ &= -(p^2-m^2)M(p^2) - m^2M(0). \end{aligned} \quad (8.21)$$

The last line here was obtained by shifting the integration variable in the first denominator ($k \to q = p+k$), and then by combining both terms together.

$$M^\mu(p) = -\big[(p^2-m^2)M(p^2) + m^2M(0)\big]p^\mu(2p^2)^{-1}. \quad (8.22)$$

Now set $q^\mu = p^\mu + k^\mu$ in the second of the Eqs.(8.20), so that

$$M^\mu(p) = \frac{i}{(2\pi)^n}\int d^nq \frac{(q-p)^\mu}{[q^2-m^2+i\epsilon][(q-p)^2+i\epsilon]}. \quad (8.23)$$

Differentiating this with respect to momentum, we find

$$\begin{aligned} \frac{\partial}{\partial p_\nu}M^\mu(p) &= \frac{-i}{(2\pi)^n}\int d^nq \frac{\left[g^{\mu\nu} - \frac{2(q-p)^\mu(q-p)^\nu}{(q-p)^2+i\epsilon}\right]}{[q^2-m^2+i\epsilon][(q-p)^2+i\epsilon]} \\ &= -g^{\mu\nu}M(p^2) + 2M^{\mu\nu}(p)\,, \end{aligned} \quad (8.24)$$

where we obtain the second line by switching back from q to k again. So

$$\begin{aligned} M^{\mu\nu}(p) &= g^{\mu\nu}[(p^2+m^2)M(p^2) - m^2M(0)][4p^2]^{-1} \\ &\quad - p^\mu p^\nu [p^2(p^2-m^2)\frac{d}{dp^2}M(p^2) \\ &\quad + m^2M(p^2) - m^2M(0)][2p^4]^{-1}\,. \end{aligned} \tag{8.25}$$

Hence we have expressed the vector $M^\mu(p)$ and the tensor $M^{\mu\nu}(p)$ in terms of the scalar $M(p^2)$, for which we have, in the limit $n \to 4$,

$$M(p^2) = I(p^2;m^2,0) = M(0) + \frac{p^2}{16\pi^2}\int_{m^2}^{\infty}\frac{ds}{p^2-s}\frac{s-m^2}{s^2}\,, \tag{8.26}$$

where $M(0)$ is a 'divergent constant', formally the meaningless limit

$$M(0) = \lim_{n\to 4}\frac{1}{16\pi^2}\frac{1}{n-4}\,. \tag{8.27}$$

We obtain M^μ from Eq.(8.22). To evaluate $M^{\mu\nu}$ from Eq.(8.25), we need

$$\begin{aligned} \frac{d}{dp^2}M(p^2) &= \frac{-1}{16\pi^2}\int_{m^2}^{\infty}\frac{ds}{(p^2-s)^2}\frac{s-m^2}{s} \\ &= \frac{1}{16\pi^2}\int_{m^2}^{\infty}\frac{ds}{p^2-s}\frac{m^2}{s^2}, \end{aligned} \tag{8.28}$$

where a partial integration has been performed. Substituting all these results into Eq.(8.19), we find

$$\Sigma(p) = A + B\gamma p + p^2\int_{m^2}^{\infty}\frac{ds}{p^2-s+i\epsilon}\frac{\gamma p\,\Omega_1(s)+\Omega_2(s)}{s} \tag{8.29}$$

where

$$A = -(3+a)me^2M(0) - (3-a)\frac{e^2}{64\pi^2}\,, \tag{8.30}$$

$$B = ae^2M(0) - (1-a)\frac{e^2}{64\pi^2}, \tag{8.31}$$

$$\Omega_1(s) = \frac{ae^2}{16\pi^2}\frac{s^2-m^4}{s^2}\,, \tag{8.32}$$

$$\Omega_2(s) = -\frac{e^2}{16\pi^2}(3+a)m\frac{s-m^2}{s}\,. \tag{8.33}$$

8.3 Vacuum Polarization

To order e^2, the full photon propagator, $D_F'^{\mu\nu}$, can be depicted as follows:

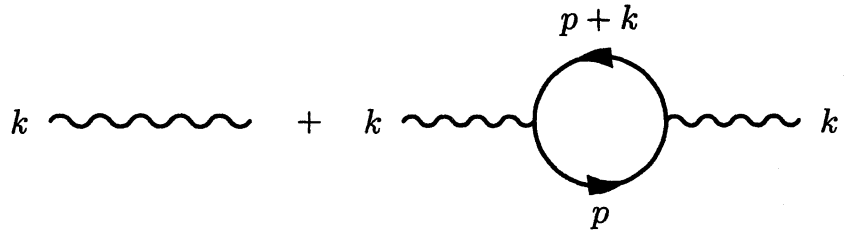


Figure 8.3 Photon propagator

and we can express this in the form

$$D_F'^{\mu\nu}(k) = D_F^{\mu\nu}(k) + D_F^{\mu\rho}(k)\Pi_{\rho\sigma}(k)D_F^{\sigma\nu}(k)\,,$$

where the bare propagator is

$$D_F^{\mu\nu}(k) = \frac{-g^{\mu\nu} + (1-a)k^\mu k^\nu/k^2}{k^2 + i\epsilon}\,,$$

and where the polarization tensor is given by

$$\Pi^{\mu\nu}(k) = \frac{ie^2}{(2\pi)^n}\int d^n p\,\mathrm{Tr}\Big\{\gamma^\mu S_F(p+k)\gamma^\nu S_F(p)\Big\}\,. \tag{8.34}$$

We have immediately written this integral in n dimensions, in anticipation of the dimensional regularization that will be performed. Note that

$$\begin{aligned}
k_\nu\Pi^{\mu\nu}(k) &= \frac{ie^2}{(2\pi)^n}\int d^n p\,\mathrm{Tr}\Big\{\gamma^\mu[\gamma(p+k)-m+i\epsilon]^{-1}\gamma k[\gamma p - m + i\epsilon]^{-1}\Big\} \\
&= \frac{ie^2}{(2\pi)^n}\int d^n p\,\mathrm{Tr}\Big\{\gamma^\mu[\gamma(p+k)-m+i\epsilon]^{-1} \\
&\qquad\qquad [\gamma(p+k)-m-(\gamma p - m)][\gamma p - m + i\epsilon]^{-1}\Big\} \\
&= \frac{ie^2}{(2\pi)^n}\int d^n p\,\mathrm{Tr}\Big\{\gamma^\mu\Big[(\gamma p - m + i\epsilon)^{-1} - (\gamma(p+k)-m+i\epsilon)^{-1}\Big]\Big\}\,.
\end{aligned}$$

The dimensional regularization recipe permits a shift of the integration variable in the second term in this integrand, $p+k \to p$, so one concludes that the right-hand side here is zero, since the two parts of the integrand cancel one another.

The justification is that the integrals are interpreted in the manner of Eqs.(8.7)-(8.9), i.e., the denominators are to be replaced by a Gaussian integral over a variable x, and the n-dimensional momentum integrals are to be performed *under* this Gaussian integral. Shifts of momentum integration variables are now allowed, because of the Gaussian convergence factors. The original interchange of order of the momentum integration variable and the Gaussian integration variable x cannot be performed in general without change in the value of the integral, but the point of dimensional regularization is that the regularized value of the integral is *defined* to be given by placing the momentum integrals under the Gaussian x-integral. This guarantees that identities between integrands remain valid between the regularized integrals. That the perturbation series which is produced by this method makes physical sense is not guaranteed *a priori*: it is strictly an *a posteriori* matter that the results obtained appear to agree very well with experiment. We conclude then that

$$k_\mu \Pi^{\mu\nu}(k) = 0 = \Pi^{\mu\nu}(k) k_\nu \,, \tag{8.35}$$

under dimensional regularization, the second identity being obtained from the first by symmetry (a consequence of the cyclicity of the trace operation). Hence it must be possible to write the polarization tensor in the form

$$\Pi^{\mu\nu}(k) = \left[g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right]\Pi(k^2)\,,$$

and therefore

$$g_{\mu\nu}\Pi^{\mu\nu}(k) = \left[\delta^\mu_\mu - \frac{k^\mu k_\mu}{k^2}\right]\Pi(k^2) = (n-1)\Pi(k^2)\,.$$

The scalar vacuum polarization is defined to be

$$\begin{aligned} \Pi(k^2) &= \frac{g_{\mu\nu}\Pi^{\mu\nu}(k)}{n-1} \qquad (8.36)\\ &= \frac{ie^2}{n-1}\frac{\mathrm{Tr}}{(2\pi)^n}\int d^n p\, \gamma^\mu [\gamma(p+k) - m + i\epsilon]^{-1}\gamma_\mu [\gamma p - m + i\epsilon]^{-1}\,. \end{aligned}$$

On rationalization of the integrand, we obtain

$$\Pi(k^2) = \frac{ie^2}{n-1}\frac{1}{(2\pi)^n}\int d^n p \frac{B(p,k)}{[(p+k)^2 - m^2 + i\epsilon][p^2 - m^2 + i\epsilon]}\,, \tag{8.37}$$

where

$$B(p,k) = \mathrm{Tr}\,\{\gamma^\mu[\gamma(p+k) + m]\gamma_\mu[\gamma p + m]\}\,. \tag{8.38}$$

In n dimensions, the Dirac anticommutation relations,

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu},$$

can be realized by $[\frac{n}{2}]$-dimensional matrices. For example, for $n = 2$, $\gamma^0 = \sigma^3$, $\gamma^1 = \sigma^1\sigma^3$ do the job, and for $n = 3$, one adds $\gamma^2 = \sigma^2\sigma^3$, where $\vec{\sigma}$ are the 2×2 Pauli matrices. For $n = 4$, one has the familiar Dirac matrices, while for $n = 5$, one adds $-i\gamma_5 = \gamma^0\gamma^1\gamma^2\gamma^3$, and so on. However, although it *would* be important if one were interested in genuinely setting up a field theory in n dimensions, for the purpose of simply splitting off infinities in four-dimensional field theories, it is actually not necessary to generalize the dimension of the γ-matrices. Thus we may retain the trace formulas,

$$\mathrm{Tr}(1) = 4 \qquad \mathrm{Tr}(\gamma^\mu) = 0 \qquad \mathrm{Tr}(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu}.$$

In some versions of dimensional regularization, the factors of 4 are replaced by $f(n)$, where f is an analytic function such that $f(4) = 4$, such as $2^{\frac{n}{2}}$. However, all discussions of renormalization can be carried out at the level where subsets of graphs with the same number of closed fermion loops are treated together. Then all factors of $f(n)$ occur with the same power, and they may simply be replaced by 4, which is what we propose to do. However, one place in which n occurs in a nontrivial manner in the γ-algebra is in the formula $\gamma^\mu\gamma_\mu = n$, as was noted in Eq.(8.17). We find then, from Eq.(8.38),

$$B(p,k) = 4\big[(2-n)(p^2+pk) + nm^2\big]\,. \tag{8.39}$$

We may therefore write

$$\Pi(k^2) = \frac{4e^2}{n-1}\left\{m^2nP(k^2) + (2-n)k_\mu P^\mu(k) + (2-n)Q(k^2)\right\} \tag{8.40}$$

where

$$\begin{aligned}
P(k^2) &= \frac{i}{(2\pi)^n}\int d^np\frac{1}{[(p+k)^2 - m^2 + i\epsilon][p^2 - m^2 + i\epsilon]} \\
P^\mu(k) &= \frac{i}{(2\pi)^n}\int d^np\frac{p^\mu}{[(p+k)^2 - m^2 + i\epsilon][p^2 - m^2 + i\epsilon]} \\
Q(k^2) &= \frac{i}{(2\pi)^n}\int d^np\frac{p^2}{[(p+k)^2 - m^2 + i\epsilon][p^2 - m^2 + i\epsilon]}\,.
\end{aligned}$$

We recognize $P(k^2)$ as $I(k^2; m^2, m^2)$. Further, by Lorentz covariance,

$$P^\mu(k) = k^\mu R(k^2),$$

and so, by contraction with the vector $2k_\mu$,

$$\begin{aligned} 2k^2R(k^2) &= 2k_\mu P^\mu(k) \qquad (8.41)\\ &= \frac{i}{(2\pi)^2}\int d^np\frac{[(p+k)^2-m^2]-p^2-k^2+m^2}{[(p+k)^2-m^2+i\epsilon][p^2-m^2+i\epsilon]}\\ &= -k^2P(k^2)+\frac{i}{(2\pi)^n}\int d^np\Big\{\frac{1}{p^2-m^2+i\epsilon}-\frac{1}{(p+k)^2-m^2+i\epsilon}\Big\}. \end{aligned}$$

The two terms under the integral cancel one another out, as we see by shifting the integration variable in the second term, $p+k\to p$ (the justification being in terms of dimensional regularization, as in the analogous case of the electron self-energy term). We find

$$\begin{aligned} P^\mu(k) &= -\tfrac{1}{2}k^\mu P(k^2) = -\tfrac{1}{2}k^\mu I(k^2;m^2,m^2)\\ Q(k^2) &= \frac{i}{(2\pi)^n}\int d^np\frac{p^2-m^2+m^2}{[(p+k)^2-m^2+i\epsilon][p^2-m^2+i\epsilon]}\\ &= m^2P(k^2)+\frac{i}{(2\pi)^n}\int d^np\frac{1}{p^2-m^2+i\epsilon}\\ &= m^2I(k^2;m^2,m^2)+(4\pi)^{-\frac{n}{2}}\Gamma\left(\frac{2-n}{2}\right)m^{n-2}, \end{aligned}$$

where the last term has been calculated by means of manipulations similar to those of Eq.(8.8). When these results are injected into Eq.(8.40), we find

$$\begin{aligned} \Pi(k^2) &= \frac{4e^2}{n-1}\left\{\left[2m^2+\frac{n-2}{2}k^2\right]I(k^2;m^2,m^2)\right.\\ &\qquad \left.-(4\pi)^{-\frac{n}{2}}(n-2)\Gamma\left(1-\frac{n}{2}\right)m^{n-2}\right\}. \qquad (8.42) \end{aligned}$$

From Eq.(8.9), one sees that $I(0;m^2,m^2)=-(4\pi)^{-\frac{n}{2}}\Gamma\left(2-\frac{1}{2}n\right)m^{n-4}$, so

$$\Pi(0)=\frac{4e^2}{n-1}(4\pi)^{-\frac{n}{2}}m^{n-2}\left\{-2\Gamma\left(2-\frac{n}{2}\right)-(n-2)\Gamma\left(1-\frac{n}{2}\right)\right\}=0. \qquad (8.43)$$

This corresponds to the requirement that the mass of the photon should not be shifted by the interaction. The dimensional regularization has in fact preserved the gauge invariance of the theory. In the limit $n\to 4$, Eq.(8.42) takes the form

$$\Pi(k^2)=\frac{4e^2}{3}k^2\left\{M(0)+\frac{1}{16\pi^2}(k^2+2m^2)\int_{4m^2}^{\infty}\frac{ds}{s(k^2-s)}\sqrt{\frac{s-4m^2}{s}}\right\}, \qquad (8.44)$$

where $M(0)$ is the divergent constant (8.27). It is straightforward to derive the

infrared behavior,

$$\frac{\Pi(k^2)}{k^2} \to \frac{4e_0^2}{3}\left[M(0) - \frac{1}{48\pi^2}\right],$$

so evidently there is no infrared divergence. The ultraviolet behavior also follows from Eq.(8.44) without difficulty:

$$\frac{\Pi(k^2)}{k^2} \sim \frac{\alpha}{3\pi}\log\frac{k^2}{m^2}. \tag{8.45}$$

8.4 Cutkosky Rule

In Sec. **8.1** we made sense of the divergent integral Eq.(8.1) by the method of dimensional regularization. The final result [see Eq.(8.13)] may be expressed

$$\begin{aligned} I(p^2; m_1^2, m_2^2) &= \frac{i}{(2\pi)^4}\int \frac{d^4k}{[(p+k)^2 - m_1^2 + i\epsilon][k^2 - m_2^2 + i\epsilon]} \\ &\doteq I(0; m_1^2, m_2^2) + \frac{p^2}{\pi}\int_{(m_1+m_2)^2}^{\infty} \frac{ds}{p^2 - s}\frac{\rho(s)}{s}, \end{aligned} \tag{8.46}$$

where the dot on $\doteq$ is to remind us that the sign is no simple equality, and where

$$\rho(s) = \frac{\{[s - (m_1+m_2)^2][s - (m_1 - m_2)^2]\}^{\frac{1}{2}}}{s}. \tag{8.47}$$

The second line of Eq.(8.46) is called a once-subtracted dispersion relation, $\rho(s)$ is called a spectral function, and $I(0; m_1^2, m_2^2)$ is a subtraction constant. Since the integral over s is absolutely convergent when p^2 is complex or real and less than the threshold value of $(m_1+m_2)^2$, it follows that $I(p^2; m_1^2, m_2^2)$, considered as a function of the complex variable p^2, is an analytic function in the plane cut along the interval $(m_1+m_2)^2 < p^2 < \infty$. Consider calculating $I(p^2+i\epsilon; m_1^2, m_2^2)$, with p^2 on the cut and ϵ very small and positive. The integration contour can be deformed as shown in Figure 8.4, and then the integral comprises two parts in the limit $\epsilon \to 0$, namely the Cauchy Principal Value, i.e.,

$$\frac{p^2}{\pi}\mathbf{P}\int_{(m_1+m_2)^2}^{\infty} \frac{ds}{p^2 - s}\frac{\rho(s)}{s} \equiv \lim_{\epsilon\to 0}\frac{p^2}{\pi}\left\{\int_{(m_1+m_2)^2}^{p^2-\epsilon} + \int_{p^2+\epsilon}^{\infty}\right\}\frac{ds}{p^2 - s}\frac{\rho(s)}{s},$$

and the contribution of the small semicircle centered at $s = p^2$. On this circle we can write $s = p^2 + \epsilon e^{i\theta}$, so that $ds/(p^2 - s) = -id\theta$.

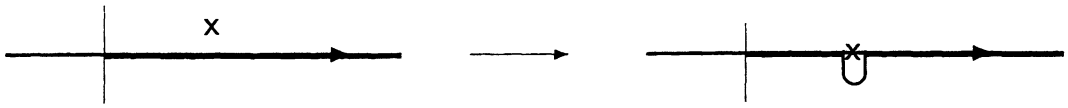


Figure 8.4 Contours in the s-plane

The semicircle is described from π to 2π, so the contribution is

$$-\frac{i}{\pi}\int_{\pi}^{2\pi} d\theta\,\rho(p^2) = -i\rho(p^2)\,.$$

The calculation of $I(p^2 - i\epsilon; m_1^2, m_2^2)$, differs only in that the integration around the point $s = p^2$ proceeds from π to 0, so in this case

$$-\frac{i}{\pi}\int_{\pi}^{0} d\theta\,\rho(p^2) = +i\rho(p^2)\,.$$

The Principal Value part is of course as before. The discontinuity of the function across the cut, i.e., the difference between the value of the function evaluated at a point just above and just below the cut, is therefore given by

$$I(p^2 + i\epsilon; m_1^2, m_2^2) - I(p^2 - i\epsilon; m_1^2, m_2^2) = -2i\rho(p^2)\,, \tag{8.48}$$

for real p^2 greater than $(m_1 + m_2)^2$. These results can be summarized by

$$\frac{1}{p^2 - s \pm i\epsilon} = \frac{\mathbf{P}}{p^2 - s} \mp i\pi\delta(p^2 - s)\,.$$

Consider next the integral

$$\mathcal{D}(p^2) \;=\; \int d^4k\,\delta[(p+k)^2 - m_1^2]\delta(k^2 - m_2^2)\,, \tag{8.49}$$

which is similar to Eq.(8.46), except that instead of propagators there are delta distributions. If p^μ is a timelike vector, we can find a Lorentz system in which the spatial components of p vanish, so $p^\mu = (p_0, 0, 0, 0)$. In this frame,

$$\begin{aligned}
\mathcal{D}(p^2) &= 2\pi\int_{-\infty}^{\infty} dk_0 \int_0^{\infty} dk^2\,\sqrt{k^2}\,\delta(p_0^2 + 2p_0k_0 + m_2^2 - m_1^2)\delta(k^2 - k_0^2 + m_2^2) \\
&= 2\pi\int_{-\infty}^{\infty} dk_0\,[k_0^2 - m_2^2]^{\frac{1}{2}}\,\theta(k_0^2 - m_2^2)\delta(2p_0k_0 + p_0^2 + m_2^2 - m_1^2) \\
&= \frac{\pi}{p_0}\left\{\left[\frac{m_1^2 - m_2^2 - p_0^2}{2p_0}\right]^2 - m_2^2\right\}^{\frac{1}{2}}\;\theta\left\{\left[\frac{m_1^2 - m_2^2 - p_0^2}{2p_0}\right]^2 - m_2^2\right\}^{\frac{1}{2}}\,.
\end{aligned}$$

We can now replace p_0^2 by p^2, to which it is equal in the special Lorentz system in which we have performed the calculation. Since the expression is now in invariant form, it is also valid in any Lorentz system. It can be rewritten

$$\mathcal{D}(p^2) \;=\; \frac{\pi}{2p^2}\Big\{\,[p^2-(m_1^2+m_2^2)^2]\,[p^2-(m_1^2-m_2^2)^2]\,\Big\}^{\frac{1}{2}}\theta\,[p^2-(m_1^2+m_2^2)^2]\;.$$

From Eqs.(8.47)-(8.48), we see that $\mathcal{D}(p^2)$ is $4\pi^2 i$ times the discontinuity of $I(p^2;m_1^2,m_2^2)$ across its cut, i.e.,

$$I(p^2+i\epsilon;m_1^2,m_2^2)-I(p^2-i\epsilon;m_1^2,m_2^2)=-\frac{i}{4\pi^2}\int d^4k\,\delta[(p-k)^2-m_1^2]\delta(k^2-m^2)\,.$$

Thus the discontinuity of $I(p^2;m_1^2,m_2^2)$ across its cut is obtained by replacing each propagator in Eq.(8.46) by $-2\pi i$ times a delta function:

$$\begin{aligned}[(p+k)^2-m_1^2+i\epsilon]^{-1} \;&\to\; -2\pi i\delta[(p+k)^2-m_1^2]\\ (k^2-m_2^2+i\epsilon)^{-1} \;&\to\; -2\pi i\delta(k^2-m_2^2)\,.\end{aligned}\qquad(8.50)$$

This recipe for obtaining the discontinuity of a Feynman integral is called the Cutkosky rule (see Problems 8.6 and 8.7).

We illustrate the use of the Cutkosky rule by reconsidering the polarization integral Eq.(8.37). The discontinuity of $\Pi(k^2)$ across the cut is

$$\Pi(k^2+i\epsilon)-\Pi(k^2-i\epsilon)=-\frac{ie^2}{3(2\pi)^2}\int d^4p\,B(p,k)\delta[(p+k)^2-m^2]\delta(p^2-m^2)\,,$$

which we have immediately written in four dimensions, and where

$$B(p,k)=4[4m^2-2pk-2p^2]\,,$$

this being simply Eq.(8.39) in the case $n=4$. We find

$$\begin{aligned}&\Pi(k^2+i\epsilon)-\Pi(k^2-i\epsilon)\\ &\qquad=-2\pi\frac{4ie^2}{3(2\pi)^2}\int_{-\infty}^{\infty}dp_0\int_0^{\infty}dp^2\,\sqrt{p^2}\,\delta(k_0^2+2p_0k_0)\delta(p^2-p_0^2+m^2)\\ &\qquad\qquad\qquad\qquad\times(4m^2-2k_0p_0-2p_0^2+2p^2)\\ &\qquad=-\frac{2ie^2}{3\pi}\int_{-\infty}^{\infty}dp_0(p_0^2-m^2)^{\frac{1}{2}}\,\theta(p_0^2-m^2)\,\delta(k_0^2+2p_0k_0)\,(2m^2-2p_0k_0)\\ &\qquad=-\frac{ie^2}{6\pi}(k_0^2+2m^2)\left[\frac{k_0^2-4m^2}{k_0^2}\right]^{\frac{1}{2}}\theta(k_0^2-4m^2)\,.\end{aligned}$$

The invariant form is obtained by replacing k_0^2 by k^2, and the spectral function by substituting s for k^2, and dividing by $-2i$ [cf., Eq.(8.48)]:

$$\rho(s) = \frac{e^2}{12\pi}(s + 2m^2)\left[\frac{s - 4m^2}{s}\right]^{\frac{1}{2}} \theta(s - 4m^2)\,.$$

A once-subtracted dispersion relation would diverge, since $\rho(s) \sim s$ as $s \to \infty$, but if we write

$$\rho(s) = \frac{e^2}{12\pi}\left[(s - k^2) + (k^2 + 2m^2)\right]\left[\frac{s - 4m^2}{s}\right]^{\frac{1}{2}} \theta(s - 4m^2)$$

then we can throw the dispersion relation into the form

$$\Pi(k^2) = A + k^2\left\{B + \frac{e^2}{12\pi^2}(k^2 + 2m^2)\int_{4m^2}^{\infty} \frac{ds}{s(k^2 - s)}\left[\frac{s - 4m^2}{s}\right]^{\frac{1}{2}}\right\}, \quad (8.51)$$

where a divergent constant, coming from $(s - k^2)$, has been replaced by a second subtraction constant, B. Note that the integral part of this expression agrees with Eq.(8.44), that we obtained by dimensional regularization. The fact that A here should vanish is not indicated by use of the Cutkosky rule, whereas in the dimensional method $\Pi(0) = 0$ is automatic.

8.5 Magnetic Moment and g-Factor of Electron

In this section we shall use the foregoing methods to calculate the anomalous magnetic moment of the electron to order α. In Sec. **2.3** we saw that the Dirac equation predicts the g-factor of the electron to be exactly 2, but we noted in Sec. **8.1** that that the experimental value of g differs from the Dirac prediction by about one part in a thousand.

The conserved electromagnetic current density of the electron is

$$j^\mu(x) = :\overline{\psi}(x)\gamma^\mu\psi(x):$$

a normal-ordered operator on Fock space. For *free* fields, we can calculate

$$\begin{aligned}\langle p', s|j^\mu(x)|p, s\rangle &= \langle 0|b(p', s)j^\mu(x)b^\dagger(p, s)|0\rangle \\ &= \frac{1}{(2\pi)^3}\frac{m}{\sqrt{\omega'_p\omega_p}}\,\overline{u}(p', s)\gamma^\mu u(p, s)\,\mathrm{e}^{ikx}\,, \end{aligned} \quad (8.52)$$

where $k = p' - p$, this form being obtained from the Fourier tranforms of the quantum fields, ψ and $\overline{\psi}$, with use of the anticommutation relations between the

creation and annihilation operators. The electric charge operator is

$$Q = e\int d^3x\, j^0(x) = e\int d^3x\ :\psi^\dagger(x)\psi(x):$$

and the Gordon identity (Problem 3.3), namely

$$\overline{u}(p',s)\gamma^\mu u(p,s) = \frac{1}{2m}\overline{u}(p',s)\big[P^\mu + i\sigma^{\mu\nu}k_\nu\big]u(p,s)\,, \tag{8.53}$$

where $P = p' + p$, and $\sigma^{\mu\nu} = \frac{1}{2}i\big[\gamma^\mu,\gamma^\nu\big]$, leads to

$$\langle p',s|Q|p,s\rangle = e\,\delta^3(\vec{p}\,' - \vec{p})\,, \tag{8.54}$$

which is the correct continuum normalization. For a normalized state,

$$|f\rangle = \int d^3p\, f(p)|p\rangle\,,$$

this corresponds to $\langle f|Q|f\rangle = e$. For *interacting* fields, Eq.(8.52) is replaced by

$$\langle p',s|j^\mu(x)|p,s\rangle \;=\; \frac{1}{(2\pi)^3}\frac{m}{\sqrt{\omega'_p\omega_p}}\,\overline{u}(p',s)\Gamma^\mu(p',p)u(p,s)\,\mathrm{e}^{ikx}\,, \tag{8.55}$$

where $\Gamma^\mu(p',p)$ is called the vertex function. Since $\partial_\mu j^\mu = 0$, it follows that

$$\overline{u}(p',s)k_\mu\Gamma^\mu(p',p)u(p,s) = 0\,. \tag{8.56}$$

In general the vertex function may be written

$$\overline{u}(p',s)\Gamma^\mu(p',p)u(p,s) = \overline{u}(p',s)\big[F(k^2)\gamma^\mu + \frac{1}{2m}G(k^2)P^\mu\big]u(p,s)\,, \tag{8.57}$$

where F and G are called form factors. A form factor multiplying k^μ is excluded by Eq.(8.56). With use of the Gordon identity (8.53), we may rewrite this

$$\begin{aligned}\overline{u}(p',s)\Gamma^\mu(p',p)u(p,s) \;&=\; \\ &\frac{1}{2m}\overline{u}(p',s)\big\{[F(k^2)+G(k^2)]P^\mu + iF(k^2)\sigma^{\mu\nu}k_\nu\big\}u(p,s)\,.\end{aligned} \tag{8.58}$$

The expectation value of the electric charge operator is

$$\begin{aligned}\langle p',s|Q|p,s\rangle \;&=\; e\int d^3x\,\langle p',s|j^0(x)|p,s\rangle \\ &= \frac{e}{2\omega_p}\overline{u}(p,s)\big\{[F(k^2)+G(k^2)]P^0 + iF(k^2)\sigma^{0j}k_j\big\}u(p,s)\delta^3(\vec{p}\,' - \vec{p}) \\ &= e[F(0)+G(0)]\delta^3(\vec{p}\,' - \vec{p})\,.\end{aligned} \tag{8.59}$$

On comparing this with the free field case Eq.(8.54), we see that

$$F(0) + G(0) = 1\,, \tag{8.60}$$

in order that e be identified with the physical charge of the electron. The interaction Hamiltonian of quantum electrodynamics is

$$H_i = e\int d^3x\ :A_\mu(x)j^\mu(x):\ = e\int d^3x\ :\overline{\psi}(x)A_\mu(x)\gamma^\mu\psi(x):$$

Consider the matrix element

$$\begin{aligned}\langle p',s|H_i|p,s\rangle &= \frac{e}{(2\pi)^3}\frac{m}{\sqrt{\omega_p'\omega_p}}\int d^3x\ \overline{u}(p',s)A_\mu(x)\Gamma^\mu(p',p)u(p,s)\,\mathrm{e}^{ikx}\\ &= \frac{e}{(2\pi)^3}\frac{1}{2\sqrt{\omega_p'\omega_p}}\int d^3x\,\overline{u}(p',s) \qquad (8.61)\\ &\quad\times\{[F(k^2)+G(k^2)]A_\mu(x)P^\mu + iF(k^2)\sigma^{\mu\nu}A_\mu(x)k_\nu\}u(p,s)\,\mathrm{e}^{ikx}\,.\end{aligned}$$

An integration by parts yields

$$\begin{aligned}\int d^3x\, i\sigma^{\mu\nu}A_\mu(x)k_\nu\,\mathrm{e}^{ikx} &= -\int d^3x\ \mathrm{e}^{ikx}\sigma^{\mu\nu}\partial_\nu A_\mu(x)\\ &= \tfrac{1}{2}\int d^3x\ \mathrm{e}^{ikx}\sigma^{\mu\nu}F_{\mu\nu}(x)\,,\end{aligned}$$

where $F_{\mu\nu}(x)$ is the electromagnetic field tensor. In terms of the electric field, $\vec{E}$, and magnetic induction, $\vec{B}$, we have

$$\tfrac{1}{2}\sigma^{\mu\nu}F_{\mu\nu}(x) = i\gamma^0\vec{\gamma}\cdot\vec{E}\,(x) - 2\vec{S}\cdot\vec{B}(x)\,,$$

where $\vec{S} = \frac{1}{2}\gamma_5\gamma^0\vec{\gamma}$, the four-dimensional spin matrix. Let us concentrate our attention on the part of the right-hand side of Eq.(8.61) that involves $\vec{B}$. It is

$$-\frac{e}{(2\pi)^3}\frac{1}{\sqrt{\omega_p'\omega_p}}F(k^2)\int d^3x\,\overline{u}(p',s)\,\vec{S}\cdot\vec{B}(x)\,u(p,s)\,\mathrm{e}^{ikx}\,. \qquad (8.62)$$

If the magnetic induction is uniform, i.e., $\vec{B}$ is independent of x, then we may write Eq.(8.62) as

$$-\frac{e}{\omega_p}F(0)\,\overline{u}(p,s)\,\vec{S}\cdot\vec{B}\,u(p,s)\delta^3(\vec{p}\,'-\vec{p})\,.$$

This is the magnetic part of the matrix element of the interaction Hamiltonian. The magnetic moment of the electron is therefore

$$\vec{\mu} = \frac{e}{\omega_p}F(0)\,\overline{u}(p,s)\vec{S}\,u(p,s)\,, \qquad (8.63)$$

and in the electron's rest system we have $\omega_p = m$, so in this system the magnetic moment has the magnitude

$$\mu = \frac{e}{2m}F(0) = \frac{e}{2m}[1-G(0)]\,,$$

where the normalization condition Eq.(8.60) has been used. We recall that the Dirac equation yields the value

$$\mu = \frac{e\hbar}{2mc} = \frac{ge}{2mc}\frac{\hbar}{2},$$

where the g-factor is precisely $g = 2$. Since we set $\hbar = 1 = c$ in this chapter, we find the improved value to be

$$g = 2\big[1 - G(0)\big] .$$

To calculate $G(0)$ to order α, we consider the Feynman diagram shown in Figure 8.5:

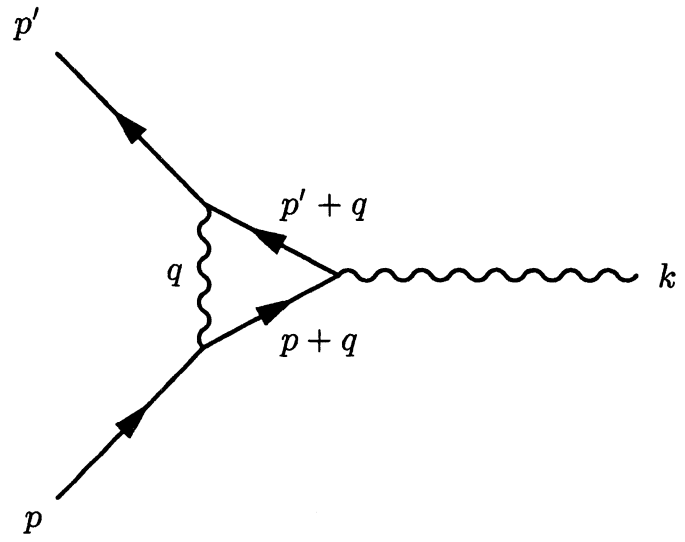


Figure 8.5
Vertex Part

To order α, we can write the vertex function as

$$\Gamma^{\mu}(p',p) = \gamma_{\mu} - \frac{ie^2}{(2\pi)^4}\int \frac{d^4q}{q^2 + i\epsilon}\gamma^{\rho}S(p'+q)\gamma^{\mu}S(p+q)\gamma_{\rho} .$$

Sandwiched between spinors, this gives

$$\bar{u}(p',s)\Big\{\Gamma^{\mu}(p',p) - \gamma^{\mu}\Big\}u(p,s) = -\frac{ie^2}{(2\pi)^4}\int d^4q \frac{\phi(p',p,q)}{[(p'+q)^2 - m^2][(p+q)^2 - m^2]q^2},$$

where the '$i\epsilon$' prescription is implicit, and

$$\phi(p',p,q) = \bar{u}(p',s)\big[\mathcal{F}\gamma^{\mu} + \mathcal{G}^{\mu}\big]u(p,s) ,$$

with

$$\mathcal{F} = P^2 - k^2 + 4Pq + 2q^2 , \qquad \mathcal{G}^{\mu} = 4[mq^{\mu} - \gamma q P^{\mu} - \gamma q\, q^{\mu}] .$$

We are interested in the form-factor $G(k^2)$, so we ignore the term $\mathcal{F}$, since this multiplies γ^μ, and thus contributes only to $F(k^2)$. Accordingly,

$$\bar{u}(p',s)\Gamma^\mu(p',p)u(p,s) = -\frac{ie^2}{4\pi^4}\int d^4q \frac{\bar{u}(p',s)\big[mq^\mu - \gamma q P^\mu - \gamma q q^\mu\big]u(p,s)}{[(p'+q)^2-m^2][(p+q)^2-m^2]q^2} + \cdots$$

where the dots indicate the omitted contributions. We may write this

$$\bar{u}(p',s)\Gamma^\mu(p',p)u(p,s) = -4e^2\bar{u}(p',s)\big[mK^\mu - P^\mu\gamma_\nu K^\nu - \gamma_\nu K^{\mu\nu}\big]u(p,s) + \cdots \quad (8.64)$$

where

$$\begin{aligned} K^\mu(p',p) &= \frac{i}{(2\pi)^4}\int d^4q \frac{q^\mu}{[(p'+q)^2-m^2][(p+q)^2-m^2]q^2} \\ K^{\mu\nu}(p',p) &= \frac{i}{(2\pi)^4}\int d^4q \frac{q^\mu q^\nu}{[(p'+q)^2-m^2][(p+q)^2-m^2]q^2}\,. \end{aligned}$$

We are interested in the limits $p'^\mu \to p^\mu$, and $p^2 \to m^2$, and we may write

$$\begin{aligned} K^\mu(p,p) &= \frac{i}{(2\pi)^4}\frac{\partial}{\partial m^2}\int d^4q \frac{q^\mu}{[(p+q)^2-m^2]q^2} = A(p^2)\,p^\mu \\ K^{\mu\nu}(p',p) &= \frac{i}{(2\pi)^4}\frac{\partial}{\partial m^2}\int d^4q \frac{q^\mu q^\nu}{[(p+q)^2-m^2]q^2} = B(p^2)\,p^\mu p^\nu + C(p^2)\,g^{\mu\nu} \end{aligned}$$

where A, B and C are scalar functions of p^2, which can be calculated, either by the Cutkosky method, or by dimensional regularization (Problem 8.9). $C(p^2)$ is not relevant to the calculation of $G(0)$, since it contributes to the term proportional to γ^μ between the spinors; and we have, in the limit $p^2 \to m^2$,

$$A(m^2) = -\frac{1}{16\pi^2 m^2} \qquad B(m^2) = \frac{1}{32\pi^2 m^2}\,. \quad (8.65)$$

The right-hand side of Eq.(8.64) can be written

$$\begin{aligned} & -4e^2\bar{u}(p',s)\big[mA(m^2)p^\mu - A(m^2)P^\mu\gamma p - B(m^2)p^\mu\gamma p\big]u(p,s) + \cdots \\ = \;& 2e^2 m\big[A(m^2)+B(m^2)\big]P^\mu + \cdots\,. \end{aligned} \quad (8.66)$$

From Eq.(8.57) and the calculated values (8.65), we obtain finally

$$G(0) = -4e^2m^2\big[A(m^2)+B(m^2)\big] = -\frac{e^2}{8\pi^2} = -\frac{\alpha}{2\pi} \approx -0.001162\,,$$

which yields the value $g = 2.002324$. This result agrees with the experimental value of 2.002319... up to the fifth decimal. To obtain better agreement, one has to extend the calculation to higher orders of the fine-structure constant, as we noted in Sec. **8.1**. This involves many Feynman diagrams and is a massive undertaking.

8.6 Exercises

Problem 1
Show that the Gamma function, defined by

$$\Gamma(a) = \int_0^\infty dz\, z^{a-1}\, \mathrm{e}^{-z}\,,$$

for $\mathrm{Re}\, a > 0$, and by analytic continuation in a for $\mathrm{Re}\, a \leq 0$, has the following properties:

(1) for n a positive integer, $\Gamma(n) = (n-1)!$
(2) for all real or complex a, $a\Gamma(a) = \Gamma(a+1)$.
(3) $\Gamma(a)$ is a meromorphic function of a, i.e., it is analytic, except for the occurrence of poles.
(4) Locate the poles of $\Gamma(a)$ and calculate their residues.

Problem 2
Show that any Lorentz vector field, $V^\mu(k)$, that is a function only of the Lorentz vector k^μ, can be written $V^\mu(k) = V(k^2)k^\mu$. Show that any second-order Lorentz tensor field, $T^{\mu\nu}(k)$, that is a function only of the Lorentz vector k^μ, can be written in the form $T^{\mu\nu}(k) = \Pi(k^2)P^{\mu\nu} + \Lambda(k^2)R^{\mu\nu}$, where the projection operators are defined by

$$P^{\mu\nu} = g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2} \qquad R^{\mu\nu} = \frac{k^\mu k^\nu}{k^2}\,.$$

Problem 3
Prove the formulae

$$\begin{aligned}
\frac{1}{AB} &= \int_0^1 \frac{dx}{[Ax + B(1-x)]^2} \\
\frac{1}{ABC} &= 2\int_0^1 dx \int_0^x \frac{dy}{[Ay + B(x-y) + C(1-x)]^3}\,.
\end{aligned}$$

Generalize these formulae to the case $(A_1 A_2 \cdots A_n)^{-1}$.

Problem 4
Write the electron self-energy formally to order e^2 as an unsubtracted dispersion integral in $W = \sqrt{s}$:

$$\Sigma(p) = \Sigma_0 + \gamma p \Sigma_1 + \frac{1}{\pi}\int_{-\infty}^\infty dW \frac{\Omega(W)}{\gamma p - W + i\epsilon(W)}\,.$$

Evaluate $\Omega(W)$ explicitly.

Problem 5

Compare the cutoff, the Pauli-Villars, the Cutkosky rule and the dimensional methods for regularizing the vacuum polarization. In which cases is it true that

(1) the tensor polarization is transverse,
(2) the scalar polarization vanishes at $k^2 = 0$?

Problem 6

Consider the integral

$$I(p^2; m_1^2, m_2^2) = \frac{i}{(2\pi)^4} \int \frac{d^4k}{[(p+k)^2 - m_1^2 + i\epsilon][k^2 - m_2^2 + i\epsilon]} .$$

Show that, as $p^2 \to (m_1 + m_2)^2$, poles of the integrand pinch the integration contour; but as $p^2 \to (m_1 - m_2)^2$, pinching does not take place. What can you conclude about the analytic properties of $I(p^2)$?

Problem 7

Derive the Cutkosky rule, $[k^2 - m^2 + i\epsilon]^{-1} \to -2\pi i\delta[k^2 - m^2]$, and use it to calculate the electron self-energy to order e^2.

Problem 8

Using dimensional regularization, calculate the scattering tensor, $\mathcal{M}_{\mu\nu\rho\sigma}$, for $\gamma + \gamma \to \gamma + \gamma$, in the limit of vanishing photon momenta. Show that the condition of gauge invariance,

$$k_1^\mu \mathcal{M}_{\mu\nu\rho\sigma}(k_1, k_2, k_3, k_4) = 0 ,$$

is automatically satisfied.

Problem 9

Work out $\overline{u}(p', s)\Gamma_\mu(p', p)u(p, s)$ to order α, using dimensional regularization. Obtain expressions for $F(0)$ and $G(0)$, the form factors in the static limit, showing that the former is divergent in the physical limit $n \to 4$, but that $G(0)$ is finite. Deduce the g-factor for the electron to order α. Show that $G(0)$ can also be obtained directly in four dimensions by the Cutkosky method.

Problem 10

Draw a parallel between the dimensional regularization method and distribution theory. In each case integrals which are initially undefined are assigned a well-defined value.

Chapter 9

Dyson-Schwinger Equations

In this chapter we shall set up the Dyson-Schwinger equations for the exact propagators and vertex function of QED. They form an alternative to the method of the Feynman perturbation series; and for some purposes they are superior, in particular for the study of relativistic bound states.

The Lagrangian of QED, with gauge parameter a, is

$$\mathcal{L} =: \overline{\psi}(i\gamma\partial - m_0)\psi - \tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2a}(\partial_\mu A^\mu)^2 - e_0\overline{\psi}\gamma A\psi : \tag{9.1}$$

where m_0 and e_0 are respectively the bare mass and bare charge of the electron. The Euler-Lagrange equations derived from this Lagrangian are

$$\left[g^{\mu\nu}\partial^2 + \frac{1-a}{a}\partial^\mu\partial^\nu\right]A_\nu = e_0\overline{\psi}\gamma^\mu\psi \tag{9.2}$$

$$[i\gamma\partial - m_0]\,\psi = e_0\gamma A\psi\,. \tag{9.3}$$

The propagators of the full theory are defined by

$$D'^{\mu\nu}_F(x-y) = -i\langle 0|T\{A^\mu(x)A^\nu(y)\}|0\rangle \tag{9.4}$$

$$S'_F(x-y) = -i\langle 0|T\{\psi(x)\overline{\psi}(y)\}|0\rangle\,, \tag{9.5}$$

where the primes have been inserted, as in Chapter 8, to distinguish these quantities from the corresponding free-field propagators. The propagators satisfy

$$\left[g^{\mu\nu}\partial^2 + \frac{1-a}{a}\partial^\mu\partial^\nu\right]D'_{F\nu\rho}(x) = \delta^\mu_\rho\delta^4(x) - ie_0\langle 0|T\{\overline{\psi}(x)\gamma^\mu\psi(x)A_\rho(0)\}|0\rangle$$

$$[i\gamma\partial - m_0]\,S'_F(x) = \delta^4(x) - ie_0\langle 0|T\{\gamma A(x)\psi(x)\overline{\psi}(0)\}|0\rangle\,.$$

In deriving these equations, we have made use of the equations of motion,

Eqs.(9.2)–(9.3). In momentum space, they can be written

$$\begin{aligned} D'^{\mu\nu}_F(k) &= D^{\mu\nu}_F(k) - ie_0 D^{\mu\rho}_F(k) \int d^4x e^{ikx} \langle 0|T\{\overline{\psi}(x)\gamma_\rho\psi(x)A^\nu(0)\}|0\rangle \\ S'_F(p) &= S_F(p) - ie_0 S_F(p) \int d^4x e^{ipx} \langle 0|T\{\gamma A(x)\psi(x)\overline{\psi}(0)\}|0\rangle\,, \end{aligned} \tag{9.6}$$

where D_F and S_F are the bare propagators, namely

$$D^{\mu\nu}_F(k) = \frac{-g^{\mu\nu} + (1-a)k^\mu k^\nu/k^2}{k^2 + i\epsilon} \tag{9.7}$$

$$S_F(p) = (\gamma p - m_0 + i\epsilon)^{-1}\,. \tag{9.8}$$

9.1 Vertex Function

We shall define the proper vertex function, Γ_μ, by the relation

$$\begin{aligned} &\langle 0|T\{\psi(x)\overline{\psi}(y)A^\nu(z)\}|0\rangle = \\ &-\frac{e_0}{(2\pi)^8}\int\int d^4p' d^4p S'_F(p')\Gamma_\mu(p',p)S'_F(p)D'^{\mu\nu}_F(p'-p)e^{-ip'x+ipy-i(p-p')z}. \end{aligned} \tag{9.9}$$

The normalization has been chosen in such a way that $\Gamma_\mu(p',p) = \gamma_\mu + O(e_0^2)$. The matrix element occurring in the first of the equations (9.6) is accordingly

$$\begin{aligned} &-\langle 0|T\{\gamma_{\rho\alpha\beta}\psi_\beta(x)\overline{\psi}_\alpha(x)A^\nu(0)\}|0\rangle = \\ &\frac{e_0}{(2\pi)^8}\int\int d^4p' d^4p \mathrm{Tr}\{\gamma_\rho S'_F(p')\Gamma_\mu(p',p)S'_F(p)\}D'^{\mu\nu}_F(p'-p)e^{i(p-p')x}, \end{aligned} \tag{9.10}$$

where the sign change occurs because the fermion fields $\overline{\psi}_a$ and ψ_β have been interchanged. The suffices α, β are spinor indices, and Tr refers to the operation of taking the trace of the 4×4 matrix in question. Substitute Eq.(9.10) into the first of the equations (9.6), do the x- and p'-integrations, and find

$$D'^{\mu\nu}_F(k) = D^{\mu\nu}_F(k) - D^{\mu\rho}_F(k)\Pi_{\rho\sigma}(k)D'^{\sigma\nu}_F(k), \tag{9.11}$$

where the vacuum polarization tensor is

$$\Pi_{\rho\sigma}(k) = \frac{ie_0^2}{(2\pi)^4}\int d^4p \mathrm{Tr}\{\gamma_\rho S'_F(k+p)\Gamma_\sigma(k+p,p)S'_F(p)\}\,. \tag{9.12}$$

We may similarly substitute Eq.(9.9), with the appropriate arguments, into the second of the equations (9.6). Performing the x- and p-integrations, we find

$$S'_F(p) = S_F(p) + S_F(p)\Sigma(p)S'_F(p)\,, \tag{9.13}$$

where the electron self-energy term is

$$\Sigma(p) = \frac{ie_0^2}{(2\pi)^4} \int d^4p' \gamma_\mu S'_F(p') \Gamma_\nu(p', p) D'^{\mu\nu}_F(p - p') \,. \tag{9.14}$$

The current coupled to the photon field, i.e., the right-hand side of Eq.(9.2), is conserved. This is easily proved:

$$i\partial_\nu \overline{\psi}\gamma^\nu\psi = \overline{\psi}(i\gamma \overleftarrow{\partial} + i\gamma \overrightarrow{\partial})\psi = \overline{\psi}\{-m_0 - e_0\gamma A + m_0 + e_0\gamma A\}\psi = 0, \tag{9.15}$$

where use has been made of Eq.(9.3) and of its adjoint. Now it is clear that

$$k^2 k_\mu D_F^{\mu\nu}(k) = -ak^\nu. \tag{9.16}$$

So if we contract the first of the equations (9.6) with k_μ, we find

$$k^2 k_\mu D'^{\mu\nu}_F(k) = -ak^\nu + iae_0 k^\rho \int d^4x e^{ikx} \langle 0|T\{\overline{\psi}(x)\gamma_\rho\psi(x)A^\nu(0)\}|0\rangle. \tag{9.17}$$

The quantity $ik^\rho e^{ikx}$ is just $\partial^\rho e^{ikx}$, and the differential operator, ∂^ρ, may be transferred from e^{ikx} to the matrix element by a partial integration. Moreover,

$$\partial^\rho T\{\overline{\psi}(x)\gamma_\rho\psi(x)A^\nu(0)\} = T\{\partial^\rho\overline{\psi}(x)\gamma_\rho\psi(x)A^\nu(0)\} + [\overline{\psi}(x)\gamma^0\psi(x), A^\nu(0)]\delta(x_0).$$

The first term on the right vanishes by current conservation, Eq.(9.15); and the equal-time commutator vanishes, by assumption. Thus we have proved that

$$k^2 k_\mu D'^{\mu\nu}_F(k) = -ak^\nu. \tag{9.18}$$

By contracting both sides of Eq.(9.11) with $k^2 k_\mu k_\nu$, we find

$$k^\rho \Pi_{\rho\sigma}(k) k^\sigma = 0, \tag{9.19}$$

where use has been made of Eq.(9.16) and Eq.(9.18). Whereas a general second order tensor could be a linear superposition of the tensors $g_{\rho\sigma} - k_\rho k_\sigma/k^2$ and $k_\rho k_\sigma/k^2$, it is clear from Eq.(9.19) that the second cannot contribute to $\Pi_{\rho\sigma}$. Accordingly, we have proved that

$$\Pi_{\rho\sigma}(k) = \left[g_{\rho\sigma} - \frac{k_\rho k_\sigma}{k^2}\right] \Pi(k^2), \tag{9.20}$$

where the scalar polarization, Π, is a function only of k^2.

9.2 Dyson-Schwinger Equations

Consider the electron propagator equation, (9.13). By multiplication from the left by S_F^{-1} and from the right by $S_F'^{-1}$, we find

$$S_F'^{-1}(p) = S_F^{-1}(p) - \Sigma(p)\,. \tag{9.21}$$

With the help of Eq.(9.8) and Eq.(9.14), we can write this

$$S_F'^{-1}(p) = \gamma p - m_0 - \frac{ie_0^2}{(2\pi)^4}\int d^4p'\gamma_\mu S_F'(p')\Gamma_\nu(p',p)D_F'^{\mu\nu}(p-p')\,. \tag{9.22}$$

This is the Dyson-Schwinger equation for the electron propagator. The '$i\epsilon$' prescription has been left implicit; in fact Eq.(9.22) defines a function analytic in a cut plane of the variable p^2. The *physical region* of this complex plane is the boundary on the upper lip of the cut, $m^2 < p^2 < \infty$, m being the physical mass (not the bare mass).

To obtain the Dyson-Schwinger equation for the photon propagator, we define first the inverse tensors, D_F^{-1} and $D_F'^{-1}$, by

$$D_F^{-1\mu\nu}(k)D_{F\nu\rho}(k) = \delta^\mu_\rho = D_F'^{-1\mu\nu}(k)D'_{F\nu\rho}(k). \tag{9.23}$$

Thus Eq.(9.11) is equivalent to

$$D_F'^{-1\mu\nu}(k) = D_F^{-1\mu\nu}(k) + \Pi^{\mu\nu}(k)\,, \tag{9.24}$$

the inverse of the bare propagator being

$$D_F^{-1\mu\nu}(k) = -k^2\left\{\left[g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right] + \frac{1}{a}\frac{k^\mu k^\nu}{k^2}\right\}. \tag{9.25}$$

From the relation (9.12), the photon propagator satisfies

$$D_F'^{-1\mu\nu}(k) = D_F^{-1\mu\nu}(k) + \frac{ie_0^2}{(2\pi)^4}\int d^4p\mathrm{Tr}\{\gamma^\mu S_F'(k+p)\Gamma^\nu(k+p,p)S_F'(p)\}\,, \tag{9.26}$$

which is the Dyson-Schwinger equation. From Eq.(9.20),

$$g^{\mu\nu}\Pi_{\mu\nu}(k) = \left[\delta^\mu_\mu - \frac{k_\mu k^\mu}{k^2}\right]\Pi(k^2) = 3\Pi(k^2)\,, \tag{9.27}$$

so the scalar polarization is

$$\Pi(k^2) = \frac{ie_0^2}{3(2\pi)^4}\int d^4p\mathrm{Tr}\{\gamma^\rho S_F'(k+p)\Gamma_\rho(k+p,p)S_F'(p)\}\,. \tag{9.28}$$

Introduce a *form factor*, $F(k^2)$, defined by

$$D'^{\mu\nu}_F(k) = \frac{-g^{\mu\nu} + \frac{k^\mu k^\nu}{k^2}}{k^2 + i\epsilon} F(k^2) + a \, \frac{k^\mu k^\nu}{(k^2 + i\epsilon)^2} \,. \tag{9.29}$$

From Eq.(9.24) we see that

$$F(k^2) = \frac{k^2}{k^2 + \Pi(k^2)} \,.$$

Let us calculate Møller scattering with exchange of the full photon propagator instead of the bare one. We will obtain the same result as in Sec. **7.3**, except that e^2, in Eq.(7.9) and Eq.(7.10), will be replaced by $e_0^2 F(k^2)$. In the present treatment only the photon corrections are taken into account, and we identify the physical coupling as $\alpha = \alpha_0 \, F(0)$, so we see that the photon form factor accounts for (part of) the renormalization of charge. We identify a *running* coupling by

$$\alpha(k^2) \equiv \frac{e_0^2}{4\pi} F(k^2) = \frac{\alpha_0}{1 + \Pi(k^2)/k^2} \,.$$

Replacing the full electron propagators in Eq.(9.28) by their bare values, we obtain the approximate result (8.44). The asymptotic behavior of $\Pi(k^2)$ was given in Eq.(8.45), and this yields

$$\alpha(k^2) \sim \frac{\alpha_0}{1 - \frac{\alpha_0}{3\pi} \log\left(-\frac{k^2}{M^2}\right)} \,, \tag{9.30}$$

for large k^2, where M is an undetermined constant. When k is spacelike, the denominator in Eq.(9.30) is real, and $\alpha(-M^2) = \alpha_0$. Because of the minus sign in front of $\frac{\alpha_0}{3\pi}$, it follows that, as $-k^2$ becomes larger than M^2, so the denominator becomes smaller, and the running coupling becomes larger. This has a physical interpretation: as the spacelike energy of the virtual photon probe increases, so the effective charge that is felt is less screened, and it is therefore larger than the low-energy value, which is the usual fine structure constant, namely $\frac{1}{137}$.

There is a problem with Eq.(9.30), namely that the denominator has a zero for very large values of k^2/M^2 — this phenomenon is called the Landau ghost, which, if it were taken seriously, would correspond to a tachyon, a particle with imaginary mass. However, it is not expected that the approximations we used to obtain Eq.(9.30) are valid at such enormous k^2, so it is not sure that QED really does possess a Landau ghost. If it does, one would have to conclude that electromagnetic forces, taken by themselves in isolation from the other forces, do not lead to a consistent quantum field theory.

9.3 Ward-Takahashi Identity

An important relation between the propagators and the vertex function can be obtained by considering the matrix element

$$W^{\mu}(x,y,z) = \langle 0|T\{\psi(x)\overline{\psi}(y)j^{\mu}(z)\}|0\rangle\,, \tag{9.31}$$

where the conserved current density,

$$j^{\mu}(z) = \overline{\psi}(z)\gamma^{\mu}\psi(z)\,, \tag{9.32}$$

is coupled to the photon field [Eq.(9.2)]. Since

$$\begin{aligned} T\{\psi_\alpha(x)\overline{\psi}_\beta(y)j^0(z)\} \;=\; & T\{\psi_\alpha(x)\overline{\psi}_\beta(y)\}j^0(z)\theta(x_0-z_0)\theta(y_0-z_0) \\ & +\psi_\alpha(x)j^0(z)\overline{\psi}_\beta(y)\theta(x_0-z_0)\theta(z_0-y_0) \\ & -\overline{\psi}_\beta(y)j^0(z)\psi_\alpha(x)\theta(z_0-x_0)\theta(y_0-z_0) \\ & +j^0(z)T\{\psi_\alpha(x)\overline{\psi}_\beta(y)\}\theta(z_0-x_0)\theta(z_0-y_0), \end{aligned} \tag{9.33}$$

one can evaluate the derivative of this time-ordered product with respect to z^0. After some careful algebra, one finds

$$\begin{aligned} &\frac{\partial}{\partial z^0}T\{\psi_\alpha(x)\overline{\psi}_\beta(y)j^0(z)\} = T\{\psi_\alpha(x)\overline{\psi}_\beta(y)\partial_0 j^0(z)\} + \\ &\{\overline{\psi}_\beta(y)[\psi_\alpha(x),j^0(z)]\theta(y_0-x_0) - [\psi_\alpha(x),j^0(z)]\overline{\psi}_\beta(y)\theta(x_0-y_0)\}\delta(x_0-z_0) + \\ &\{[\overline{\psi}_\beta(y),j^0(z)]\psi_\alpha(x)\theta(y_0-x_0) - \psi_\alpha(x)[\overline{\psi}_\beta(y),j^0(z)]\theta(x_0-y_0)\}\delta(y_0-z_0). \end{aligned} \tag{9.34}$$

Now one can prove easily from the equal-time anticommutators that

$$\begin{aligned} [\psi_\alpha(x),j^0(z)]\delta(x_0-z_0) &= \psi_\alpha(x)\delta^4(x-z), \\ \left[\overline{\psi}_\beta(y),j^0(z)\right]\delta(y_0-z_0) &= -\overline{\psi}_\beta(y)\delta^4(y-z). \end{aligned} \tag{9.35}$$

Moreover, since spatial differentiations do not affect the time-ordering operation,

$$\begin{aligned} \frac{\partial}{\partial z^\mu}T\{\psi_\alpha(x)\overline{\psi}_\beta(y)j^\mu(z)\} \;=\; & T\{\psi_\alpha(x)\overline{\psi}_\beta(y)\partial_\mu j^\mu(z)\} \\ & +[\delta^4(y-z)-\delta^4(x-z)]T\{\psi_\alpha(x)\overline{\psi}_\beta(y)\}. \end{aligned} \tag{9.36}$$

The first term on the right vanishes by current conservation (Eq.(9.15)), and so the vacuum expectation value of Eq.(9.36) is

$$\frac{\partial}{\partial z^\mu}W^{\mu}(x,y,z) = i[\delta^4(y-z)-\delta^4(x-z)]S'_F(x-y), \tag{9.37}$$

where we have used the definition (9.5). From the equation of motion Eq.(9.2), we know that Eq.(9.31) can be rewritten

$$\begin{aligned} W^\mu(x,y,z) &= \frac{1}{e_0}\langle 0|T\{\psi(x)\overline{\psi}(y)[g^{\mu\nu}\partial^2 + \frac{1-a}{a}\partial^\mu\partial^\nu]A_\nu(z)\}|0\rangle \qquad (9.38)\\ &= \frac{1}{e_0}\left[g^{\mu\nu}\partial_z^2 + \frac{1-a}{a}\partial_z^\mu\partial_z^\nu\right]\langle 0|T\{\psi(x)\overline{\psi}(y)A_\nu(z)\}|0\rangle\,, \end{aligned}$$

where the second line follows from the first, because A_ν and $\dot{A}_\nu$ are assumed to commute at equal times with ψ and $\overline{\psi}$. With the help of Eq.(9.9), we find

$$\begin{aligned} W^\mu(x,y,z) &= \frac{1}{(2\pi)^8}\iint d^4p'd^4pS'_F(p')\Gamma_\rho(p',p)S'_F(p)D'^\rho_\nu(p-p') \qquad (9.39)\\ &\left[g^{\mu\nu}(p-p')^2 + \frac{1-a}{a}(p-p')^\mu(p-p')^\nu\right]e^{-ip'x+ipy-i(p-p')z}\,. \end{aligned}$$

On differentiating with respect to z, we obtain

$$\begin{aligned} \frac{\partial}{\partial z^\mu}W^\mu(x,y,z) &= \frac{-i}{(2\pi)^8}\iint d^4p'd^4pS'_F(p')\Gamma_\rho(p',p)S'_F(p)D'^{\nu\rho}(p-p')\\ &\qquad \frac{1}{a}(p-p')^2(p-p')_\nu e^{-ip'x+ipy-i(p-p')z} \qquad (9.40)\\ &= \frac{i}{(2\pi)^8}\iint d^4p'd^4pS'_F(p')\Gamma_\rho(p',p)S'_F(p)(p-p')^\rho e^{-ip'x+ipy-i(p-p')z}\,, \end{aligned}$$

where we have used Eq.(9.18) to obtain the second line.

The two expressions (9.37) and (9.40) must be equal to one another, and if we set $z=0$ and take Fourier transforms, we find

$$\begin{aligned} S'_F(p')\Gamma_\mu(p',p)S'_F(p)(p-p')^\mu &= \iint d^4xd^4ye^{ip'x-ipy}\left[\delta^4(y)-\delta^4(x)\right]S'_F(x-y)\\ &= S'_F(p') - S'_F(p), \qquad (9.41) \end{aligned}$$

or equivalently,

$$(p'-p)^\mu\Gamma_\mu(p',p) = S'^{-1}_F(p') - S'^{-1}_F(p). \qquad (9.42)$$

This relation, which connects the longitudinal part of the vertex function to the electron propagator, is called the Ward-Takahashi identity. Now let $p' = p+k$, with k infinitesimal and independent of p. Then we see from Eq.(9.42) that

$$\begin{aligned} k^\mu\Gamma_\mu(p+k,p) &= k^\mu\left[\frac{\partial}{\partial k^\mu}S'^{-1}_F(p+k)\right]_{k=0} + O(k^2)\\ &= k^\mu\left[\frac{\partial}{\partial p^\mu}S'^{-1}_F(p+k)\right]_{k=0} + O(k^2). \qquad (9.43) \end{aligned}$$

Thus

$$k^\mu \Gamma_\mu(p,p) = k^\mu \frac{\partial}{\partial p^\mu} S_F'^{-1}(p) + O(k^2); \tag{9.44}$$

and since k is independent of p, one must have

$$\Gamma_\mu(p,p) = \frac{\partial}{\partial p^\mu} S_F'^{-1}(p), \tag{9.45}$$

which is called Ward's identity. This can also be written

$$S_F'(p)\Gamma_\mu(p,p)S_F'(p) = -\frac{\partial}{\partial p^\mu} S_F'(p)\,, \tag{9.46}$$

where the identity

$$\partial_\mu A = -A(\partial_\mu A^{-1})A$$

has been used.

The full electron propagator can be expressed as a subtracted dispersion relation. Leaving the subtraction implicit, we write

$$S_F'(p) = \int_0^\infty \frac{ds}{p^2 - s + i\epsilon}\left[\gamma p\, \rho_1(s) + \rho_2(s)\right], \tag{9.47}$$

which is called the Lehmann representation. The support of the integrand is $[m^2, \infty)$, and there may be a delta distribution at $s = m^2$, depending on the value of the gauge parameter, a (see below). The spectral functions, ρ_1 and ρ_2, were calculated to order α in Eq.(8.29), but here we envisage going beyond perturbation theory.

It is convenient to write Eq.(9.47) in the form

$$S_F'(p) = \int_{-\infty}^\infty dW\, \rho(W)\left[\gamma p - W + i\epsilon(W)\right]^{-1}, \tag{9.48}$$

where $s = W^2$, $\epsilon(W) = \pm\epsilon$, and

$$\rho(W) = \pm\left[W\rho_1(W^2) + \rho_2(W^2)\right],$$

the plus sign applying to $W > 0$, and the minus sign to $W < 0$. The Ansatz of Delbourgo and Salam consists in guessing a dispersion relation for the full vertex function, with the fermionic legs attached:

$$S_F'(p')\Gamma_\mu(p',p)S_F'(p) = \int_{-\infty}^\infty dW\, \rho(W)\left[\gamma p' - W\right]^{-1}\gamma_\mu\left[\gamma p - W\right]^{-1}, \tag{9.49}$$

where the '$i\epsilon(W)$' prescription has been left implicit. This form satisfies the Ward-Takahashi identity for any spectral function, ρ, for

$$\begin{aligned}(p'-p)^\mu S'_F(p')\Gamma_\mu(p',p)S'_F(p) &= \int_{-\infty}^{\infty} dW\,\rho(W)\big[\gamma p' - W\big]^{-1}\Big\{\big[\gamma p' - W\big] - \big[\gamma p - W\big]\Big\}\big[\gamma p - W\big]^{-1} \\ &= \int_{-\infty}^{\infty} dW\,\rho(W)\Big\{\big[\gamma p - W\big]^{-1} - \big[\gamma p' - W\big]^{-1}\Big\} \\ &= S'_F(p) - S'_F(p')\,.\end{aligned}$$

The Ansatz (9.49) is not the only form that guarantees satisfaction of the Ward-Takahashi identity, for one can add any longitudinal term, $\Gamma^{\rm long}_\mu(p',p)$, such that $(p'-p)^\mu\Gamma^{\rm long}_\mu(p',p) = 0$; but it is hoped that the Delbourgo-Salam Ansatz is good at low energies.

By multiplying both sides of the Dyson-Schwinger equation (9.22) from the right by $S'_F(p)$, one obtains

$$(\gamma p - m_0)S'_F(p) = 1 + \frac{ie_0^2}{(2\pi)^4}\int d^4p'\gamma_\mu S'_F(p')\Gamma_\nu(p',p)S'_F(p)D'^{\mu\nu}_F(p'-p)\,. \quad (9.50)$$

On substituting the Lehmann representation, Eq.(9.48), into the left-hand side of this equation, and the Delbourgo-Salam Ansatz, Eq.(9.49), into the right-hand side, one obtains a *linear* equation for the spectral function, ρ.

To proceed further, we would need the full photon propagator, since this occurs in Eq.(9.50). The simplest option is just to replace it by its bare form, Eq.(9.7). The bare charge, e_0, is similarly replaced by the physical charge, e. After performing the four-dimensional integral in Eq.(9.50), under the dispersion integral (9.49), we find

$$(W-m)\rho(W) = \int_{-\infty}^{\infty} dW'\,\rho(W')\frac{\Omega(W,W')}{W-W'}\,, \quad (9.51)$$

$\Omega(W,W') = \pm\alpha\theta(W^2-W'^2)(W^2-W'^2)\big[a(W'^2+W^2)-(3+a)W'W\big]/(4\pi W^3)$, where the necessary subtractions have been left implicit. It is left as an exercise (Problem 9.6) to show that the integral equation (9.51) has a solution, and that this leads to the infrared behavior ($p^2 \to m^2$),

$$S_F(p) \sim (\gamma p - m)^{-1+\frac{\alpha}{2\pi}(a-3)}\,.$$

Evidently the naively expected pole behavior at $p^2 = m^2$ is only true if $a = 3$ (the Yennie gauge). In the Feynman gauge ($a = 1$), and the Landau gauge ($a = 0$), the branch point at $\gamma p = m$ is more singular than a pole.

9.4 Chiral Symmetry Breaking in Strong QED

Is it possible that the electron's mass is purely of electromagnetic origin? A way to investigate this idea is to consider the Dyson-Schwinger equation for the electron propagator, Eq.(9.22), in which the bare mass is set equal to zero. The free Dirac equation for the electron has thus no mass term, and so the field possesses chiral invariance [see Eq.(3.94) et seq.]; but this does not imply that the full electron propagator, obtained as a solution of the Dyson-Schwinger equation, corresponds to zero physical mass. If this mass is nonzero, we speak of dynamical chiral symmetry breaking and of the generation of an electron mass.

We shall show that the nonlinear form of the Dyson-Schwinger equation allows for such a dynamical generation, but that it occurs only for large values of the coupling, $\alpha = \frac{e^2}{4\pi} \approx 1$, and not for the physical value, $\alpha \approx \frac{1}{137}$. The conclusion is that the mass of the electron cannot be purely of electromagnetic origin. Nevertheless, the result of the analysis is not wholly negative, for it serves as a model for a similar treatment of the quark Dyson-Schwinger equations of quantum chromodynamics. In this theory, which is beyond the scope of this book, the equivalent strong interaction coupling, α_s, is of order unity at low energies, so that the dynamical generation of quark masses is indeed possible.

We propose to make some cavalier approximations which, while they are not expected to have small numerical consequences, do not destroy the crucial nonlinearity of the Dyson-Schwinger equation for the electron propagator. The purpose is to show how the phenomenon of dynamical chiral symmetry breaking can occur. We shall approximate the vertex function by its bare value, $\Gamma^\mu(p',p) \to \gamma^\mu$, the bare charge, e_0, by the physical charge, e, and the photon propagator by its bare form, which we choose to write in the Landau gauge,

$$D'^{\mu\nu}_F(k) \to D^{\mu\nu}_F(k) = \frac{-g^{\mu\nu} + k^\mu k^\nu/k^2}{k^2 + i\epsilon} .$$

With these approximations, the Dyson-Schwinger equation for the propagator of the electron reads

$$S'^{-1}_F(p) = \gamma p - \frac{ie^2}{(2\pi)^4} \int d^4p' \gamma_\mu S'_F(p') \gamma_\nu D^{\mu\nu}_F(p-p') . \tag{9.52}$$

Now $S'^{-1}_F(p)$ can be regarded as a function of the matrix γp, so we can write it

$$S'^{-1}_F(p) = Z^{-1}(-p^2) \left[\gamma p - m(-p^2)\right] ,$$

where the wave-function renormalization, $Z(-p^2)$, and the mass function, $m(-p^2)$,

are to be determined. Since the matrices γ^μ are traceless, it follows that

$$\mathrm{Tr}\, S_F'^{-1}(p) = -4Z^{-1}(-p^2)m(-p^2) \qquad \mathrm{Tr}\,\gamma p\, S_F'^{-1}(p) = 4p^2 Z^{-1}(-p^2)\,;$$

hence by taking traces of the right-hand side of Eq.(9.52), we can obtain coupled scalar integral equations for Z^{-1} and $Z^{-1}m$:

$$\begin{aligned} Z^{-1}(-p^2)m(-p^2) &= \frac{ie^2}{4(2\pi)^4}\mathrm{Tr}\int d^4p'\gamma_\mu S_F'(p')\gamma_\nu D_F^{\mu\nu}(p-p') \qquad (9.53)\\ Z^{-1}(-p^2) &= 1 - \frac{ie^2}{4p^2(2\pi)^4}\mathrm{Tr}\,\gamma^\rho p_\rho \int d^4p'\gamma_\mu S_F'(p')\gamma_\nu D_F^{\mu\nu}(p-p')\,. \end{aligned}$$

After computing the traces of the integrands, and then making the Wick rotation to Euclidean space, so that $-p^2_{\mathsf{Lor}} \to p^2_{\mathsf{Euc}}$, and finally performing the angular integrations, we find (Problem 9.3) that $Z(p^2) \equiv 1$ and

$$m(p^2) = \frac{3\alpha}{4\pi}\int_0^\infty \frac{dp'^2}{\max(p^2,p'^2)}\,\frac{p'^2 m(p'^2)}{p'^2 + m^2(p'^2)}\,, \qquad (9.54)$$

where $\alpha = \frac{e^2}{4\pi}$ is the fine-structure constant.

It is clear that $m(p^2) \equiv 0$ is always a solution of the nonlinear equation (9.54), in which case $S_F'^{-1}(p) = \gamma p$, corresponding to a massless Dirac particle without interaction. The interesting question is whether the equation can have a nontrivial solution too, which would correspond to the dynamical generation of mass. The Dyson-Schwinger equations must however be regularized. This can be done by the dimensional method (Problem 9.4); but here it will prove adequate, for the approximate equation (9.54), simply to insert a cut-off, Λ^2:

$$m(s) = \frac{3\alpha}{4\pi}\int_0^{\Lambda^2} \frac{dt}{\max(s,t)}\,\frac{t\,m(t)}{t + m^2(t)}\,, \qquad (9.55)$$

where $s = p^2$ and $t = p'^2$. We shall now show that Eq.(9.55) has no nontrivial solution if $\alpha < \frac{\pi}{3}$. First note that, for any real $m(t)$,

$$\left|\frac{t\,m(t)}{t+m^2(t)}\right| = \sqrt{\frac{t}{t+m^2(t)}}\,\frac{|m(t)|}{\sqrt{t+m^2(t)}}\,\sqrt{t} \le \sqrt{t}\,.$$

From Eq.(9.55) it follows that, for $0 \le s \le \Lambda^2$,

$$|m(s)| \le \frac{3\alpha}{4\pi}\left\{\int_0^s dt\,\frac{\sqrt{t}}{s} + \int_s^{\Lambda^2}\frac{dt}{\sqrt{t}}\right\} = \frac{3\alpha}{2\pi}\left\{\Lambda - \tfrac{2}{3}\sqrt{s}\right\} < \frac{3\alpha\Lambda}{2\pi}\,, \qquad (9.56)$$

so $m(s)$ is necessarily a bounded function, although the bound does explode as $\Lambda \to \infty$.

Since any function that satisfies Eq.(9.55) is bounded, as we have just proved, it must have a finite norm of the following kind:

$$||m|| = \sup_{0 \le s \le \Lambda^2} \left\{ \sqrt{s}\, |m(s)| \right\} . \tag{9.57}$$

Of course, sup $|m(s)|$ also exists, but Eq.(9.57) is the most useful norm, since from Eq.(9.55),

$$\begin{aligned} |m(s)| &\le \frac{3\alpha}{4\pi} \left\{ \frac{1}{s} \int_0^s dt\, t^{-\frac{1}{2}} + \int_s^{\Lambda^2} dt\, t^{-\frac{3}{2}} \right\} ||m|| \\ &= \frac{3\alpha}{4\pi} \left\{ \frac{4}{\sqrt{s}} - \frac{2}{\Lambda} \right\} ||m|| < \frac{3\alpha}{\pi\sqrt{s}} ||m|| . \end{aligned}$$

Therefore

$$||m|| \le \frac{3\alpha}{\pi} ||m|| .$$

If $m(s)$ is not identically zero, $||m|| \neq 0$, and so we conclude

$$\frac{3\alpha}{\pi} \ge 1 , \tag{9.58}$$

as a necessary condition that Eq.(9.55) have a nontrivial solution. It is significant that the inequality (9.58) does not contain the cut-off, Λ. For the proof to work, there must be a finite cut-off (more generally, a regularization), but it may be as large as we please.

Although the above analysis was based upon an approximate form of the Dyson-Schwinger equation, it is generally felt that the idea of creating an electron mass by breaking chiral symmetry in QED, and the electroweak theory that is its generalization (see the Appendix), is doomed to failure, since the physical value of α is two orders of magnitude too low to satisfy Eq.(9.58). Nevertheless, we shall proceed with this 'Strong QED' model, in which $\alpha > \frac{\pi}{3}$, in order to introduce a method that is of wider use.

A general technique for the analysis of nonlinear equations like Eq.(9.55) is bifurcation analysis. We have remarked that this equation always has the trivial solution. The question to be answered is whether, as the coupling is increased from zero, there occurs a critical value, at which a nontrivial solution splits off, or *bifurcates* from the trivial one. In an infinitesimal neighborhood of such a bifurcation point, one may evaluate the derivative of the right-hand side of the equation with respect to m, at the point $m = 0$. If the resultant linear equation has a solution, the nonlinear equation has indeed a bifurcation point,

the solution of the linear equation providing the starting point. To see how this works for Eq.(9.55), consider

$$\frac{\partial}{\partial m}\left\{\frac{tm}{t+m^2}\right\} = \frac{t}{t+m^2} - \frac{2m^2t}{(t+m^2)^2}\,,$$

which reduces to 1 at the point $m = 0$. The bifurcation equation appertaining to Eq.(9.55) is therefore

$$\delta m(s) = \frac{\lambda}{4}\int_{\epsilon^2}^{\Lambda^2} \frac{dt}{\max(s,t)}\,\delta m(t)\,, \tag{9.59}$$

where δm is an infinitesimal, where we have introduced an infrared cut-off for technical reasons, and where we have set $\lambda = 3\alpha/\pi$ for convenience.

By judicious differentiation, we find that any solution of Eq.(9.59) must also satisfy the differential equation

$$s^2\delta m''(s) + s\delta m'(s) + \tfrac{1}{4}\lambda\delta m(s) = 0\,,$$

which has solutions for any λ, indeed the most general solution is

$$\delta m(s) = A_+ s^{\alpha+} + A_- s^{\alpha-}\,,$$

where $\alpha_\pm = -\frac{1}{2} \pm \frac{1}{2}\sqrt{1-\lambda}$. This general solution of the differential equation does not satisfy the integral equation (9.59), because of infrared and ultraviolet boundary conditions, at $s = \epsilon^2$ and at $s = \Lambda^2$, respectively. These conditions can be simultaneously satisfied only if

$$\left(\frac{\Lambda}{\epsilon}\right)^{\sqrt{1-\lambda}} = \frac{1-\sqrt{1-\lambda}}{1+\sqrt{1-\lambda}}\,.$$

For $0 \leq \lambda \leq 1$, the left-hand side is greater than 1, while the right-hand side is less than 1, which means that the integral equation has no solution for λ in this range. When $\lambda > 1$, there are solutions, because then $\sqrt{1-\lambda}$ is imaginary. The smallest value of the coupling for which the bifurcation equation (9.59) has a solution is

$$\lambda \approx 1 + \frac{4\pi^2}{\log^2 \frac{\Lambda}{\epsilon}}\,,$$

which tends to unity from above as $\Lambda \to \infty$ and/or $\epsilon \to 0$. The details of the above calculation have been relegated to Problem 9.5.

9.5 Exercises

Problem 1

Derive the Euler-Lagrange equations for QED and fill in all the details in the derivation of the following exact integrals for the propagators:

$$\begin{aligned} D_F'^{\mu\nu}(k) &= D_F^{\mu\nu}(k) - ie_0 D_F^{\mu\rho}(k) \int d^4x e^{ikx} \langle 0|T\{\overline{\psi}(x)\gamma_\rho\psi(x)A^\nu(0)\}|0\rangle \\ S_F'(p) &= S_F(p) - ie_0 S_F(p) \int d^4x e^{ipx} \langle 0|T\{\gamma A(x)\psi(x)\overline{\psi}(0)|0\rangle \,. \end{aligned}$$

Problem 2

Demonstrate the equality

$$\begin{aligned} &\frac{\partial}{\partial z^0} T\{\psi_\alpha(x)\overline{\psi}_\beta(y)j^0(z)\} = T\{\psi_\alpha(x)\overline{\psi}_\beta(y)\partial_0 j^0(z)\} + \\ &\{\overline{\psi}_\beta(y)[\psi_\alpha(x), j^0(z)]\theta(y_0 - x_0) - [\psi_\alpha(x), j^0(z)]\overline{\psi}_\beta(y)\theta(x_0 - y_0)\}\delta(x_0 - z_0) + \\ &\{[\overline{\psi}_\beta(y), j^0(z)]\psi_\alpha(x)\theta(y_0 - x_0) - \psi_\alpha(x)[\overline{\psi}_\beta(y), j^0(z)]\theta(x_0 - y_0)\}\delta(y_0 - z_0). \end{aligned}$$

Problem 3

If the electron propagator satisfies

$$\begin{aligned} S_F'^{-1}(p) &= Z^{-1}(-p^2)\left[\gamma p - m(-p^2)\right] \\ &= \gamma p - \frac{ie^2}{(2\pi)^4} \int d^4p' \gamma_\mu S_F'(p')\gamma_\nu D_F^{\mu\nu}(p - p') \,, \end{aligned}$$

where $D_F^{\mu\nu}$ is the bare photon propagator, obtain coupled integral equations for $Z(-p^2)$ and $m(-p^2)$. Show that $Z(-p^2) = 1$ in Landau gauge, and write down the resulting integral equation for $m(-p^2)$.

Problem 4

Repeat the calculation of the previous question in n dimensions. Show that dimensional regularization leads to $Z(-p^2) = 1$ in Landau gauge, independently of the value of n.

Problem 5

Find the smallest value of λ for which

$$\delta m(s) = \frac{\lambda}{4} \int_{\epsilon^2}^{\Lambda^2} \frac{dt}{\max(s,t)} \, \delta m(t) \,,$$

has a nontrivial solution. Show that this eigenvalue tends to unity as $\epsilon \to 0$ or $\Lambda \to \infty$.

Problem 6

Show that the Delbourgo-Salam Ansatz for the vertex function in QED leads to an integral equation, for the spectral function of the electron propagator, of the following kind:

$$(W - m)\rho(W) = \int_{-\infty}^{\infty} dW'\, \rho(W') \frac{\Omega(W, W')}{W - W'} .$$

Calculate $\Omega(W, W')$, and investigate the infrared behavior of the propagator.

Problem 7

Calculate the inverse of the electron propagator,

$$\begin{aligned} S_F^{-1}(p) &= \alpha(-p^2)\gamma p + \beta(-p^2) \\ &= \frac{ie^2}{(2\pi)^4} \int d^4k\, D_F^{\mu\nu}(k)\gamma_\mu S_F(p+k)\gamma_\nu , \end{aligned}$$

where D_F is a regularized photon propagator,

$$D_F^{\mu\nu}(k) = \left[-g^{\mu\nu} + (1-a)\frac{k^\mu k^\nu}{k^2}\right] \frac{1}{k^2 - \mu^2} \frac{-\Lambda^2}{k^2 - \Lambda^2} .$$

Here μ is a fictitious photon mass that serves as an infrared regulator, and Λ is a Pauli-Villars ultraviolet regulator. Derive coupled integral equations for α and β. Consider the limits in which the regulators are removed, $\mu \to 0$ and $\Lambda \to \infty$. Under what conditions is $\beta(p^2) = 1$ in the Landau gauge?

Problem 8

Consider the approximate Dyson-Schwinger equation for the electron's mass-function in Landau gauge,

$$m(s) = m(0) + \frac{\lambda}{4} \int_0^s dt \left(\frac{t}{s} - 1\right) \frac{m(t)}{t + m^2(t)} ,$$

where $\lambda = 3\alpha/\pi > 1$, and where $m(0)$ is assumed to be nonzero.

(1) In the integral equation, replace the kernel, $\left(\frac{t}{s} - 1\right)$, by its average, $-\frac{1}{2}$, and show that the solutions of the approximate equation have complex singularities on the first Riemann sheet of the variable s.

(2) Suggest a method to locate the singularities of $m(s)$, without the above averaging of the kernel.

(3) What is the objection to having complex singularities? What is their origin? How may they be removed?

Problem 9

Derive the Dyson-Schwinger equation for the electron mass function, $m(p^2)$, in massless QED(3) (i.e., quantum electrodynamics in two space and one time dimension, with zero bare electron mass). In the approximation of taking a free vertex and a free photon propagator, show that this equation reduces, after Wick rotation to Euclidean momenta, to

$$m(p^2) = \frac{2e^2}{(2\pi)^3} \int \frac{d^3q}{(q-p)^2} \frac{m(q^2)}{m^2(q^2)+q^2} .$$

Approximate this equation further by making the replacement $m^2(q^2) \to m_0^2$ in the denominator of the integrand, with the subsidiary requirement,

$$\lim_{p^2 \to 0} m(p^2) = m_0 .$$

Solve this equation and show that chiral symmetry is dynamically broken down for arbitarily small coupling in this model. Does this result change after inclusion of the contribution of an electron loop to the vacuum polarization?

Problem 10

In QCD, the non-Abelian gauge theory of the strong interaction, the running mass-function of a quark is believed to be well described by the approximate Dyson-Schwinger equation, in the Landau gauge,

$$m(s) = m_0(\Lambda) + \sigma \int_0^\Lambda \frac{dt}{\tau(s,t)} \frac{tm(t)}{t+m^2(t)} ,$$

where $\tau(s,t) = \max(s,t) \log\left[1 + \max(s,t)\right]$. The logarithm factor accounts for asymptotic freedom, i.e., the fact that the interaction between quarks vanishes in the extreme ultraviolet; and σ has been calculated in perturbation theory, with the result $\sigma = \frac{12}{21}$, given that there are six quarks with three colors. Λ is an ultraviolet cut-off, and $m_0(\Lambda)$ is the bare mass of the quark, which is allowed to depend upon the cut-off.

(1) Show that the general solution has the ultraviolet behavior

$$m(s) \sim \frac{A}{s} \log^{-1+\sigma} s + B \log^{-\sigma} s .$$

(2) Prove that there is no solution for $m(s)$ if $\Lambda = \infty$ and $m_0(\infty) \neq 0$.

(3) Why does the ultraviolet boundary condition not determine a unique solution, up to normalization?

Appendix A

Electroweak Interactions

We shall consider the first generation of leptons, the electron and its neutrino. Both particles interact weakly, and the electron interacts electromagnetically because of its charge. The electron has mass, whereas the electron-neutrino mass is extremely small; indeed for a long time the neutrino was thought to be massless. We shall present the electroweak theory as if the neutrino were exactly massless. Inclusion of a mass term can be implemented by the method given in Sec. **A.3**, the details being relegated to Problem A.8.

The electron has both left-handed and right-handed chiral components

$$\psi_e = P_L\psi_e + P_R\psi_e\,,$$

whereas the neutrino was thought to have only a left-handed chiral component,

$$\psi_\nu = P_L\psi_\nu\,.$$

As in Eq.(3.96), the right and left chiral projection operators are

$$P_R = \tfrac{1}{2}(1+\gamma_5) \qquad P_L = \tfrac{1}{2}(1-\gamma_5)\,. \tag{A.1}$$

As we showed at the end of Chapter 3, chirality and helicity coincide for a massless particle, and for this and other reasons the electroweak theory is set up in the first place for massless particles. This allows the Lagrangian to have a high degree of local symmetry, but at the expense of neglecting all masses. By the magic of spontaneous symmetry breaking, it will be shown how an effective mass term can be generated for the electron, and, optionally, for the neutrino. The left chiral parts of the electron and the neutrino fields are put into a doublet, while the remaining right chiral part of the electron is put into a singlet:

$$L_e = \begin{pmatrix} P_L\psi_\nu \\ P_L\psi_e \end{pmatrix} \qquad R_e = P_R\psi_e\,. \tag{A.2}$$

The Lagrangian density for a free, massless theory with a left-handed doublet and a right-handed singlet is

$$\mathcal{L} = i : \overline{L_e}\gamma^\rho \partial_\rho L_e : + i : \overline{R_e}\gamma^\rho \partial_\rho R_e :$$

The terms are invariant under independent, global $U(2)$ transformations,

$$U(\theta, \vec{\theta}) = \exp[-i\theta] \exp[-i\vec{t} \cdot \vec{\theta}] . \tag{A.3}$$

The first factor is a phase that corresponds to the Abelian group, $U(1)$, while the second factor constitutes a nontrivial $SU(2)$ mixing of the components, $\vec{t}$ being the generator with the appropriate dimensionality. For the right-handed singlet, this dimensionality is 1, and $\vec{t} = 0$ in this case; but for the left-handed doublet, the dimensionality is 2, and $\vec{t}$ is represented by the Pauli matrices, which we write $\vec{\tau}$ rather than $\vec{\sigma}$, to avoid confusion with spin. The degree of freedom generated by $\vec{t}$ is called (weak) isospin. For the right-handed singlet,

$$\begin{aligned} R_e &\rightarrow U R_e U^\dagger = u_R R_e \\ u_R &= \exp[-i\theta_R] , \end{aligned}$$

whereas, for the left-handed doublet,

$$\begin{aligned} L_e &\rightarrow U L_e U^\dagger = u_L L_e \\ u_L &= \exp[-i\theta_L] \exp[-i\vec{\theta} \cdot \vec{\tau}] . \end{aligned}$$

In general the $U(1)$ phases, θ_L and θ_R, may be different. It is convenient to replace these phases by $\theta_L = Y_L\tilde{\theta}$ and $\theta_R = Y_R\tilde{\theta}$. The general $U(2)$ matrix can now be written

$$u = \exp[-iY\tilde{\theta}] \exp[-i\vec{\theta} \cdot \vec{t}] ,$$

where Y is the generator of $U(1)$, taking the values Y_L and Y_R for the doublet and the singlet, respectively. Y is called the (weak) hypercharge, and it proves useful to restrict it by requiring

$$Q = \tfrac{1}{2}Y + T_3 , \tag{A.4}$$

where Q is the electric charge and T_3 is the third component of the isospin, this being 0 for $e_R = P_R\psi_e$ and $\pm\frac{1}{2}$ for $\nu_L = P_R\psi_\nu$ and $e_L = P_R\psi_e$ respectively. The hypercharge can then be calculated from Eq.(A.4). The following table shows the values of the electroweak quantum numbers:

	T	T_3	Q	Y
ν_L	$\frac{1}{2}$	$\frac{1}{2}$	0	-1
e_L	$\frac{1}{2}$	$-\frac{1}{2}$	-1	-1
e_R	0	0	-1	-2

Following Yang and Mills, we now extend the global symmetry to a local one, i.e., we allow $\tilde{\theta}$ and $\vec{\theta}$ to depend on space and time. This local symmetry is called gauge symmetry, on the analogy of the electromagnetic case, and it requires the existence of vector gauge fields, like the photon. For the right-handed singlet, we introduce the isospin scalar field B_ρ, which changes under the gauge transformation Eq.(A.3) as follows:

$$B_\rho \to B'_\rho = B_\rho - \frac{1}{\tilde{g}}\partial_\rho \tilde{\theta}\,. \tag{A.5}$$

Next we define the covariant derivative operator

$$\mathcal{D}_\rho = \partial_\rho - i\tilde{g}YB_\rho\,, \tag{A.6}$$

so that the term $\overline{R}_e\gamma^\rho\mathcal{D}_\rho R_e$ is invariant under this transformation, the term arising from differentiating R_e being canceled by the change in B_ρ.

The case of the left-handed doublet is more complicated, because of its more intricate transformation properties in the original global theory. We proceed by analogy with the singlet case by extending the covariant differentiation Eq.(A.6) to include isospin:

$$\mathcal{D}_\rho = \partial_\rho - ig\,\vec{t}\cdot\vec{A}_\rho - i\tilde{g}YB_\rho\,. \tag{A.7}$$

We have defined the coupling, g, to an isospin vector field $\vec{A}_\rho$, with $g \neq \tilde{g}$. In order that the term $\overline{L}_e\gamma^\rho D_\rho L_e$ be invariant under the gauge transformation, the vector field must undergo the transformation

$$\vec{\tau}\cdot\vec{A}_\rho \to u_L\left[\vec{\tau}\cdot\vec{A}_\rho\right]u_L^{-1} - \frac{i}{g}(\partial_\rho u_L)u_L^{-1}$$

With $\vec{\theta}$ infinitesimal, $u_L \approx 1 - i\vec{\theta}\cdot\vec{\tau}$, and one obtains (Problem A.1)

$$A_{\rho b} \to A'_{\rho b} = A_{\rho b} - \frac{1}{g}\partial_\rho\theta_b + \epsilon_{bcd}\theta_c A_{\rho d}\,. \tag{A.8}$$

We are thus led to consider the following Lagrangian:

$$\mathcal{L}_e =: \overline{L}_e\gamma^\rho[i\partial_\rho + g\vec{\tau}\cdot\vec{A}_\rho - \tilde{g}B_\rho]L_e + \overline{R}_e\gamma^\rho[i\partial_\rho - 2\tilde{g}B_\rho]R_e : \tag{A.9}$$

where Y has been replaced by its eigenvalues ($Y_L = -1$ for the doublet and $Y_R = -2$ for the singlet). The Lagrangian density (A.9) is invariant under

the gauge group $U(2) = SU(2)_{\text{iso}} \times U(1)_{\text{hyp}}$. The isoscalar Abelian gauge field B_ρ propagates just like the photon in QED; that is, the Lagrangian density is $-\frac{1}{4}B_{\rho\sigma}B^{\rho\sigma}$, where $B_{\rho\sigma} = \partial_\rho B_\sigma - \partial_\sigma B_\rho$. For the isovector (non-Abelian) gauge field, the Lagrangian density is $-\frac{1}{4}A_{\rho\sigma b}A_b^{\rho\sigma}$, with summation over the $SU(2)$ index, $b = 1, 2, 3$. Here

$$A_{\rho\sigma b} = \partial_\rho A_{\sigma b} - \partial_\sigma A_{\rho b} + g\epsilon_{bcd}A_{\rho c}A_{\sigma d}\,, \tag{A.10}$$

and the inclusion of the last term in this expression is essential in maintaining gauge invariance (Problem A.2). It produces cubic and quartic self-interaction terms in the Lagrangian density.

Now we introduce scalar fields that constitute the 'Higgs sector' of the electroweak theory, which will induce spontaneous symmetry breaking. In the Weinberg-Salam theory, the complex $SU(2)_L$ doublet of complex scalar fields,

$$\phi = \begin{bmatrix} \phi^+ \\ \phi^0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} \phi_1^+ + i\phi_2^+ \\ \phi_1^0 + i\phi_2^0 \end{bmatrix},$$

is defined. The complex field ϕ^+ annihilates charge $+1$ states, the field ϕ^0 charge 0 states. Since the isospin components are $T_3 = +\frac{1}{2}$ for ϕ^+ and $T_3 = -\frac{1}{2}$ for ϕ^0, the weak hypercharge, $Y = 2(Q - T_3)$, is $+1$ for ϕ^+ as well as ϕ^0, i.e., for the Higgs doublet. The scalar Lagrangian density is defined by

$$\mathcal{L}_s = :(\mathcal{D}_\rho\phi)^\dagger(\mathcal{D}^\rho\phi) - \lambda(\phi^\dagger\phi - \mu)^2: \tag{A.11}$$

where couplings of the scalar field to the isovector field $\vec{A}_\rho$ and the isoscalar field B_ρ occur, thanks to the covariant derivative terms,

$$\mathcal{D}_\rho\phi = (\partial_\rho - ig\vec{\tau}\cdot\vec{A}_\rho - i\tilde{g}B_\rho)\phi\,, \tag{A.12}$$

where the eigenvalue $Y_s = 1$ has been inserted. The above equations specify the couplings in the theory, and they reflect the underlying group structure.

A.1 Spontaneous Symmetry Breaking

By way of orientation, we first set every field except ϕ to zero, and consider the pure scalar Lagrangian density

$$\mathcal{L}_s = :(\partial_\rho\phi)^\dagger(\partial^\rho\phi) - \lambda(\phi^\dagger\phi - \mu)^2:$$

Classically, the state of minimum energy, the vacuum, is obtained by minimizing the corresponding Hamiltonian, as a function of the scalar field, ϕ. If $\mu > 0$, the

minimum energy is reached when the scalar field is constant and such that

$$\mu = \phi^\dagger \phi = \tfrac{1}{2}\{\,[\phi_1^+]^2 + [\phi_2^+]^2 + [\phi_1^0]^2 + [\phi_2^0]^2\,\}.$$

We implement this relation by the choice

$$\phi_1^+ = \phi_2^+ = \phi_2^0 = 0 \qquad \phi_1^0 = \sqrt{2\mu}\,.$$

Now imagine that the fields depart just a little from the above values, and for convenience replace the four fields $\phi_1^+, \phi_2^+, \phi_1^0, \phi_2^0$ by four new ones, an isoscalar field Φ_0 and an isovector field $\vec{\Phi}$, according to the parametrization

$$\phi = \frac{1}{\sqrt{2}} \begin{bmatrix} \phi_1^+ + i\phi_2^+ \\ \phi_1^0 + i\phi_2^0 \end{bmatrix} = \frac{1}{\sqrt{2}} M(\vec{\Phi}) \begin{bmatrix} 0 \\ \Phi_0 + \sqrt{2\mu} \end{bmatrix},$$

where M is the two dimensional matrix

$$M(\vec{\Phi}) = \exp\left[\frac{i}{\sqrt{2\mu}} \vec{\tau} \cdot \vec{\Phi}\right].$$

The vacuum corresponds to the vanishing of the fields Φ_0 and $\vec{\Phi}$.

We will express the scalar Lagrangian density in terms of these fields, after making a gauge transformation. Under such a transformation the lepton fields undergo the change

$$\begin{aligned} L_e &\to U L_e U^{-1} = u_L L_e \\ R_e &\to U R_e U^{-1} = u_R R_e\,. \end{aligned}$$

We make the special choice

$$\begin{aligned} u_L &= M^{-1}(\vec{\Phi}) \\ u_R &= 1\,. \end{aligned}$$

The isoscalar field B_ρ remains unchanged, and the isovector field $\vec{A}_\rho$ undergoes the change

$$\vec{\tau} \cdot \vec{A}_\rho \to \vec{\tau} \cdot \vec{A}'_\rho = u_L[\vec{\tau} \cdot \vec{A}_\rho]u_L^{-1} - \frac{i}{g}(\partial_\rho u_L)u_L^{-1}\,.$$

Under this transformation the field ϕ becomes

$$\phi \to u_L \phi = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ \Phi_0 + \sqrt{2\mu} \end{bmatrix}, \tag{A.13}$$

and the fields $\vec{\Phi}$ have disappeared completely in this, the 't Hooft gauge.

After these preliminaries, we return to the scalar Lagrangian density (A.11), now with the gauge fields $\vec{A}_\rho$ and B_ρ restored, but after the gauge transformation that left the scalar field, ϕ, in the simple form (A.13). As we noted, the gauge field B_ρ is unchanged, but $\vec{A}_\rho$ is changed into $\vec{A}'_\rho$. We agree to drop the prime on $\vec{A}'_\rho$, on the understanding that the Lagrangian density is to be written in the new gauge. Since the Lagrangian density is gauge invariant, it is not altered by making a gauge transformation on the fields it contains.

On writing Eq.(A.11) explicitly, in terms of the fields $\vec{A}_\rho$, B_ρ and Φ_0, we find terms which are quadratic, cubic and quartic in the fields, considered together. The quadratic terms belong to the 'free' part of the Lagrangian, the cubic and quartic terms to the 'interaction' part, according to the standard nomenclature of perturbation theory. It turns out that there are, in the quadratic part, besides terms of the sort $B^\rho B_\rho$ and $A_3^\rho A_{\rho 3}$, also crossed terms of the sort $A_3^\rho B_\rho$. To remove these undesirable terms, we introduce the linear combinations of fields,

$$\begin{aligned} A_\rho &= A_{\rho 3}\cos\theta_W + B_\rho \sin\theta_W \\ Z_\rho &= B_\rho \cos\theta_W - A_{\rho 3}\sin\theta_W\,, \end{aligned} \tag{A.14}$$

where the Weinberg angle, θ_W, is defined by

$$\tan\theta_W = \frac{2\tilde{g}}{g}\,.$$

There are no crossed quadratic terms for these new fields, nor for the fields $A_{\rho 1}$ and $A_{\rho 2}$, and we use these to define complex fields:

$$W_\rho = \frac{1}{\sqrt{2}}[A_{\rho 1} + iA_{\rho 2}] \qquad\qquad W_\rho^\dagger = \frac{1}{\sqrt{2}}[A_{\rho 1} - iA_{\rho 2}]\,. \tag{A.15}$$

After all these manipulations, we discover, after some calculation, that the scalar Lagrangian density can be written (Problem A.3)

$$\begin{aligned} \mathcal{L}_s &= \tfrac{1}{2} : (\partial_\rho \Phi_0)(\partial^\rho \Phi_0) - 4\lambda\mu\Phi_0^2 : + \tfrac{1}{4}\mu g^2 : W_\rho^\dagger W^\rho + \frac{Z_\rho Z^\rho}{\cos^2\theta_W} : \\ &+ : \tfrac{1}{4}\lambda\Phi_0^4 - \lambda\sqrt{2\mu}\Phi_0^3 + \tfrac{1}{8}g^2\left[\Phi_0^2 + \sqrt{8\mu}\Phi_0\right]\left[2W_\rho^\dagger W^\rho + \sec^2\theta_W Z_\rho Z^\rho\right] : \end{aligned} \tag{A.16}$$

There are no quadratic cross-terms between the fields W and Z, so that these are indeed physical mass eigenstates. The quadratic terms Φ_0^2, $W_\rho^\dagger W^\rho$ and $Z_\rho Z^\rho$ have just the right form to make the Higgs, the W and the Z particles massive, whereas the photon fortunately remains massless, since there is no term $A_\rho A^\rho$.

A.2 Leptonic Sector

As we see from Eq.(A.9), the lepton fields couple to the gauge fields $\vec{A}_\rho$ and B_ρ. This Lagrangian density can be rewritten in terms of the physical fields, A_ρ, W_ρ and Z_ρ as follows (Problem A.4):

$$\mathcal{L}_e = i : \overline{\psi}_e \gamma^\rho \partial_\rho \psi_e : + i : \overline{\psi}_{\nu_e} \gamma^\rho \partial_\rho P_L \psi_{\nu_e} : - e : J^\rho_{\mathsf{em}} A_\rho : \\ + \frac{e}{\sin 2\theta_W} : J^\rho_{\mathsf{neut}} Z_\rho : + \frac{e}{\sqrt{2}\sin\theta_W} : [J^{\dagger\rho}_{\mathsf{ch}} W_\rho + J^\rho_{\mathsf{ch}} W^\dagger_\rho] : \qquad \text{(A.17)}$$

where the $SU(2)$ coupling constant, g, has been removed in favor of

$$e = g \sin\theta_W \,,$$

so that the coupling of the photon field to the electromagnetic current,

$$J^\sigma_{\mathsf{em}} =: \overline{\psi}_e \gamma^\sigma \psi_{\nu_e} : \qquad \text{(A.18)}$$

is $-e$. In Eq.(A.17), the charged weak current is

$$J^\sigma_{\mathsf{ch}} =: \overline{\psi}_e \gamma^\sigma P_L \psi_{\nu_e} : \qquad \text{(A.19)}$$

which has a purely $V - A$ form, and the neutral weak current is

$$\begin{aligned} J^\rho_{\mathsf{neut}} &= 2\sin^2\theta_W J^\rho_{\mathsf{em}} - : \overline{\psi}_e \gamma^\rho P_L \psi_e : + : \overline{\psi}_{\nu_e} \gamma^\rho P_L \psi_{\nu_e} : \\ &= : \overline{\psi}_{\nu_e} \gamma^\rho P_L \psi_{\nu_e} : - : \overline{\psi}_e [\hat{C}_V - \hat{C}_A \gamma^5] \psi_e : \end{aligned} \qquad \text{(A.20)}$$

where the vector and axial vector coupling factors are respectively given by $\hat{C}_V = \frac{1}{2} - 2\sin^2\theta_W$ and $\hat{C}_A = \frac{1}{2}$. The charged fields W_ρ and $W^\dagger_\rho$ create vector particles that are carriers of the weak force, and they couple to left-handed charged currents only. The field Z_ρ creates vector particles that are also carriers of the weak force, and they couple to neutral weak currents.

So far the electron has remained massless, but a mass term can be produced by adding the following gauge-invariant coupling of the original scalar field ϕ and the electron field:

$$-g_i : [R^\dagger_e \phi^\dagger L_e + L^\dagger_e \phi R_e] :$$

where g_i is a real. For the gauge choice Eq.(A.13), this becomes (Problem A.5),

$$-g_i\sqrt{\mu} : \overline{\psi_e}\psi_e : - \frac{g_i}{\sqrt{2}} : \overline{\psi_e}\psi_e \Phi_0 : \,.$$

We thereby obtain an electron mass $m_e = g_i\sqrt{\mu}$ and an electron-positron-Higgs vertex interaction term.

In terms of the physical field tensors,

$$A_{\rho\sigma} = \partial_\rho A_\sigma - \partial_\sigma A_\rho \qquad W_{\rho\sigma} = \partial_\rho W_\sigma - \partial_\sigma W_\rho \qquad Z_{\rho\sigma} = \partial_\rho Z_\sigma - \partial_\sigma Z_\rho \,,$$

we can write the gauge part of the Lagrangian density as

$$-\tfrac{1}{4} B_{\rho\sigma} B^{\rho\sigma} - \tfrac{1}{4} A_{\rho\sigma b} A_b^{\rho\sigma} = -\tfrac{1}{4} A_{\rho\sigma} A^{\rho\sigma} - \tfrac{1}{4} Z_{\rho\sigma} Z^{\rho\sigma} - \tfrac{1}{2} W^\dagger_{\rho\sigma} W^{\rho\sigma} + \cdots$$

the omitted terms being cubic and quartic in the gauge fields. They represent three-gauge and four-gauge interaction vertices. This gauge part contains no mass terms, but, together with the relevant terms from the scalar Lagrangian density (A.17), we can write the terms quadratic in the gauge fields in the form

$$-\tfrac{1}{4} : A_{\rho\sigma} A^{\rho\sigma} : -\tfrac{1}{4} : Z_{\rho\sigma} Z^{\rho\sigma} : + \tfrac{1}{2} m_Z^2 : Z_\rho Z^\rho : - \tfrac{1}{2} : W^\dagger_{\rho\sigma} W^{\rho\sigma} : + m_W^2 : W_\rho W^\rho :$$

where the masses, m_Z and m_W, can be identified by reference to Eq.(A.17). After spontaneous symmetry breaking and the gauge transformation, the photon and the neutrino remain massless, and the other particles acquire masses as shown in the table:

particle	electron	charged vector	neutral vector	Higgs scalar
mass	m_e	m_W	m_Z	m_0
equivalence	$g_i\sqrt{\mu}$	$g\sqrt{\mu/2}$	$m_W \sec\theta_W$	$2\sqrt{\lambda\mu}$

In this model of the electroweak interaction, gauge invariance was maintained. We have three kinds of fields: leptons, gauge bosons, and a Higgs scalar. There are five parameters in the theory, the gauge couplings, g and $\tilde{g}$, and the Higgs-sector parameters λ, μ and g_i. Equivalently, the five parameters can be taken to be the charge and mass of the electron, as well as the masses of the the W-boson, the Z-boson and the Higgs particle.

There is a second generation of leptons, namely the muon and its neutrino. The whole $SU(2)_L \times U(1)$ theory is simply repeated for a left-handed doublet and a right-handed singlet of muonic leptons. The gauge bosons, γ, W and Z, are the same as those in the electronic sector, and it is via these gauge particles that muons and electrons, and their neutrinos, interact with each other. The Higgs sector is also the same, except that Φ_0 has a stronger coupling to the muon than it does to the electron, say $m_\mu = \tilde{g}_i\sqrt{\mu}$. Muon and electron numbers are separately conserved. A third generation of leptons, the τ and its neutrino, have been discovered, and, by measuring the decay width of the Z gauge boson, it has been determined that there are no more generations (at least, so long as the neutrinos are approximately massless).

A.3 Quark Sector

Next we incorporate the electroweak interactions of quarks into the formalism. We limit our presentation to the four lightest quarks: up, down, strange, and charmed, with respective electric charges $\frac{2}{3}$, $-\frac{1}{3}$, $-\frac{1}{3}$ and $\frac{2}{3}$ times the proton charge. We introduce a linear combination of the down and strange quark fields,

$$\psi_{d_C} = \cos\theta_C\,\psi_d + \sin\theta_C\,\psi_s\,, \tag{A.21}$$

where the Cabibbo angle, θ_C, is determined experimentally to be about 13°. We put the left chiral projections of the up quark field, ψ_u, and the Cabibbo-rotated down field, ψ_{d_C}, into an electroweak $SU(2)_L$ doublet:

$$L_u = \begin{bmatrix} P_L\psi_u \\ P_L\psi_{d_C} \end{bmatrix}.$$

The superposition orthogonal to Eq.(A.21),

$$\psi_{s_C} = -\sin\theta_C\,\psi_d + \cos\theta_C\,\psi_s\,, \tag{A.22}$$

along with the charmed quark field ψ_c, comprises a second left-handed doublet:

$$L_c = \begin{bmatrix} P_L\psi_c \\ P_L\psi_{s_C} \end{bmatrix}.$$

The right-handed projections of all four quark fields are put into $SU(2)_L$ singlets. We may write the quark Lagrangian in the form

$$\begin{aligned} \mathcal{L}_q \;=\; & i:\overline{L}_u\gamma^\rho D_\rho L_u: + i:\overline{L}_c\gamma^\rho D_\rho L_c: + i:\overline{R}_u\gamma^\rho D_\rho R_u: + \\ & i:\overline{R}_d\gamma^\rho D_\rho R_d: + i:\overline{R}_c\gamma^\rho D_\rho R_c: + i:\overline{R}_s\gamma^\rho D_\rho R_s: \end{aligned} \tag{A.23}$$

where $R_u = P_R\psi_u$, and so on. Note that, for the right-handed singlets, rotational symmetry implies $\overline{R}_{d_C}\gamma^\rho D_\rho R_{d_C} + \overline{R}_{s_C}\gamma^\rho D_\rho R_{s_C} = \overline{R}_d\gamma^\rho D_\rho R_d + \overline{R}_s\gamma^\rho D_\rho R_s$, so the Cabibbo rotation has been undone for the singlets. The general form of the covariant derivative, given in Eq.(A.12), provides kinetic energy terms and ensures couplings to the electroweak fields. For the doublets, L_u and L_c, the isospin generator $\vec{t}$ is $\vec{\tau}$, whereas $\vec{t} = 0$ for the right-handed singlets.

We now consider the Higgs-quark coupling, which will produce quark mass terms after spontaneous symmetry breaking. Since there are two left-handed doublets and four right-handed singlets of quark fields, as well as a $Y = 1$ doublet of Higgs scalar fields, we have the following invariant Lagrangian density:

$$\mathcal{L}_a = -G_1\overline{L}_u\phi R_d - G_2\overline{L}_u\phi R_s - G_3\overline{L}_c\phi R_d - G_4\overline{L}_c\phi R_s + \text{h.c.}\,,$$

where 'h.c.' means 'Hermitian conjugate'. This is not the most general invariant form possible, since the transformed scalar field, $\tilde{\phi} = 2i\tau_2\phi^*$, is independent of ϕ and has hypercharge -1 (Problem A.7). So we may add also

$$\mathcal{L}_b = -G_5\overline{L}_u\tilde{\phi}R_u - G_6\overline{L}_u\tilde{\phi}R_c - G_7\overline{L}_c\tilde{\phi}R_u - G_8\overline{L}_c\tilde{\phi}R_c + \text{h.c.}$$

The charge and weak isospin of the various fields are known, and the weak hypercharge is calculated from $Y = 2(Q - T_3)$. As we have already noted, $Y = 1$ for ϕ and $Y = -1$ for $\tilde{\phi}$, and the values for the quark fields are

Quark	T	T_3	Q	Y
$u_L,\ c_L$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{1}{3}$
$d_{Lc},\ s_{Lc}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{3}$	$\frac{1}{3}$
$u_R,\ c_R$	0	0	$\frac{2}{3}$	$\frac{4}{3}$
$d_R,\ s_R$	0	0	$-\frac{1}{3}$	$-\frac{2}{3}$

Note that $Y = 0$ for all terms in $\mathcal{L}_q$, $\mathcal{L}_a$ and $\mathcal{L}_b$, as is required by the gauge invariance of the Lagrangian under the group $U(1)_{\text{hyp}}$. Also, all terms are isospin singlets, as required by invariance under the group $SU(2)_{\text{iso}}$.

After the gauge transformation, the scalar fields reduce to the form Eq.(A.13), and the vacuum point is given by

$$\phi_V = \sqrt{\mu}\begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad , \quad \tilde{\phi}_V = \sqrt{\mu}\begin{bmatrix} 1 \\ 0 \end{bmatrix} .$$

This leads to quadratic mass contributions in $\mathcal{L}_a$ and $\mathcal{L}_b$, and in general there will be unwanted cross-terms between different quark fields. These cross-terms are eliminated by choosing the couplings to satisfy the relations

$$\sqrt{\mu}\begin{bmatrix} \cos\theta_c & -\sin\theta_c \\ \sin\theta_c & \cos\theta_c \end{bmatrix}\begin{bmatrix} G_1 & G_2 \\ G_3 & G_4 \end{bmatrix} = \begin{bmatrix} m_d & 0 \\ 0 & m_s \end{bmatrix}$$

$$\sqrt{\mu}\begin{bmatrix} \cos\theta_c & -\sin\theta_c \\ \sin\theta_c & \cos\theta_c \end{bmatrix}\begin{bmatrix} G_5 & G_6 \\ G_7 & G_8 \end{bmatrix} = \begin{bmatrix} m_u & 0 \\ 0 & m_c \end{bmatrix}$$

so that the quark mass terms are diagonal:

$$\mathcal{L}_a + \mathcal{L}_b = -m_u : \overline{\psi}_u\psi_u : -m_d : \overline{\psi}_d\psi_d : -m_c : \overline{\psi}_c\psi_c : -m_s : \overline{\psi}_s\psi_s : + \cdots$$

where the omitted terms are interactions between the quarks and the Higgs particle. With the same omission, we find (Problem A.7)

$$\begin{aligned}\mathcal{L}_a + \mathcal{L}_b + \mathcal{L}_q &= \sum_q : \overline{\psi}_q(i\gamma\partial - m_q)\psi_q : + \frac{e}{\sqrt{2}\sin\theta_W} : [J^{\dagger\rho}_{\text{ch}} W_\rho + J^\rho_{\text{ch}} W^\dagger_\rho] : \\ &\quad + \frac{e}{\sin 2\theta_W} : J^\rho_{\text{neut}} Z_\rho : + e : J^\rho_{\text{em}} A_\rho : + \cdots \qquad \text{(A.24)}\end{aligned}$$

The quark sum, q, extends over u, d, c, s, and physical gauge fields have been introduced through Eqs.(A.14)–(A.15). The (vector) electromagnetic current is

$$J^\rho_{\text{em}} = \tfrac{2}{3}\overline{\psi}_u\gamma^\rho\psi_u - \tfrac{1}{3}\overline{\psi}_d\gamma^\rho\psi_d + \tfrac{2}{3}\overline{\psi}_c\gamma^\rho\psi_c - \tfrac{1}{3}\overline{\psi}_s\gamma^\rho\psi_s \qquad \text{(A.25)}$$

The charged weak current (V−A as expected) is

$$J^\rho_{\text{ch}} = \overline{\psi}_d\gamma^\rho P_L[\psi_d\cos\theta_C + \psi_s\sin\theta_C] + \overline{\psi}_c\gamma^\rho P_L[-\psi_d\sin\theta_C + \psi_s\cos\theta_C]\,. \qquad \text{(A.26)}$$

The weak neutral current has the form

$$\begin{aligned}J^\rho_{\text{neut}} &= -\sin^2\theta_W J_{\text{em}} + \overline{\psi}_u\gamma^\rho P_L\psi_u - \overline{\psi}_d\gamma^\rho P_L\psi_u + \overline{\psi}_c\gamma^\rho P_L\psi_c - \overline{\psi}_s\gamma^\rho P_L\psi_s \\ &= \overline{\psi}_u\gamma^\rho(C_V - \gamma^5 C_A)\psi_u - \overline{\psi}_d\gamma^\rho(\overline{C}_V - \gamma^5\overline{C}_A)\psi_d + \\ &\quad \overline{\psi}_c\gamma^\rho(C_V - \gamma^5 C_A)\psi_c - \overline{\psi}_s\gamma^\rho(\overline{C}_V - \gamma^5\overline{C}_A)\psi_s \qquad \text{(A.27)}\end{aligned}$$

where

$$C_V = \tfrac{1}{2} - \tfrac{4}{3}\sin^2\theta_W \qquad \overline{C}_V = \tfrac{1}{2} - \tfrac{2}{3}\sin^2\theta_W \qquad C_A = \tfrac{1}{2} = \overline{C}_A\,.$$

The full interaction between the fermions and the gauge particles can be expressed by adding electronic, muonic, and quark contributions. A compact expression is obtained by introducing left-handed weak-isospin currents, namely

$$J^\rho_{L,k} = 2\{\,\overline{L}_e\gamma^\rho\tau_k L_e + \overline{L}_\mu\gamma^\rho\tau_k L_\mu + \overline{L}_u\gamma^\rho\tau_k L_u + \overline{L}_c\gamma^\rho\tau_k L_c\}\,,$$

for $k = 1, 2, 3$. The full charged weak current is

$$J^\rho_{\text{ch}} = \tfrac{1}{2}J^\rho_{L,+} = \tfrac{1}{2}\{\,J^\rho_{L,1} + iJ^\rho_{L,2}\}\,.$$

The full electromagnetic current is

$$J^\rho_{\text{em}} = \sum_j Q_j\,\overline{\psi}_i\gamma^\rho\psi_i\,,$$

where the index j runs over all the particles, e, μ, u, d, s, c, and where Q_j is the electric charge of the particle in units of $|e|$. The neutral weak current has the elegant form

$$J^\rho_{\text{neut}} = J^\rho_{L,3} - 2\sin^2\theta_W J_{\text{em}}\,.$$

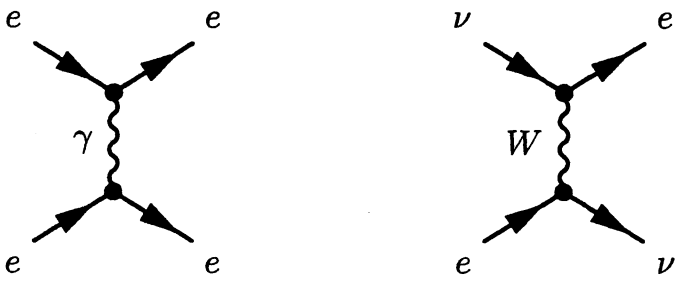
A.4 Experimental Tests

Let us first consider $O(e^2)$ Feynman diagrams for electron-electron, and for electron-neutrino scattering:

$$e + e \to e + e \qquad\qquad e + \nu \to e + \nu\,.$$

Instead of giving the momenta of the lines, as we did in Chapter 7, we indicate here only the species of particle in question:

Figure A.1



The first diagram is driven by the electromagnetic part of the interaction, the coupling between the photon field and the electromagnetic current being $-e$. The contribution of the left-hand diagram can be written

$$e^2 J^{\rho}_{\mathsf{em}} \left\{ \frac{-g_{\rho\sigma}}{k^2} \right\} J^{\sigma}_{\mathsf{em}}\,,$$

k being the momentum of the virtual photon that is exchanged between the electron lines. The photon propagator has been given in the Feynman gauge, $a = 1$, and the currents are to be understood in momentum space, which means that they reduce simply to Dirac γ matrices, enclosed by the appropriate spinors. The corresponding expression for electron-neutrino scattering is

$$\frac{e^2}{2\sin^2\theta_W} J^{\rho}_{\mathsf{ch}} \left\{ \frac{-g_{\rho\sigma}}{k^2 - m_W^2} \right\} J^{\dagger\sigma}_{\mathsf{ch}}\,,$$

where the charged weak current was given in configuration space in Eq.(A.19). In momentum space it is

$$\gamma^{\rho} P_L = \tfrac{1}{2}\gamma^{\rho}(1 - \gamma_5)\,,$$

sandwiched again by spinors. The propagator of the W boson has been written in the Feynman gauge, m_W being the mass that was engendered by spontaneous symmetry breaking. For values of k^2 much smaller than m_W^2, this reduces

effectively to the current-current iteraction

$$\frac{e^2}{2m_W^2 \sin^2\theta_W} J^\rho_{\text{ch}} J^{\dagger\rho}_{\text{ch}} \,. \tag{A.28}$$

The old Fermi interaction, in the form due to Feynman and Gell-Mann, was

$$\frac{G}{\sqrt{2}} \overline{\psi}_\nu \gamma^\rho (1-\gamma_5)\psi_e \overline{\psi}_e \gamma^\rho (1-\gamma_5)\psi_\nu \tag{A.29}$$

where G is the low-energy, or Fermi weak interaction coupling constant, which is known experimentally. On comparing this with Eq.(A.28), we find

$$m_W = \frac{e}{2^{\frac{5}{4}} G^{\frac{1}{2}} \sin\theta_W} \approx \frac{38\,\text{GeV}}{\sin\theta_W} \,.$$

The weak interaction is so much feebler at low energies than the electromagnetic interaction because the W boson mass is so great.

Two points should be made in connection with the above discussion. In Møller scattering, there are actually two diagrams of the form shown at the left of Figure A.1, corresponding to the exchange of the external electron lines, as we saw in Sec. **7.3**. In electron-neutrino scattering, one can also exchange the final particles, but one must then replace the virtual W boson by a virtual Z particle, as shown below

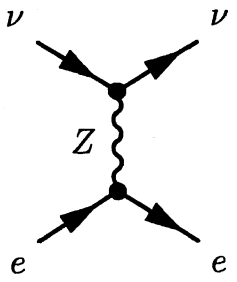


Figure A.2
Neutral current contribution to electron-neutrino scattering

The contribution of this diagram can be written

$$\frac{e^2}{\sin^2 2\theta_W} J^\rho_{\text{neut}} \left\{ \frac{-g_{\rho\sigma}}{k^2 - m_Z^2} \right\} J^{\dagger\sigma}_{\text{neut}} \,.$$

In the low energy limit, recalling that $m_Z = m_W \sec\theta_W$, we find

$$\frac{e^2}{4m_W^2 \sin^2\theta_W} J^\rho_{\text{neut}} J^{\dagger\rho}_{\text{neut}} \,. \tag{A.30}$$

With the help of Eq.(A.20), we can write this more explicitly as

$$-\frac{G}{\sqrt{2}}\overline{\psi}_\nu\gamma^\rho(1-\gamma_5)\psi_\nu\overline{\psi}_e\gamma^\rho(\hat{C}_V-\hat{C}_A\gamma_5)\psi_e\ , \tag{A.31}$$

where we have included only the cross-terms, since the others do not contribute to the reaction depicted in Figure A.2. This neutral current term must be added to the charged current term, and we can do that most expeditiously by recalling the Fierz transformation (Problem 3.7) and applying it to Eq.(A.29), thereby casting the charged current term into the equivalent neutral current form,

$$\frac{G}{\sqrt{2}}\overline{\psi}_\nu\gamma^\rho(1-\gamma_5)\psi_\nu\overline{\psi}_e\gamma^\rho(1-\gamma_5)\psi_e\ . \tag{A.32}$$

The sum of the Z term (A.31) and the W term (A.32) is

$$\frac{G}{\sqrt{2}}\overline{\psi}_\nu\gamma^\rho(1-\gamma_5)\psi_\nu\overline{\psi}_e\gamma^\rho(\overline{C}_V-\overline{C}_A\gamma_5)\psi_e\ , \tag{A.33}$$

where

$$\overline{C}_V=1-\hat{C}_V=\tfrac{1}{2}+2\sin^2\theta_W \qquad\qquad \overline{C}_A=1-\hat{C}_A=\tfrac{1}{2}\ .$$

The amplitude (A.33) can be used to calculate the scattering cross-section for electron-neutrino scattering. Note that, if the old Fermi-Feynman-Gell-Mann theory had been correct, or if only the W boson existed, then we would have had $\overline{C}_V=1=\overline{C}_A$. Thus the above expressions constitute a definite prediction of the electroweak theory, which has been tested with success by analyzing the results of electron-neutrino scattering.

An even more striking success of the electroweak theory is its application to the scattering of electrons by muon neutrinos. The latter are produced in the decay of pions ($\pi^+\to\mu^++\nu_\mu$). To order e^2, the scattering is wholly due to exchange of the neutral boson, Z, for the right-hand part of Figure A.1 is now forbidden, due to the separate conservation of electron and muon number. Hence the occurrence of this scattering process, at the calculated rate, is a very clean test of the standard model, which, as we have seen, requires the existence of the neutral gauge boson. Experimentally, it is found that the scattering does take place as predicted. A comparison of muon and electron neutrino and antineutrino scattering on electrons leads to estimates of the Weinberg angle, θ_W, and hence, from the measured value of the Fermi constant, G, of the masses m_W and m_Z. This work was done in the 1970's, and the values obtained were 0.23 for the Weinberg angle, and for the masses, $m_W=79\pm8$ GeV/c^2 and $m_Z=90\pm9$ GeV/c^2.

On 20 January 1983, it was reported at CERN that W bosons had been created as resonances in proton-antiproton scattering; and on 1 June of the same year, it was announced that the Z boson had been found. The values quoted by the Particle Data Group in 2002 were

$$\begin{aligned} \sin^2\theta_W &= 0.2315 \pm 0.0002 \\ m_W &= 80.42 \pm 0.06\,\mathrm{GeV}/c^2 \\ m_Z &= 91.188 \pm 0.002\,\mathrm{GeV}/c^2\,. \end{aligned}$$

In the original form of the quark sector of the theory, only three quarks, u, d and s were taken into account, and the theory predicted that the neutral gauge boson, Z, should couple to strangeness changing currents of the form

$$J^{\rho}_{\text{change}} = \overline{\psi}_d \gamma^{\rho}(\check{C}_V - \gamma^5 \check{C}_A)\psi_s + \text{h.c.}\,,$$

as well as strangeness conserving currents (Problem A.9). However, this implied the possible decay, to lowest order in the weak coupling, of strange particles into particles without strangeness, for example, the neutral kaon was predicted to decay into a muon pair. This decay mode, and other reactions involving a strangeness-changing neutral weak current, were not observed experimentally. In 1970, Glashow, Iliopolous and Maiani took the bold and brilliant step of postulating the existence of a fourth, then unsuspected fourth quark, c, with a new quantum number, charm. Because of the symmetry of the right-handed chiral part of $\mathcal{L}_q$ under the Cabibbo rotation, Eq.(A.23), there are no crossed terms like that in Eq.(A.34), the unwanted term being suppressed in a natural and elegant manner.

The hypothesized new quark was introduced to solve a problem, so it cannot be claimed that the absence of strangeness-changing, neutral weak interactions verifies the existence of the charmed quark. However, its postulated existence implies the existence of new physical particles, namely bound states including charmed quarks. The first such state was observed in 1974, the ground state of charmonium, a bound state of c and its antiparticle, $\overline{c}$, analogous to positronium. This state, and many others that have been found since 1974, in particular bound states $c\overline{u}$, $c\overline{d}$ and $c\overline{s}$, as well as charmed baryonic states, constitute indeed a genuine verification of the four-quark theory of Glashow, Iliopolous and Maiani.

The major lacuna in the experimental verification of the electroweak theory is that the predicted Higgs scalar had not been observed, as of 2002. Many hope that it will be detected by what may prove to be the ultimate particle accelerator, LHC at CERN, before the end of the first decade of the 21st century.

A.5 Exercises

Problem 1
Given that

$$\vec{\tau} \cdot \vec{A}'_\rho = u_L \left[\tau \cdot \vec{A}_\rho - \frac{i}{g} u_L^{-1} (\partial_\rho u_L) \right] u_L^{-1} ,$$

where $u_L = \exp[-i\vec{\tau} \cdot \vec{\theta}\,]$, show that

$$A'_{\rho b} = A_{\rho b} - \frac{1}{g} \partial_\rho \theta_b + \epsilon_{bcd} \theta_c A_{\rho d} + \mathrm{O}(\theta^2) .$$

Problem 2
Given that

$$A_{\rho\sigma b} = \partial_\rho A_{\sigma b} - \partial_\sigma A_{\rho b} + g \epsilon_{bcd} A_{\rho c} A_{\sigma d}$$

show that $A_{\rho\sigma b} A_b^{\rho\sigma}$ is gauge invariant.

Problem 3
Express the scalar Lagrangian,

$$\mathcal{L}_s = : (D_\rho \phi)^\dagger (D^\rho \phi) - \lambda (\phi^\dagger \phi - \mu)^2 :$$

in terms of the physical fields W^ρ, Z^ρ and Φ_0. Obtain expressions for the masses associated with these fields.

Problem 4
Express the leptonic Lagrangian,

$$\mathcal{L}_e = i : \overline{L_e} \gamma^\rho [\partial_\rho - ig\vec{\tau} \cdot \vec{A}_\rho + i\tilde{g} B_\rho] L_e : + i : \overline{R_e} \gamma^\rho [\partial_\rho + 2i\tilde{g} B_\rho] R_e :$$

in terms of the physical fields and currents.

Problem 5
Express the interaction Lagrangian,

$$\mathcal{L}_i = -g_i : [\overline{R}_e \phi^\dagger L_e + \overline{L}_e \phi R_e] :$$

in terms of the physical fields ψ_e and Φ_0.

Problem 6

Express the gauge boson Lagrangian,

$$\mathcal{L}_g = -\tfrac{1}{4}B_{\rho\sigma}B^{\rho\sigma} - \tfrac{1}{4}A_{\rho\sigma b}A_b^{\rho\sigma}$$

in terms of the physical fields A^ρ, W^ρ, Z^ρ. Calculate the three-gauge and four-gauge interaction terms explicitly. What extra terms are needed to generate masses for the W and Z fields?

Problem 7

Given that ψ is an electroweak isospinor, show that $\tilde{\psi} = i\tau_2\psi^*$ is an independent electroweak isospinor. Express the sum of the quark-gauge-vector and Higgs-quark Lagrangians in terms of physical fields.

Problem 8

Set up the $SU_L(2)\otimes U(1)$ electronic part of the electroweak Lagrangian, allowing for the fact that the electron neutrino has a small mass.

Problem 9

Show that, in the absence of the charmed quark, strangeness-changing neutral weak currents must occur in lowest order.

Problem 10

Generalize the electroweak theory by assuming that there are right-handed, as well as left-handed weak currents, and that the Lagrangian is invariant under parity reversal. Take as the gauge group $SU(2)_L \otimes SU(2)_R \otimes U(1)$, and explain how spontaneous symmetry breaking can give rise to parity breakdown in this theory. How might one test this theory experimentally?

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