Advanced Quantum Gauge Field Theory

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Preface

Modern quantum field theory for gauge theories should be based on path integrals and BRST symmetry, and not (in first instance at least) on Feynman graphs and operator methods. This is the point of view which forms the basis for this book. Renormalizability and unitarity of QED, QCD and electroweak gauge theory, background field methods, and anomalies will all be discussed by using Ward identities and functional methods which follow from path integrals and BRST symmetry.

Quantum gauge field theory is a vast subject, and only by both working out general ideas in concrete examples, and explaining concrete problems by placing them in a general context, can one begin to understand this enormous edifice. Therefore, it is equally important to reach the level of Feynman graphs, and to work out concrete problems in gauge field theory and applications to particle physics.

We shall give both detailed derivations of path integrals and their symmetry properties, but also discussions of regularization issues of chiral gauge theories and infrared divergences in QED and QCD. We prove unitarity by using cutting rules for Feynman graphs and simplify Feynman graph calculations by using background field methods. We even present some ongoing experiments which test the Standard Model. We have incorporated into our presentation of quantum gauge field theory some new concepts which were developed under the stimulus of string theory but we do not discuss string theory. We discuss supersymmetric gauge theories because they give much insight into gauge theory in general, but we do not discuss supersymmetric phenomenology. We shall discuss several subjects which usually are not covered in textbooks. Our hope is that new graduate students, those specializing in string theory as well as those engaged in higher-loop calculations or parton distribution functions or modern nuclear physics or modern statistical mechanics, will enjoy this broader outlook as much as the author who taught these subjects for two and a half decades.

Chapter 1

A brief history of quantum gauge field theory

All fundamental interactions between particles except gravitation are nowadays very well described by quantum gauge field theories: the electroweak theory and quantum chromodynamics. The principles of gauge invariance and Lorentz invariance, together with the choice of gauge groups and some discrete symmetries, and the requirement of renormalizability, determine all interactions up to the numerical value of the coupling constants. Masses are due to spontaneous gauge symmetry breaking and are determined by the coupling constants of the Yukawa interactions. Renormalizability excludes particles with spins larger than one, and only admits minimal gauge interactions introduced by covariant derivatives $D_\mu = \partial_\mu + gA_\mu^a T_a$, but not, for example, Pauli couplings of the form $g\bar{\psi}\gamma^{\mu\nu}\psi F_{\mu\nu}$. Particles belong to multiplets which form representations of semisimple nonabelian Lie algebras with generators T_a, and for a given internal symmetry group each multiplet couples with the same nonabelian coupling constant g. Electromagnetism is at present the only abelian gauge theory known to exist in nature, and particles couple with their electric charge as coupling constant. The result is the Standard Model (perhaps one might even call it the Standard Theory) with 27 free parameters: the 6 quark masses, the 3 charged lepton masses, the 3 gauge coupling constants, the 4 quark mixing parameters, the Higgs mass and its vacuum expectation value, and further the 3 neutrino masses, the 4 neutrino mixing parameters, the θ angle of QCD, and, to describe classical gravity, the cosmological constant.

Path integrals are used to describe quantum effects, and the quantum action for these gauge theories (the action which appears in the path integral) is obtained by adding a gauge fixing term and a ghost action to the classical gauge action, which also contains a matter part with scalars and spinors coupled to gauge fields and to themselves. There are powerful formal arguments which explain why the quantum action has this form, in particular the observation that it has a rigid symmetry which is an extension of classical gauge symmetry to the quantum level called BRST symmetry (a kind of "quantum gauge symmetry"). Furthermore, there are very good reasons for using path integrals: they allow one to describe nonperturbative as well as perturbative physics and yield simple and general methods to deduce the consequences of BRST symmetry for the correlation functions (the Ward-Takahashi-Slavnov-Taylor identities, called Ward identities in this book).

However, in this chapter we shall follow the historical path and recall the experimental indications and theoretical developments which led to these gauge theories and this structure of the quantum action. This will explain how the concept of gauge theories grew out of studies of gravity, why the nongravitational gauge groups are $SU(3)$ and $SU(2)$ and $U(1)$, why the gauge fields of the weak interactions couple to chiral fermions, why also the nuclear forces are described by gauge theory, why this gauge theory couples quarks to gluons, and why one moved from canonical quantization and the Schrödinger equation in the Coulomb or temporal gauge, via covariant quantization with Heisenberg fields, to the manifestly relativistic path integral approach. In the next chapter we shall introduce BRST symmetry.

One may divide the history of particle physics [1,2] and the development of quantum gauge field theory into three periods:[1]

(i) "**the birth of particle physics**" from 1926–1950 [3], beginning with the quantization of the electromagnetic field, and ending with the renormalization of QED. Early on, the Dirac equation led to the prediction of antiparticles, and within a year the positron was discovered.[2] In this period also Fermi's nonrenormalizable four-fermion vector theory of the weak interactions appeared. Yukawa introduced a new heavy particle (the later pion) to describe the nuclear forces (and, less successfully, the weak forces). The neutrino, neutron, and pion were proposed, but instead the neutron, muon and pion were discovered. At the end of this period, the magnetic radiative corrections to QED explained the observed anomalous magnetic moment of electrons in atoms, and the electric radiative corrections to QED explained the Lamb shift of energy levels which should be degenerate according to quantum mechanics and the Dirac equation (see the appendix). As a result, abelian quantum gauge field theory enjoyed a few years of glory;

(ii) "**from pions to quarks**", the period from 1950 to the late 1960's [9]. This was initially a confusing period but it led to particle physics in its modern form. New hadronic particles were discovered in cosmic ray experiments: first the "V particles" (the $K^{\pm}$ (494) and the Λ (1116)), so called because their tracks formed a V or an inverted V, and then the $\Sigma^{\pm}$ (1190) and Ξ^{-} (1320) "hyperons" (baryons heavier than

[1] This division suits our purposes, but is, of course, arbitrary. For example, Marshak [2] considers the period from Newton to 1945 as early particle physics, and divides the period of modern particle physics into fifteen-year periods, the "startup period" (1945-1960), the "heroic period" (1960-1975), and the "period of consolidation and speculation" (1975-1990). The period from 1970 till present has been called the period of super-physics, but any characterization of any period by one word is perhaps somewhat superficial.

[2] Before this period 3 elementary particles had already been discovered: the electron by J.J. Thomson in 1897 (or 1899) [4], the proton by Rutherford (and van den Broek) in 1911 (or 1913) [5], and the photon by Compton in 1923. [6] The next elementary particles were discovered in one remarkable year, the "annus mirabilis" 1932: the deuteron (an elementary particle in nuclear physics) in January, the neutron by Chadwick in February [7] and the positron by Anderson in August [8]. (The neutrino had already been proposed by Pauli in 1930 but was only discovered in the 1950's).

nucleons) [10].

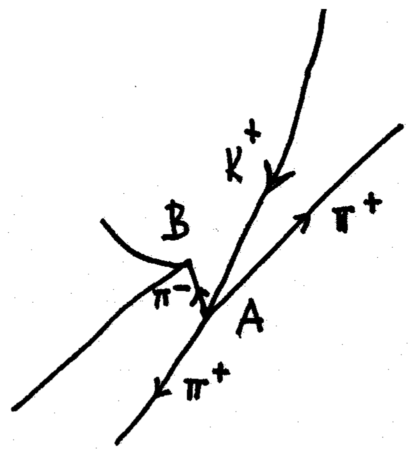


Figure 1: A V-particle. The K^+, entering from above, decays at $A : K^+ \to \pi^+ + \pi^+ + \pi^-$. (The π^- subsequently causes a nuclear disintegration at B). For a typical picture of Λ decay, see figure 17.

In subsequent accelerator experiments many more particles were discovered: first the spin 3/2 pion-nucleon resonance Δ (1232) by Fermi at the Chicago proton cyclotron in 1952, then the Σ^0 (1190) and Ξ^0 (1320) at the Brookhaven "Cosmotron" (a proton synchrotron), and in the early 1960's at the Brookhaven and Berkeley proton synchrotrons the pion-kaon resonance K^* (890), the pion-pion resonance ρ (770), and the three-pion resonances η (550) and ω (780). In 1962 the φ (1020) was discovered at the "Alternating gradient synchrotron" at Brookhaven, and in 1964 the η' (958) was discovered both at Brookhaven and at Berkeley.[3] Theorists and experimentalists were equally baffled by the large amount of data without any theoretical framework to describe it. Nonrenormalizable "mesotron field theories" with derivative couplings were proposed to describe these new particles. The V-particles were copiously produced if there was enough energy available, but they took a very long time to decay. This was explained by introducing a new quantum number S that is conserved by the strong interactions, but not the weak interactions, called strangeness.[4] The weak

[3]The η' is much heavier than the other 8 pseudoscalar mesons of this nonet, because it is not a Goldstone boson for the broken axial $U(1)$ symmetry; rather, the divergence of the corresponding axial vector current is equal to the instanton density $F_{\mu\nu}{}^*F^{\mu\nu}$.

[4]According to this selection rule, a single strange particle could only decay by means of the weak

decays of strange particles were found to satisfy certain selection rules, the nonleptonic $\Delta I = 1/2$ rule, and the leptonic $\Delta Q = \Delta S$ rule. (The latter was later explained by the weak interactions of quarks in the Standard Model.) A rigid $SU(3)$ flavor symmetry brought some order into this chaos by grouping hadrons into multiplets. Its realization in terms of quarks was successful in explaining and predicting some properties of the new mesons and baryons. A new degree of freedom (rigid $SU(3)$ color symmetry) was found to be needed to prevent the spin-statistics relation for hadrons from breaking down. The electron neutrino was discovered in a reactor experiment. Intermediate vector bosons for the weak interactions (the later W and Z bosons) were proposed. Following the discovery of parity violation in 1956, the universal four-fermion $V - A$ theory of weak interactions was constructed as a refinement of Fermi's vector theory. Baryons and leptons formed chiral currents which coupled to each other and to themselves, and this theory fully agreed with experiments. Absence of the decay $\mu \to e\gamma$ suggested that muon-number and electron-number are separately conserved, and this required the existence of a second neutrino, the muon neutrino, which was subsequently discovered. At the theoretical front a momentous development took place: a gauge theory for nonabelian gauge groups was proposed by Yang and Mills in 1954, but it was not clear whether it could describe massive particles and its renormalizability posed a formidable problem. Field theory was in decline, and alternative approaches, first and foremost the S-matrix program and Regge theory, were proposed as alternatives;

(iii) "**the Rise of the Standard Model**" in the late 1960's and 1970's [11]. The renormalization of nonabelian gauge theory in general was achieved. A perturbatively renormalizable theory of the electroweak interactions with W and Z gauge fields was

interactions, but pairs of V particles with opposite strangeness could be easily produced by the strong interactions. This concept of "associated production" was introduced by A. Pais. M. Gell-Mann introduced the quantum number strangeness; assuming it to be preserved by the strong and electromagnetic interactions but not the weak interactions, he could explain in more detail the slow decay of strange particles [12]. The empirical Nakano-Nishijima-Gell-Mann relation of 1955 [13] related the electric charge Q of hadrons to their isospin, strangeness and baryon number by $Q = I_3 + \frac{1}{2}Y, Y = S + B$.

constructed based on a particular nonabelian gauge field theory with chiral couplings to fermions. The corresponding W and Z bosons acquired a mass without breaking the gauge invariance of the action by the Higgs mechanism which uses spontaneous symmetry breaking in gauge theories. Absence (or at least strong suppression) of flavour-changing neutral currents could be explained by postulating the existence of a charmed quark, which was indeed in due time discovered. A new lepton was found, the τ lepton. Absence of chiral anomalies (needed for the renormalizability of the electroweak gauge theory) required two new quarks, the top and bottom quark. The bottom quark was soon discovered. QCD as the field theory of another nonabelian gauge theory based on $SU(3)$ color symmetry was proposed, and also shown to be renormalizable. Asymptotic freedom implied that perturbation theory made sense for the strong interactions at high energies. By the end of the 1970's the Standard Model, as we know it today, was established, and quantum field theory had been welcomed back. In the 1980's and 1990's the picture was completed: the W and Z bosons, the top quark and the τ neutrino were found. For the Standard Model only the Higgs boson remains to be detected.

Although these three periods cover the development of particle physics and gauge field theory, we should add a fourth period which preceded them, and in which studies of gravitation and electromagnetism paved the way for gauge field theory:
(iv) "**the dawning of gauge theory**" from 1915–1930 [14]. Striking analogies between gravitation and electromagnetism were noticed. Several ingenious attempts at unifying gravity with electromagnetism were proposed. The Kaluza–Klein–Fock program offered a spacetime explanation of internal symmetries. The relation between conservation laws and symmetries was established. Most importantly, the concept of gauge theories was proposed by Weyl as a result of his attempts to arrive at a unified geometrical description of gravity and electromagnetism. The connection which was used by Einstein for parallel transport in general relativity, was first generalized by Weyl to contain the electromagnetic vector potential A_μ, but later he dropped

this relation to the electromagnetic field, and identified the connection for parallel transport in general relativity with the gauge field for local Lorentz gauge symmetry. The potential A_μ was identified as an independent gauge field for electromagnetism. At the end of this period it was realized that gravity and electromagnetism are both gauge theories, each with its own connection; connections were reinterpreted as gauge fields which couple to matter in a minimal way, through covariant derivatives.

Writing a history of scientific events (even a brief history) is fraught with problems. One practical problem is that students are usually not interested in history, but rather prefer "getting on with the job" and doing research using the modern approach. They should turn to chapter 2 at this point. The author had the same attitude when he was a student, but during a life devoted to a new type of gauge field theories he got curious how fundamental problems were solved in the past. The short history that follows is thus written from the point of view of a practicing researcher who assesses the past, rather than a historian of science, with all obvious advantages and disadvantages. As a researcher, one is interested as much in the struggles, mistakes and false leads of the past as in the smooth presentations of contemporary textbooks, but these alternatives are not as fully developed and may confuse the reader. A related problem is that the way yesterday's discoveries appear in the original articles is often quite different from the way they are nowadays presented; sometimes one can only understand the reasoning by projecting it onto today's framework of thinking. As a consequence, modern derivations and modern approaches often seem clearer and more complete then the original works, but some interesting ideas may have been lost. Often the notation and symbols used then are quite different from those we use nowadays. Also discoveries were (and are) sometimes made before the problems which they solve had been clearly recognized. In addition, a new idea often appears in the original literature as part of a whole series of interconnected arguments, and posterity has only extracted and developed that part which seems most relevant or correct. Looking back in time at the great discoveries of the past, one is apt to focus on the peaks honored by Nobel prizes, and forget the valleys (and subpeaks called collaborators) which connect these peaks and which are also full of interesting physics and physicists. The modern edifice of particle physics and quantum field theory is an end-product which by itself gives little insight into why and how it was constructed, but if one studies the original literature one discovers long-forgotten articles which fill in the gaps and slowly the foundations of the modern edifice are revealed. In more recent times, physicists have written accounts, or books have appeared based on interviews with physicists, which contradict accounts by other physicists or other books. As time goes on, one reads and learns more and more, sometimes from the people directly involved, but it is perhaps impossible to discover the whole truth; in fact, there may be more than one truth in more than one mind. Apart from the difficult task of attributing discoveries correctly to the right people (a hazardous task in view of the strong emotions and personalities involved), one runs the risk of looking with today's eyes at yesterday's discoveries, and by doing so, not seeing the alternative developments which also took place and the confusion which reigned at that time. If one begins to study some of these alternatives, they seem less logical today because they do not fit in as well with the current way of thinking, but that may be because, as has been said [15], the current way of thinking was developed by the victorious. Even today, alternative approaches are developed all the time, and, as Feynman has pointed out in his Nobel lecture, although they may seem to be mathematically equivalent, some may be more useful than others for future developments.

We shall trace those historical developments which have led to the current description of Nature in terms of a renormalized abelian and nonabelian quantum gauge field theory and the Standard Model. Sometimes we shall interrupt the historical account with critical assessments based on the modern point of view about quantum gauge field theory. We have organized the material as a series of logical developments by interested happy physicists. This makes for pleasant reading, but the reality was very different. It took protracted struggles by many of the leading physicists of the last century to create these new concepts. Yet, the result is one of mankind's great achievements.

1 QED

After Heisenberg and Schrödinger had written down the matrix approach and the wave function approach to quantum mechanics in 1925 and early 1926, respectively, and Schrödinger and Dirac had shown that they were equivalent, the problem arose how to apply quantum mechanics to the electromagnetic field. Dirac was able in 1926 to reproduce Einstein's B–coefficient for induced absorption and emission of radiation by atoms in thermal equilibrium with the electromagnetic field, but although he used quantum mechanics for atoms, he still treated the electromagnetic field classically (semiclassical radiation theory) [16]. However, early in 1927 he constructed a theory of spontaneous emission of radiation and was able to compute also Einstein's A–coefficient [17]. Hamiltonian believer that he was, he began with $(\vec{p}c - e\vec{A})^2 + (mc^2)^2 = (E - e\phi)^2$ and took the square root to obtain the Hamiltonian $H = E$ for the Schrödinger equation of a charged point particle without spin in an electromagnetic field.[5] Expanding the square root to first order, and using transverse vector potentials, he found a nonrelativistic interaction term $H_{\text{int}} = -\frac{e}{c}\vec{A} \cdot \vec{p}/m = -\frac{e}{c}\vec{A}(\vec{x},t) \cdot \frac{d\vec{x}}{dt}$. Next he expanded the free transverse classical electromagnetic vector potential $\vec{A}(\vec{x},t)$ into a sum of plane waves whose coefficients he took as operators. Instead of the nowadays standard annihilation and creation operators a_r and $a_r^\dagger$, he used number operators N_r and phase operators θ_r, where r denoted the momentum and transversal polarization. In order that $|a_r|^2 = a_r^\dagger a_r$

[5] Schrödinger and others had already in 1926 proposed a "relativistic Schrödinger equation" without spin, given by $\{(\vec{p}c - e\vec{A})^2 + (mc^2)^2 - (E - e\phi)^2\}u = 0$ for a one-component wave function u. [18] Spin had been introduced by Goudsmit and Uhlenbeck in 1925. They proposed that electrons have a gyromagnetic ratio $g = 2$ instead of the classical value $g = 1$ to fit experimental data on the anomalous Zeeman splitting of spectral lines [19]. We discuss these issues in more detail in the Appendix.

be equal to the number N_r of particles with label r, he set $a_r = e^{-i\theta_r/\hbar}\sqrt{N_r}$ and also $a_r^\dagger = \sqrt{N_r}e^{i\theta_r/\hbar}$. Then $a_r^\dagger a_r = N_r$ but $a_r a_r^\dagger = e^{-i\theta_r/\hbar}N_r e^{i\theta_r/\hbar}$. One might at this point be inclined to assume that N_r and θ_r are hermitian operators, which are canonically conjugate variables. This would justify the expression for $a_r^\dagger$, and would yield $a_r a_r^\dagger = N_r + 1$. However, this would be incorrect because θ_r is not hermitian[6]. Dirac proceeded differently. In order that a_r and $a_r^\dagger$ are canonically conjugate, they should satisfy $a_r a_r^\dagger - a_r^\dagger a_r = 1$, and to achieve this, he proposed two other expressions for a_r and $a_r^\dagger$

$$\begin{aligned} a_r &= e^{-i\theta_r/\hbar}N_r^{1/2} = (N_r+1)^{1/2}e^{-i\theta_r/\hbar}; \\ a_r^\dagger &= (N_r)^{1/2}e^{i\theta_r/\hbar} = e^{i\theta_r/\hbar}(N_r+1)^{1/2} \end{aligned} \tag{1.1.1}$$

From these expressions one obtains $a_r a_r^\dagger = N_r + 1$, and with $a_r^\dagger a_r = N_r$ this showed that $[a_r, a_r^\dagger] = 1$ and that N_r was positive definite with eigenvalues $0, 1, 2 \ldots$. One obtains the same results if one treats the θ_r and N_r as if they are canonically conjugate variables, $[\theta_r, N_s] = i\hbar\delta_{rs}$. (We shall soon see that Jordan used the same starting point for fermions, but derived an anticommutation relation for N_r and $\exp i\theta_s/\hbar$). Dirac took the Schrödinger wave function to depend on the maximal commuting set of N_r, while θ_r was represented by $i\hbar\partial/\partial N_r$. This is one of the first times second quantization was used. Of course, the formulation with a and $a^\dagger$ is nowadays universally used in relativistic quantum gauge field theory, but in quantum optics the formulation in terms of N and θ is used.[6]

[6]The relation between the number operator and the phase operator is different for Dirac fermions as we shall soon discuss. It is well-known for nonrelativistic bosons in condensed matter physics. [20] If one takes for example $\lambda(\varphi^*\varphi)^2$ theory with a mass term $m^2\varphi^*\varphi$, and one takes the nonrelativistic limit by writing $\varphi = \frac{1}{\sqrt{2m}}[\exp -imt]\chi$ and omitting $(\partial_t\chi)^2$ terms, one obtains $\mathcal{L} = i\chi^*\partial_t\chi - \frac{1}{2m}\partial_i\chi^*\partial_i\chi - (\lambda/4m^2)(\chi^*\chi)^2$. If one then decomposes χ as $\chi = e^{-i\theta}\sqrt{\rho}$, one finds $\mathcal{L} = \frac{i}{2}\partial_t\rho + \rho\partial_t\theta + \cdots$. The first term is a total derivative but the second term shows that θ is canonically conjugate to ρ, hence $[\theta, \rho] = i\hbar$. For relativistic bosons, in particular photons, one decomposes the modes as in (1.1.1). There are some subleties: from the first line in (1.1.1) it follows that $e^{-i\theta_r/\hbar}$ annihilates the vacuum, hence $e^{-i\theta_r/\hbar}e^{i\theta_r/\hbar} = 1$ but $e^{i\theta_r/\hbar}e^{-i\theta_r/\hbar}$ is not unity. So $e^{-i\theta_r/\hbar}$ is not really an exponent and not unitary. As a consequence, θ_r is not hermitian. However, $a_r^\dagger a_r = \sqrt{N_r}e^{i\theta_r/\hbar}e^{-i\theta_r/\hbar}\sqrt{N_r}$ is still equal to N_r, even though $e^{i\theta_r/\hbar}e^{-i\theta_r/\hbar}$ is

Second quantization was first used in full-fledged form by Born, Heisenberg and Jordan in 1925 (the "Dreimänner Arbeit", the three-man work), who applied it to the oscillators of a one-dimensional model of the free electromagnetic field [22]. They noted that the photons introduced in their work satisfy Bose–Einstein statistics.[7] Their article solved a fundamental problem in quantum physics. It should have convinced sceptics that quantum field theory was a complete consistent theory that could solve quantum problems all by its own, without having to fall back on classical concepts in statistical mechanics modified by new quantum principles. Unfortunately, not many physicists paid much attention to this article at the time. Since (free) quantum field theory started with this article, we shall consider it in some detail, and resume the discussion of Dirac's work above (1.1.10).

Jordan, using second-quantized fields, derived in this paper a formula for the energy fluctuations $\overline{(E-\bar{E})^2}$ which Einstein had obtained two decades earlier from thermodynamical considerations (see [1] for an interesting discussion). Einstein had considered the fluctuations in the energy of the electromagnetic field in a small volume v which was in thermal equilibrium with a large cavity [24]. He had used thermodynamics and Planck's quantization of energy to derive his celebrated fluctuation

not unity, because both N_r and $e^{-i\theta_r/\hbar}$ annihilate the vacuum. These issues are well-known in quantum optics. [21]

[7]Bose had already in 1924 studied the statistics for photons in his derivation of Planck's radiation law. He considered each assembly of photons as one state, and therefore photons of the same frequency were truly indistinguishable. His article had been rejected by the "Philosophical Magazine" of the Royal Society in London, but Einstein translated his paper from English into German and submitted it for publication in the "Zeitschrift für Physik", and later applied the new statistics to material gases and predicted Bose-Einstein condensation. Fermi began what is now known as Fermi-Dirac statistics in 1925 in a study of the statistical mechanics of identical systems and the quantization of the ideal mono-atomic gas, stimulated by Pauli's exclusion principle. Heisenberg studied two coupled harmonic oscillators, and found a few relations between the symmetry properties of wave functions and statistics for N particle systems. Applying these ideas to the helium atom he found the exchange interactions, and could solve the long-standing problem of the helium spectrum. Dirac gave the complete solution: particles have either totally symmetric or totally antisymmetric wave functions; photons realize the former possibility and satisfy Bose-Einstein statistical mechanics, while electrons in an atom have antisymmetrical eigenfunctions. See [23].

formula

$$\overline{(E-\bar{E})^2} = \overline{E^2} - (\bar{E})^2 = h\nu\bar{E} + \frac{\bar{E}^2}{\rho_\nu d\nu} \tag{1.1.2}$$

where E is the total thermal energy in the small volume v at a given time and in the frequency range $(\nu, \nu + d\nu)$, $\bar{E}$ is its time average, and $\overline{(\Delta E)^2} = \overline{(E-\bar{E})^2}$ describes the fluctuations in E. The expression for $\bar{E}$ can be obtained in a simple way by using a method which Debye had developed to derive Planck's formula [25]. Starting from the assumption that the only allowed energies of an oscillator of the electromagnetic field with frequency ν are $E_n(\nu) = nh\nu$ and that the probabilities for these energies are given by the Boltzmann factor $y^n/\sum y^n$ with $y = \exp(-h\nu/kT)$, Debye found for the time-average of the total thermal energy U of the radiation (summing over the two polarizations) in the small volume with frequencies between ν and $\nu + d\nu$

$$\bar{E} = \bar{U}_\nu d\nu = \frac{h\nu}{e^{h\nu/kT}-1}\rho_\nu d\nu, \rho_\nu = \left(\frac{8\pi\nu^2}{c^3}\right)v \tag{1.1.3}$$

This is Planck's law (without zero point energy).

Einstein noticed that according to the thermodynamics of Boltzmann

$$\bar{E} = \frac{\int E e^{-E/kT} dpdq}{\int e^{-E/kT} dpdq} \tag{1.1.4}$$

However, to implement Planck quantization he interpreted $dpdq$ to mean that one should sum over all states which are created by all oscillators in the interval $(\nu, \nu+d\nu)$ with discrete energies $nh\nu$. A given state would then have an energy E which was the sum of all oscillator energies. Differentiating this formal expression for $\bar{E}$ in (1.1.4) w.r.t. $-\frac{1}{kT}$ yields $\bar{E^2} - (\bar{E})^2$, whereas differentiation of the explicit expression in (1.1.3) yields (1.1.2).[8] On general grounds the left-hand side of (1.1.2) should be proportional to $d\nu$, while it is clear from the explicit expressions in (1.1.2) that the right-hand side is indeed linear in $d\nu$. Einstein first derived (1.1.2) for the energy

[8] Since the probability $P = \sum_{n_1,n_2,n_3} e^{-E/kT}$ for states in the interval $d\nu$ factorizes into a product of terms $\sum_n e^{-nh\nu_i/kT}$ for oscillators with frequency ν_i, it is correct that differentiation of $\ln P$ w.r.t. $\frac{-1}{kT}$ gives Debye's expression for $\bar{E}$. However, it would be incorrect to use this formula to obtain an expression

fluctuations in the whole volume V, but later he claimed that it also applied to the energy fluctuations in any subvolume v of the large volume V. Several authors have disputed this latter claim [26], even up to present times [27]. The problem for a successful quantum theory of radiation was to give a derivation of (1.1.2) using only quantum mechanics.

Jordan approached this problem from the field theoretic side. As Lagrangian density he took the same expression as nowadays used in string theory, namely the Klein-Gordon Lagrangian density in $1+1$ dimensions, $\mathcal{L} = \frac{1}{2}\frac{1}{c^2}(\partial_t u)^2 - \frac{1}{2}(\partial_x u)^2$, and he considered a small "volume" $0 \leq x \leq a$ in a large "cavity" $0 \leq x \leq l$. (This small volume was thus not separated from the large cavity by boundary walls, see footnote 9). He imposed the boundary conditions $u(x=0,t) = u(x=l,t) = 0$, and expanded $u(x,t)$ into modes as follows: $u(x,t) = \sum_{k=1}^{\infty} q_k(t) \sin \frac{k\pi}{l} x$ (corresponding to $u(x,t) = \sum_k \left(\frac{\hbar c^2}{2\omega_k l}\right)^{1/2} (a_k e^{-i\omega_k t} + \text{h.c})\sqrt{2} \sin \frac{k\pi}{l} x$ in the modern notation). Substituting this result into the formula for the total energy density $\mathcal{E} = \frac{1}{2}\frac{1}{c^2}(\partial_t u)^2 + \frac{1}{2}(\partial_x u)^2$, he obtained a sum over j and k. The **time**-average $\bar{\mathcal{E}}$ corresponded to the terms with $j = k$, so $\mathcal{E} - \bar{\mathcal{E}}$ corresponded to the sum of terms with j not equal to k. For the energy fluctuations in the small volume $E - \bar{E} = \int_0^a dx(\mathcal{E} - \bar{\mathcal{E}})$ he obtained

$$\begin{aligned} \Delta E = E - \bar{E} &= \frac{1}{4} \sum_{\substack{j,k=1 \\ j \neq k}}^{\infty} \left(\frac{1}{c^2} \dot{q}_j \dot{q}_k K^-_{jk} + \frac{j\pi}{l}\frac{k\pi}{l} q_j q_k K^+_{jk} \right) \\ K^{\pm}_{jk} &= \frac{\sin(u_j - u_k)a}{u_j - u_k} \pm \frac{\sin(u_j + u_k)a}{u_j + u_k}, u_j = \frac{j\pi}{l}. \end{aligned} \tag{1.1.5}$$

The square of these fluctuations $(\Delta E)^2 \equiv (E - \bar{E})^2$ contained terms with four q's, and became proportional to $(\delta_{jm}\delta_{kn} + \delta_{jn}\delta_{km})$ upon averaging over time. Converting the sum over $j \neq k$ into an integral (thereby implicitly adding terms with $j = k$),

for the fluctuations by differentiating once more and writing

$$\bar{E}^2 - (\bar{E})^2 = \left[\frac{\left(\sum_n (nh\nu)^2 e^{-nh\nu/kT}\right)}{\sum_n e^{-nh\nu/kT}} \right] \rho_\nu d\nu$$

because already for two oscillators $\overline{(E_1 + E_2)^2}$ is not equal to $\overline{E_1^2} + \overline{E_2^2}$. Einstein's interpretation of the Boltzmann equation with Planck quantization as discussed below (1.1.4) avoided this pitfall.

yielded for the time average of $(\Delta E)^2$

$$\overline{(\Delta E)^2} = \frac{1}{8}\int_0^\infty \int_0^\infty du_j du_k \left(\frac{l}{\pi}\right)^2 \left\{ \frac{1}{c^4}\overline{\dot{q}_j^2}\,\overline{\dot{q}_k^2}(K_{jk}^-)^2 \right.$$
$$\left. +j^2k^2\left(\frac{\pi}{l}\right)^4 \overline{q_j^2}\,\overline{q_k^2}(K_{jk}^+)^2 + \frac{1}{c^2}jk\left(\frac{\pi}{l}\right)^2\left(\overline{q_j\dot{q}_j}\,\overline{q_k\dot{q}_k} + \overline{\dot{q}_jq_j}\,\overline{\dot{q}_kq_k}\right)K_{jk}^-K_{jk}^+ \right\} \tag{1.1.6}$$

Using $\sin^2(u_j - u_k)a/(u_j - u_k)^2 = a\pi\delta(u_j - u_k)$ and dropping the terms with $\sin(u_j + u_k)a/(u_j + u_k)$, Jordan obtained, defining $\omega = uc$,

$$\overline{(\Delta E)^2} = \frac{a\pi l^2}{4\pi}\int_0^\infty \frac{d\omega}{2\pi c}\left\{ \frac{\left(\overline{\dot{q}_\omega^2}\right)^2}{c^4} + \omega^4\left(\overline{q_\omega^2}\right)^2 + \frac{\omega^2}{c^4}\left(\overline{q_\omega\dot{q}_\omega}\right)^2 + \frac{\omega^2}{c^4}\left(\overline{\dot{q}_\omega q_\omega}\right)^2 \right\} \tag{1.1.7}$$

From here on, the journey home was easy. For harmonic oscillators equipartition of energy holds, $\frac{1}{c^2}\overline{\dot{q}_\omega^2} = \omega^2\overline{q_\omega^2}$, and the time-average of the energy is given by $\bar{E} = \int_0^\infty \left(d\omega\frac{l}{\pi}\right)\left(\frac{1}{2}\frac{1}{c^2}\overline{\dot{q}_\omega^2} + \frac{1}{2}\omega^2\overline{q_\omega^2}\right)\int_0^a \sin^2\omega x\, dx$. Hence, using $\int\frac{d\omega}{2\pi} = \int d\nu$, and $\int_0^a \sin^2\omega x\, dx = \frac{1}{2}a$, one obtains $\bar{E}_\nu = \frac{al}{2c}\left(\frac{1}{c^2}\overline{\dot{q}_\omega^2} + \omega^2\overline{q_\omega^2}\right) = \frac{al}{c}\frac{1}{c^2}\overline{\dot{q}_\omega^2} = \frac{al}{c}\omega^2\overline{q_\omega^2}$ for $\frac{\omega}{c} \gg 1/a$. Moreover $\overline{q_\omega\dot{q}_\omega} = -\overline{\dot{q}_\omega q_\omega} = i\hbar c^2/l$, as follows most easily from using the modern expansion in terms of annihilation and creation operators

$$q_{\omega_k}\dot{q}_{\omega_k} = \frac{\hbar c^2}{2\omega_k l}(a_k e^{-i\omega_k t} + a_k^\dagger e^{i\omega_k t})(-i\omega_k a_k e^{-i\omega_k t} + i\omega_k a_k^\dagger e^{i\omega_k t})(\sqrt{2})^2$$
$$\overline{q_{\omega_k}\dot{q}_{\omega_k}} = \frac{\hbar c^2}{\omega_k l}i\omega_k(a_ka_k^\dagger - a_k^\dagger a_k) = \frac{i\hbar c^2}{l} \tag{1.1.8}$$

Substituting these results yielded finally

$$\left(\overline{(\Delta E)^2}\right)_\nu \equiv \left(\overline{E^2}\right)_\nu - \left((\bar{E})^2\right)_\nu = \frac{(\bar{E}_\nu)^2}{2a/c} - \frac{(h\nu)^2}{2}(a/c) \tag{1.1.9}$$

(The dimensions are correct: a/c has the dimension of $1/d\nu$).

At this point Jordan identified the total energy $\overline{E_\nu}$ with the sum of the thermal energy U_ν Einstein had used, and the zero-point energy: $\bar{E}_\nu = U_\nu + h\nu a/c$. Substituting this expression into (1.1.9), he obtained complete agreement with Einstein's formula. (Because $\omega_k = \frac{k\pi}{l}c$ and $\omega_k = 2\pi\nu_k$ one has $dk\frac{\pi}{l}c = 2\pi d\nu$, so $dk =$

$\frac{2l}{c}d\nu$, hence $\rho_\nu = 2a/c$. The zero-point energy is then indeed $(\frac{1}{2}h\nu)(2a/c) = h\nu a/c)$. Hence, quantum field theory could explain at least one nontrivial thermodynamical result, but not many physicists paid attention to this very clever derivation.[9]

We now return to Dirac's work on spontaneous emission of radiation by an atom [17]. The total Hamiltonian Dirac took was given by

$$H = H^{(0)}_{\text{matter}} + H^{(0)}_{\text{rad}} + H_{\text{int}} + H_C \tag{1.1.10}$$

namely, the Hamiltonian for free matter and free transverse electric and magnetic fields + the coupling of matter to the transverse vector potential given above + the instantaneous Coulomb interactions. From a modern perspective he was using the Coulomb gauge $\partial^k A_k = 0$, whereas the scalar potential $A_0 = -\phi$ had been integrated out from the path integral. (One can of course not choose a gauge in which both $\partial^k A_k = 0$ and $A_0 = 0$). He then wrote down the Schrödinger equation for this system, expanded the wave function into a complete set of solutions for $H^{(0)}_{\text{matter}} + H^{(0)}_{\text{rad}} + H_C$, and developed time-dependent perturbation theory. Together with the article by Born, Heisenberg and Jordan, this article by Dirac was the beginning of field theory, so field theory began with QED.

The result for spontaneous emission which Dirac derived, agreed with a result Born and Jordan [28] had obtained from an analogy with classical electromagnetism. In this "Zweimänner Arbeit" (two-man work) which preceded the "Dreimänner Arbeit" by a few months they proposed certain rules how turn the formulas of classical radiation by a dipole into formulas for the quantum theory. They assumed that the radiation with frequency ω which is emitted by an atom if it makes a transition from a state α to a state β can be described by a harmonic oscillator with position

[9] In fact, Heisenberg later critized his coauthor Jordan for using the formula $\sin^2 \omega a/\omega^2 = a\pi\delta(\omega)$; he noted that it was multiplied by $\overline{\dot{q}^2}$ in (1.1.6) which was proportional to ω^2, and this led to a divergent result! He redid the problem with a **smooth** cut-off of the energy at $x = 0$ and $x = a$; this was in his opinion a better model for a small volume in thermodynamical equilibrium and this indeed resolved this problem [22].

$\vec{r}_{\beta\alpha} e^{-i\omega t} + h.c.$, where $E_\alpha - E_\beta = \hbar\omega$ and $\vec{r}_{\beta\alpha}$ is the matrix element of the operator $\vec{r}$ between the states α and β. They replaced $| \vec{r} |^2$ by $| \vec{r}_{\beta\alpha} |^2$ in the classical formula for the amount of radiation emitted per second by a harmonic oscillator and they interpreted this as the probability for spontaneous emission. In this article they also showed that the complicated formulas of Heisenberg involving summations such as $\sum_m \sum_n \sum_p \sum_q$ (Heisenberg's antics in the words of A. Pais [1]) amounted to simple matrix algebra for the electric and magnetic fields [28]. Here the commutation relation $qp - pq = i\hbar$ first appeared. Shortly afterwards, and independently, Dirac obtained the same results in his first paper on quantum mechanics. [29]

Taking further terms in the expansion of the square root which defined the Hamiltonian would give higher-derivative interactions with $p^2, p^3, \ldots$. Dirac did initially not include them. Later he did include the $\vec{A}^2$ term (a four-point coupling in modern terms), and used it to compute the scattering of a real photon by an electron in an atom using second order perturbation theory [30]. So Dirac's approach was nonrelativistic. Moreover, his approach based on (1.1.10) was a combination of field theory and quantum mechanics which could not have explained pair creation.

The problem then arose to construct a relativistically invariant quantum theory of radiation. Dirac did not think that this was a difficult problem; as he wrote in 1929: "The general theory of quantum mechanics is now almost complete, the imperfections that still remain being in connection with the fitting of the theory with relativity ideas" [31]. He was too optimistic. We now know that combining quantum mechanics with special relativity inevitably leads to quantum field theory with its infinitely many degrees of freedom, and a whole host of new problems arises at the level of radiative corrections.

Heisenberg and Pauli decided in 1928 to tackle this problem [32]. Their purpose was to construct a full-fledged relativistically invariant field theory of QED from first principles based on Lagrangians, and not, as Dirac had done, combine field theory with Hamiltonian quantum mechanics. Dirac had used the nonrelativistic coupling $H_{int} = -\frac{e}{c}\vec{A}_{tr}(\vec{x},t)\frac{d\vec{x}}{dt}$ where $\vec{x}$ is the position of an electron in quantum mechanics (his relativistic equation for the electron came only one year later, namely in 1928 [33]), whereas Heisenberg and Pauli used in their papers the meanwhile published Dirac equation for electrons. They started from the (relativistically invariant) classical Maxwell action with variables $Q_\alpha(t)$ (the fields A_μ at all $\vec{x}$), and went to the Hamiltonian formulation, imposing equal-time canonical commutation relations between $Q_\alpha(t)$ and their canonically conjugate momenta $P^\alpha(t)$. Dirac had used the Coulomb gauge, but they decided not to impose this gauge as it violated manifest relativistic invariance. They then ran into a huge problem: the canonical momentum of A_0 $(= -\phi)$ vanished. It took them quite some time before they found a solution to this problem[10], and this solution led to the nowadays familiar procedure of adding a gauge fixing term to the classical action.

The main reason given nowadays for adding a gauge fixing term to the classical gauge action is that then the kinetic operator of the gauge fields becomes invertible and a free field propagator exists. For Heisenberg and Pauli, and Fermi, who developed QED in the late 1920's and early 1930's, the motivations were different. Fermi, as we shall discuss later in more detail, started with the gauge-fixed field equations $\Box A_\mu = -\frac{4\pi}{c}j_\mu$ instead of the gauge invariant Maxwell equations $\partial^\nu F_{\nu\mu} = -\frac{4\pi}{c}j_\mu$, so that the issue of gauge fixing was of secondary importance to him. For Heisenberg and Pauli, the gauge fixing term was needed to solve the problems with canonical quantization. Heisenberg and Pauli proposed in 1929 (page 30 of their first paper) "einen formalen Kunstgriff" (a formal trick), namely to add a "Zusatzglied" (an extra term) $-\frac{1}{2}\,\epsilon(\partial^\mu A_\mu)^2$ to the Maxwell gauge action, and to send $\epsilon \to 0$ at the end in physical applications. The conjugate momentum for A_0 is then $p(A_0) = \epsilon\partial^\mu A_\mu$, and

[10] In this year Heisenberg took a sabbatical from field theory. "In order not to be forever irritated with Dirac, I have done something else" he wrote to Pauli. That something else was his theory of ferromagnetism [34]. When he saw early in 1929 a solution to the problem with the canonical momentum of A_0, he returned to the strait (not straight) and narrow path of quantum field theory.

in this way they circumvented the problem that the canonical momentum conjugate to A_0 vanishes. This procedure maintained manifest relativistic invariance, but later in 1930 (page 171 of their second paper), they admitted "ein Schönheitsfehler der Theorie" (a blemish of the theory), namely for $\epsilon \neq 0$ gauge invariance of the action is lost. Blemish or not, this is exactly the procedure followed today, with $-\frac{1}{2}\epsilon(\partial^\mu A_\mu)^2$ the gauge fixing term. However, back in 1930 they came to the conclusion that violation of gauge invariance (even at intermediary stages) is too high a price to pay[11] and they decided to follow an alternative approach, namely to choose the temporal gauge $A_0 = 0$. Note that this is a different gauge from the Coulomb gauge $\partial^k A_k = 0$ which Dirac had chosen. Although they had now sacrificed manifest relativistic invariance, the problem that the canonically conjugated momentum of A_0 vanished was still solved because there was no longer an A_0. In fact, although the gauge $A_0 = 0$ broke manifest Lorentz invariance, they showed that their approach was nevertheless relativistically invariant [32] because after a Lorentz transformation one could always make a compensating gauge transformation to restore the temporal gauge ("umeichen"). The three components of $\vec{A}$ yielded three Maxwell equations, but the fourth equation, the Gauss law $C \equiv \mathrm{div}\vec{E} - 4\pi\rho = 0$ seemed to be absent. They observed, however, that the operator C commutes with the Hamiltonian, hence it is constant. Moreover, they noted that upon using the equal-time canonical commutation relations C generated residual gauge transformations (gauge transformations with a time-independent parameter. These preserve the gauge $A_0 = 0$). When taken as an operator equation the Maxwell equation $\mathrm{div}\,\vec{E} = 4\pi\rho$ would lead to a constraint on the canonical variables ($\vec{E}$ is the canonical momentum of $\vec{A}$). This would create huge difficulties. Since physical states should be gauge invariant, Heisenberg and Pauli proposed (page 174 of their second paper) to impose the Gauss law instead as a subsidiary condition ("Nebenbedingung") which selects physical (gauge-invariant)

[11] They apparently did not consider the possibility that the S matrix might still be gauge-choice independent. In fact, the concept of the S matrix was only introduced by Heisenberg years later.

states.[12] (They did not actually use the term physical states but noted that the Gauss operator is diagonal on gauge invariant quantities, and thus can be given any numerical value. They chose of course the value zero). Subsidiary conditions on states had been introduced in the meantime by Fermi [35], to whom they refer.

Another problem one encounters if one tries to construct a quantum field theory for photons and electrons is that imposing the standard equal-time canonical commutation relations[13] on the fields for the electrons, the Pauli exclusion principle of 1925 is violated [39]. In a work of stunning originality, Jordan showed in 1927, even before the Dirac equation was invented, that one should use equal-time **anti**commutation relations for an ideal Fermi-Dirac gas [40]. Following Dirac, he began with $b_r^\dagger b_r = N_r$, but now he required that as a consequence of Pauli's exclusion principle N_r should only have eigenvalues 0 and 1. Setting again $b_r^\dagger = \sqrt{N_r} e^{i\theta_r/\hbar}$ and $b_r = e^{-i\theta_r/\hbar}\sqrt{N_r}$, he required that also $e^{-i\theta_r/\hbar} N_r e^{i\theta_r/\hbar}$ have only eigenvalues 0 and 1 because this is a similarity transformation of N_r. Then the only possibility is $e^{-i\theta_r/\hbar} N_r e^{i\theta_r/\hbar} = 1 - N_r$, and thus (because $N_r^2 = N_r$ and $(1-N_r)^2 = 1 - N_r$)

$$\begin{aligned} b_r &= e^{-i\theta_r/\hbar}\sqrt{N_r} = \sqrt{1-N_r}\, e^{-i\theta_r/\hbar} \\ b_r^\dagger &= \sqrt{N_r}\, e^{i\theta_r/\hbar} = e^{i\theta_r/\hbar}\sqrt{1-N_r} \end{aligned} \tag{1.1.11}$$

It followed that $b_r^\dagger b_r = N_r$ and $b_r b_r^\dagger = 1 - N_r$. Hence he arrived at an anticommutation relation, $\{b_r, b_r^\dagger\} = 1$. Furthermore, he could also prove that $b_r b_r = 0$ and $b_r^\dagger b_r^\dagger = 0$ since $N_r(1-N_r) = 0$. Jordan also obtained a 2×2 matrix representation for N_r and θ_r. Because $\{N_r, e^{i\theta_r/\hbar}\} = e^{i\theta_r/\hbar}$ with $N_r = \frac{1}{2}(1-\sigma_z)$ he deduced that $e^{i\theta_r/\hbar} = \frac{1}{2}(\sigma_x - i\sigma_y)$

[12] The Gauss condition div $\vec{E} = 4\pi\rho$ follows from the field equations $\Box A_\mu = -\frac{4\pi}{c} j_\mu$ if $\partial^\mu A_\mu = 0$, but recall that Heisenberg and Pauli worked in the gauge $A_0 = 0$ and did not impose $\partial^\mu A_\mu = 0$.

[13] From a relativistic point of view, equal-time commutation relations are not natural. Jordan and Pauli derived the unequal-time commutation relations for free electric and magnetic fields (gauge invariant objects) [36] and showed that these relations were relativistically invariant. Bohr and Rosenberg studied the physical meaning of these unequal-time commutation relations (the relation between causality and the uncertainty principle). [37] Finally Heisenberg showed [38] that the uncertainty in the averages of the electric and magnetic fields over a small spacetime region with volume L^4 satisfied $\Delta E_i \Delta H_j \geq ch/L^4$ for $i \neq j$.

and obtained $b_r = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, b_r^\dagger = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Note that θ_r cannot be represented by $\partial/\partial N_r$ because the commutator of θ_r and N_r is not proportional to the unit matrix. This bold proposal not only explained the Pauli exclusion principle, but it also gave a derivation of the antisymmetry of wave functions of electrons which Dirac had postulated earlier.

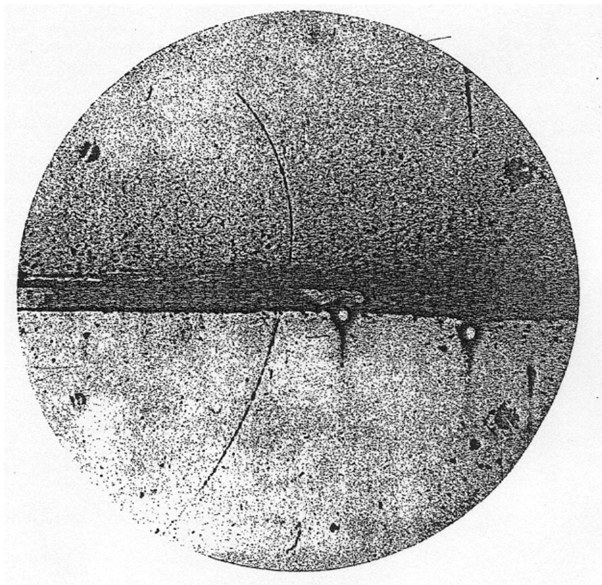
A year later (1928) Dirac invented the relativistic equation for the electron which bears his name. [33] Here words like stunning originality are insufficient to convey the depth of creativity involved. He began with an observation which later turned out to be a side issue: the conserved current $j_\mu \sim \varphi^* \overleftrightarrow{\partial}_\mu \varphi$ of Klein–Gordon theory [18] leads to a density $\rho = \varphi^* \overleftrightarrow{\partial}_t \varphi$ which is not positive definite, and therefore cannot be interpreted as a probability. (As we now know, j_μ is the charge density current, and then ρ need not be positive definite). He identified the problem as being due to the derivative $\frac{\partial}{\partial t}$ in ρ, and set out to write down a field theory with one time derivative less. Special relativity then required also one space derivative less, and in this way Dirac factorized the Klein–Gordon equation,[14] and deduced anticommutation relations for the 4×4 matrices which bear his name and which are a suitable generalization of the Pauli matrices. The magnetic moment of the electron came out correctly, namely twice as large as would be obtained from a model in which the spin of the electron would be due to small internal classical currents. However, he still used quantum mechanics instead of second-quantized quantum field theory, so he did not reach the same point for electrons which he had reached for photons, namely the concept of creation and annihilation operators for electrons. In fact, Dirac viewed electrons as so different from bosons that he found this difference in quantizing bosons and fermions quite natural. He discovered, however, solutions of his equation for the electron with negative energy. These solutions were a direct consequence of relativity; for example

[14]The relativistic wave equation for a scalar field (the "Klein–Gordon equation") had already been proposed two year earlier [18]. A year later, Pauli proposed a 2-component wave equation for the electron which described spin and which contained the 2×2 matrices which bear his name [41]. We discuss the relations between these various approaches in the appendix.

the nonrelativistic 2-component Pauli theory (see the appendix) was free from these "unphysical solutions". Weyl proposed that the 4-component Dirac spinor consists of a 2-component spinor for the electron and a 2-component spinor for the proton [43] but Dirac noticed that the negative energy solutions still had negative charge and so could not correspond to protons. To remedy the problem with charge he introduced the notion of the Dirac sea such that holes in this sea correspond to particles with positive energy and positive charge. Late in 1929 he proposed that when an electron is removed from a negative-energy solution, a proton-electron pair (not a positron-electron pair!) would be created [42]. Since there were only 3 elementary particles known at that time (electron, proton, and photon; the nuclei were considered to consist of protons and electrons, the neutron was discovered only later in 1932) this was an attractive idea. Dirac hoped that the electromagnetic interactions would explain why the proton is heavier than the electron. However, Oppenheimer and Tamm noted that according to Dirac's proposal a hydrogen atom would rapidly decay into two photons (Dirac's hydrogen atom was actually today's positronium), and Weyl noted that charge symmetry required that also with electromagnetic interactions the proton mass should be the same as the electron mass [44]. In 1931 Dirac bit the bullet (he made, in his words, "a small step forward") and proposed a new elementary particle, the anti-electron[15] [46]. In 1932 Anderson discovered these positrons in cosmic rays [8]. (The name positron is due to the editor of Science News Letters where Anderson first published his discovery). In England, Blackett and Occhialini decided to study cosmic rays, and as soon as they saw Anderson's article they looked for positron events in their own plates, and found many, including several cases in which

[15] Einstein had already in 1925 come close to predicting antimatter [45]. He had constructed a unified field theory for gravity and electromagnetism with the antisymmetric part of the metric equal to the Maxwell curvature, and had derived from P and T symmetry that a particle with the same mass but opposite charge to the electron should exist. Like initially Dirac had thought, Einstein believed for a while that this positively charged particle was the proton, but he soon realized the difficulties with this interpretation and abondoned the whole idea.

electron-positron pairs were created. [47] This clinched the case for antiparticles.



(1.1.12)

Figure 2: The discovery of the positron by Anderson in 1932. [8] The track shows a charged particle in a cosmic ray burst which moves in a magnetic field of 15000 Gauss and passes through a horizontal lead plate. The particle is moving upwards because the curvature of its track is larger above the lead plate. It has positive charge because the magnetic field points away from the reader. A proton with this curvature would have a very short track (5mm) whereas the track is much longer (5cm). The small amount of ionization along the track, combined with the length of the track, led Anderson to estimate that this new particle had a mass of less than 20 times the electron mass. Dirac had predicted positrons in 1931, but his theory involved such unusual ideas (an infinite sea of negative energy states) that it did not play a role in the discovery (Anderson did not refer to Dirac, but Blackett and Occhialini did so).

The quantum mechanical Dirac equation contained such unusual ideas that Dirac could not bring himself to check that it agreed with the data for the spectrum of the hydrogen atom. Soon others performed these calculations [48], and they found that the Dirac equation gave a derivation of all the relativistic corrections that were known to be necessary, and predicted new corrections which agreed with the data. In the appendix we derive all these relativistic corrections to the nonrelativistic Schrödinger equation by starting from the Dirac equation itself and making various approximations. We also discuss there the relation to Pauli's 2-component spinor formalism for relativistic quantum mechanics.

Despite its enormous successes, the Dirac equation was still quantum mechanics, and a field theory for fermions was still lacking. Others embarked on the construction of a quantum field theory approach which treated all fields on equal footing. In this approach, following Jordan [40], one imposed equal-time "canonical" anticommutation relations for the electrons. Pauli and Weisskopf studied spinless particles and found that a relativistic quantum treatment made sense by applying second quantization to the Klein-Gordon equation, [49] but with commutation instead anticommutation relations; instead of negative energy states one dealt with particles and antiparticles, and instead of negative probabilities one dealt with charge densities. In fact, in later years physicists started asking the question whether all fermions satisfy anticommutation relations, and all bosons commutation relations. This became the celebrated spin-statistics theorem [50] from which the exclusion principle and Fermi–Dirac or Bose-Einstein statistics follow. For fermions it was found that only with anticommutation relations the energy is positive definite, but with commutation relations there are states with negative energy. For bosons anticommutation relations for a spin 0 field still yield positive energy but the relation $\{\phi(x,t), \phi^\dagger(x',t)\} = 0$ leads to $\sum_\chi |\langle\chi|\phi|\mu\rangle|^2 = 0$ for the matrix elements in a Hilbert space with states $|\chi\rangle$. This is unacceptable since it means that the field operator ϕ vanishes. So for free spin 0 and 1/2 fields the spin-statistics theorem can easily be verified. Furthermore, it was found that imposing commutation relations on fermions, and anticommutation relations on bosons, fields no longer did (anti) commute for spacelike separations [52]. This resulted in the following necessary and sufficient conditions for the spin-statistics connection for physical particles to be satisfied [50]:

(1) the vacuum should be the lowest energy state

(2) fields either commute or anticommute at spacelike separations

(3) the norm in Hilbert space should be positive definite

A generalized spin-statistics theorem was obtained [51] by only requiring that physical observables commute at spacelike separations; this allowed the standard spin-statistics connection, but also parastatistics (see section 3). When Feynman introduced in the

1950's ghost particles to rescue unitarity for nonabelian gauge theories, the spin-statistics relation entered a new interesting phase. In the Pauli-Villars regularization method heavy extra fields appear with negative probabilities but no difference in statistics. For the ghost fields of Feynman eventually opposite statistics was used and it was found that energy was still positive, and also causality ((anti) commutativity at spacelike separations) was satisfied, but the norm of ghost states was negative. In time the spin-statistics theorem was proven for higher spins, and for interacting fields, but we leave here these issues and return to the quantization of gauge fields in the 1920's. For a fascinating history of the spin-statistics theorem see [53].

In the 1950's, Pauli and Luders, and Zumino, noted that any local Lagrangian field theory which is relativistic (Lorentz invariant) and satisfies the spin-statistics connection, is necessarily invariant under CPT (the combined operation of C = charge conjugation, P= parity and T= time reversal). For a simple proof see [54]. Conversely, any theory in which CPT is broken, necessarily also must break Lorentz invariance. (The converse is not true: one can break Lorentz invariance while conserving CPT symmetry, an example being Galileo invariant local Lagrangian field theories). There are several ways to violate CPT in local Lagrangian field theory, for example by adding external fields.[16]

We now briefly comment on problems with Heisenberg and Pauli's temporal gauge $A_0 = 0$ and Dirac's Coulomb gauge $\partial^k A_k = 0$ which persist till today. Choosing the gauge $A_0 = 0$ is different from integrating out A_0 in a path integral. If one integrates out A_0 one obtains instantaneous Coulomb forces in a theory which is still gauge invariant. Choosing in this theory the Coulomb gauge one recovers Dirac's theory in (1.1.10). One is then left with only two independent field components for A_μ. Having

[16]As an example, consider 4-dimensional Maxwell theory $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ to which a 3-dimensional Chern-Simons term $\mu\epsilon^{ijk}F_{ij}A_k$ is added where μ is an arbitrary mass parameter. The latter term can be written as $\theta^* F_{\mu\nu}F^{\mu\nu} = v_\mu{}^* F^{\mu\nu}A_\nu$ where $v_\mu = \partial_\mu\theta$ with $\theta = \mu t$. This theory violates CPT because if v_μ were a vector field, it would need to transform as CPT $v_\mu = -v_\mu$ in order to conserve CPT. Hence, if v_μ is an external field, CPT is violated.

only two components implies that one cannot define instantons and Chern–Simons invariants for nonabelian gauge theories (instantons and Chern-Simons invariants will be discussed later in this book). Moreover, till today nobody has been able to prove that abelian or nonabelian gauge field theories in the Coulomb gauge are renormalizable. Thus the Coulomb gauge may have had its uses in perturbative physics, but for nonperturbative physics it has serious drawbacks. On the other hand there are also problems with algebraic noncovariant gauges such as $A_0 = 0$. As we shall discuss, renormalization in algebraic noncovariant gauges $n \cdot A = 0$ with $n^2 \neq 0$ is also an unsolved problem. The gauge $A_0 = 0$ is often used to study the role of instantons in tunneling, and in high-temperature field theory. It has also been used to study leading logarithms in QCD, and at this level the problems with renormalization do not yet surface. However, at nonleading loop level there are problems, and one is led to the covariant gauges.

A manifestly relativistically covariant approach to quantum field theory was developed by Fermi [35] at about the same time as Heisenberg and Pauli.[17] The reason we only now discuss the important parallel work of Fermi is that we wanted first to introduce the main theoretical problems and ideas; having identified these, we now describe his approach. Fermi was more pragmatic and less worried about gauge invariance. (He had a practical outlook on physics; for example, he was not interested in discussing the philosophical meaning of measurements in quantum mechanics). He just started from the equations $\Box A_\mu = -\frac{4\pi}{c} j_\mu$, and noticed that they are equivalent to Maxwell's equations provided that, as he stated in his review article of 1932, "... [The scalar potential] V and [the vector potential] U are not completely independent of each other; they satisfy the relation $\text{div} U + \frac{1}{c}\frac{\partial V}{\partial t} = 0$, which is closely related to the equation of continuity for the electricity" [35]. However, Fermi did not impose $\partial^\mu A_\mu = 0$ as an

[17]Pauli (1900–1958), Fermi (1901–1954), Heisenberg (1901–1976), Jordan (1902-1980), and Dirac (1902–1984) were all in their late twenties when they made their discoveries. Pauli called their work "Knabenphysik" (boys physics).

operator equation because it would have been incompatible with the canonical commutation relations. Rather he proposed (although this is not very explicitly stated) to view the relation $\partial^\mu A_\mu = 0$ as a subsidiary condition which physical states must satisfy, thus $\partial^\mu A_\mu(x)|\text{phys}\rangle = 0$ for all $\vec{x}$ and t. Heisenberg and Pauli (see the footnote on page 174 of their second paper) gave a Lagrangian reformulation of Fermi's approach, and observed that the action $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 - \frac{1}{2}(\partial^\mu A_\mu)^2 + \mathcal{L}(\text{minimal coupling of matter})$ yields the equations of motion $\Box A_\mu = -\frac{4\pi}{c}\, j_\mu$, which agrees with their own approach if one takes $\epsilon = 1$. Fermi also noted that the subsidiary Lorentz condition implied Gauss's law: if the Maxwell equations hold, the condition $\partial^\mu A_\mu = 0$ at all times t is equivalent to the two relations $\partial^\mu A_\mu = 0$ and $\partial^\mu \dot{A}_\mu = \text{div}E - 4\pi\rho = 0$ at $t = 0$ [55]. One can simultaneously impose these two relations as subsidiary conditions for physical states because the corresponding operators $p(A_0)$ and $\text{div}\vec{E} - 4\pi\rho$ commute (recall that $\partial^\mu A_\mu = p(A_0)$ in Fermi's approach). The two constraints $p(A_0) = 0$ and the Gauss law are called primary and secondary constraints, respectively, in Dirac's 1952 terminology [56]. They are both first class constraints which generate separate gauge transformations of A_k and A_0, respectively. Thus Fermi in fact required that physical states had to be gauge invariant, a very natural requirement. The fact that they commute with each other and with themselves means that they are themselves also gauge invariant.

Of course already at the end of the 1920's the question arose how Dirac's approach with a transverse radiation field and an instantaneous Coulomb force was related to Heisenberg and Pauli's and Fermi's covariant approach with four fields A_μ. It seems not well-known that this problem was solved by Fermi in a manner which is surprisingly modern [35]. He started from the Schrödinger equation in the covariant theory, but then he factorized the wave function into a part which depended on the longitudinal and timelike quanta, and a reduced part with Dirac's transverse quanta. This led to a Schrödinger equation for the reduced wave function which reproduced Dirac's results. In particular he derived in this manner the instantaneous Coulomb force, whereas Dirac had just added this term by hand. (Fermi was very proud of this demonstration ... "what took Heisenberg and Pauli 60 pages, I did in 4"). From a modern perspective one would say that Fermi made a gauge transformation from the covariant to the Coulomb gauge.

The approach of Dirac on the one hand, and of Heisenberg and Pauli on the other hand, reflected a deep difference in points of view, with Fermi taking a position in between. Dirac's approach was based on quantum mechanics and particles; when he needed to describe more than one electron, he introduced the concept of a Dirac-sea filled with electrons and the total number of particles never changed. Initially, in his first paper on QED, he treated the electromagnetic field semiclassically, so this was the extreme of the particle point of view: both electrons and photons were treated with quantum mechanics, excluding particle creation and annihilation. Later, he quantized the electromagnetic field with second quantization, and at that point "hole theoretic QED [became] a particle theory as far as matter was concerned, but a field theory in its treatment of radiation" (see Schweber [2]). In later years this approach culminated in Feynman's pictorial description in which particles run forward and backward in time, and in which Feynman diagrams depict how particles interact with each other at specific spacetime points. Heisenberg and Pauli took a completely opposite point of view. Both electrons and photons were described in terms of quantum field theory. In their approach, particle creation and annihilation emerged in a natural way, without the need of introducing a sea. Fermi developed a compromise: for pragmatic reasons he used quantum field theory for photons, but in his work in QED and weak interactions he treated electrons, photons and neutrons as particles. (He modeled his theory of the weak interctions on QED, and although a neutron could turn into a proton, the number of nucleonic particles did not change). The history of particle physics till 1950 is a constant struggle between these two ideologies. One group of physicists (de Broglie, Schrödinger, Jordan, Pauli, Heisenberg, Weisskopf, Schwinger) developed quantum field theory. Another group (Dirac, Heither, Feynman) worked out the particle approach. A third group (Fermi, Bethe) synthesized an approach that combined the virtures of each. Of course, in time, the distinctions between both approaches got blurred, and eventually they were shown to be mathematically equivalent by Dyson.

In addition to a dichotomy into "particles" and "fields", there was also the dichotomy into the manifestly covariant approach and the approach based on the Coulomb gauge. The manifestly relativistic approach to QED was worked out in Wentzel's book [55], but as time went on, it seemed less useful for practical applications to QED. It was much easier to follow Dirac and use the Coulomb gauge and transverse radiation fields. For decades, till the end of the 1940's, physicists used the Coulomb gauge formulation of QED. The standard treatise is Heitler's book [57]. Even today many textbooks on quantum mechanics follow this approach.

Only in the late 1940's when renormalization was studied, did one return to the relativistic Lorentz gauge. It was then discovered that there seemed to be complications with Fermi's subsidiary condition. In 1949 Ma [58] showed that Fermi's condition $\partial^\mu A_\mu|phys\rangle = 0$ for physical states $|phys\rangle$ is too strong: such states cannot be normalized[18] if one uses a vacuum which is annihilated by $a_\ell(\vec{k})$ for $\ell = 1, 2, 3$ and $a_0^\dagger(\vec{k})$. The Fock space built from this vacuum yields states with positive norm but states with timelike photons have negative expectation value of the energy.[19] Furthermore, this vacuum breaks Lorentz invariance because the conditions $a_0^\dagger(\vec{k})|vac\rangle = 0$ and $a_i(\vec{k})|vac\rangle = 0$ do not transform into each other under Lorentz transformations. The alternative is to choose the vacuum defined by $a_\mu(\vec{k})|vac\rangle = 0$ for $\mu = 0, 1, 2, 3$.

[18]The proof is simple. States in Fock space have the form $|\psi\rangle = \sum c(n_0, n_1, n_2, n_3) a_0^{n_0} (a_1^\dagger)^{n_1} (a_2^\dagger)^{n_2} (a_3^\dagger)^{n_3} (n_0! n_1! n_2! n_3!)^{-1/2} |0\rangle$ where a summation over all possible momenta $\vec{k}$ has been suppressed and $[a_\mu(\vec{k}), a_\nu^\dagger(\vec{k}')] = \eta_{\mu\nu}\delta(\vec{k} - \vec{k}')$. The operator $a_3^\dagger(\vec{k})$ generates longitudinal photons and $a_0(\vec{k})$ generates timelike photons. For momenta $k^\mu = (1, 0, 0, 1)$ the condition $\partial^\mu A_\mu|phys\rangle = 0$ implies that $[a_3(\vec{k}) + a_0(\vec{k})]|\psi\rangle = 0$ and $[a_3^\dagger(\vec{k}) + a_0^\dagger(\vec{k})]|\psi\rangle = 0$. The solution is $c(n_0, n_1, n_2, n_3) = 0$ if $n_0 \neq n_3$, and $c(0, n_1, n_2, 0) = c(1, n_1, n_2, 1) = c(2, n_1, n_2, 2) = \cdots$ are all equal. Hence $|\psi\rangle$ is not normalizable.

[19]Note that a **state** with a timelike photon has positive energy, but the zero point energy of timelike photons (the vacuum expectation value of their stress tensor) is negative.

This vacuum is Lorentz invariant and leads to a Fock space with positive energy, but there are "ghosts", namely negative-norm states. The condition $\partial^\mu A_\mu |phys\rangle = 0$ is equivalent to the conditions $[a_0(\vec{k}) + a_3(\vec{k})]|phys\rangle = 0$ and $[a_0^\dagger(\vec{k}) + a_3^\dagger(\vec{k})]|phys\rangle = 0$ for all $\vec{k}$. (By a_3 and a_0 we denote the longitudinal and timelike annihilation operators). The general solution of the first of these equations are the states of the form $(a_0^\dagger(\vec{k}) + a_3^\dagger(\vec{k}))^\ell |transverse\rangle$ for any $\ell = 0, 1, 2, \ldots$ (note that $a_0 + a_3$ and $a_0^\dagger + a_3^\dagger$ commute. By $|transverse\rangle$ we mean a state obtained by acting with $a_1^\dagger$ and $a_2^\dagger$ on the vacuum). However, the second condition, $[a_0^\dagger(\vec{k}) + a_3^\dagger(\vec{k})]|phys\rangle = 0$, leaves no physical states at all.

Thus the condition $\partial^\mu A_\mu(x)|phys\rangle = 0$ was too strong. This led Gupta and Bleuler in 1950 to propose the weaker condition[20] $(\partial^\mu A_\mu)^+|phys\rangle = 0$ where the superscript $+$ denotes the annihilation part [59]. They showed that this condition led to normalizable physical states. In fact, physical states were found to contain a finite number of transverse quanta, and equal admixtures of longitudinal and timelike photons

$$\begin{aligned} |\, phys\rangle &= \prod_{\vec{k}} (a_1^\dagger(\vec{k}))^{n_1} (a_2^\dagger(\vec{k}))^{n_2} N|0\rangle \\ N &= \sum_{l=0}^{\infty} c_l \prod_{\vec{k}} (a_0^\dagger(\vec{k}) + a_3^\dagger(\vec{k}))^l \end{aligned} \tag{1.1.13}$$

The notation $\prod_{\vec{k}}(a_1^\dagger(\vec{k}))^{n_1}$ is shorthand notation for $a_1^\dagger(\vec{k}_1)^{n_1} a_1^\dagger(\vec{k}_2)^{n_2} \cdots a_1^\dagger(\vec{k}_m)^{n_m}$ and c_l are constants. Because the operators $a_0(\vec{k}) + a_3(\vec{k})$ in $(\partial^\mu A_\mu)^+$ commute with the operators $a_0^\dagger(\vec{k}) + a_3^\dagger(\vec{k})$ in $|phys\rangle$, it is clear that these states are solutions of the Gupta-Bleuler condition. It is not difficult to prove that they are the only solutions. For $l \neq 0$ the states $N|0\rangle$ are null states: they are orthogonal to themselves, to other null states $N'|0\rangle$ of the same form, and to purely transverse states. Physical states

[20] For QED with interactions, the positive frequency part of the field A_μ itself is difficult to write down in closed form, but the field $\partial^\mu A_\mu$ is a free field satisfying $\Box \partial \cdot A = 0$. This follows directly from the action $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 - \frac{1}{2\xi}(\partial \cdot A)^2 + A^\mu j_\mu + \mathcal{L}\,(\text{Dirac})$. Also for nonabelian gauge theories, $\partial^\mu A_\mu$ is a free field.

are identified with equivalence classes of states with the same transverse part. Hence, in matrix elements for physical states, all contributions from the null states $N|0\rangle$ cancel. Thus one can also define physical states as states with $N = 1$ (the natural representatives of the equivalence classes).

This concludes our discussion of how the original ideas of Dirac, Heisenberg, Pauli, Fermi and others were modified in the course of time, why a gauge fixing term was added to the Maxwell action[21], and how physical states were defined. The final definition of physical states by Gupta and Bleuler did not add much content to the work of Fermi, but its generalization to nonabelian gauge theories became of crucial importance and led to the BRST approach.

The modern BRST approach, which we introduce in the next chapter in more detail, associates to every real local gauge symmetry a real ghost field c and a real (or purely imaginary) antighost field b, with statistics opposite to the gauge parameter. Even in QED one needs ghosts, although they are free fields in this case, and do not contribute to S matrix elements.[22] The fundamental condition for a state $|\psi\rangle$ in a gauge theory to be physical is that $Q|\psi\rangle = 0$ where Q is the BRST charge [61]. It is a hermitian Heisenberg operator which commutes with the full Hamiltonian, $[Q, H] = 0$. The main property of Q is its nilpotency, $Q^2 = 0$. Since the S matrix (see below) is obtained from the Hamiltonian, it also commutes with the BRST charge, $[Q, S] = 0$. Despite their simple form, the relations $[Q, H] = 0$ and $Q^2 = 0$ are very complicated equations because they deal with composite operators constructed from interacting Heisenberg fields.[23] Fortunately, there exists a basis in the space of fields on which

[21] Another argument for adding a gauge fixing term to the classical gauge action came from a canonical analysis of QED in the 1950's, which showed that the Maxwell field equations are inconsistent as operator equations, but adding a gauge-fixing term by hand, the corresponding Heisenberg field equations become consistent [60].

[22] But at finite temperature one even needs ghosts in QED. For example in the calculation of the free energy the field A_μ yields four times the result for one real scalar, and the ghosts subtract twice the result of one real scalar. Furthermore, in nonlinear gauges such as the Dirac gauge $\partial^\mu A_\mu + cA^\mu A_\mu = 0$, ghosts do couple even in QED.

[23] In string theory one uses in many cases a free action, and then Q is relatively simple, so simple in

Q and H take on a much simpler form, the so-called in- and out-states.

We now proceed with a discussion of the relation between physical states, gauge conditions and BRST symmetry. This discussion is more technical than the rest of this chapter, but for a true assessment of the earlier work on QED from a modern perspective it is indispensable.

In the Heisenberg picture states are time-independent while operators are in general time-dependent. Applying this to $Q|\psi\rangle = 0$, where also Q is time-independent, one can solve the equation $Q|\psi\rangle = 0$ most easily by using a basis in Hilbert space in terms of which fields take on a very simple form at asymptotic times $t_0 \to \pm\infty$. This basis are the in- and out-states, corresponding to free in- and out- fields. The BRST operator (or, in fact, any other conserved charge [62] unless spontaneously broken) becomes bilinear in asymptotic fields, and the BRST transformations reduce to linear transformations on the in- and out fields. The S matrix connects in- and out-states: $S|\psi, \mathrm{out}\rangle = |\psi, \mathrm{in}\rangle$, and since the set of in-fields is the same as the set of out-fields, one may identify the in- and out states (and fields) and speak of asymptotic states and fields. The condition $Q|\psi\rangle = 0$ (BRST closed states) selects then asymptotic states with the following properties:

(i) states $|\psi\rangle$ have either strictly positive norm or vanishing norm. States with positive norm correspond to one or more transversely polarized gauge bosons. These are the physical states. States with vanishing norm may or may not contain transverse gauge bosons but they are BRST exact: they can be written as $|\psi\rangle = Q|\psi'\rangle$. For example $|\psi\rangle = a_1^\dagger(\vec{k})Q|\chi\rangle = Q(a_1^\dagger(\vec{k})|\chi\rangle$. Since $Q^2 = 0$, BRST-exact states are orthogonal to themselves and to BRST-closed states: $\langle Q\psi'|Q\psi''\rangle = 0$ and $\langle Q\psi'|\psi\rangle = 0$ if $Q|\psi\rangle = 0$.

(ii) the longitudinal and time-like polarizations together with the ghosts and anti-ghosts form multiplets called Kugo-Ojima quartets [62] (if one uses relativistic gauges with $\partial^\mu A_\mu = 0$). A physical state is only determined up to terms of the form $Q|\chi\rangle$.

fact that one can solve the equations $Q|\psi\rangle = 0$ in closed form.

Thus physical states form equivalence classes, and there is always one state in an equivalence class which only depends on transversal modes and which is without any quartet states.

(iii) physical states have vanishing ghost number, so they have quartet admixtures with equal numbers of ghost and antighost oscillators. On the other hand, BRST closed states with nonvanishing ghost number are BRST exact ("pure gauge").

Having defined physical states by $Q|\psi\rangle = 0$ but $|\psi\rangle \neq Q|\chi\rangle$, and discussed the solutions in terms of asymptotic states, it becomes plausible that the S matrix for nonabelian (as well as abelian) gauge theories is unitary. Consider the unitarity relation (discussed in much more detail later in this book)

$$\sum_C \langle\psi', phys|S^\dagger|C\rangle\langle C|S|\psi, phys\rangle = \langle\psi', phys|\psi, phys\rangle \tag{1.1.14}$$

where $|\psi, phys\rangle$ and $|\psi', phys\rangle$ are annihilated by Q. In order that the S matrix remains unitary in the subspace of physical states with positive norm, one should be allowed to restrict the sum over states $|C\rangle$ to a sum over only physical states with positive norm. That this is allowed follows from the following two observations

(i) Since $[S, Q] = 0$ and $Q|\psi, phys\rangle = 0$, one may restrict the states $|C\rangle$ to states satisfying $Q|C\rangle = 0$,

(ii) The states satisfying $Q|C\rangle = 0$ with vanishing norm are Q-exact, hence they drop out of the inner product. Thus only physical states $|C\rangle$ with positive norm remain. However, the precise form of these BRST exact states is a complicated issue. For example, in the sector with two quartet modes the state $(s^\dagger l^\dagger - b^\dagger c^\dagger)|0>$ is BRST exact ($s^\dagger$ denotes the scalar modes in $A_0, l^\dagger$ the longitudinal modes, and $b^\dagger$ and $c^\dagger$ the antighost and ghost modes), but neither $s^\dagger l^\dagger|0>$ or $b^\dagger c^\dagger|0>$ are BRST closed.

It should be stressed that the proof that Q becomes linear in asymptotic states at asymptotic times is not at all trivial for nonabelian gauge theories, see [62][24], although the result is plausible if a conserved charge is not spontaneously broken, because in that case states form representations ("multiplets") of the algebra satisfied by the charges. Thus one-particle states are transformed linearly into each other, and since one-particle states are created by annihilated by asymptotic fields, the charges, in particular the BRST charge, is bilinear in asymptotic fields.[25] In this book we shall present a more explicit proof of unitarity which holds to each order in the coupling constant g, and which is based on cutting rules.

Let us now come back to the various proposals for a definition of physical states made in the early days of QED. Physical states should, of course, not depend on the gauge fixing term. In a Hamiltonian approach, one finds indeed that the BRST charge Q is independent of the gauge fixing term and for QED it is given by $Q = \int [c(\text{ div } E - 4\pi\rho) - p(b)p(A_0)]d^3x$, where c is the ghost field of QED (a free field) and $p(b)$ the conjugate momentum of the antighost field b of QED (also a free field). Eliminating $p(A_0)$ from its own nonpropagating field equation, and substituting the result into Q, the BRST charge Q becomes dependent on the gauge fixing term. For the Lorentz gauge the field equation for $p(A_0)$ reads $p(A_0) = \partial^\mu A_\mu$ as Heisenberg and Pauli already noticed. States without c and $p(b)$ quanta are then annihilated by Q if

[24]The single BRST condition $Q|phys\rangle = 0$ is also the complete condition which defines physical states in nonabelian gauge theories [61] but in nonabelian gauge theory the form of Q in terms of Heisenberg fields is more complicated. There appear cubic terms in Q of the form $p(b)cc$ and other terms. However, in terms of in- and out-fields Q becomes bilinear. The BRST formalism is more complicated in nonabelian gauge theories than in string theory, because string theory is usually formulated in terms of free fields and (anti)ghosts in a $1+1$ dimensional world, whereas the fields of nonabelian gauge theories are of course interacting. In the latter case one must for that reason determine how the concept of a BRST charge Q_0 acts on in- and out-states and one finds that the renormalized BRST charge maps A_μ^{ren} into $\partial_\mu c^{\text{ren}}$, and b^{ren} into d^{ren} [61]. In string theory the notion of in- and out-states loses meaning because, due to conformal invariance, large times are equivalent to small times. One can perhaps still define in- and out-states for quarks and gluons, but since they are not color singlets, they can not be observed.

[25]Of course, the observed asymptotic fields (for example protons and pions) need not correspond in $1-1$ manner with the fields in the action (for example quarks and gluons).

and only if[26]

$$(\mathrm{div}E(x) - 4\pi\rho)^+ \mid phys\rangle = 0, \quad \partial^\mu A_\mu(x)^+ \mid phys\rangle = 0 \tag{1.1.15}$$

These are precisely the subsidiary conditions imposed on physical states by Heisenberg and Pauli, and Fermi, but weakened according to Gupta and Bleuler. In gauges with gauge fixing term $\mathcal{L}(\mathrm{fix}) = -\frac{1}{2\xi}(\partial^\mu A_\mu)^2$ the field equation reads $\partial^\mu F_{\mu\nu} + \frac{1}{\xi}\partial_\nu\partial^\mu A_\mu = 0$ (recall that Heisenberg fields satisfy their field equations), hence the transverse modes still satisfy the on-shell condition $k^2 = 0$ although the longitudinal or timelike components satisfy a more complicated equation if $\xi \neq 1$. The latter develop "dipole ghosts", and it becomes very useful to introduce an auxiliary field (called d in this book) [60, 63]. In modern applications of BRST symmetry, this auxiliary field plays a crucial role.

Dirac's original approach in [17] did not need any subsidiary conditions on physical states. By using transverse vector potentials he had, in effect, solved the constraint $\mathrm{div}\vec{A} = 0$. Using the Dirac equation for electrons, his approach to quantizing the electromagnetic field became relativistically invariant, although not manifestly so.[27]

Dirac, Heisenberg and Pauli, Fermi and others, all used operator methods and canonical quantization, thus a Hamiltonian approach, to quantize QED, and they all

[26]The ghost and antighost fields are free fields in QED, and expanding them into modes, the annihilation operators vanish on states without c and $p(b)$ quanta. The creation operators do not annihilate these states, and because they are multiplied by $(\mathrm{div}E - 4\pi\rho)^+$ and $(\partial^\mu A_\mu)^+$, this yields (1.1.15).

[27]A manifestly relativistic treatment of N particles coupled to the radiation field, which extended (I.1.1), was proposed by Dirac in 1932 [65]. (In fact, even earlier Fermi had considered this approach [35]). He considered wave functions which depend on the coordinates $\vec{x}_1$,..., $\vec{x}_N$ of the particles as well as on the time coordinates t_1,...,t_N which were needed to obtain a relativistic description with four-vectors x_1^μ,...,x_N^μ. The wave function satisfied general Dirac equations, but if one set all times t_n equal to the same time t, one came back to Fermi's and Dirac's radiation theory. This so-called many-time formalism remains a combination of quantum mechanics for electrons and field theory for photons, but it can describe pair creation if one uses the concept of the Dirac sea. It was shown to be equivalent to earlier approaches [66], but because it was very complicated, equivalent to other approaches, and a mixture of quantum mechanics and field theory, it was soon abandoned by all physicists, including Dirac himself.

applied the Schrödinger equation to field theory. This had the drawback that manifest Lorentz invariance was lost when one switched from the relativistic action to the Hamiltonian operator. However, in the late 1940's it was shown that if one uses the interaction picture, one may use minus the interaction Lagrangian $\mathcal{L}_{int}$ instead of plus the interaction Hamiltonian $\mathcal{H}_{int}$ to compute amplitudes [67]. If the interactions contain derivatives (as in some mesotron theories[28], and in QCD or scalar electrodynamics, but not in QED), there are in general extra terms in $\mathcal{H}_{int}$ as compared to $-\mathcal{L}_{int}$, but also the propagators in the Hamiltonian approach contain extra terms compared to the naive (and relativistically covariant) propagators of the Lagrangian approach[29], and both kinds of extra terms cancel each other if one computes Green functions. This allowed one to use the manifestly relativistic Lagrangian approach instead of the older, more cumbersome Hamiltonian methods based on the Coulomb gauge. Thus around 1950 a shift occurred from the Hamiltonian to the Lagrangian approach, and the era of the Coulomb gauge came to an end.

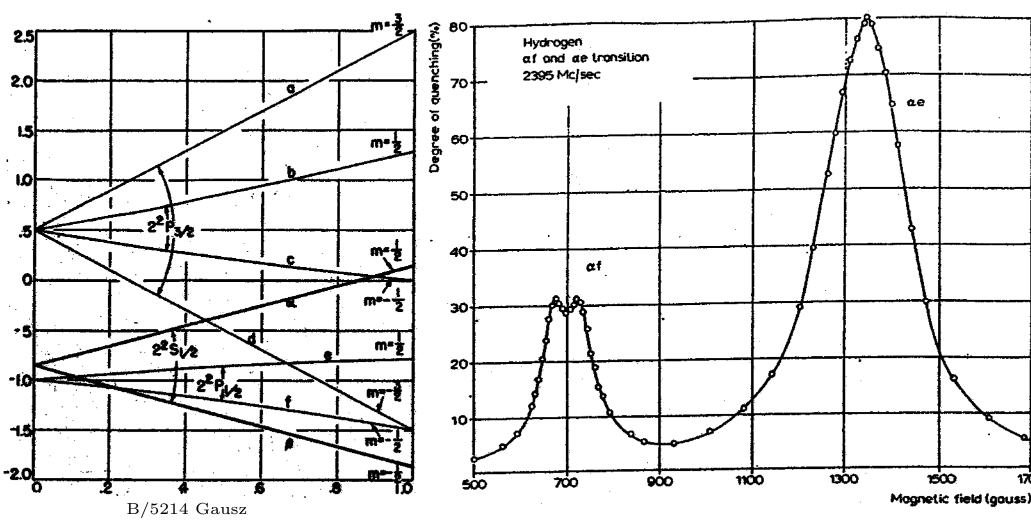
In the 1930's calculations of radiative corrections to physical processes revealed what was to become a crisis: divergences. Dirac encountered them in 1934 when he calculated the polarization of the vacuum (extended by Ühling and Serber in

[28]Mesotron theories were attempts towards a field theory of the strong interactions by exchange of spin 0 and spin 1 bosons. The first attempt was Yukawa's meson theory of 1934.

[29]For example, for scalar QED there are derivative couplings $eA^\mu\varphi^* \overleftrightarrow{\partial}_\mu \varphi$, and the propagator $\langle 0|T\partial_\mu\varphi(x)\partial_\nu\varphi^*(y)|0\rangle$ of the Hamiltonian approach differs from the covariant expression $\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial y^\nu}\langle 0|T\varphi(x)\varphi^*(y)|0\rangle$ which one uses in the Lagrangian approach by the noncovariant term $i\hbar\delta^0_\mu\delta^0_\nu\delta^4(x-y)$. (When a time derivative hits the theta function in the time-ordering symbol one obtains the equal-time commutator $[\varphi(x),\partial_\nu\varphi^*(y)]$ which is nonvanishing for $\nu=0$). One may distinguish between a covariant time-ordering T^* used in Lagrangian approaches and Lagrangian path integrals, and the T symbol of canonical approaches. In the commutation relations of currents one found extra noncovariant terms on the right-hand side (called Schwinger terms) if one used T ordering, but covariant results if one used T^* ordering. (The commutators of two time-components of currents were in general free from Schwinger terms, but in commutators involving one or two space components one found Schwinger terms.) In fact, one could prove that they were needed for Lorentz covariance and positivity. When the shift occurred from Hamiltonian methods to path integrals, the T ordering became obsolete and in much of the modern literature one no longer even bothers to write the star on the time-ordering symbol T^*, it being understood that one uses covariant propagators and vertices.

1935 to the case of an electron in a hydrogen–like atom). [68] He discarded those infinities which were field-independent, and thus developed the beginnings of charge renormalization. Weisskopf noted that the antiparticles of Dirac decreased the degree of divergences. Methods to drop the divergences and keep only the finite parts were proposed but a justification for this procedure required a theory of renormalization which was lacking[30]. We discuss this further in section I.5.

We now reach the crucial period when QED was renormalized. In 1947 several important experimental papers were published, one by Kusch and Foley [70] announcing a value for the magnetic moment of the electron which deviated from Dirac's value, another by Lamb and Retherford [71] announcing a shift between the energy of the $2\ ^2s_{1/2}$ and $2\ ^2p_{1/2}$ levels of the hydrogen atom (in Dirac theory these levels are degenerate), and further some papers on deviations from Dirac theory in the hyperfine structure of hydrogen and deuterium [72].



(1.1.16)

[30]The author met Dirac in Florida in 1978 after a seminar on supergravity. Dirac had asked at the end of the seminar "How many anticommuting variables has your theory?" Since the theory contained a local 4-component spinor $\epsilon^\alpha(x)$, the author answered "Infinity to the fourth power". Dirac only replied "That is a large number". Afterwards he told the author that doing loop calculations (in supergravity) was the wrong approach, as it would lead to the same problems that he had encountered in the 1930's. Rather, in his words, one had to first invent a whole new approach. For further information on this very logical, original but shy man, see [69].

Figure 3: The Lamb shift. In this experiment, excited hydrogen atoms in the $2\ ^2s_{1/2}$ state were made by bombarding hydrogen atoms by electrons. These excited states are metastable, whereas the $2\ ^2p_{1/2}$ and $2\ ^2p_{3/2}$ states decay rapidly (in 10^{-9} sec.) to the ground state. The $2\ ^2s_{1/2}$ atoms were led into a cavity with a magnetic field B, resulting in the Zeeman splitting indicated in the figure on the left. They were also exposed to radiowaves in the cavity. By varying the magnetic field B while keeping the frequency of the radio waves fixed, for certain values of B transitions to the p-levels (indicated by αf and αe in the figure) took place which then rapidly decayed to the ground state ($\Delta l = 1$ and $\Delta m = 1, 0$ dipole transitions, respectively). The remaining $2\ ^2s_{1/2}$ atoms were detected by moving them to a metal surface from which they could eject electrons (thereby returning to their ground state) and so be detected. In the figure on the left the slopes of the various energy levels as a function of B differ from each other due to Landé factors, and due to mixing of the two $2^2p_{1/2}$ and $2^2p_{3/2}$ levels with the same $m_j = \pm 1/2$. In the figure on the right a resonance curve is shown for hydrogen. (The splitting of the first resonance is due to hyperfine interactions [73] which split each level into a singlet and a triplet, amounting to $1420/n^3$ Mc/sec ($= 21cm$ for $n = 1$) for s levels. The hyperfine splitting of the transition αe it too small to be seen). By careful experimental study and theoretical analysis of such resonance curves, the energy separations $2\ ^2s_{1/2} - 2\ ^2p_{1/2}$ and $2\ ^2p_{3/2} - 2\ ^2p_{1/2}$ in zero magnetic field B could be determined. It was found that the $2\ ^2s_{1/2}$ level lies 1059.0 Mc/sec above the $2\ ^2p_{1/2}$ level, as shown in the figure on the left-hand side. (In this figure the energy is given in units of 7300 Mc/sec). Quantum field theory reproduces this number, but a qualitative explanation is also possible: due to the fluctuating zero-point electric field, an electron in a hydrogen atom is moved about its unperturbed position. The electron feels an effective potential $V + \Delta V$ where $V = -e^2/r$ and $\Delta V \sim \frac{1}{6}\langle(\Delta x)^2\rangle\nabla^2 V$ which only lifts the s-states. [74]

Later that same year, a conference of leading theoretical physicists was mostly devoted to discussions of these results [75]. Kramers explained there his earlier ideas of mass renormalization [76]. (As early as 1938 he had come to the conclusion that there existed radiative corrections which did not follow from Dirac theory. However, he used a nonrelativistic approach which introduces quadratic divergences (in the relativistic approach, the electron mass has only a logarithmic divergence). In his opinion "the only legitimate starting point for a theory of electrons and radiation was a classical description of an extended electron in a classical electromagnetic field. Both relativity and quantum theory were modifications and refinements grafted onto an essentially classical pictorial world view" [76]. As a result he did not discover QED). A few weeks later Bethe was able to explain most of the Lamb shift by a nonrelativistic approximation of a calculation in quantum field theory. Several others tried to include relativistic corrections. One of them was Schwinger. He started from the Dirac electron coupled to the transverse radiation field and an external Coulomb field and made a series of canonical transformations to remove the interaction with the transverse radiation field. These calculations were not yet **manifestly** relativistically invariant because he used a Hamiltonian approach to hole-theoretic calculations. As a by-product he calculated the one-loop correction to the anomalous magnetic moment, and quickly found at the end of 1947 the famous result $\alpha/2\pi$, in perfect agreement with experiment [77]. The Lamb shift was subsequently correctly computed by Fukuda, Miyamoto and Tomonoga, and Kroll and Lamb, and French and Weisskopf. However, Schwinger encountered problems during the subtraction of infinities in the calculation of the Lamb shift which were caused by the fact that his approach was not manifestly covariant.[31] The results he initially obtained for the Lamb shift did not agree with the experimental results. In the meantime, others [80] using covariant or noncovariant methods, found derivations of the Lamb shift which agreed with the experimental data. There occurred some unfortunate misunderstandings [75] but in the end all scientific questions were resolved.

In early 1948, Schwinger worked out a manifestly relativistic and gauge invariant formulation, developing a Schrödinger equation on a general spacelike surface which was manifestly Lorentz invariant. The manifest Lorentz invariance and gauge invariance resolved ambiguities in the subtraction procedure of infinities which had been

[31] For a modern derivation of the Lamb shift see [78] and [79]. One writes the exact propagator $S(\vec{x},\vec{y},E)$ in terms of wave functions for electrons and positrons as $\sum_N u_N(\vec{x})\ \bar{u}_N(\vec{y})/(E_N - E - i\epsilon) - \sum_N v_N(\vec{x})\ \bar{v}_N(\vec{y})/(E_N + E - i\epsilon)$, where $u_N = u_N^{(0)} + \delta u_N$, $v_N = v_N^{(0)} + \delta v_N$ and $E_N = E_N^{(0)} + \delta E_N$. The superscript (0) denotes quantities in Dirac theory, and δ denotes radiative corrections. One next writes S as $S^{(0)} + S^{(0)}\Sigma S^{(0)} + \ldots$. Expanding $S(\vec{x},\vec{y},E)$ the terms with $(E_N - E)^{-2}$ lead to the equation $\delta E_N = -\int d^3p\ d^3p'\ \bar{u}_N^{(0)}(\vec{p})\ \Sigma(\vec{p},\vec{p}\,',E)\ u_N^{(0)}(\vec{p}\,')$. One then calculates the proper self-energy Σ with standard Feynman rules, but with order α radiatively corrected propagators $S^{(0)}(\vec{p}\,',\vec{p}) = \frac{-i\not{p}+m}{p^2+m^2-i\epsilon}\delta^4(p-p') - ie\frac{-i\not{p}'+m}{p'^2+m^2-i\epsilon}\not{A}\frac{-i\not{p}+m}{p^2+m^2-i\epsilon} + \ldots$. Due to the approximation to order α, infrared divergences seem introduced. They cancel if one regulates properly, either by cutting the momentum interval into a high-energy and a low-energy part [78], or by using dimensional regularization and computing once with $n > 4$ and once with $n < 4$, and then joining the results at $n = 4$ [79]. Both Schwinger and Feynman made initially mistakes in the joining of these two corrections, see footnote 33.

present before,[32] and now he also found a numerical result for the energy shift that agreed with experiment. These calculations of the anomalous magnetic moment and the Lamb shift established QED as a consistent field theory [75]. For more details, see the chapter on the anomalous magnetic moment of the muon and electron.

Schwinger pioneered many other developments in quantum field theory [81]. He introduced the proper time technique (based on the heat kernel method which was developed by Fock [82]) in a study of the gauge invariance of vacuum polarization (perhaps his best paper). We discuss this further in the chapter on anomalies. He introduced the parameters to combine propagators in momentum space (related but not equal to the parameters introduced later by Feynman, which are nowadays called Feynman parameters). He developed a self-contained quantum dynamical principle, the so-called action principle, from which the equations of motion and commutation relations could be deduced, and which forms the basis of much modern fundamental research. In a study of properties of Green's functions he introduced external sources for the electromagnetic field, which we call Schwinger sources in this book, and arrived at the first exact definitions of QED in terms of differential equations, the counterpart of Feynman's definition in terms of path integrals. He also introduced Grassmann variables and coherent states into quantum field theory. For further information about him and his work see [83].

Feynman developed a completely different formalism for QED, and for quantum field theory in general. Already as a student he had been fascinated by the principle of least action in classical mechanics. With his advisor J.A. Wheeler he had tried to develop an approach to QED without the selfenergy problem of the electron by integrating out the gauge fields. The resulting theory was very complicated and did not achieve the desired result, but it provided a more global spacetime approach

[32] In addition to the magnetic interaction $\vec{\sigma} \cdot \vec{B}$ which yields the anomalous magnetic moment, there is an electric interaction $\vec{\sigma} \cdot \vec{E}$ which contributes to the Lamb shift. Special relativity fixes the coefficient of the latter in terms of the former, but Schwinger found a different value from his noncovariant calculations.

than the localized field equations. Feynman set out to find a Lagrangian formalism for QED which was based on a global spacetime description. He reinterpreted the antiparticles as particles running backwards in time (an idea first put forward by Stückelberg) whose existence was a direct consequence of special relativity, and he derived the spin-statistics relation from the requirement that the total probability of scattering processes cannot exceed unity [9]. In this way he arrived at the path integral formalism in the way it is used today. We discuss further historical aspects of path integrals in section 5. One advantage of his approach was that it no longer needed arbitrary spacelike surfaces for relativistic invariance. Another advantage was that the perturbation expansion of path integrals led to a formulation of quantum field theory in terms of simple space-time pictures, which became known as Feynman diagrams. Most importantly, the simple pictorial formulation combined with its manifestly relativistic invariance made it easy to perform calculations[33] and to subtract divergences in an unambiguous way [88]. For further information about Feynman and his work, see [89].

Earlier, Tomonaga and collaborators had independently arrived at the same results as Schwinger and Feynman, but due to the absence of suitable communication during and after the second World War their work initially went unnoticed [75]. The approaches of Tomonaga, Schwinger and Feynman were shown to be equivalent by Dyson [90] who not only showed how Feynman graphs followed from Schwinger's approach, but, more importantly, showed how to renormalize to all orders, something

[33] Yet even Feynman made a mistake which has been repeated in more recent times. [25] He gave the photon a small mass to regularize the infrared divergences, but he failed to sum over 3 instead of 2 polarizations of this massive vector boson. In fact, Schwinger made the same mistake. Later, Bass and Schrödinger [86] wrote a very readable article in which they explained that for physical observables the limit of vanishing photon mass is continuous. For example, a massless photon leads to an energy kT and a massive photon yields $3/2\ kT$, but in a cavity the longitudinal polarization ceases to interact with the walls as the mass tends to zero and leaks out of the cavity, yielding a continuous limit for vanishing photon mass. For gravity, the zero-mass limit is discontinuous; the helicity-zero part of a massive spin 2 field in a space that is asymptotically Minkowskian does not decouple if the mass tends to zero [87].

which nobody else had achieved. He made a systematic analysis of divergences in general Feynman graphs, and derived criteria when all infinities could be absorbed by redefining the parameters of the theory (the masses, coupling constants and wave functions). It was Dyson who introduced the concepts of primitive divergences, skeleton graphs, and overlapping divergences, and with his work a systematic approach to renormalization became available.

With the work of Tomonaga, Schwinger, Feynman, Dyson, Bethe, Weisskopf, French, Lamb, and many other theorists, the early wrestlings of Dirac, Heisenberg, Pauli, Fermi and their successors with the infinities in loop calculations in QED got a happy ending. QED became the queen of theoretical physics.[34]

However, "mesotron" theories were studied to explain the strong interactions, and they still yielded divergences. Since the coupling constant of the strong interactions was not small, perturbation theory seemed inapplicable. First dispersion relations, which were supposed to describe nonperturbative field theory, were studied. In a few cases they yielded quantitative successes, but it was not a full-fledged theory. Many physicists turned away from quantum field theory, and started developing S-matrix theories. In the 1950's and 1960's it seemed that QED was a happy exception to a general breakdown of quantum field theory.

2 Weak interactions

Nuclear β decay has an interesting history by itself, and has led to the modern gauge theory of electro-weak interactions. In the beginning of 1896, a few months after

[34]More than a decade later, trouble for field theory seemed again to arise. Anomalies in processes involving chiral fermions were discovered [91] and it initially seemed these could not be described by a path integral approach; in fact, some well-known physicists even claimed anomalies did not exist. It was only through the work of Fujikawa [92] that anomalies were found to reside in the Jacobian for a gauge transformation of the integration variables of the path integrals for gauge theories. In QED some of these anomalies were even welcome because they explained why the π^0 and η meson decay into two photons. We discuss this further in the chapter on anomalies.

Röntgen had discovered X-rays (at the end of 1895), Becquerel discovered by accident β rays during his investigations of X-rays. He placed a sample of uranium ore in the same drawer as a photographic plate and observed that "The phosphorescent substance in question emits radiations which traverse paper opaque to light".[35] Later β rays were identified with electrons; for example, unlike the X-rays, they were deflected by magnetic fields. Rutherford found that the α particles emitted from nuclei were mono-energetic, and for some time it was believed that the β decay electrons were also mono-energetic. Subsequently, measurements showed that the electrons had a continuous spectrum, but it was not clear whether the electrons were emitted with a unique energy from the nucleus and then lost energy due to Bremsstrahlung, or whether they were emitted with a continuous β spectrum. However, in 1914 Chadwick (the same who discovered the neutron in 1932) showed that the electrons were emitted with a continuous spectrum. Some unorthodox explanations were put forth (for example, violation of energy conservation by Bohr, Kramers and Slater). In 1930 Pauli proposed that energy is conserved in nuclear β decay but a new neutral spin 1/2 particle is emitted, which he called the neutron. (The name neutrino was later introduced by Fermi, to distinguish it from the real neutron which was discovered in the beginning of 1932 by Chadwick. [7]) With the discovery of the neutron and the proposal of a neutrino, the problems with the spin-statistics properties of nuclei disappeared,[36] and one now needed a field theory for β decay.

In order to be able to construct a field theory for β decay a fundamental new

[35]W.C. Röntgen was awarded the first Nobel prize in physics in 1901 "for his discovery of the remarkable rays subsequently named after him", and H.A. Becquerel got half of the third Nobel prize in 1903 "for his discovery of spontaneous radioactivity". (The other half went to P. and Mme. Marie Curie "for their joint researches on the Radiation phenomena discovered by Professor Henri Becquerel". The second Nobel prize went in 1902 to H.A. Lorentz and P. Zeeman).

[36]Before the neutrino was conceived, nuclei were thought to be bound states of protons and electrons. For example, the nitrogen nucleus with $A = 14$ and $Z = 7$ required 14 protons and 7 electrons, and should thus have half-integer spin. However, band spectra showed that it had spin 1. Another problem with the proton-electron model of the nucleus was that nuclei would have far too large magnetic moments (a factor 1000 too large) if they contained electrons. See ref. [1], section 14.b.

concept in field theory was introduced by Fermi in the 1930's: the notion of particle creation. In Schrödinger and Heisenberg's theory of quantum mechanics, an electron could make a transition from one level to another in the atom, but it remained an electron. Transitions could be induced by an external classical field but that external field was treated as an external field and not as a collection of photons. The concept of particle creation started with Born, Heisenberg and Jordan's 1925 one-dimensional model of photons [22] and Dirac's 1927 transverse quantized radiation field [17] which could create or annihilate photons. Since moving charges produced radiation in classical electrodynamics, it became clear that photon creation and annihilation should also be part of a realistic quantum theory of radiation. This raised a problem: where did the radiated photons come from? Dirac solved this problem by introducing the concept of the "zero state" which contained an infinite number of zero-momentum zero-energy photons. "When a light-quantum is absorbed it can be considered to jump into this zero state, and when one is emitted it can be considered to jump from this zero state to one in which it is in physical evidence, so that it appears to be created ..." [30]. In Dirac's hole theory of the electron coupled to the transverse radiation field [42], the electrons were not created or annihilated; rather, an electron in a negative energy state could be elevated to a state with positive energy, thus creating an electron-positron pair. On the other hand, β decay offered great conceptual problems, as we now discuss.

In nuclear β decay $n \to p+e^-+\bar{\nu}_e$ the transition $n \to p$ in a nucleus did not change the number of nucleons and could perhaps still be compared with the transition of an electron from one bound state to another in an atom, but the emission of an electron and a neutrino posed a real puzzle. If one tried to use Dirac's hole theory, the hole left by the negative-energy electron would have to correspond to the neutrino which would violate charge conservation. The electron and neutrino which are emitted in β decay could not already be present in the nucleus before the β decay took place because this would require too much energy due to the uncertainty principle. Therefore Fermi did not introduced a "zero state" for electrons and neutrinos, but postulated that at the moment of β decay the electron and neutrino are **created**. He wrote down 4-component Dirac spinors for both, expanded them into plane waves, and, following Jordan and Wigner [40], interpreted their coefficients as annihilation and creation operators which satisfied anticommutation relations.

To describe the β decay $n \to p + e^- + \bar{\nu}$ Fermi decided in 1933 and 1934 [93] to mimic the electromagnetic interactions $j^\mu A_\mu$. Since the proton and neutron are almost at rest in the nucleus, he began by restricting his attention to the static nonrelativistic part $\rho\phi$. He replaced ρ by the matrix element of the Pauli spin matrix τ^+ (introduced by Heisenberg in 1932 to describe the nuclear forces) between isotopic doublets of proton and neutron wave functions. Next he replaced ϕ by the product of spinors $\psi_\nu^\dagger \psi_e$ $(= \bar{\psi}_\nu \gamma_4 \psi_e)$. To make his theory relativistically invariant, he then replaced the densities by vector currents. In modern notation he wrote[37]

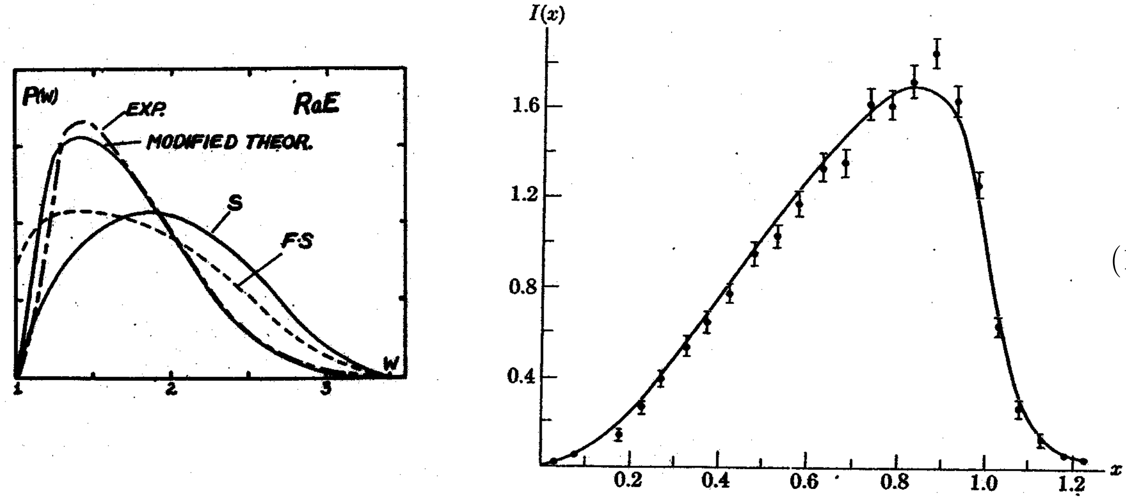
$$\mathcal{L} = G_F(\bar{\psi}_N \gamma^\mu \tau^+ \psi_N)(\bar{\psi}_e \gamma_\mu \psi_\nu) \text{ with } \psi_N = \begin{pmatrix} \psi_p \\ \psi_n \end{pmatrix} \tag{1.2.1}$$

(Actually, he did not distinguish between neutrinos and antineutrinos, and he mentioned the possibility of other relativistic invariants). So parity was assumed from the beginning, as were baryon and lepton number conservation, and this theory described very well those nuclear β decays in which the spin of the nucleus did not change.[38]

[37] Although posterity has focused on his current-current interaction, the real work for Fermi was to calculate the relativistic Coulomb functions for the electrons. Because only natural radioactive elements were available to him (artificial radioactivity was not yet discovered) which have high Z, the Coulomb corrections were large.

[38] Fermi constructed his current-current interaction for nuclear β decay by mimicking the minimal couplings to currents in QED, and from this perspective it is natural that he did not include derivatives in his four-fermion theory. However, experiments at that time indicated that Fermi's model predicted too few low energy electrons in β decay. To remedy this defect, another model was proposed, with a derivative on the neutrino field, namely $\bar{\psi}_e \gamma_\mu \psi_\nu$ was replaced by $\bar{\psi}_e \partial_\mu \psi_\nu$. This favoured higher-energy neutrinos, and thus lower-energy electrons. For several years this KU theory [94] dominated β decay, but then it was found that all experiments had been wrong: they had included secondary effects due to thick sources [1]. With thinner sources, Fermi's theory reemerged as the correct model, and the KU theory disappeared from the stage.

Most importantly, he checked the universality of G_F: different radioactive elements had absolute decay rates which agreed with the same value of G_F.



(1.2.2)

Figure 4: On the left: The energy distribution of the electrons in β decay of R_aE. The areas under the curves have been made equal to each other. S denotes the statistical distribution (phase space), FS the results according to Fermi's theory and Modified Theory refers to the results predicted by the KU theory. With thinner sources the Fermi theory agrees best with experiment. On the right: the electron distribution in μ-decay according to Fermi's proposal in (1.2.1), but with radiative corrections included.

To describe β decay processes in which the spin of the initial nucleus differed from that of the final nucleus, Gamov and Teller proposed a year later also to use axial vector currents $\bar{\psi}_p\gamma_\mu\gamma_5\psi_n$ and tensor currents $\bar{\psi}_p\gamma_{[\mu}\gamma_{\nu]}\psi_n$ [95] to describe transitions in which the nuclear spin changed. This in turn suggested to begin with the most general interaction invariant under Lorentz transformation and parity, and without derivatives

$$H_{\text{int}} = \frac{G_F}{\sqrt{2}} \sum_I C_I(\bar{\psi}_p O_I \psi_n)(\bar{\psi}_e O_I \psi_\nu) + h.c. \tag{1.2.3}$$

where $O_I = \left\{ I, \gamma_\mu, i\gamma_\mu\gamma_5, \frac{i}{2}[\gamma_\mu, \gamma_\nu], \gamma_5 \right\}$ and C_I are arbitrary constants (the normalization of G_F was fixed by taking $C_I = 1$ for $O_I = \gamma_\mu$).

The Fermi-Gamov-Teller theory (also called the four-fermion theory) did describe nuclear β decay very well, and later also the decay of the muon (discovered in 1937, but erroneously first identified as the pion, see below). The coupling constant for μ decay was of the same order of magnitude as the coupling constant for nuclear β decay, and this led people to propose in the late 1940's the notion of universality of the weak interactions: electrons, muons and nucleons (later quarks) all couple in the same way, and their weak interactions are given by

$$\begin{aligned} H_{\text{int}} &= \frac{G_F}{\sqrt{2}} \sum_I C_I [\bar{\psi}_p O_I \psi_n + \bar{\psi}_\nu O_I \psi_e + \bar{\psi}_\nu O_I \psi_\mu] \\ & \quad [\bar{\psi}_n O_I \psi_p + \bar{\psi}_e O_I \psi_\nu + \bar{\psi}_\mu O_I \psi_\nu] \end{aligned} \tag{1.2.4}$$

One could now describe all weak interactions by this theory

$(\bar{p}n)(\bar{e}\nu)$	nuclear β decay
$(\bar{\nu}e)(\bar{n}p)$	proton decay in nuclei
$(\bar{p}n)(\bar{n}p)$	weak nuclear forces
$(\bar{\nu}e)(\bar{e}\nu)$	electron-neutrino scattering
$(\bar{e}\nu)(\bar{\nu}\mu)$	μ decay
$(\bar{n}p)(\bar{\nu}\mu)$	μ-capture in nuclei

This universal four-fermion theory was very successful, despite the fact that it was nonrenormalizable. Also an understanding of nonleptonic weak decays was developed, and applied to such decays as $\Lambda \to p + \pi$.

In 1956 a puzzle arose: charged K mesons were found to decay into two pions, but also into three pions. One explanation was that there were really two particles, one (θ) decaying into two pions, the other (τ) into three pions. The puzzle then was that they had nearly the same mass (the $\tau - \theta$ puzzle). Another explanation was that there was only one particle but that its decay would violate parity. Lee and Yang realized that no experiments had ever demonstrated that parity is conserved in radioactive decays, and proposed experiments in which to check whether parity might be violated by the weak interactions [96]. Soon afterwards parity was shown indeed to be violated in weak interactions [97].

It was known from the electron spectrum in μ decay that "the" neutrino[39] is massless (or nearly massless). Several physicists [98] suggested to use Weyl's 1929

[39]That there are separate muon neutrinos and electron neutrinos was subsequently discovered in 1962

theory of two-component fermions.[40] Not only did this theory explain why neutrinos had to be massless, but it also implied parity violation, charge conjugation violation but conservation of CP symmetry,[41] and led to a universal theory of pion decay and muon decay. An immediate argument in favor of the chirality of the neutrino was provided by pion decay: π^+ decays almost entirely into $\mu^+ + \nu_\mu$ but very little into $e^+ + \nu_e$. Since the positron can be viewed as almost massless in this decay, the channel $\pi^+ \to e^+ + \nu_e$ would be strongly suppressed if e^+ and ν_e have opposite helicities.[42] They should then occur in the V, A combination $\bar{\psi}_\nu(1 \pm \gamma_5)\gamma_\mu\psi_e$, but not in the S, P, T combination $\bar{\psi}_\nu(1 \pm \gamma_5)\psi_e$ and $\bar{\psi}_\nu(1 \pm \gamma_5)\gamma_{\rho\sigma}\psi_e$. After a period of confusion, pion decay and He^6 β-decay pointed indeed to a combination of V and A interactions.

In a classic experiment, using nuclei which emit neutrinos and thereby decay into other nuclei which emit photons, M. Goldhaber et al. confirmed that the helicity of the electron neutrino is antiparallel to its momentum [100]. This experiment is even for abstract theorists such a delight that we cannot resist describing it here briefly. A spin zero nucleus ${}^{152}Eu$ decays to the excited spin 1 nucleus ${}^{152}Sm^*$ by electron capture from the K-shell under emission of a neutrino with energy $0.840 MeV$. This excited nucleus decays in 3.10^{-14} seconds to its spin 0 ground state under emission of a photon with energy $0.960 MeV$. Because the energies of the neutrino and the

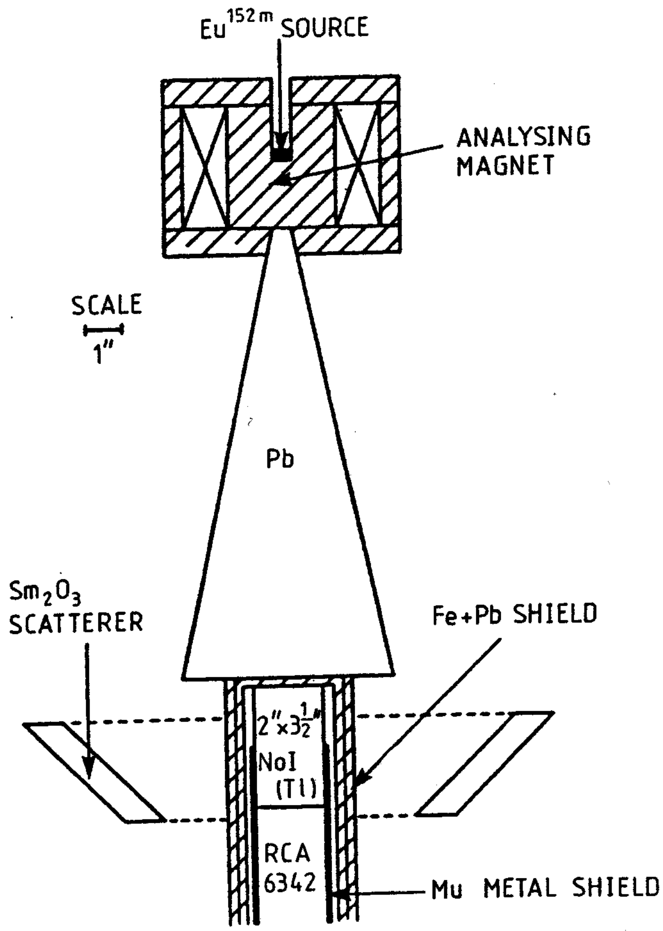
at Brookhaven, by letting the neutrinos from the π-decay $\pi^+ \to \mu^+ + \nu_\mu$ collide with matter. It was found that the process $\nu_\mu + p \to n + \mu^+$ occurs but $\nu_\mu + p \to n + e^+$ did not occur. [99]

[40] Weyl considered a massless Dirac spinor whose first two components described the electron and whose last two components described the proton. Under Lorentz transformations they do not mix, so one could study chiral components separately. A mass term could not be written down, but Weyl believed that with a better understanding of gravity that problem might be cured [43]. However, parity was violated in Weyl's theory, and thus this theory was, as Pauli erroneously wrote in the Handbuch der Physik, "inapplicable to physical reality".

[41] Under a parity transformation P, the direction of momentum is reversed but not its spin direction, hence P maps a physical state to a nonexistent state. This violates mirror symmetry. Under charge conjugation C, a neutrino changes into an antineutrino, but its momentum and spin are not changed, hence C maps a neutrino to an antineutrino with the wrong helicity. So, C is also violated. It is clear from these arguments that CP is conserved.

[42] Two-component neutrinos are necessarily massless (at least if Majorana mass terms $\epsilon_{\alpha\beta}\psi^\alpha\psi^\beta$ with $\alpha, \beta = 1, 2$ are excluded; these violate fermion number) and have definite handedness.

photon are almost equal, if the photon is emitted in the same direction as $^{152}Sm^*$ is travelling[43], its energy is boosted just enough that it can scatter resonantly from a target with Sm_2O_3 molecules at rest; the scattering angle is then 90 degrees.



$$\begin{array}{l} \nu \\ Eu \rightarrow Sm^* + \nu \\ Sm^* \\ Sm^* \rightarrow Sm + \gamma \\ \gamma \\ Sm + \gamma \rightarrow Sm^* \\ \qquad\qquad\quad \swarrow \\ Sm + \gamma \end{array} \tag{1.2.5}$$

Figure 5: Apparatus in the experiment of Goldhaber, Grodzins and Sunyar to determine the neutrino helicity by measuring circular polarization of resonant scattered γ-rays. The apparatus has cylindrical symmetry about the vertical axis.

The helicity of the photon was determined by passing it through the magnetized iron of an analyzing magnet. (The absorption cross section of a circularly polarized photon depends on the spin of the scattering electrons. One helicity is maximally absorbed, so all photons which emerge from the iron have a unique helicity which follows from the theory of absorption and has been verified experimentally). Conservation of

[43]The nucleus travels over a distance of an Angstrom, and can thus be considered to be free.

angular momentum then showed that neutrinos are left-handed.[44]

The result for the helicity of the neutrino and the fact that the weak interactions contain only vector and axial-vector interactions and violate parity maximally, led to the universal $V-A$ theory. Marshak and Sudarshan studied all the data (both the off-diagonal interactions between hadrons and leptons, and the diagonal terms) and found that the $V-A$ interaction gave the best fit [101]. This theory could be deduced by requiring invariance under $\psi \to \gamma_5\psi$ for any of the fermions. Two subsequent papers, one by them and one by Feynman and Gell-Mann, obtained the following interaction Lagrangian

$$\begin{aligned}\mathcal{L}_{\text{int}} = \quad & \frac{G_\mu}{\sqrt{2}}[\bar{\psi}_p(1-\gamma_5)\gamma^\sigma\psi_n + \bar{\psi}_\nu(1-\gamma_5)\gamma^\sigma\psi_e + \bar{\psi}_\nu(1-\gamma_5)\gamma^\sigma\psi_\mu] \\ & [\bar{\psi}_n(1-\gamma_5)\gamma_\sigma\psi_p + \bar{\psi}_e(1-\gamma_5)\gamma_\sigma\psi_\nu + \bar{\psi}_\mu(1-\gamma_5)\gamma_\sigma\psi_\nu] \qquad (1.2.6)\end{aligned}$$

where G_μ was determined by μ decay.

The Hamiltonian in (1.2.6) was not yet quite correct. The weak hadronic current $\frac{G_\mu}{\sqrt{2}}\bar{\psi}_p(1-\gamma_5)\gamma^\sigma\psi_n$ was replaced by $\frac{1}{\sqrt{2}}\bar{\psi}_p(G_V - G_A\gamma_5)\gamma^\sigma\psi_n$ where the experimental value $G_A/G_V = 1.22 \pm 0.02$ was attributed to renormalization effects of the axial vector coupling constant due to strong interactions. The vector coupling constant G_V was believed not to be renormalized by the strong interactions because it was assumed to be conserved. In fact, the charged hadronic vector current of the weak

[44]The z-component of the spin of the captured electron is equal to the sum of the z-components of the spins of the neutrino and the $^{152}Sm^*$ nucleus. However, the z component of the $^{152}Sm^*$ nucleus can only be ± 1 if it emits a photon into the forward direction. So the z-components of the spins of the neutrino and the $^{152}Sm^*$ nucleus are opposite. Since the $^{152}Sm^*$ nucleus and the neutrino move in opposite directions, **the helicity of $^{152}Sm^*$ is the same as the helicity of the neutrino**. The helicity of the $^{152}Sm^*$ nucleus is the same as that of the photon. (The helicity of the nucleus is $m = \pm 1$, but it cannot be $m = 0$, because then no photon can be emitted because the final nucleus has spin zero). Hence the helicity of the photon is equal to the helicity of the neutrino, and measuring the former showed that neutrinos are left-handed. Notice that in the region around the source there are lots of photons with all kinds of energies but using resonant scattering selected only photons which were emitted in the forward direction. The metal shield around the detector served to stop soft x-rays, and also to prevent the magnetic field near the source from interphering with the photon detector.

interactions and the isospin-one part of the electromagnetic current were assumed to be all conserved and to form an isospin-one triplet (conserved vector current) hypothesis[45] [103]. For a while it was believed that universality demanded $G_V = G_\mu$.

Careful measurements of the Fermi constant G_μ in μ decay and Fermi constant G_V for the vector current contribution to ^{14}O decay revealed, however, that G_V was about 3% lower than G_μ. This posed a problem because the vector current was conserved, and should therefore not be renormalized by the strong interactions. Gell-Mann and Levy [104] proposed that the vector current was of the form $G_\mu p\gamma_5\gamma^\mu(n\cos\theta+\Lambda\sin\theta)$, with $G_V = G_\mu\cos\theta$ for the strangeness conserving current (pn), and $G_\mu\sin\theta$ for the strangeness violating current $(p\Lambda)$. With $G_V/G_\mu \approx 0.97$, the strong suppression for $\Lambda \to p+e^-+\nu_e$ was explained. Cabibbo went a step further, and proposed [105] that these currents were members of an $SU(3)$ octet. Thus was universality of the weak interactions rescued; the angle $\theta \simeq 0.25$ became known as the Cabibbo angle.

Fermi's 1932 idea of mimicking the electromagnetic interactions $j^\mu A_\mu$ had come a long way[46], but the $V-A$ theory was nonrenormalizable. To obtain a renormalizable theory of the weak interactions, intermediate vector bosons had been proposed at the end of the 1940's (W bosons) [106], which played a similar role as A_μ. These W_μ fields coupled to the charged current $j_\mu^- = \bar{\psi}_\nu\gamma_\mu(1+\gamma_5)\psi_e$ and its hermitian conjugate j_μ^+, and the electromagnetic fields A_μ coupled to the electromagnetic current $j_\mu^{\mathrm{EM}} = \bar{\psi}_e\gamma_\mu\psi_e$.

If the $W^\pm$ bosons would belong to a $U(1)$ gauge group, one could not explain why they coupled with equal strength to electrons and muons, but if the gauge group was nonabelian, the coupling strength should depend only on the multiplets in which the fermions were, and should thus be the same for electrons and muons. This

[45]These conserved vector currents contained not only contributions from the nucleons, but also from pions and other hadrons. This allowed a prediction of the weak decay $\pi^+ \to \pi^0 + e^+ + \nu_e$.

[46]Fermi had sent his paper originally to the prestigious journal Nature, but it was rejected for publication because, as the referee wrote "... it contained speculations too remote from reality to be of interest to the reader" (see Fermi's collected papers).

was indeed what experimentalist found. An action and transformation rules for a general nonabelian gauge theory had already been written down by Yang and Mills in 1954 [107]. Schwinger [108] proposed a $SU(2)$ gauge group, with A_μ and $W^\pm_\mu$ forming a triplet of gauge fields, and (μ^+, ν_e, e^-) and (e^+, ν_μ, μ^-) forming triplets of leptons. The problem with this theory was that it violated electron-number and muon-number. Bludman [109] proposed another $SU(2)$, in which $W^\pm_\mu$ formed a triplet not with A_μ but with a new neutral massive vector boson Z_μ which should couple weakly to neutral currents. Glashow [110] proposed a model with $SU(2) \times U(1)$ as gauge group and four gauge bosons, $W^\pm_\mu$, A_μ and Z_μ; in his model the leptons formed doublets instead of triplets.

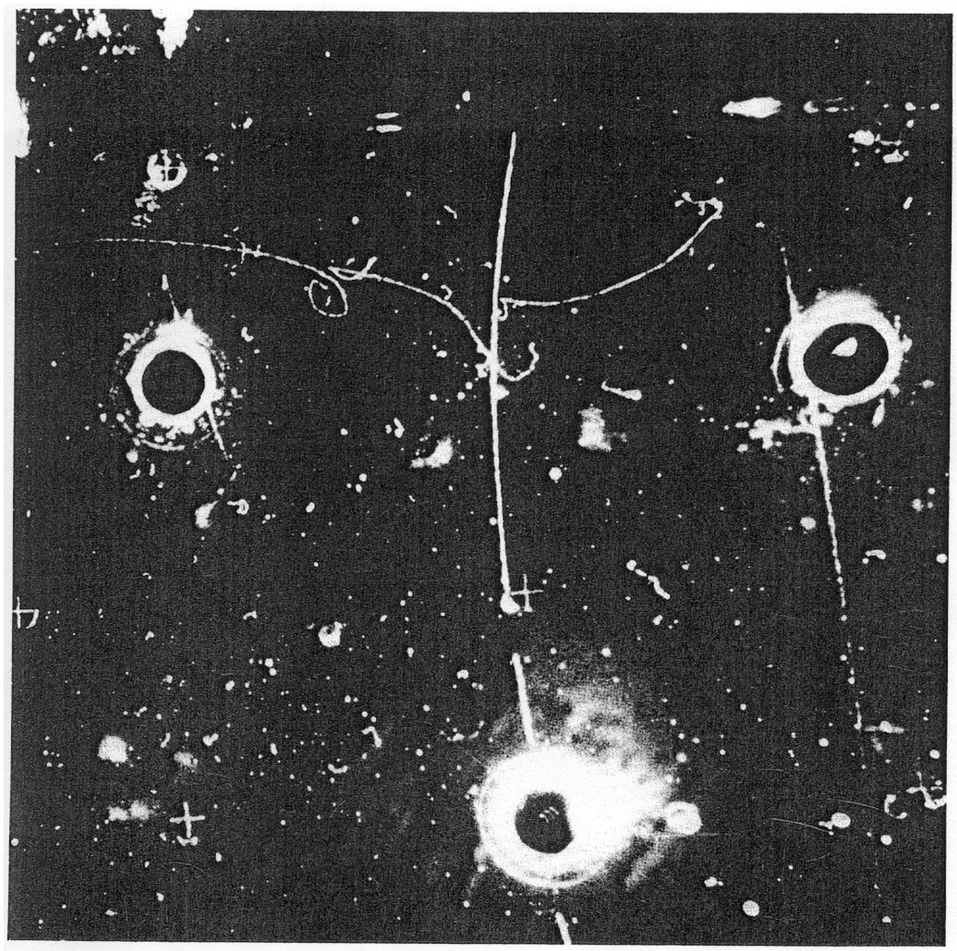
Current commutation relations seemed to favor the $SU(2) \times U(1)$ model. The commutator of j^+_μ with j^-_μ produced not j^{EM}_μ but a new neutral current $\bar{\psi}_e\gamma_\mu(1 + \gamma_5)\psi_e - \bar{\psi}_\nu\gamma_\mu(1 + \gamma_5)\psi_\nu$, to which a new neutral vector boson (the Z boson) should couple. More specifically, the current commutation relations for the weak currents j^+_μ and j^-_μ and the electromagnetic current j^{EM}_μ formed the group $SU(2)\times U(1)$, and this suggested that the Z and W bosons belonged to a nonabelian gauge theory with gauge group $SU(2) \times SU(1)$.

However, at that time neutral currents had not yet been detected. It seemed more natural to construct a model without neutral currents. The obvious choice was the $SU(2)$ model of Schwinger with $W^\pm$ and the photon A_μ in one triplet. Fermions were also put into triplets. One could not use doublets instead of triplets for the leptons, because if the vector boson triplets would couple to lepton doublets as $\bar{\mathcal{E}}\ \gamma^\mu \vec{W}_\mu \cdot \vec{\tau}\ \mathcal{E}$, the photon would couple to neutrinos (just as Z bosons couple to neutrinos in the Standard Model). Furthermore one needed new heavy leptons E^+ and E^0 to avoid violating lepton number; for example, replacing E^+ by the positron e^+ would violate electron number. To achieve that $W^\pm$ couple much weaker to the weak current $\bar{\nu}_L(1+\gamma_5)e^-$ than photons to the QED current, a weak mixing angle β was introduced. This weak triplet with the electron was then $\mathcal{E} = (E^+,\ E^0_R + \cos\beta\ E^0_L + \sin\beta\ \nu_L,\ e^-)$.

The result was the $SO(3)$ model of Georgi and Glashow [111] with the following coupling to electrons.

$$e\, \vec{\vec{\mathcal{E}}} \times \vec{\mathcal{E}} \cdot \vec{W} = eA_\mu(\bar{E}\gamma^\mu E - \bar{e}\gamma^\mu e) + \left[\frac{1}{2}e\,\sin\beta\, W_\mu^-\, \bar{e}\gamma^\mu(1+\gamma_5)\nu + h.c.\right] \tag{1.2.7}$$

The question in those days was thus: neutral currents or heavy leptons? Experiments found later neutral currents and $W^\pm$ bosons with a mass of 83 GeV, ruling out the $SO(3)$ model.[47]



(1.2.8)

Figure 6: A neutral current event, the process $e+\nu \to e+\nu$ taken in the CERN heavy-liquid bubble chamber Gargamelle. On top an electron is accelerated which emits 5 cm lower a photon which in

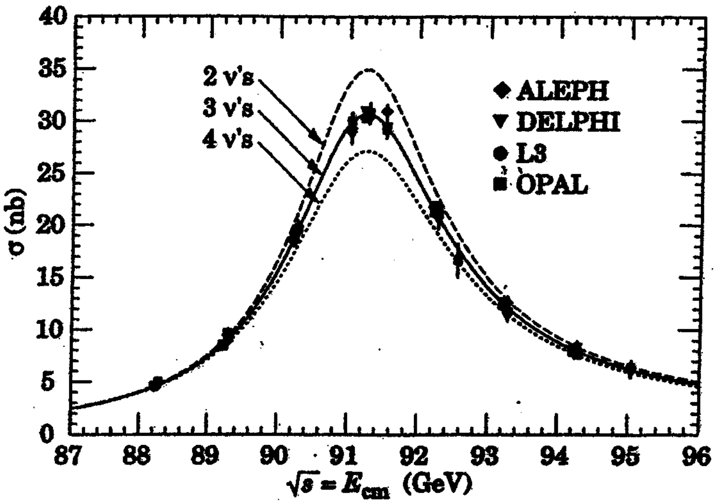
[47]The mass of the vector bosons $W^\pm$ follows from equating the tree graph for $e^-\ \nu \to W^- \to e^-\ \nu$ to the four-fermi coupling $(G_F/\sqrt{2})\ \bar{e}\gamma^\mu(1+\gamma_5)\nu\ \bar{\nu}\gamma_\mu(1+\gamma_5)e$ with $G_F = 1.1\cdot 10^{-5}/(GeV)^2$. This predicted that the intermediate vector bosons of the $SO(3)$ model had a mass of $M_W = \frac{1}{2}e\,\sin\beta\ (G_F/\sqrt{2})^{-1/2}$, so less than 53 GeV (the value when $\sin\beta = 1$).

turn produces an e^+e^- pair on the right (with a long e^- track and a short e^+ track). This identifies the event as $e + \nu \to e + \nu$ because protons emit far fewer photons (being heavier). A confirmation that one is dealing with an electron is the thickening of the track near the bottom where the electron slows down (protons take longer to slow down). (Photo courtesy CERN.)

A model for the weak interactions of leptons coupled to $SU(2) \times U(1)$ gauge fields was proposed by Weinberg [112] generalizing earlier proposals of Schwinger [108] and Glashow [110]. In this model, leptons formed $SU(2) \times U(1)$ doublets. Quarks were put into $SU(2) \times U(1)$ doublets in [113], and coupled to the electroweak gauge fields in [114]. Salam put the leptons in a triplet of $SU(3)$, which was equivalent to the $SU(2) \times U(1)$ model as long as no tau lepton and tau neutrino had been discovered. [115]

Despite these developments in the construction of the action for the weak interactions, there remained the problem with divergences in loop corrections. In the absence of a consistent fundamental theory of the weak and strong interactions, one used currents which were supposed to satisfy certain current commutation relations ("current algebra", introduced by Gell-Mann [5]). For currents constructed from free fields one could justify these relations by the canonical (anti) commutation relations of the constituent fields. These latter were also supposed to hold for interacting Heisenberg fields but here problems with renormalization were found to occur. Also extra singular terms in the current commutation relations were found (Schwinger terms). In particular it was found in the process $\pi^0 \to \gamma + \gamma$ that at the quantum level one could not maintain simultaneously local vector gauge invariance and local axial vector gauge invariance. This was clearly exhibited in the simple linear σ model[48] of Gell-Mann and Levy [104], and led to the concept of chiral anomalies [117]. Chiral anomalies gave the correct decay rate for $\pi^0 \to \gamma + \gamma$ (to obtain the correct decay rate in the quark model, one needed 3 colors). However, anomalies in the chiral gauge symmetry of the weak interactions would violate unitarity, and were to be avoided.

[48] This beautiful little model played also a role in the renormalization of spontaneously broken gauge theories as we shall discuss in section I.5



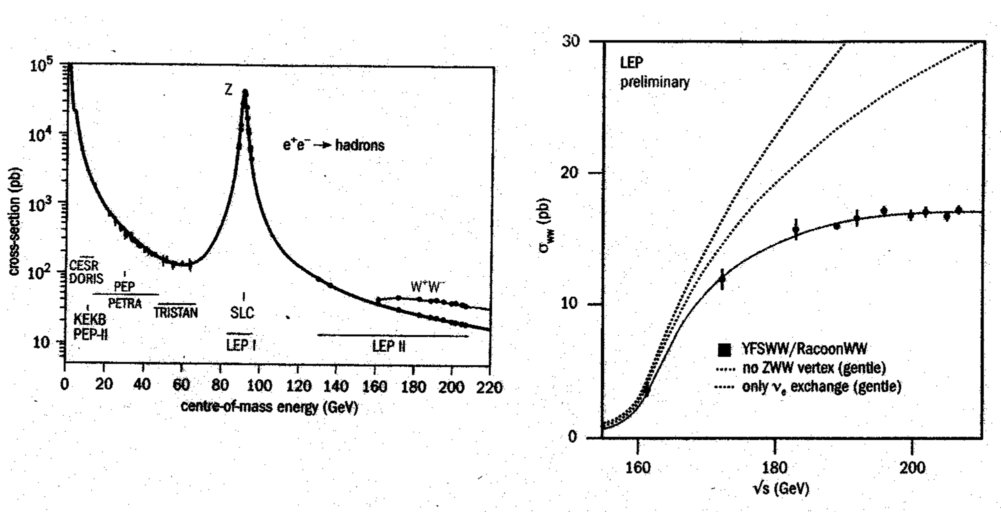
(1.2.9)

Figure 7: Fit to Z resonance for $e^+e^- \to$ hadrons within the Standard Model. Note that the hadronic cross section at its peak comes down when there are more neutrinos because the total width Γ_Z of the Z resonance in the relativistic Breit-Wigner formula $\sigma_{had} = \frac{12\pi}{m_Z^2} \frac{s\Gamma_{ee}\Gamma_{had}}{(s-m_Z^2)^2+\Gamma_Z^2 m_Z^2}$ increases while Γ_{ee} and $\Gamma_{\rm had}$ stay fixed. At the maximum, $s = m_Z^2 + \frac{1}{2}\Gamma_Z^2$ or $E = m_Z + \frac{1}{4}\Gamma_Z^2/m_Z$, and the full width at half maximum is Γ_Z for any decay mode, and for any number of neutrinos. From these data one has found: $m_Z = 91,187.5 \pm 2.1$ Mev, $\Gamma_Z = 2495.2 \pm 2.3$ MeV and $\sin^2\theta = 0.23147 \pm 0.00016$.

Finally, in 1971, 't Hooft and Veltman [118] proved the renormalizability of non-abelian gauge theories in general. For the electroweak interactions, renormalizability and unitarity required that chiral anomalies due to fermion triangle graphs cancelled, and this could be achieved if the sum of all charges of leptons and quarks of one family vanished [119]. In the $V - A$ theory, the discrete symmetries C (charge conjugation) and P (parity) were violated, but CP was preserved. For example, a left-handed neutrino state is transformed under C into left-handed antineutrino state, and under P into right-handed neutrino state, both of which do not exist, but under CP it is transformed into right-handed antineutrino, which exists. Also the $V-A$ interactions preserve CP symmetry. However, in 1964 experiments showed that CP is violated in the $K^0 - \bar{K}^0$ system [120]. A K^0 produced by a strong interaction process can be expanded into CP eigenstates as $K^0 = \frac{1}{2}(K^0 + \bar{K}^0) + \frac{1}{2}(K^0 - \bar{K}^0)$. The combinations $K_S = K^0 + \bar{K}^0$ and $K_L = K^0 - \bar{K}^0$ have $CP = +1$ and $CP = -1$, respectively, and K_S decays quickly into two pions while K_L decays later into three pions. However, CP

violating $K^0\bar{K}^0$ oscillations occur, similar to neutrino oscillations, according to which a K_L could oscillate back into K_S and decay into two pions. Subsequently other CP violating processes were found to occur: for example, the decay $K^0_L \to e^+ + \nu_e + \pi^-$ occurs 0.33% more often than the CP conjugated process $K^0_L \to e^- + \bar{\nu}_e + \pi^+$. Much experimental work is going on to study CP violation in the decays of B mesons (mesons with bottom quarks).

Theoretically it was shown that in the new renormalizable electroweak gauge theory CP violation can only occur if there are at least three families [121]. Careful measurements of the width of the Z boson at CERN and SLAC showed that there are exactly three families in the SM (more precisely: there are no more than three nearly massless neutrinos in the Standard Model). Direct experimental evidence of a 3-gluon coupling was extracted from data on four-jet final states of Z-boson decays. Charged W^+W^- pairs produced in e^+e^- collisions by neutrino exchange and photon and Z-boson resonances showed a steeply rising cross-section near threshold, and from these data the mass and width of the W boson could be determined. They were in excellent agreement with values extracted from radiative corrections. Thus a consistent quantum theory for QED and the weak interactions, based on renormalizable gauge field theory, was constructed in a decade.



(1.2.10)

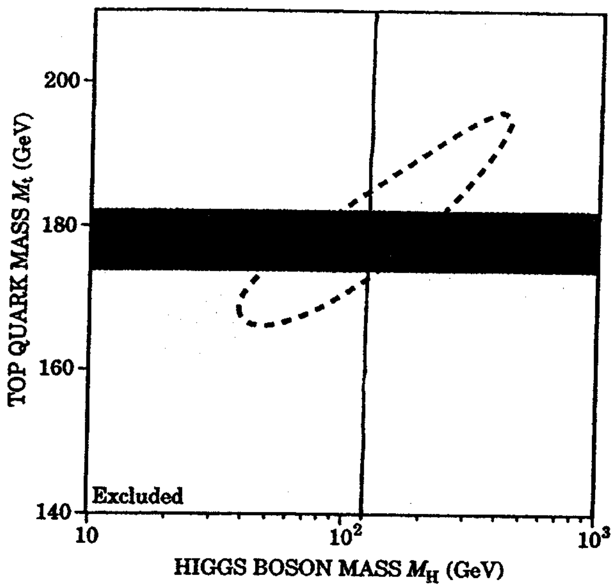
Figure 8: From the data on W^+W^- production near threshold the mass and width of the W boson was found to be $m_W = 80.412 \pm 0.042$ GeV and $\Gamma_W = 2.150 \pm 0.091$ GeV. Omitting the three-boson self-couplings gives a poor fit to the data, confirming the nonabelian nature of the interactions.

The W and Z bosons are massive spin 1 gauge bosons. They are believed to get their mass from the mechanism of spontaneous symmetry breaking $[BE, H]$. One assumes the existence of a complex $SU(2)$ doublet $\phi = (\phi^0, \phi^-)$ where $\phi^0 = \frac{1}{\sqrt{2}}(H + i\varphi)$ with real Higgs field H and three nonphysical scalars φ, ϕ^- and $(\phi^-)^*$ which give a mass to Z, W^- and W^+. The Higgs potential is assumed to be of the form

$$V = -\mu^2|\phi|^2 + \frac{1}{2}\lambda|\phi|^4 = \frac{1}{2}\lambda\left(|\phi|^2 - v^2/2\right)^2 - \frac{1}{8}\lambda v^4 \tag{1.2.11}$$

where the mass term is assumed to have a negative sign. (In supersymmetric models, there are at least two Higgs doublets, and then one can show that one of the physical Higgs scalars has a running $(\text{mass})^2$ which becomes negative at low energies). The gauge bosons couple minimally, schematically as $|(\partial - gW)\phi|^2$, and the term $\frac{1}{2}g^2 W^* W \langle H\rangle\langle H\rangle$ yields then a mass for the W and Z bosons. Knowing the $SU(2)$ gauge coupling constant and the W mass, $m_W = gv/\sqrt{2}$, one finds that the vacuum expectation value of H is $\langle H\rangle = v = 250$ GeV experimentally. This leaves then only one parameter to be fixed, for which one takes the Higgs mass M_H. Quarks and leptons couple with Yukawa couplings, schematically as $\lambda_Y \bar{\psi}\psi H$, and $m = \lambda_Y\langle H\rangle$ is then their mass. Rewriting this interaction as $(m/\langle H\rangle)\bar{\psi}\psi H$, it is clear that the strength of the couplings of quarks and leptons to the Higgs particle is proportional to the quark and lepton masses.

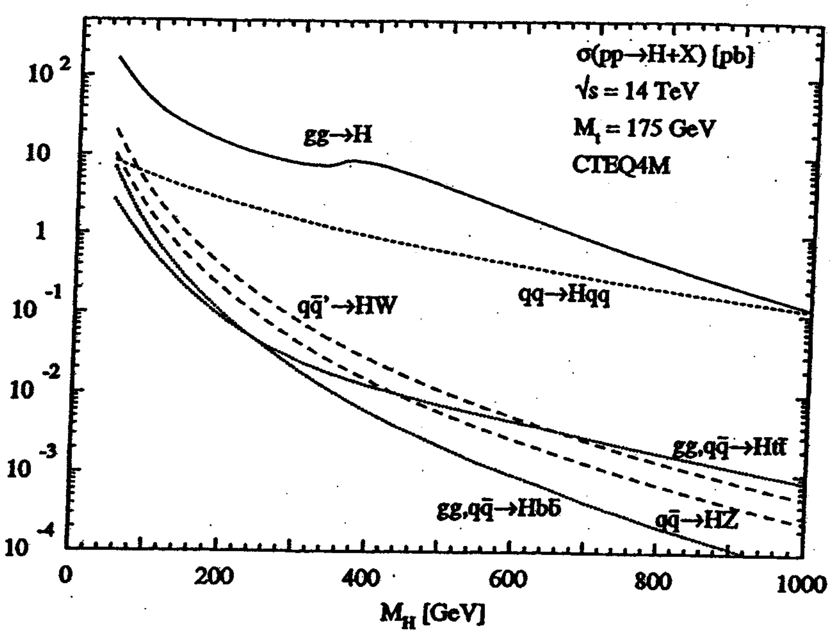
At the Large Hadron Collider (LHC) at CERN two detector groups (ATLAS and CMS) will be looking for the Higgs particle in the coming years. This LHC is a proton-proton collider with 7 TeV per beam. From LEP (Large Electron Positron collider) experiments one has a lower bound on the Higgs mass, $M_H > 114$ GeV. A global fit to data of the Standard Model yields as best estimate $M_H = 117$ GeV, with an upper limit of 250 GeV.



(1.2.12)

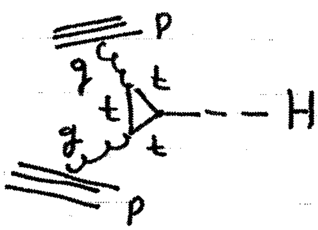
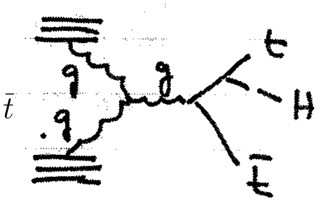
Figure 9: A global standard-model fit to well-measured low-energy parameters restricts the masses of the top quark and Higgs boson to lie within the ellipse, with a confidence level of 68%. The horizontal band shows the measured value of M_t and the region of M_H to the left is already excluded by direct searches for the Higgs at LEP. The present best standard-model guess for M_H from such fits is 117 GeV, with an upper limit of about 250 GeV.

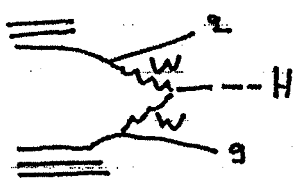
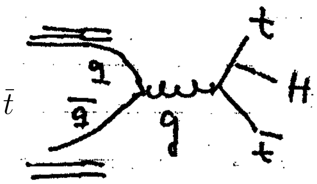
In the next figure the theoretical predictions for the production cross sections of the Higgs particle at the LHC are plotted as a function of M_H.

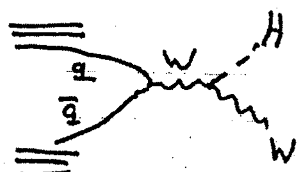
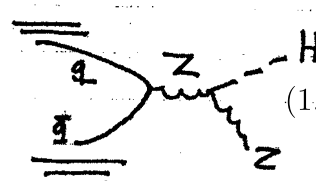


(1.2.13)

Figure 10: The production cross sections for the Higgs particle in the Standard Model at the LHC at CERN. Clearly the production by two gluons is dominant. The lowest-order Feynman graphs for these reactions are as follows

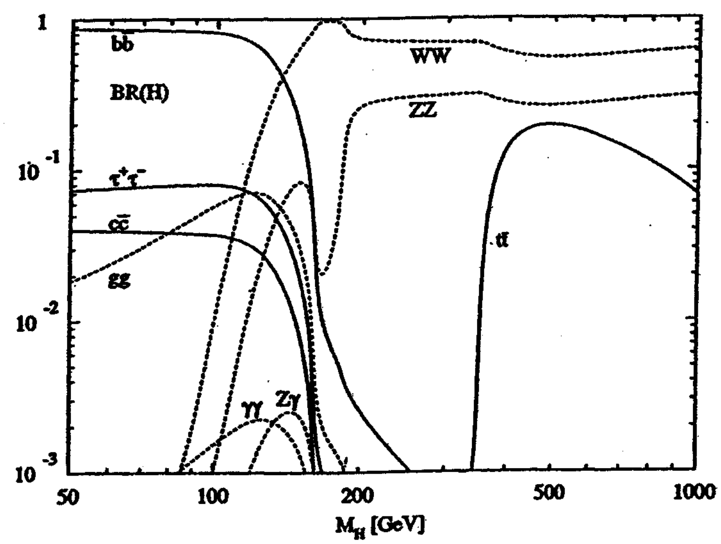
$g + g \to H$  $g + g \to H + t + \bar{t}$ 

$q + q \to H + q + q$  $q + \bar{q} \to H + t + \bar{t}$ 

$q + \bar{q} \to H + W$  $q + \bar{q} \to H + Z$  (1.2.14)

The processes $g+g \to H+b+\bar{b}$ and $q+\bar{q} \to H+b+\bar{b}$ are similar to $g+g \to H+t+\bar{t}$ and $q+\bar{q} \to H+t+\bar{t}$, but less frequent because the mass of the bottom quark is much less than the mass of the top quark. Clearly production by gluons $g+g \to H$ is dominant [why does one mention $W \to W+H$ and $Z \to Z+H$ so often?].

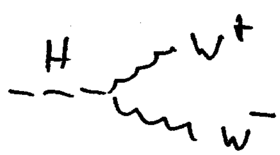
The theoretical predictions for the branching ratios of the decay of the Higgs boson into specific channels are given in the next figure, in which both scales are logarithmic.



(1.2.15)

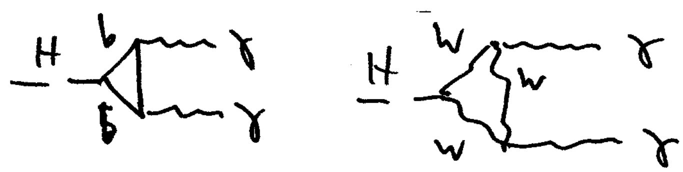
Figure 11: The branching ratios for the decay of the Higgs particle as a function of its mass M_H.

Below the threshhold for $H \to W+W$ or $H \to Z+Z$, the Higgs can only decay into quarks or leptons, mostly into $b\bar{b}$, $\tau^+\tau^-$ or $c\bar{c}$. Decays into lighter quarks or leptons are much suppressed since the Higgs couples proportional to the mass of the quarks and leptons. The decay $H \to g+g$ into gluons has in this region a large cross section, but this process will be drowned by ordinary QCD processes which will produce lots of gluons. If the Higgs is heavier than 160 GeV, one will be looking for $H \to W^+W^-$.



(1.2.16)

Below 160 GeV, one will be looking for back-to-back $\gamma\gamma$ pairs with a large invariant mass.



(1.2.17)

Hopefully, there will be a bump in the $\gamma\gamma$ mass distribution, indicating the discovery of the Higgs particle.

Even if one assumed that a Higgs boson did exist, there remain some puzzles: why is the neutrino mass zero? Why are electron- number, muon- number and tau- number preserved? A massless neutrino field describes in the Standard Model left-handed neutrinos and right-handed antineutrinos, and these helicities are indeed observed. Neutrino capture $\bar{\nu}_e + p \to e^+ + n$ is allowed, but for example $\bar{\nu}_e + {}^{37}Cl \to e^- + {}^{37}Ar$ is forbidden, again in agreement with the data. The shape of the μ-decay spectrum agrees with $\mu \to e+\nu+\bar{\nu}$ but rules out $\mu \to e+\nu+\nu$ or $\mu \to e+\bar{\nu}+\bar{\nu}$. Lepton-number conservation allows $\mu \to e + \gamma$ but separate muon-number and electron- number conservation rule out this decay, again as observed. The reactions $\bar{\nu}_\mu + p \to e^+ + n$ and $\nu_\mu+n \to e^-+p$ should be forbidden but $\bar{\nu}_\mu+p \to \mu^++n$ and $\nu_\mu+n \to \mu^-+p$ should be allowed, and indeed only $\mu^\pm$ but no $e^\pm$ were seen in this classic experiment [99]. Yet, a very small neutrino mass and small violations of electron- , muon- and tau- number conservation could not be ruled out on theoretical grounds. For example, supersymmetry could not put neutrinos and photons in the same massless multiplet and thereby explain why neutrinos are massless, because their $SU(2)\times U(1)$ quantum number are different.

Experiments on neutrinos have in recent decades led to a revolution: neutrinos do have small masses, and as a consequence neutrino oscillations which convert one neutrino species into another do occur. We consider three types of experiments: solar neutrino experiments, atmospheric neutrino experiments, and reactor neutrino experiments, which we now briefly discuss. We begin with solar neutrinos.

The sun shines by converting protons into α-particles, photons, e^+ and ν_e, and hence it only produces electron-neutrinos. The dominant production mode is the $p-p$ chain

$$\begin{array}{lll} p+p \rightarrow {}^2H + e^+ + \nu_e & & \\ \quad\searrow & & \\ & {}^2H + p \rightarrow {}^3He + \gamma & \\ & \quad\searrow & \\ & & {}^3He + {}^3He \rightarrow {}^4He + 2p \end{array} \tag{1.2.18}$$

The net result is that four protons convert to a helium nucleus, two positrons, two electron neutrinos, and an energy of 26.7 MeV. These neutrinos have energies below 0.4 MeV and account for 98.4% of the total number of neutrinos produced. Although there are various other reactions, all solar neutrinos have energies below 20 MeV.

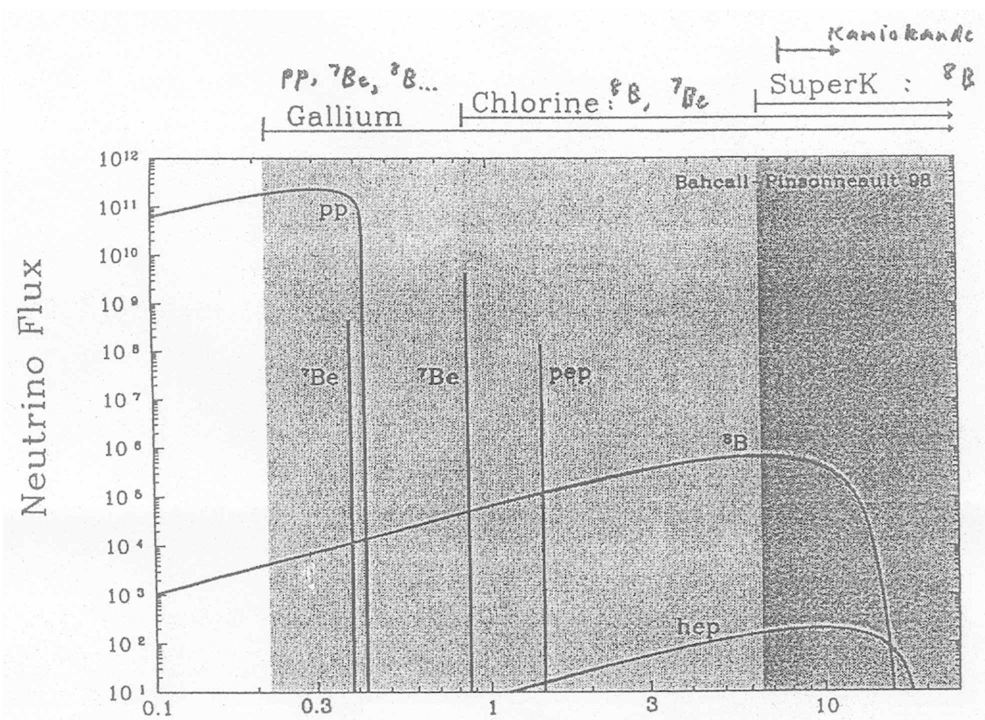


Figure 12: The solar neutrino energy spectrum. The neutrino flux per cm^2 per second per MeV on earth is plotted as a function of the neutrino energy in MeV.

The electron neutrino flux can be rather accurately calculated from the "Standard

Solar Model", but only about half of the predicted ν_e seem to reach the earth. For energies above 5 MeV, even less than one-half of the electron neutrinos reach earth (actually 1/3). Neutrino oscillations in vacuum can at most yield a decrease of 50% (when averaged over the production point in the core of the sun), hence another mechanism is at work. It is currently believed that this is the MSW effect [123] which is based on resonance neutrino oscillations generated by W and Z exchange with matter. This indicates a conversion of ν_e into another species, for which ν_μ and ν_τ are the most natural candidates.

Another source of neutrinos is due to the bombardment of the atmosphere of the earth by cosmic particles. High-energy protons produce pions and kaons, which decay into muons, which in turn decay into electrons

$$\begin{array}{l} \pi^+ \to \mu^+ + \nu_\mu \\ \qquad\quad \searrow \\ \qquad\qquad e^+ + \bar{\nu}_\mu + \nu_e \end{array} \qquad \text{idem } \pi^-, K^+, K^- \tag{1.2.19}$$

These neutrinos have energies which peak at a few hundred MeV, certainly above the cut-off of 20 MeV for solar neutrinos. Thus for not too high energies (so that the pions and muons have time to decay before reaching the detector) but still above 20 MeV, there should be twice as many muon-neutrinos as electron-neutrinos. The total flux of atmospheric neutrinos integrated over all energies is approximately isotropic and equal to about $10/sec/cm^2$, while the corresponding flux of solar neutrinos is of course not isotropic and equal to $6.9 \times 10^{10}/sec/cm^2$. (In addition there are about three times 112 neutrinos (three for the 3 species) per cm^3 which are due to the Big Bang; these neutrinos are of very low energy, $1.95\,K(1.6\,10^{-4}eV)$, and cannot be detected by the detectors used for solar or atmospheric neutrinos). Because the energy of the atmospheric neutrinos is far above the energy of solar neutrinos, atmospheric neutrino experiments are feasible.

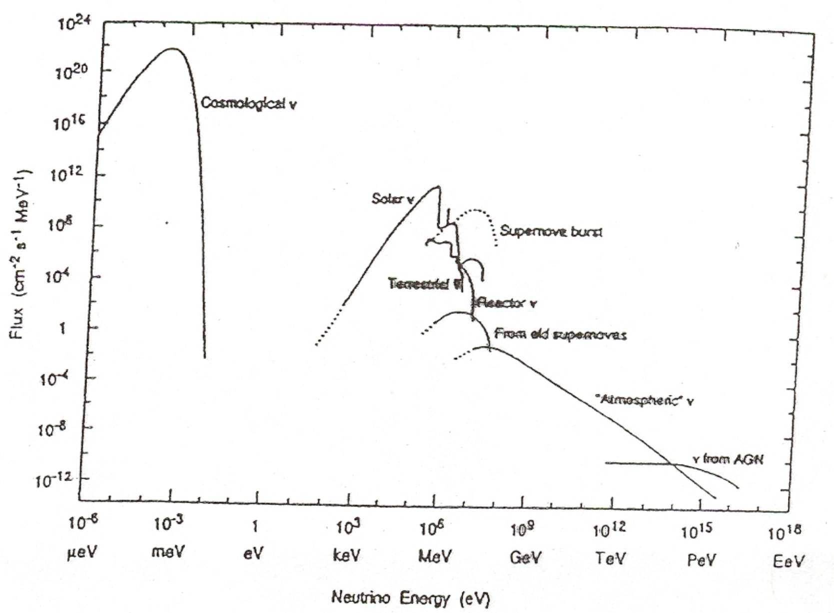


Figure 13: The various neutrino fluxes per cm^2 per second per MeV observed on earth as a function of the neutrino energy. The solar neutrinos correspond to figure 12.

Whereas experiment confirmed the theoretical prediction of the electron-neutrino flux of atmospheric neutrinos[49], the Kamiokande group in Japan found that the observed muon-neutrino flux of neutrinos produced at the other side of the earth is only 60% of the theoretical prediction. Since there was no excess electron-neutrino flux, this indicated a conversion of ν_μ into another species, for which ν_τ is the most natural choice.

The theory of neutrino oscillations is simple. Assuming two-species mixing with a mixing angle θ and mass eigenstates $|\nu_1\rangle$ and $|\nu_2\rangle$ with masses m_1 and m_2, a muon- and tau-neutrino (or electron- and muon- neutrino) have the following time dependence

$$\begin{aligned} |\nu_\mu\rangle &= \cos\theta e^{-\frac{i}{\hbar}E_1 t}|\nu_1\rangle + \sin\theta e^{-\frac{i}{\hbar}E_2 t}|\nu_2\rangle \\ |\nu_\tau\rangle &= -\sin\theta e^{-\frac{i}{\hbar}E_1 t}|\nu_1\rangle + \cos\theta e^{-\frac{i}{\hbar}E_2 t}|\nu_2\rangle \end{aligned} \tag{1.2.20}$$

[49]The oscillation wave length for atmospheric electron neutrinos of 1 GeV is 25 times the diameter of the earth, see (1.2.22). Hence, one cannot observe oscillations of these atmospheric electron neutrinos.

At $t = 0$ these states describe a muon or tau lepton as produced by the weak decays in (1.2.19). If a ν_μ is produced at $t = 0$ in the upper atmosphere, the probability that at a later time t a ν_τ is detected is

$$P_{\nu_\mu \to \nu_\tau} = |\langle \nu_\tau | \nu_\mu(t) \rangle|^2 = \sin^2 2\theta \sin^2 \frac{(E_1 - E_2)t}{2\hbar} \tag{1.2.21}$$

with $E_j = \sqrt{p^2c^2 + m_j^2c^4} \simeq pc + \frac{m_j^2c^4}{2pc}$ where $E_j \sim pc \sim 1$ GeV. If a ν_μ would travel a distance $ct = L$ and a ν_τ would be detected, this would give an estimate of the difference of the masses squared. The oscillation length λ follows from the probability for $\nu_\mu \to \nu_\tau$ conversion

$$\begin{aligned}
&P_{\nu_\mu} \to \nu_\tau = \sin^2 2\theta \sin^2 \left[\frac{\Delta m^2 c^4 L}{4\hbar c E}\right] \\
&\lambda = \frac{4\pi\hbar c E}{\Delta m^2 c^4} = 2.47 m \frac{E}{MeV} \frac{(eV)^2}{\Delta m^2} \\
&\Delta m_{23}^2 c^4 = |m_2^2 - m_3^2| c^4
\end{aligned} \tag{1.2.22}$$

Several atmospheric neutrino experiments, using different E and L, have yielded as best fit for the mixing angle θ and the $(\text{mass})^2$ difference

$$\sin^2 2\theta_{23} = 1 \text{ and } \Delta m_{23}^2 = 2.3\, 10^{-3} (eV)^2 \tag{1.2.23}$$

Half the oscillation length for 10 GeV muon neutrinos is then approximately equal to the diameter of the earth, so that $\mu - \tau$ oscillation is maximal at this energy.

In principle one should describe neutrino mixing as a 3×3 matrix problem. However, one can approximate it as two 2×2 problems. Let ν_α denote e, μ, τ and ν_i the three mass eigenstates with masses m_i. Mixing at $t = 0$ implies $\nu_\alpha = \Sigma_i U_{\alpha i} \nu_i$. Then (1.2.21) generalizes to

$$\begin{aligned}
P_{\nu_\alpha \nu_\beta} &= \sum_{i,j} (U_{\alpha i} U_{\beta i})(U_{\alpha j} U_{\beta j}) \cos(E_i - E_j)t/\hbar \\
&= \delta_{\alpha\beta} - 4 \sum_{j>i} U_{\alpha i} U_{\beta i} U_{\alpha j} U_{\beta j} \sin^2 \frac{\Delta m_{ij}^2 c^4 L}{4\hbar c E}
\end{aligned} \tag{1.2.24}$$

where we assumed that U are real (the CP violating phase is neglected). The unitary matrix mixing $U_{\alpha i}$ may be parametrized as follows

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1.2.25}$$

where $c_{ij} = \cos\theta_{ij}$ and $s_{ij} = \sin\theta_{ij}$ while δ is the CP violating phase in the neutrino sector. Using that

(i) $\sin\theta_{23} = \cos\theta_{23} = \frac{1}{\sqrt{2}}$ (maximal $\mu\tau$ mixing).

(ii) $\Delta m_{12}^2 \ll |\Delta m_{23}^2|$ (hence also $\Delta m_{12}^2 \ll |\Delta m_{13}^2|$)[50]

(iii) $\sin\theta_{13} \simeq 0$ (from reactor experiments one has $\sin_2^2\theta_{13} < 0.13$)

one finds that the solar and atmospheric neutrino problems reduce to simple 2-neutrino problems in the 1-2 and 2-3 sector, respectively. Namely,

$$U = \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12}/\sqrt{2} & c_{12}/\sqrt{2} & 1/\sqrt{2} \\ s_{12}/\sqrt{2} & -c_{12}/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \tag{1.2.26}$$

and one finds

$$\begin{aligned} P_{\nu_e\nu_\mu} &= -4U_{11}U_{21}U_{12}U_{22}\sin^2\frac{\Delta m_{12}^2c^4L}{4\hbar cE} \text{ since } U_{13}=0 \\ P_{\nu_e\nu_\tau} &= -4U_{11}U_{31}U_{12}U_{32}\sin^2\frac{\Delta m_{12}^2c^4L}{4\hbar cE} \text{ since } U_{13}=0 \\ P_{\nu_\mu\nu_\tau} &= -4U_{22}U_{32}U_{23}U_{33}\sin^2\frac{\Delta m_{23}^2c^4L}{4\hbar cE} \text{ since } m_{23}^2 \ll m_{12}^2, m_{13}^2 \end{aligned} \tag{1.2.27}$$

Then $|\nu_e\rangle$ oscillates into $|\nu_+\rangle = \frac{1}{\sqrt{2}}(|\nu_\mu\rangle + |\nu_\tau\rangle)$ (because $\theta_{23} = \pi/4$ according to (1.2.23)), whereas the orthogonal combination $\nu_- = \frac{1}{\sqrt{2}}(|\nu_\mu\rangle - |\nu_\tau\rangle)$ does not take part in this process.

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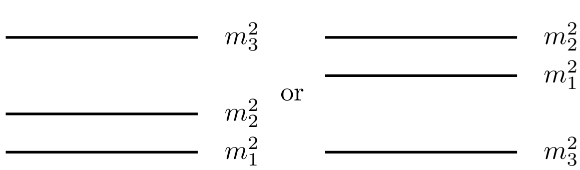
[50] In a picture



One knows that $m_2^2 > m_1^2$ due to the MSW effect.

For solar neutrinos, the best data are

$$tg^2\theta_{1+} \sim 0.4 \text{ (or } \theta_{1+} \sim 35^0\text{) and } \Delta m^2_{1+} \sim 8 \times 10^{-5}(eV)^2 \tag{1.2.28}$$

Thus the oscillation length for solar neutrinos (or electron antineutrinos from reactors) of 10 MeV is about 300 km, and for 0.4 MeV neutrinos from the $p-p$ chain in the sun it is 12 km, while the oscillation length for atmospheric muon neutrinos of 10 GeV is about 10000 km.

One of the crucial experiments on solar neutrinos is the SNO experiment (Sudbury Neutrino Observatory in Canada). It uses heavy water (D_2O). Solar neutrinos can lead in heavy water to 3 reactions: elastic scattering off electrons, charged current reactions, and neutral current reactions

$$\begin{aligned}
\text{elastic scattering :} \quad & \nu_e + e^- \to \nu_e + e^-, W \text{ and } Z \text{ exchange} \\
& \nu_\mu + e^- \to \nu_\mu + e^-,\ Z \text{ exchange} \\
& \nu_\tau + e^- \to \nu_\tau + e^-,\ Z \text{ exchange} \\
\text{charged currents :} \quad & \nu_e + (d) \to e^- + p + p, W \text{ exchange} \\
\text{neutral currents :} \quad & \hat{\nu} + (d) \to \hat{\nu} + n + p, Z \text{ exchange}
\end{aligned} \tag{1.2.29}$$

The notation (d) denotes a deuteron, while $\hat{\nu}$ can be ν_e, ν_μ or ν_τ. At Super-Kamiokande in Japan only elastic scattering is observed in pure water (neutral current interactions of solar neutrinos with protons also occur but the scattered protons are nonrelativistic and do not produce Cherenkov radiation). The rate of $\hat{\nu} + (d) \to \hat{\nu} + n + p$ is independent of whether neutrino oscillations have occurred because the total number of incoming neutrinos remains the same, but the rate of elastic electron scattering is smaller when some fraction of the ν_e have oscillated away (because for ν_e scattering both W and Z contribute, while for ν_μ and ν_τ scattering only Z contributes). The SNO data in the neutral current sector is consistent with the predictions of the Standard Solar Model. One third of the observed solar neutrinos are electron-neutrinos;

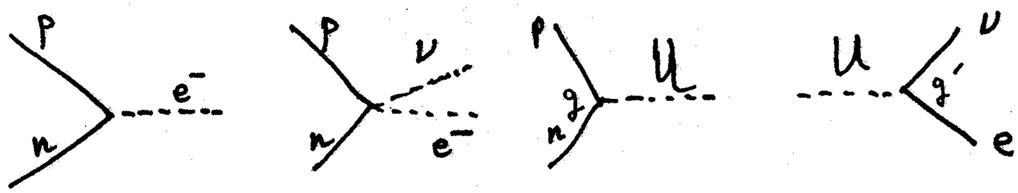
obviously the rest are muon-neutrinos and tau-neutrinos. In this way the existence of solar neutrino oscillations has been proven beyond doubt.

Over the last three years, the oscillation evidence from solar and atmospheric neutrinos has been tested with experiments using "man made" neutrino beams from reactors and accelerators. In particular, the KamLAND experiment in Japan has detected the disappearance of reactor electron antineutrinos of energies of several MeV's over distances of 180 km [124]. This is a direct confirmation of the interpretation of the solar neutrino deficit in terms of oscillations of electron neutrinos with the parameters given in Eq.(1.2.28). Also in Japan, the K2K collaboration has observed the disapperance of muon neutrinos of about 1 GeV energy – produced at the KEK laboratory – by the time they reach the SuperKamiokande detector [125] located at 250 Km. This confirms the oscillation of muon neutrinos with the parameters inferred from analysis of the atmospheric neutrino signals (Eq.(1.2.23))

Whereas two decades ago the theory of electroweak interactions was considered to be in its final form, at present all bets are off. New physics beyond the Standard Model awaits us.

3 QCD

The explosion of theoretical and experimental developments in the 1960's and the early 1970's which led to a realistic renormalizable gauge theory of the electroweak interactions produced also, as an equally important by-product, a gauge theory of the strong interactions. The earliest proposal for a field theory of the nuclear forces was made in the 1930's after the emergence of QED in the late 1920's, and at about the same time as the construction of a field theory for nuclear β decay by Fermi. Namely in 1934 Yukawa realized that the fact that the nuclear force is only effective at very short distances could be explained by the exchange of a new particle that was very heavy [126]. Until 1932 nuclei were thought to be bound states of protons and electrons, but in 1932 the discovery of the neutron laid these models to rest. Heisenberg tried to derive the nuclear force from exchange of electrons between protons and neutrons, but this ran into the problem that it violated conservation of angular momentum. When Fermi published his theory of β decay, Heisenberg tried to derive the nuclear force from exchange of a neutrino-electron pair. The problem with angular momentum was now solved, but the strength of the nuclear forces which this model predicted was far too weak.[51]



(1.3.1)

Figure 14: Heisenberg's electron-exchange model for the nuclear forces, his modified model based on Fermi's field theory of β decay, and Yukawa's unified model of the strong and weak forces. The Fermi coupling constant G_F is equal to gg'/m_U^2 in Yukawa's model, while the strong interactions are proportional to g^2/m_U^2.

Yukawa decided to replace the neutrino-electron pair by a new particle, called U by him and called mesotron in later years. He even proposed that his U particle would couple with different strengths to the $p-n$ current and the $\nu - e$ current, thus proposing a model that would unify the strong and weak interactions. He considered both the case that U was a scalar particle, and the case that U was a vector field, whose scalar part U_0 should dominate at low energies. He assumed the wave equation $(\Box - m^2)U = g\rho$ for this new particle, with $\rho = \psi^\dagger \tau_- \psi$ and $\psi = \binom{p}{n}$.

By using $2 \cdot 10^{-13}$ cm for the range of the nuclear force, he found a mass of the U particle of the order of 200 times the electron mass. In 1937 Neddermeyer and Anderson, using the same magnetic cloud chamber which the latter had used in 1932

[51]Years later (in 1962-1963) Feynman studied whether one can give a microscopical explanation of gravity due to exchange of a pair of massless neutrinos instead of a graviton. He did obtain an $1/r$ potential if one considers exchange between three instead of two masses, and speculated that the third mass represented some effective average over all other masses in the universe. However, these other masses led to other problems [127]. Moreover, we now believe that neutrinos are massive, and then one does not get an $1/r$ potential.

to discover the positron, found a new charged particle in cosmic rays with a mass between the electron mass and the proton mass called meson. [128] This seemed to be Yukawa's mesotron. However, later analysis made clear that this particle could not be Yukawa's carrier of the strong forces because most of the time it came to rest in carbon and then decayed, instead of rapidly being absorbed by the nuclei [129]. Also its mass was 50% too low, and its lifetime 100 times too long. A period of confusion followed. The issue was "meson = mesotron?", where meson referred to the experimentally observed particle (which turned to be the muon) and mesotron to the theoretical particle needed for Yukawa's theory (the pion). However, in 1947 Perkins found another meson which had strong interactions [130], and the Bristol group discovered tracks in photographic emulsions of a particle coming to rest at the end of its range, and then producing a big kink which implied that a lighter charged particle had been emitted [131]. This resolved the contradictions: the heavier particle was Yukawa's carrier of the nuclear force, called π-meson by Powell, and it decays into a kind of heavy electron called μ-meson. (The names pion and muon were given by Fermi). With hindsight it was clear that the particle of 1937 had been the muon. With its existence firmly established, Yukawa's pion seemed to lead to the beginning of a field theory for the strong interactions.

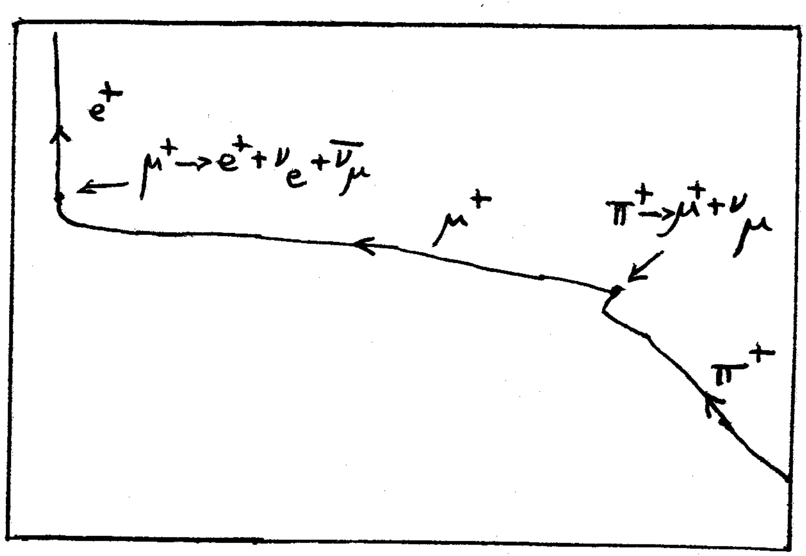


Figure 15: A “double meson track” of 1947 in which a pion (discovered in 1947) and a muon (discovered in 1937) decay in a two-step process. The pion and muon come to rest before decaying which explains the curving of their tracks just before the decays [2].

In the 1940’s, 1950’s and 1960’s many new resonances were discovered. However, no obvious candidate for a field theory which could describe all these new particles was available. For example, assuming a Yukawa interaction, the coupling constants of the strong interactions were of order unity or larger, so that perturbation theory seemed out of the question. Faced with this very confusing situation, two approaches were developed to bring some insight. On the one hand, one tried to find symmetries which combined sets of particles into multiplets, and on the other hand models were proposed in which the new particles were bound states of a few basic constituent particles. We now briefly discuss both approaches, beginning with the approach based on symmetries.

In the thirties the fact that nuclei with approximately the same mass have the same numbers of protons and neutrons, led Heisenberg to suggest that the forces between two protons cannot be very different from the forces between two neutrons. Studies of H^3 (tritium, pnn) and He^3 (ppn) showed that if one subtracted the Coulomb energy, the pp and nn interactions were actually the same. Proton scattering experiments showed that the pp and pn forces were also the same. This led to the concept of isospin invariance of the nuclear forces, with the proton and neutron forming an isospin doublet. Towards the end of the 1930’s models for meson interactions were studied (before the actual discovery of the pions in 1947). A charged scalar pion was coupled to nucleons as $H_{\text{int}} = g\bar{\psi}(\tau_- \phi + \tau_+ \phi^*)\psi$. Kemmer [133] noticed that by introducing a new neutral meson π^0, one can rescue isospin invariance: $H_{\text{int}} = g\bar{\psi}\vec{\tau} \cdot \vec{\pi}\psi$. (The π^0 was discovered only in 1950 in the reaction $\pi^- + p \to \pi^0 + n$ and $\pi^0 \to \gamma + \gamma$). He also made the crucial observation that isospin invariance holds to any order in perturbation theory. Here thus is the beginning of the use of symmetries instead of detailed dynamics to describe the strong interactions. The next step was to combine charge conjugation invariance C with isospin invariance. It is easy to check that the isospin generators (T_1, T_2, T_3) do not commute with the charge conjugation operator C; for example, for protons and neutrons, $T_3 = N(p) - N(\bar{p}) - N(n) + N(\bar{n}))$, and $CT_3C^{-1} = -T_3$, C and T_3 anticommute rather than commute. However, one can construct an operator G from C and $\vec{T}$ which commutes with $\vec{T}$. This was done by L. Michel [134] who knew that the two-dimensional representation of $SU(2)$ is pseudoreal (the complex conjugate of the **2** representation is related to the representation itself by a similarity transformation). He proposed the operator $C(\exp i\pi T_2)$, called G-parity by Lee and Yang, which maps pions $\vec{\pi}$ into minus themselves[52], and this explained why only an even number of pions can be produced (as in $\eta' \to \eta + \pi^+ + \pi^-$ or $\eta' \to \eta + \pi^0 + \pi^0$). The two symmetries strangeness (S) and isospin (with generators T_1, T_2, T_3) were going to play a fundamental role later in the quark picture, as we shall discuss.

The approach according to which the new particles were bound states of only a few basic constituent particles was started by Fermi and Yang [135].[53] In an article entitled "Are mesons elementary particles?" they studied as a test case the assumption that pions are bound states of a nucleon and an antinucleon. This was only the beginning of a new idea with more problems than questions, so they wrote: "We will try to work out a special example more as an illustration of a possible program . . . than in the hope that what we suggest may actually correspond to reality". (However, the idea that a pion is a bound state of two fermions is now considered correct, namely a bound state of a quark and an antiquark). To accommodate also strangeness, Sakata [139] proposed that the lightest hadrons (the baryon octet and the pseudoscalar meson

[52]The π^0 meson has $C = +1$ because it decays into two photons, and it rotates into minus itself under a rotation over 180 degrees along the y-axis.

[53]Actually, already de Broglie had entertained the possibility that photons are bound states of a neutrino and an antineutrino. [136] (In his paper the term antiparticle first appears). However, these and other efforts came to an end when it was noted that these composite photons had helicity 0 instead of the required helicity ± 1 [137]. In more modern times, Weinberg and Witten have shown that gravitons (and gravitinos in supergravity) cannot be bound states of lower spins [138].

octet) were composed of 3 basic elementary particles: the proton (p), neutron (n) and lambda (Λ). With collaborators he went even further, and proposed that these 3 particles were in turn a bound state of the 4 known leptons ($e^-, \nu_e, \mu^-, \nu_\mu$) and a hypothetical baryon B^+. Because there was one more lepton then basic baryons, they proposed neutrino mixing: $\nu_e = \nu_1 \cos\delta - \nu_2 \sin\delta$ and $\nu_\mu = \nu_1 \sin\delta + \nu_2 \cos\delta$. This should then lead to the bound states $p = (\nu_1 B^+), n = (e^- B^+)$ and $\Lambda = (\mu^- B^+)$. (If they had considered the fourth bound state ($\nu_2 B^+$), they might have proposed charmed baryons). Their scheme produced a hadronic weak $V - A$ current from the leptonic weak current $j^\mu_{lep} = \bar{e}^- \gamma^\mu (1 + \gamma_5)\nu_e + \bar{\mu}^- \gamma^\mu (1 + \gamma_5)\nu_\mu$: by adding a pair of $(B^+, \bar{B}^+)$ fields to the leptons they obtained $j^\mu_{had} = \bar{n}\gamma^\mu (1 + \gamma_5) p \cos\delta + \bar{\Lambda}\gamma^\mu (1 + \gamma_5) \sin\delta$. So they introduced the notion of lepton mixing and constructed an early precursor of the Cabibbo angle which was a year later proposed by Cabibbo based on the $SU(3)$ scheme (see below). An octet of pseudoscalar mesons was obtained (because (p, n, Λ) formed a triplet and $(\bar{p}, \bar{n}, \bar{\Lambda})$ and antitriplet), and seven of these mesons were identified with the pions and kaons (the η meson was not yet discovered). Further work [140] proposed to consider the Sakata triplet as a realization of the fundamental representation of the group $SU(3)$, and led to the beginning of a classification scheme based on $SU(3)$, but "the Sakata model" broke down when applied to bound states of two baryons and one antibaryon. Instead of the observed decuplet they obtained baryons with incorrect strangeness and electric charges. With hindsight it is clear where the Sakata model went wrong: it did not use quarks with fractional electric and baryon charges.

Around the same time, in 1961, Gell-Mann and Ne'eman [141] also tried to put the low-lying mesons and baryons into multiplets. Particles of a given multiplet should all have the same spin, parity, charge conjugation number, and a low-lying octet of baryons was clearly present. The question was of which group this octet was a representation. They also settled on $SU(3)$. (To avoid confusion with the color group $SU(3)$ to be discussed later, note that this $SU(3)$ is a rigid symmetry of the flavours up, down and strange). The crucial difference with earlier quarks models was the assumption that quarks had fractional charges; apparently, Serber mentioned this casually as a possibility to Gell-Mann during lunch at Columbia. The lowest-lying mesons were all pseudoscalars but only a 7-plet consisting of $(\pi, K, \bar{K})$ was known, and the rival group G_2 did have a 7-dimensional representation. However, Gell-Mann predicted the existence of an 8th meson which would complete the octet, and a few months later this $\eta(547)$ was indeed found [10]. The $SU(3)$-singlet $\eta'(958)$ which according to the $SU(3)$ scheme should also exist because mesons form nonets instead of octets, was found later, and the corresponding $U(1)$ problem (which got a solution in quantum gauge field theory in the 1970's by the discovery of instantons) had not yet been identified as a problem. A year later, the vector mesons K^*, ρ, ω and ϕ found at Berkeley, Brookhaven and Cern could also be grouped into an octet and a singlet. The lowest-lying baryons $(p, n, \Lambda, \Sigma$ and $\Xi)$ clearly formed a spin 1/2 octet.

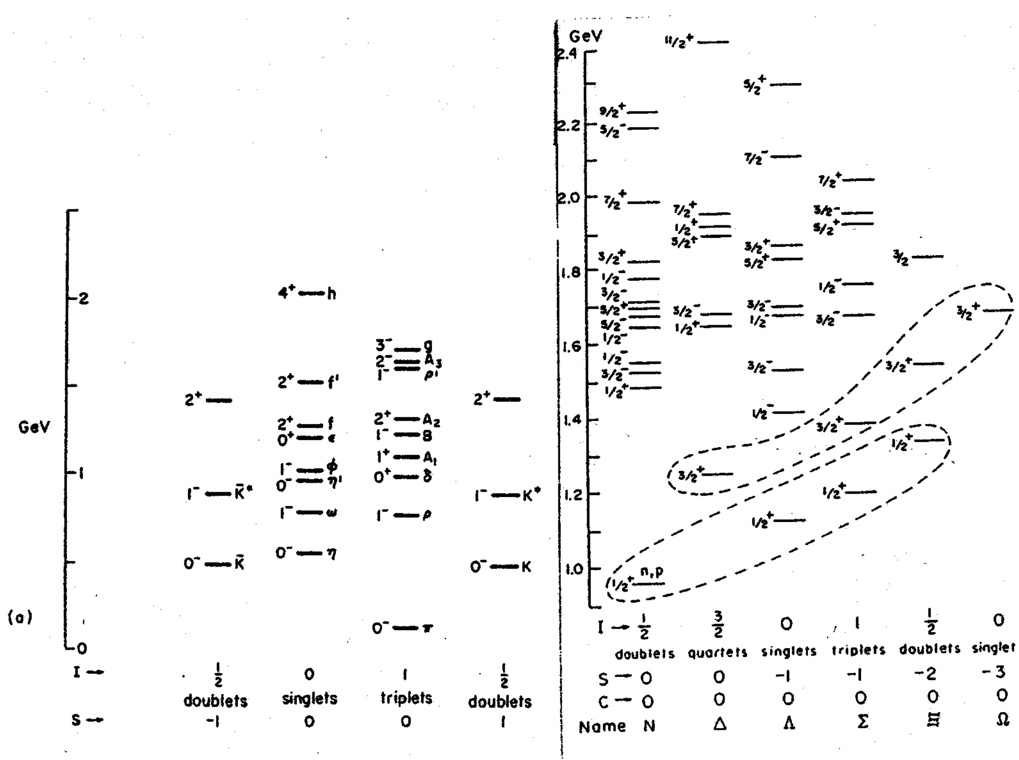


Figure 16: meson masses and baryon masses. The baryon spin $1/2^+$ octet and the spin $3/2^+$ decouplet are clearly recognizable. The spin-zero negative-parity 0^- octet $(\pi, K, \bar{K}, \eta$ and $\eta')$ and

the spin-one negative- parity 1^- nonet ($\rho, \omega, K^*, \bar{K}^*$ and φ) are easily identified. However, the spin 0^- pseudoscalar nonet consists of a low-lying octet and a much heavier ninth member, the η'. Instantons are responsible for the high mass of the latter.

At a conference at Cern in 1962 the discovery of new spin 3/2 resonances Σ^* and Ξ^* were announced. Together with the Δ resonance they seemed to belong to a decuplet, but its tenth member was lacking. Using Okubo's mass formula [142] for $SU(3)$ multiplets $M = a + bY + c[I(I+1) - \frac{1}{4}Y^2]$, Gell-Mann and Ne'eman predicted its existence, and the Ω^- (1986) was a half year later found at Brookhaven. Although this is a book on field theory and not on particle physics, and certainly not on detector methods, we make an exception for the Ω^- discovery because it has become one of the great experimental discoveries of particle physics. Consider the following figure.

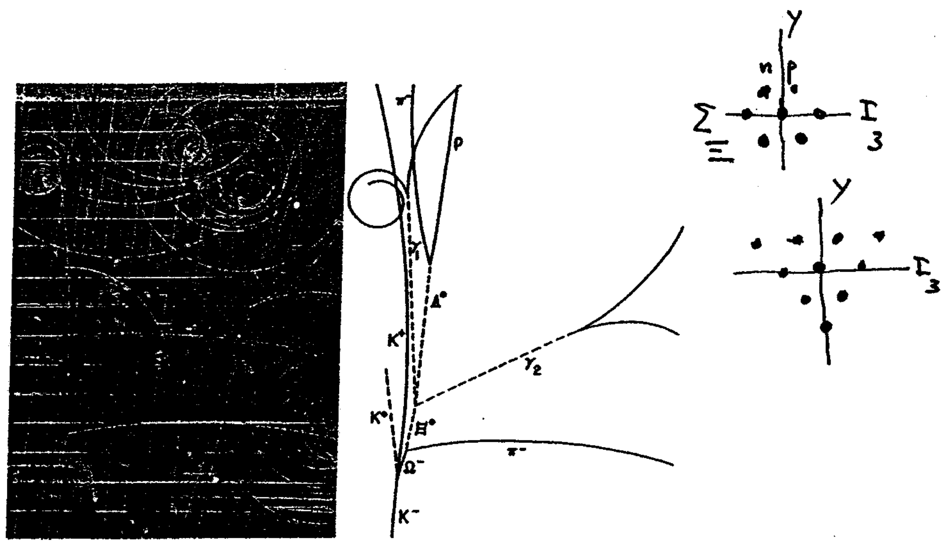


Figure 17: An example of the production of an Ω^- particle in the 80″ Brookhaven bubble chamber filled with hydrogen. $K^-p \to \Omega^- K^+ K^0; \Omega^- \to \Xi^0\pi^-, \Xi^0 \to \Lambda^0\pi^0; \Lambda^0 \to p\pi^-; \pi^0 \to \gamma\gamma$.

At first sight it seems incomprehensible that one could detect the Ω^- from this baffling plethora of particle tracks. Input for the discovery was that the two lines at the top of the picture (the p and π^-) which constitute the decay of a "V particle" (the Λ^0) determined a line of flight which did not coincide with the vertex for the decay $K^-(+p) \to K^+\pi^-$. Thus something else had happened in between. Conservation of strangeness by the strong interactions allowed $K^-(+p) \to K^+ + \Xi^{*-}$, but also

$K^-(+p) \to K^+ + K^0 + \Omega^-$ and $K^-(+p) \to K^+ + \bar{K}^0 + \Sigma^-$. Furthermore it was noted that the two e^-e^+ pairs in the picture were produced by photons whose lines of flight intersected. This suggested that a π^0 had decayed.

The strong reaction $K^-(+p) \to K^+ + K^0 + \Omega^-$ could fit all the data. The Ω^- with strangeness -3 had to decay weakly to a particle with strangeness -2, hence $\Omega^- \to \Xi^0 + \pi^-$ and the π^- was seen. The Ξ^0 had to decay weakly to a $\sum^0$ or Λ^0 but the phase space for the $\sum^0$ decay was too small, hence $\Xi^0 \to \Lambda^0 + \pi^0$. The π^0 should instanteously decay into the two photons whose e^+e^- pairs were indeed seen. The Λ^0 had to decay weakly to $p + \pi^-$ which were also seen. All nonleptonic weak decays satisfied the empirical $\Delta I = 1/2$ rule and $|\Delta S| = 1$. Thus the plethora of particles helped instead of complicated the Ω^- discovery.

Since all representations of $SU(3)$ can be obtained by combining the fundamental representation **3** and its complex conjugate $\mathbf{3}^*$, it was tempting to give a dynamical meaning to these **3** and $\mathbf{3}^*$, and conjecture that all mesons and baryons are made from quarks (as Gell-Mann called them), or aces (as Zweig called them) [143]. If one assumed a unification of the internal flavor-symmetry $SU(3)$ with the spin symmetry $SU(2)$, one obtained the group $SU(6)$, which ought to describe the low-energy spectrum[54]. Indeed, the pseudoscalar octet plus the vector meson nonet (plus the η') combined into the $\mathbf{6} \times \mathbf{6}^* = \mathbf{1} + \mathbf{35}$ of $SU(6)$. Also the baryon spin 1/2 octet and the baryon spin 3/2 decuplet were part of an $SU(6)$ representation, namely[55]

[54]This group was not based on the direct product $SU(3) \times SU(2)$ (which formed the starting point for the grand-unified $SU(5)$ theory) but it was defined by its fundamental six-dimensional representation $(u^+, u^-, d^+, d^-, s^+, s^-)$. The indices $\pm$ denote the helicities $\pm\frac{1}{2}$, and all orbital momenta were assumed to be vanishing for the lowest mass states.

[55]The totally antisymmetric p-index tensor in $SU(n)$ has $\begin{pmatrix} n \\ p \end{pmatrix}$ components, and this yields $\begin{pmatrix} 6 \\ 3 \end{pmatrix} = \mathbf{20}$. A totally symmetric tensor has $\begin{pmatrix} n+p-1 \\ n-1 \end{pmatrix}$ components, and this yields $\begin{pmatrix} 8 \\ 5 \end{pmatrix} = \mathbf{56}$. Young tableaux method show that the mixed representation **70** appear twice. The quarks should form totally antisymmetric wave functions, whereas the **56** is totally symmetric. This required the introduction of a new quantum number, color, as we shall discuss.

$\mathbf{6} \times \mathbf{6} \times \mathbf{6} = \mathbf{56} + \mathbf{70} + \mathbf{70} + \mathbf{20}$ where $\mathbf{56} = (\mathbf{8}, \frac{\mathbf{1}}{\mathbf{2}}) + (\mathbf{10}, \frac{\mathbf{3}}{\mathbf{2}})$. If one assumed that the strange quark was heavier than the up and down quarks, the Gell-Mann-Okubo mass formula [142] could even be derived. However, quarks had fractional baryon number and electric charges. Since this was very unusual, quarks were initially seen as useful mathematical objects without physical reality. The 1960's were the time of Chew's bootstrap program, based on "nuclear democracy" according to which all observed particles were bound states of each other, and none was more fundamental then another. (In fact, this idea, apart from the name, was already proposed by Heisenberg, who tried to incorporate it into his nonlinear spinor theory). The $SU(3)$ classification scheme could be viewed as **how** particles formed multiplets, and the bootstrap program should explain **why** they formed these multiplets. Moreover, the interactions between quarks seemed to pose another daunting problem: should one use field theory?

The quark model of hadrons did for elementary particles what Mendelejev had done for chemistry. It was not clear in the beginning that quarks really existed, because despite enormous experimental effort no single free quark was ever discovered. However, the model explained so clearly and simply some properties of hadrons that a growing number of physicists became convinced it had to be right. We present here a few of the successes of the simplest version of the model: the nonrelativistic quark model, according to which baryons consist of only 3 quarks, and mesons of a quark-antiquark pair. [145] In time, of course, it was realized that this picture was too simple, and that next to these "constituent quarks" there were virtual quark pairs and gluons which formed a "sea of partons".

The magnetic moments of the proton and neutron differ from the value $g = 2$ of Dirac theory which corresponds to the value of one Bohr magneton, $\mu_B = e\hbar/(2m_N c)$ where m_N is the nucleon mass. This is to be expected because due to strong interactions with gluons, part of the time the proton or neutron is replaced by virtual particles with different momenta which couple differently to the magnetic field. Since gluons have $g = 0$ (they are electrically neutral) the average g is not 2 but experimentally one has $g(\text{proton}) = 2.8$ and $g(\text{neutron}) = -1.9$. The nonrelativistic quark model which neglects QCD corrections, gives a surprisingly good approximation, as we now show.

The proton wave function in spin-isospin space should be totally symmetric (being a color-singlet, it is totally antisymmetric in color space), and since it should also be orthogonal to the wave function of the $(I = 3/2, I_3 = 1/2, J = 3/2, J_3 = \frac{1}{2})$ Δ^+ resonance, it is given by[56]

$$\psi(I = I_3 = 1/2; J = J_3 = 1/2) = \frac{1}{\sqrt{18}}[2 \mid u_\uparrow u_\uparrow d_\downarrow\rangle + 2 \mid u_\uparrow d_\downarrow u_\uparrow\rangle + 2 \mid d_\downarrow u_\uparrow u_\uparrow\rangle$$
$$- \mid u_\uparrow u_\downarrow d_\uparrow\rangle - \mid u_\uparrow d_\uparrow u_\downarrow\rangle - \mid d_\uparrow u_\uparrow u_\downarrow\rangle - \mid u_\downarrow u_\uparrow d_\uparrow\rangle - \mid u_\downarrow d_\uparrow u_\uparrow\rangle - \mid d_\uparrow u_\downarrow u_\uparrow\rangle] \qquad (1.3.2)$$

The wave function of the Δ^+ resonance contains the same 9 states but all with a coefficient $\frac{1}{3}$. The neutron wave function is obtained from the proton wave function by interchanging up and down quarks.

The magnetic moment of the proton is then given by taking the expectation value of the magnetic moments of the quarks. This yields $\mu(p)/\mu(B,q) = \frac{2}{3}(\frac{4}{3} + \frac{1}{3}) + \frac{1}{3}(-\frac{1}{3}) = 1$ where $\mu(B,q) = (e\hbar/2m_q c)$ is the Bohr magneton for a spin 1/2 particle with mass m_q and unit charge.[57] For the neutron one finds in the same way $\mu(n)/\mu(B,q) = \frac{2}{3}(-\frac{2}{3} - \frac{2}{3}) + \frac{1}{3}(\frac{2}{3}) = -2/3$. The prediction $\mu(n)/\mu(p) = -2/3$ agrees surprisingly well with the experimental result -0.685. The absolute values agree also rather well if one assumes that each quark has a constituent mass which is 1/3 of the nucleon mass: $m_q = \frac{1}{3}m_N$. One obtains then $\mu(p) = 3\mu_B$ and $\mu(n) = -2\mu_B$, close to the experimental values $2.8\mu_B$ and $-1.9\mu_B$. Of course, a real explanation requires detailed and complicated calculations in QCD.

[56] One can write these states much more simply if one uses products of creation operators for quarks [144].

[57] The factors $\frac{2}{3}$ and $\frac{1}{3}$ the probabilities for the first 3 and last 6 states, respectively, while the terms within parentheses give the magnetic moments of the up-quark and down-quarks, respectively.

As we have discussed, at the end of the 1940's QED was renormalized and at the beginning of the 1970's the electroweak interactions were also shown to be renormalizable. However a renormalizable field theory of the nuclear forces with which one could calculate could not yet be constructed in a similar way because the strong coupling constant was so large that perturbation theory seemed useless. Furthermore, it was not clear which fields to use for the strong interactions, only pion fields, or also ρ-meson fields, kaon fields etc., or perhaps quarks, or other fields? As we have discussed, quarks with fractional electric charges were invented in 1964 by Gell-Mann and Zweig [143] to explain the observed rigid $SU(3)$ flavor symmetry of the hadrons [141], but gauging this symmetry by using Yang-Mills theory of nonabelian symmetry seemed problematic because the $SU(3)$ flavor symmetry was found to be only approximate. Furthermore, nonabelian gauge fields should be massless whereas the nuclear forces had a very short range and thus required heavy particles. (At least, so it seemed; confinement was not yet conceived).

The problems with understanding the nuclear forces as arising from conventional field theory, and the apparent absence of fundamental physical constituents, led many physicists in the 1950's and 1960's to the belief that lack of a solution was not so much due to shortcoming of the theoretical physics community, but rather that a solution to these problems simply did not exist. As an alternative, the bootstrap mechanism was developed, as we already mentioned. However, in the 1970's a solution to all problems was found, and all that it took was a new symmetry, color symmetry. It began with the observation that another quantum number of quarks was needed to satisfy the spin-statistics relation for the quarks inside protons and neutrons and other baryons. The Δ^{++} baryon with spin $J = 3/2, J_3 = 3/2$ and isospin $I = 3/2, I_3 = 3/2$ contains three up-quarks. It is symmetric in spin and isospin, and if it corresponded to a ground state with vanishing orbital angular momentum, its wave function should also be totally symmetric in the three quarks. We already showed this when we discussed the low-energy group $SU(6)$, but one can also construct the quark states directly, using only their spin and isospin.[58] This would violate Fermi-Dirac statistics. Greenberg [144] introduced the concept of color; he suggested that quarks do not satisfy Fermi-Dirac statistics but parastatistics ("parafermions of order 3"), according to which quarks would occur in three versions, baryons consisting of one of each. Parastatistics allowed states with exotic statistics, so the question was why only the qqq and $q\bar{q}$ combinations occur. Parastatistics and $SU(3)$-color were shown to be equivalent in the following way: states that are bosons or fermions in the parastatistics theory are in one-to-one correspondence with the states that are singlets in the $SU(3)$-color theory [146]. Minimal gauge coupling to parastatistics quarks was shown to be impossible [147], but the idea that quarks have order three (three colors) remained.

Conservative quark theories were in turn constructed in which quarks had integral electric charges but appeared with multiplicity 3. In particular Han and Nambu constructed in 1965 a model of quarks with three flavors and three colors with integral charges and with ordinary statistics, and with a double $SU(3)$ group [148]. There were two u-quarks with charge one and one u-quark with charge zero, yielding an average charge 2/3, while there were one down quark with charge -1 and two down quarks with charge zero, yielding an average charge $-1/3$. The same triplet as for the down quarks was proposed for strange quarks. Baryons would again have one quark of each type. On the other hand, the model of fractionally charged quarks of Gell-Mann and Zweig could also easily be extended to contain three colors. Both models resolved the spin-statistics problems. Also the decay rate of $\pi^0 \to \gamma + \gamma$ agreed with both models, but e^+e^- annihilation into hadrons agreed with the fractionally charged quark model

[58] Up and down quarks form the fundamental representation **4** of $SU(4)$. The generators of $SU(4)$ consist of the generators of $SU(2)_{\text{spin}}$ and $SU(2)_{\text{isospin}}$ and direct products of these. In the space of up and down quarks, the spin-isospin wave functions for the product of 3 quarks span a $\mathbf{4} \times \mathbf{4} \times \mathbf{4} = \mathbf{64}$ dimensional space. The totally symmetric representation has $\binom{6}{3} = \mathbf{20}$ components and it splits into a **16** dimensional space for the $(3/2, 3/2)$ Δ resonance, and a **4** dimensional space for the $(1/2, 1/2)$ system of the proton and neutron. The other combinations are not totally symmetric and hence play no role in the construction of mesons and baryons since these are supposed to be color singlets.

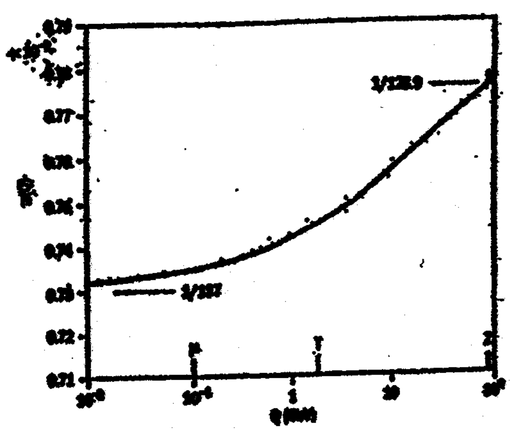
but not with the integrally charged quark model[59]. Thus fractionally charged quarks with a rigid $SU(3)$ color symmetry seemed to be the elementary constitutents of hadrons. Forces between these quarks should be due to fields called gluons, but the precise nature of these gluons was initially not clear.

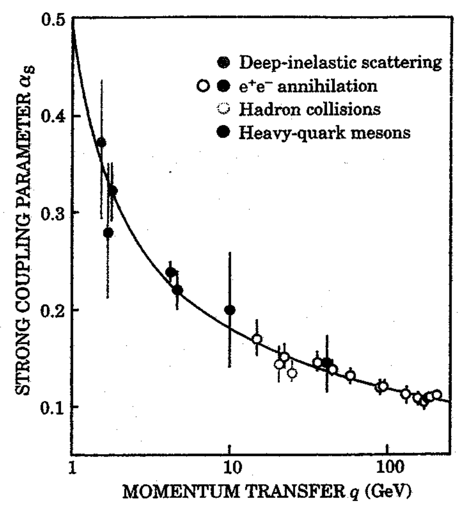
The quarks provide the electric charges of the nucleons, and the carriers of the strong interactions, the gluons, should be electrically neutral, because otherwise emission of a gluon would change the flavor of a quark, and the color symmetry would interfere with the rigid flavor symmetry. Initially it was not clear whether the gluons should carry color charge. The successes of gauge theory for the electroweak interactions suggested also to apply the concept of gauge theory to strong interactions. A gauge theory turns a rigid symmetry into a local symmetry, so (with hindsight!) what was more natural in 1971-1972 than to take the rigid color symmetry and gauge it? Since quarks contain 3 colors, the gauge group must have a 3 dimensional representation, hence $SU(3)$ was the obvious choice. (Another choice would be $SO(3)$, but this group was ruled out because not only color singlets with one quark and one antiquark would be allowed, but also color singlets with two quarks, and these were experimentally not observed). Han and Nambu had already in 1965 proposed that the interactions between quarks were due to exchange of an octet of vector bosons belonging to the adjoint representation of the $SU(3)$ color group. [149] The gauge fields for the $SU(3)$ color group, called gluons, would then be spin 1 vector fields which carry themselves color charge. Bardeen, Fritzsch and Gell-Mann constructed in 1972 an extension of current algebra called light-cone current algebra, in order to be able

[59]The contribution to π^0 decay due to proton and neutron triangles coupled to a π^0 and two photons yields the same result as up and down quarks with 3 colors and charges $\frac{2}{3}$ and $-\frac{1}{3}$ because pions couple to any fermion doublet as $g\vec{\pi} \cdot (\bar{\psi}\gamma_5\vec{\tau}\psi)$ and $1^2 - 0^2 = 3(\frac{2}{3})^2 - 3(\frac{1}{3})^2$.The strange quarks play no roles in pion decay because their mass is higher than that of pions. Also, the Han-Nambu model predicted the same rate: $(1^2 + 1^2 + 0^2) - (0^2 + 1^2 + 0^2) = 1$. However, for e^+e^- annihilation into hadrons using up, down and strange quarks, the prediction of the fractionally charged quark model $3[(\frac{2}{3})^2 + (\frac{1}{3})^2 + (\frac{1}{3})^2] = 2$ agreed with the data, while the Han-Nambu model yielded a result a factor 2 too much: $(1^2 + 1^2 + 0^2) + (0^2 + 1^2 + 0^2) + (0^2 + 1^2 + 0^2) = 4$.

to describe the scaling observed in the deep inelastic electron scattering experiments at SLAC, and these articles contained further ideas which led to QCD [150]. In this way the interactions between quarks and gluons became based on a model with octet gluons [151], which received the name quantum chromodynamics (QCD)[60].

In 1973 a major breakthrough occurred: the concept of running coupling constant was applied to QCD, and one-loop calculations revealed that the effective coupling constant of QCD at high energies decreases (asymptotic freedom [152]), and that only nonabelian gauge theories can be asymptotically free [153].



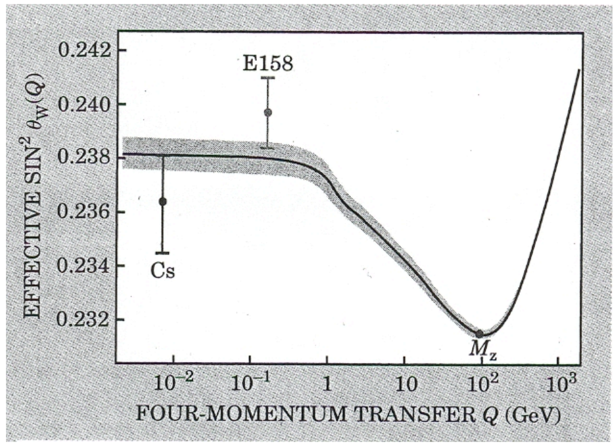


(1.3.3)

Figure 18: The effective coupling $\alpha_1(Q^2)$ of QED. The monotonically rising theoretical curve agrees with precise measurements at the Z mass at CERN's LEP electron-positron collider. Virtual gluons lead to anti-shielding.

Figure 19: The effective coupling $\alpha_3(Q^2)$ of QCD. The data for the monotonically falling curve are due to deep-inelastic scattering, e^+e^- annihilation, heavy-quark mesons, and hadron collisions. [154]. Virtual e^+e^- and quark-antiquark pairs lead to shielding.

[60]The name quantum chromodynamics, or QCD, for this model, is due to Gell-Mann. Chromos means color in Greek



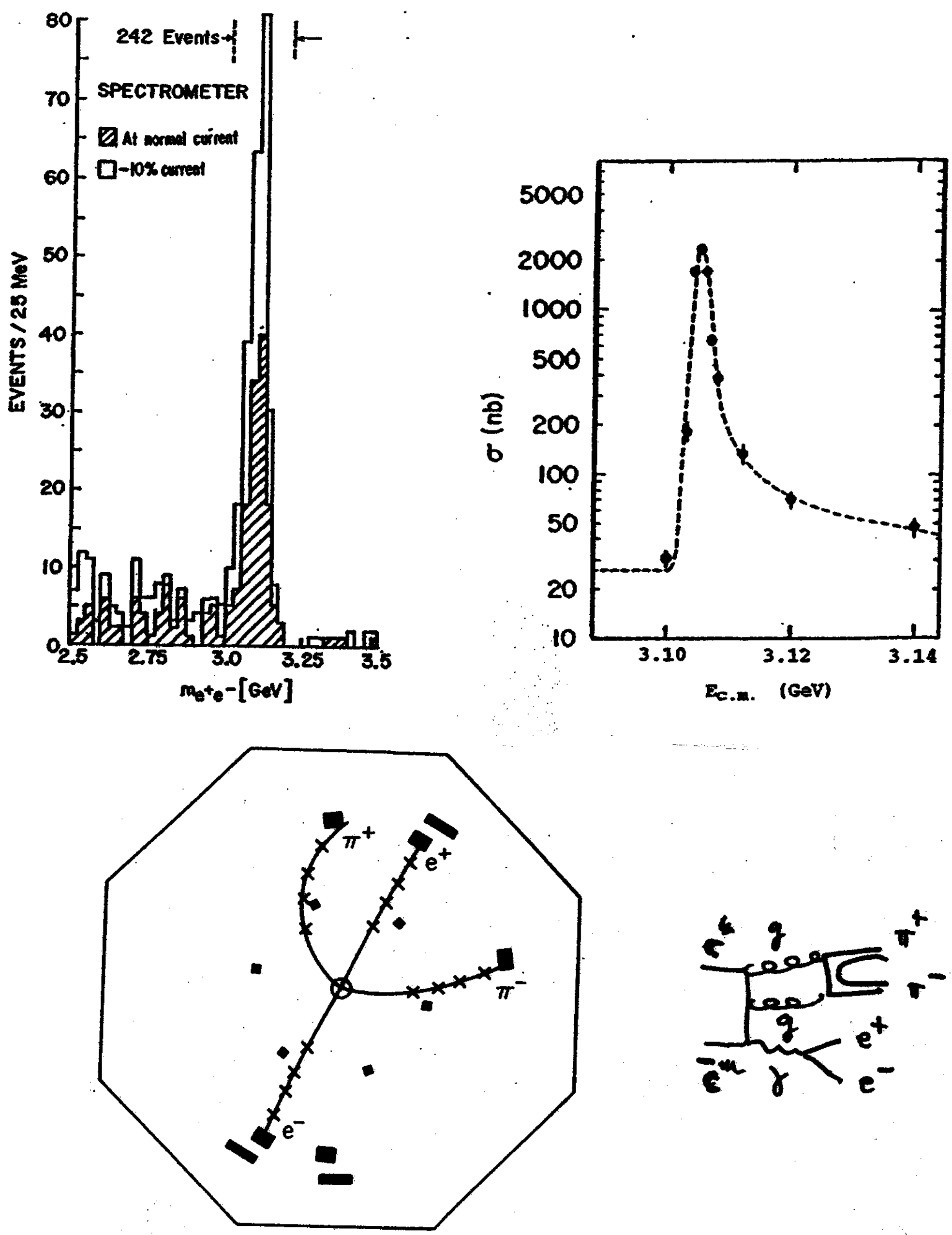
(1.3.4)

Figure 20: The effective weak mixing parameter $\sin^2\theta_w = e^2(Q)/g_{\text{weak}}(Q^2)$ of the electroweak gauge theory. Below the Z-mass, virtual quark-antiquark pairs dominate and yield shielding, but above it, virtual W pairs yield antishielding. The data at 10^{-2} GeV is due to measurements of parity violation in cesium atoms, and is not conclusive about running, being only one standard deviation away from the value at the Z-mass. However, the data at about 2.10^{-1} GeV is due to the scattering of 50 GeV polarized electrons off an unpolarized liquid-hydrogen target, and clearly exhibits running since this data point lies 6 standard deviations above the value of M_Z. One measures the left-right asymmetry $A = (\sigma_L - \sigma_R)/(\sigma_L + \sigma_R)$ for scattering over 90^0 in the COM frame [155], and theory yields $A(Q) = \sqrt{2}[1 - 4\sin^2\theta(Q)]G_F Q^2/e^2(Q)$.

The concept of a running constant is the main result of the renormalization group method, and grew out of early studies of Stückelberg and Petermann, Gell-Mann and Low, and Bogoliubov and Shirkov [156]. In QED, the running coupling constant is just the Fourier transform of the effective charge squared, as measured at a distance r away from the center of the nucleon, $e^2(r)$. In order to remove the large logarithms $\ln(Q^2/\Lambda)$ from perturbation theory (Λ is the renormalization mass and Q^2 a typical energy squared), one redefines the coupling constant such that it absorbs these logarithms [159]. The result of this redefinition is that the effective coupling in QCD decreases at higher energies (and vanishes at infinite energy: asymptotic freedom). Earlier, it had already been noted that the sign of the charge renormalization in QCD is opposite to that in QED, but in these articles the implications of asymptotic freedom are not explicitly mentioned [158]. QCD could be renormalized with the methods of 't Hooft and Veltman. This showed that all difficulties with perturbation theory for the strong interactions encountered since Yukawa, were due to the low energies considered, and that at high energies perturbation theory for QCD becomes as reliable as for QED and electroweak theory at laboratory energies.

Deep-inelastic scattering experiments revealed a pointlike substructure of nucleons which was later recognized to be the experimental discovery of quarks. In the early 1970's people started believing that quark and gluon confinement follows from QCD. In 1979 three-jet events in e^+e^- experiments at DESY were shown to consist of two quark jets and one gluon jet. Thus a second gauge field, the gluon, had been discovered, fifty years after Compton had demonstrated the particle behaviour of photons in 1923, which were predicted by Einstein in 1905. In 1983 the W and Z gauge bosons were discovered in $p\bar{p}$ collisions at CERN, and this concluded the search for the gauge bosons of the Standard Model.

New quarks and leptons were discovered soon after QCD had been established. The charmed quarks were discovered by accident in 1974 in $p{-}p$ collision in Brookhaven, and in e^+e^- experiments at SLAC.



(1.3.5)

Figure 21: The discovery of charm in the fall of 1974. Upper left: the spectrum of e^+e^- pairs, produced and observed at Brookhaven in collisions of protons with a beryllium target. Upper right: the reaction $e^+e^- \to \gamma \to c\bar{c} \to$ anything observed at SLAC at energies near the J/ψ (a bound state of a c and a $\bar{c}$ quark). Lower left: a typical hadronic decay of the J/ψ into $e^+e^-\pi^+\pi^-$; the pattern suggested the name ψ. Lower right: a possible Feynman graph for this process. (One needs at least two gluons because the π^+ and π^- are colorless).

The tau leptons were discovered at SLAC in the later part of the 1970's by carefully studying the results of e^+e^- annihilation experiments. Bottom quarks were discovered in 1977 in pp collisions and top quarks in 1995 in $p\bar{p}$ collisions, both at Fermilab. Finally the tau-neutrinos were discovered just before the end of the century [157]. This concluded the search for the fermions of the Standard Model. That there are not more than three families of fermions followed from studies at CERN of the width of the Z boson at CERN and SLAC, following earlier indications from studies in astrophysics on big bang cosmology. We already discussed this in the previous section.

For small coupling constants (the coupling constant for processes at high energies), perturbation theory could be used for QCD, and gave excellent agrement with the experimental data. However, at lower energies where the QCD coupling constant becomes large, field theory was unable to obtain equally impressive results. As a remedy, K. Wilson started a lattice approach, in which space is divided into a lattice, and discrete Feynmann graphs can be constructed with path-ordered Wilson lines $P\exp\int A_\mu dx^\mu$ as a propagator. The discretization of space into a lattice is similar to the time-slicing used for path integrals. No ghosts are needed, hence gauge-invariance is maintained at all intermediate stages. The great advantage of this approach is that it allows realistic nonperturbative physics to be calculated.

Although this is a book on continuum gauge field theory and not on lattice gauge field theory, we mention one of the most impressive results of the lattice approach: a realistic spectrum of baryons and mesons. In QCD with 3 flavors, there are only 4 parameters used as input: the gauge coupling constant g and the 3 current quark masses (m_{up}, m_{down} and $m_{strange}$). All depend on the renormalization scale μ. Instead of g one uses the nonperturbative parameter Λ of QCD. For vanishing quark masses ($m_{quark} = 0$) it is related to g by the following nonperturbative formula

$$\Lambda = \mu e^{-\frac{1}{2\beta_0^2 g^2}}(\beta_0^2 g^2) e^{-\frac{\beta_1}{2\beta_0}}(1 + \mathcal{O}(g^2)) \tag{1.3.6}$$

Then one finds from lattice calculations that all masses can be expressed in terms of Λ

$$\begin{aligned}
&m_p = c_p\Lambda, \quad \text{same mass for} \quad n, \Lambda, \Sigma, \Xi \\
&m_\rho = c_\rho\Lambda, \quad \text{same mass for} \quad K^*, \omega, \phi \\
&m_\Delta = c_\Delta\Lambda, \quad \text{same mass for} \quad \Sigma^*, \Xi^*, \Omega \\
&m_\pi = m_K = m_\eta = 0 \\
&m_{\eta'} = c_{\eta'}\Lambda
\end{aligned} \tag{1.3.7}$$

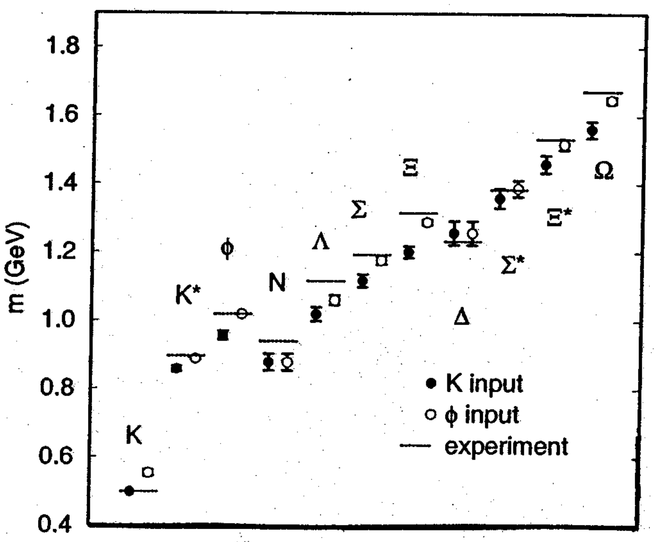
The nonzero value for $m_{\eta'}$ is due to instantons; the Venezianno-Witten formula yields

$$m_{\eta'}^2 - \frac{1}{2}(m_{\pi^0}^2 + m_\eta^2) = 3\lambda \tag{1.3.8}$$

$$\lambda \approx \frac{1}{2f_\pi^2}\frac{\langle Q_{top}^2\rangle}{L^4}; \quad Q_{top} = \int d^4x \frac{TrF_{\mu\nu}\tilde{F}^{\mu\nu}}{16\pi^2} \tag{1.3.9}$$

So, the mass of η' is proportional to the expectation value of the winding operator Q_{top} , and contains a sum over all winding numbers.

Letting the 3 quark masses take nonvanishing values, these results are modified in calculable ways. In this way lattice gauge theory predicts the masses of baryons and mesons in terms of Λ and 3 quark masses. To fix the 4 free parameters, one uses as input the experimental values of 4 convenient masses, usually m_p, m_{π^+}, m_{K^+}(or m_ϕ) and m_n. A good approximation is to set $m_{up} = m_{down}$, in which case one does not need the neutron mass m_n as input. The resulting mass spectrum, obtained without taking fermion loops into account, is given in the following figure, where m_p, m_{π^+} and either m_K or m_ϕ have been chosen as input.



(1.3.10)

Figure 22: Spectrum of baryons and mesons as obtained from lattice gauge field theory without taking fermion loops into account.

We close this subsection with some brief comments on modern developments in gauge field theory and beyond. The most exciting theme of all this work is that for a deeper understanding of the quantum properties of the nongravitational world one needs gravity. In the next section we discuss that one also arrives at these modern developments if one starts with gravity, and extends it by adding fermi-bose symmetry (supergravity) and introducing extended objects (strings). It becomes increasingly difficult to separate gravity from nongravitational quantum theory. Ongoing research in these areas produces as a by-product new perspectives on quantum gauge field theory, just as the study of nonabelian gauge theories clarified some of the formal aspects of QED.

The concept of running coupling constants in QCD can also be applied to the coupling constants of the $SU(2)$ and $U(1)$ gauge theories for the electroweak sector. Assuming there are no new particles between 10^3 and 10^{16} GeV (the "desert scenario"), it was found that these three coupling constants come approximately together at an energy of $10^{14} - 10^{16}$ GeV. Closer inspection revealed that for the

Standard Model this unification of the 3 coupling constants is not very good, but for the minimally supersymmetric standard model (the MSSM model) the unification is much improved.

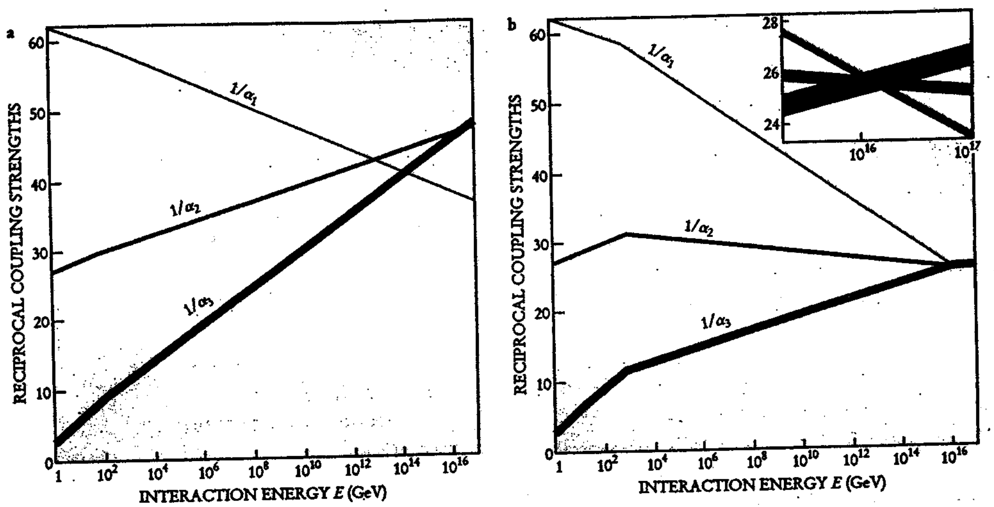


Figure 23: The three coupling "constants" of the standard model of particle theory are, in fact, "running" functions of the energy E at which particles interact. Plotted here are the reciprocals of the running coupling constants for the electroweak interactions (α_1 and α_2) and the strong interaction (α_3), theoretically extrapolated to energies far above our experimental knowledge. In the minimal standard model (a) the three couplings fail to converge. But the supersymmetric extension of the standard model (b) does succeed in uniting the extrapolated couplings to within a few percent (see insert) at about 10^{16} GeV. The kinks near 10^3 GeV, crucial to that convergence, reflect the predicted appearance of supersymmetric particles at those energies. The width of the trajectories indicates the experimental uncertainty.

Supersymmetry (susy), a symmetry of the action between bosonic and fermionic fields, was constructed in the early 1970's. It is sometimes criticized for needing (at least) as many new particles as particles which are known (for every particle there should be a supersymmetric partner, but none of the existing particles have the quantum numbers which allow them to be superpartners.) However, this situation is not new in physics; when the Dirac equation was proposed it predicted also a doubling of the number of charged fermions, namely particles and antiparticles. In a few years experiments at CERN should detect supersymmetric particles if they exist.

One can unify the 3 nongravitational gauge theories $SU(3) \times SU(2) \times U(1)$ into

one simple renormalizable nonabelian gauge group ($SU(5)$ or $SO(10)$ for example). One can make these "Grand Unified Theories" [159] supersymmetric by adding supersymmetric partner particles. These supersymmetric extensions of the grand-unified theories (GUT's) have had some successes. They
(i) predict the relative strengths of the 3 "low-energy" coupling constants of $SU(3) \times SU(2) \times U(1)$ as we already discussed; hence they predict the weak mixing angle θ_w between the gauge fields of $SU(2)$ and $U(1)$ reasonably well;
(ii) both GUT theories and susy GUT theories explain why protons, electrons, etc. all have the same charge (this comes about because quarks and leptons belong to the same representations of the GUT);
(iii) in the MSSM model one of the 5 physical Higgs bosons has a $(\text{mass})^2$ which turns negative as one runs from a common mass at the unification scale down to laboratory energies. This offers an explanation how spontaneous symmetry breaking of $SU(2) \times U(1)$ to the electromagnetic $U(1)$ might occur;
(iv) these theories can also explain the ratio between the masses of some quarks and leptons.

The GUT model of Pati and Salam [160] is based on the left-right symmetric group $G(2,2,4) \equiv SU(2)_L \otimes SU(2)_R \otimes SU(4)_c$ which acts on a left-handed fermion multiplet F_L in the $(2,1,4)$ representation, and a right-handed multiplet F_R in the $(1,2,4)$ representation. The group $SU(2)_L$ is the $SU(2)_L$ of the Standard Model, but $SU(2)_R$ is an extension of the $U(1)_Y$ of the Standard Model. Also $SU(4)_c$ is an extension of the color group $SU(3)_c$ of the Standard Model. The fermion multiplets F_L and F_R contain the 15 particles of one family of the Standard Model and an extra right-handed neutrino. Denoting the colors by r(ed), y(ellow) and b(lue), one has

$$F_L = \begin{pmatrix} u_r & u_y & u_b & \nu_e \\ d_r & d_y & d_b & e^- \end{pmatrix}_L \text{ idem } F_R \tag{1.3.11}$$

Thus $SU(4)_c$ treats leptons as a fourth color. The quark-lepton unification of this scheme is manifest, in contrast to the $SU(5)$ scheme where there is also quark-lepton unification but in two different multiplets (the **5** and **10**). If one embeds this model into the simple GUT group $SO(10)$ (or into string theory), one obtains gauge coupling constant unification. By itself $G(2,2,4)$ does not predict this, and it also treats each family separately (no family unification). But it yields two predictions which are experimentally satisfied: the neutrino mass $\Delta m_{23} \sim \frac{1}{20}eV$ and the value of the neutrino mixing angle $\sin^2\theta_{23}$ (obtained from the mass mixing matrix element $V_{cb} = 0.04$).

The electric charge Q should be a linear combination of the generators of G. Since quarks with different colors all have the same electric charge, Q should commute with $SU(3)_c$. Any $SU(4)$ generator in Q should then be proportional to the unit matrix in the first 3×3 submatrix, and this fixes it uniquely. There is an $SU(4)$ generator in Q, and its eigenvalues are just $B - L$ (baryon number minus lepton number).

$$Q = I_{3,L} + I_{3,R} + \frac{1}{2}(B-L); B - L = \begin{pmatrix} 1/3 & & & \\ & 1/3 & & \\ & & 1/3 & \\ & & & -1 \end{pmatrix} \tag{1.3.12}$$

One may check that the electric charges of quarks and lepton come out correctly.

To break the group G down to the group $SU(2)_L \otimes U(1)_Y \otimes SU(3)_c$ of the Standard Model, one uses two complex Higgs multiplets ϕ_L and ϕ_R in the same fundamental representation as the fermions

$$\phi_L = (2,1,4); \phi_R = (1,2,4) \tag{1.3.13}$$

The potential

$$V = -\frac{1}{2}\mu^2(|\phi_L|^2 + |\phi_R|^2) + \frac{1}{4}\lambda_1(|\phi_L|^4 + |\phi_R|^4) + \frac{1}{2}\lambda_2|\phi_L|^2|\phi_R|^2 \tag{1.3.14}$$

has minima at

$$\begin{aligned} &\phi_L = 0 \text{ or } -\mu^2 + \lambda_1|\phi_L|^2 + \lambda_2|\phi_R|^2 = 0, \text{ and} \\ &\phi_R = 0 \text{ or } -\mu^2 + \lambda_1|\phi_R|^2 + \lambda_2|\phi_L|^2 = 0 \end{aligned} \tag{1.3.15}$$

The local minimum at the origin $\phi_L = \phi_R = 0$ has $V = 0$, but the minima at $\phi_L = 0, \phi_R \neq 0$ and $\phi_L \neq 0, \phi_R = 0$ have $V = -\frac{1}{4}$, while the minimum at $\phi_L \neq 0, \phi_R \neq 0$ has $|\phi_L|^2 = |\phi_R|^2$ and $V = -\frac{1}{2}\mu^4/(\lambda_1 + \lambda_2)$. For $\lambda_1 < \lambda_2$ the minimum at $\phi_L = 0, \phi_R \neq 0$ (or $\phi_R = 0, \phi_L \neq 0$) is the absolute minimum, and hence $L - R$ symmetry is spontaneously broken. The only scalar in ϕ_R without an electric charge is the scalar at the same place as ν_R

$$< \phi_R >= \begin{pmatrix} 0 & 0 & 0 & v \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{1.3.16}$$

and this shows that $SU(4)_c$ is broken down to $SU(3)_c$. Below the GUT scale of $M_{\mathrm{GUT}} = 2.10^{16} GeV$, the $B-L$ symmetry is broken, but above this scale it is unbroken, and this means that the right-handed neutrinos ν_R cannot acquire Majorana masses of the order of the Planck scale. This will be important for the seesaw mechanism which we shall discuss shortly. The $SU(2)_R$ group is broken down to the 2×2 generator under which $\binom{v}{0}$ is invariant (τ^+) and this generator we identify with the $U(1)$ hypercharge generator Y. Thus, G is broken down to the group of the Standard Model.

The same running of masses which leads in $SU(5)$ to the relation $m_b(M_{GUT}) = m_\tau(M_{GUT})$ (satisfied up to 10% with susy) also predicts in $G(2,2,4)$ that $m_{\nu_\tau}(M_{GUT}) = m_{top}(M_{GUT})$. Actually, the τ leptons and τ neutrino do not run very much, hence the bottom and top quark mass at the GUT scale are small, a result of asymptotic freedom.

With a right-handed neutrino present, the neutrinos can acquire Dirac masses $m_D = \bar{\nu}\nu$ with $\bar{\nu} = \nu^\dagger i\gamma^0$. However, the right-handed neutrinos can also get a Majorana mass $m_{Maj} = \bar{\bar{\nu}}\ \nu$ with $\bar{\bar{\nu}} = \nu^T C$. This mass term violates $B - L$ symmetry by 2 units, hence it cannot be present at the Planck scale but only below the GUT scale where $B - L$ gets broken by the Higgs effect. The mass matrix, of the heaviest neutrino for simplicity, and its diagonal form read then

$$M = \begin{pmatrix} 0 & m_D \\ m_D & m_{Maj} \end{pmatrix} \rightarrow \begin{pmatrix} \frac{-m_D^2}{m_{Maj}} & 0 \\ 0 & m_{Maj} \end{pmatrix} \tag{1.3.17}$$

So, if m_{Maj} is huge, the neutrino mass $m(\nu) = -m_D^2/m_{Maj}$ is tiny. This is called the see-saw mechanism. The Dirac mass at the GUT scale is equal to the top quark mass at the GUT scale, as we already mentioned, and the latter is 120 GeV. The Majorana mass is due to an effective coupling $\mathcal{L}_{Maj} = g^2 f_{ij} \mathbf{16}_i \mathbf{16}_j \mathbf{16}_H^* \mathbf{16}_H^*/M$, where i, j are family indices, and $\mathbf{16}_i$ denote the fermion multiplets while $\mathbf{16}_H$ denote the Higgs multiplets. The effective scale M is argued to lie between the string scale $M_{string} \approx 4.10^{17} GeV$ and the Planck scale $m_{Pl} \approx 2.10^{18}$. So one may take $M \approx 10^{18} GeV$. This gives then a Majorana mass of the following order

$$m_{Maj} \simeq g^2 < \mathbf{16}_H >< \mathbf{16}_H > /M = \frac{(2.10^{16})^2}{10^{18}} GeV \tag{1.3.18}$$

One finds then for the tau neutrino mass ($i = j = 3$, setting $f_{33} = 1$) $m(\nu_\tau) = m_D^2/m_{Maj} = (120GeV)^2/4.10^{14}GeV \simeq \frac{1}{28}eV$ which is precisely the experimental value!

Quantum gauge field theory itself might be an approximation of a deeper truth. String theory as a theory of extended objects is an extension of quantum field theory which reproduces in the low-energy limit quantum gauge field theory. One of the main achievements of string theory has been that it points to a finite theory of quantum gravity. Quantum field theory, at least at the perturbative level, leads to nonrenormalizable divergences in quantum gravity because the gravitational coupling constant has the wrong dimension. One might try to solve the problem of quantum gravity by introducing a new kind of intermediate vector bosons for gravity similar to the W and Z bosons which made the nonrenormalizable 4-fermion $V - A$ interactions renormalizable. However, this idea has never been successful. String theory solves this problem in another way: a closed string contains left-moving vectors and right-moving vectors, and the two indices μ and ν of the metric $g_{\mu\nu}$ correspond to one of each of them.

Strings contain in general tachyons (particles with negative mass-squared which move faster than light). However, if one incorporates supersymmetry into string theory, the tachyons are eliminated. Supersymmetry is also needed to cancel the divergences in string loop graphs, and to achieve that physical predictions do not depend on which coordinates one chooses on the two-dimensional worldsheet swept out by closed strings (modular invariance). So, supersymmetry is crucial for the consistency of string theory.

In fact, there exists an ordinary gauge field theory for local supersymmetry, called supergravity. [161] It is a supersymmetric extension of Einstein's theory of general relativity, with a massless spin 3/2 fermionic partner for the graviton called gravitino. A supergravity theory with two gravitinos even realizes Einstein's dream of unifying QED and gravity [162]. Conversely, if one starts constructing a gauge theory of local supersymmetry, one discovers that one needs general relativity in the bosonic sector of this theory. Thus, local supersymmetry and supergravity are equivalent. They are the low energy limit of supersymmetric string theory, and any connection between these very new theories and physical reality will have to be formulated in the framework of supergravity.

We leave here (with some regret) the fascinating modern developments in supergravity, superstring theory and other theories of quantum gravity but we make more comments about them in the next section. However, this might be an occasion to close the circle and make a journey back in time to a period before even quantum mechanics was established, when there were only two field theories (Maxwell theory and Einstein's general relativity), and 3 elementary particles (electrons, photons and protons). The problem physicists faced in those times was what relation there was between these two field theories: was one of them contained in the other, or were both manifestations of something more profound?

4 Gravity

The concept of gauge invariance grew out of studies of gravitation and electromagnetism, and attempts at unifying these two fundamental interactions of the 19th century. The quest for unification is not new. Already three centuries earlier, Newton had unified terrestrial gravity and celestial gravity, as presented in his "Principia" of 1686. Two centuries later, Maxwell and others had achieved a similar unification of electricity and magnetism, or rather, as presented in Maxwell's 1864 treatise "A dynamical theory of the electromagnetic field", a unification of light and electromagnetism. The next milestone was Einstein's 1905 special theory of relativity [163] which unified space and time. In 1915 his general theory of relativity unified dynamical forces with geometry, and from then on spacetime was no longer a fixed arena in which physics took place, but rather an active part of physics. The attempts at unifying gravity and electromagnetism which we are going to discuss were developed after the construction of general relativity, and all are based on notions of spacetime geometry in one way or another. By the 1920's Maxwell theory and general relativity were, of course, established as successful physical theories describing electromagnetism and gravity, but the fact that both were gauge theories was initially not appreciated. The principle of gauge invariance was developed in the time period between 1918 and 1930, and is largely due to Weyl (1918, 1929) and Fock (1927, 1929), with important contributions by Kaluza (1921) and Klein (1926), Schrödinger (1922) and London (1927), with other contributions by Nordström (1912), Noether (1918), Cartan (1922) and others. The role of gauge invariance for the strong and weak nuclear interactions was further developed by Klein (1939), and Pauli (1953), and culminated in the gauge theory for nonabelian groups by Yang and Mills (1954)[61].

[61] At about the same time as Yang and Mills, but unaware of their work, Utiyama developed an approach both to nonabelian spin one fields and to gravity. Upon arrival at the Institute for Advanced Studies in Princeton for a stay, he was told of the article by Yang and Mills, and he was so disappointed that he put his own paper away. Only a year later he noticed that Yang and Mills had not discussed gravity, and

We trace here the early part of these developments.

One of the earliest attempts to unify gravity and electromagnetism is due to Nordström [165]. He proposed in 1912 in Helsinki to start in five dimensions with Maxwell theory. The 5-dimensional vector potential $A_{\hat{\mu}}$ decomposed into a 4-component part A_μ and a scalar part $A_5 = \phi$, and the scalar field ϕ was to be the gravitational field.

$$A_{\hat{\mu}}(x^\mu, x^5 = 0) = \{A_\mu(x^\mu), \phi(x^\mu)\} \tag{1.4.1}$$

The coordinates x^μ were to be identified with the physical coordinates, not at all obvious in view of the difficulties in identifying x^μ with the physical position of a Dirac electron. Exchange of massless even-spin particles between static sources leads to an attractive long-range force, so this theory correctly predicted that gravity is attractive. Instead of coupling to mass as in Newton's theory, it became clear after the theory of special relativity of 1905 that one should rather couple to energy than to mass, and relativity required then coupling to the energy- momentum tensor of matter. Nordström coupled his scalar field to the trace $T_\mu{}^\mu$, but the latter vanishes for electromagnetism. Thus in his theory light would not be deflected by the sun, in contradiction with the later sun-eclipse experiments of 1919. Furthermore, it was later noted his theory predicted a recession instead of the experimentally observed precession of the planet Mercury in its orbit around the sun, and it could only account for 1/6 of the observed magnitude. Still, the very idea of unifying two physical field theories in a relativistically correct way by introducing an extra dimension of spacetime was a bold endeavour.

Einstein noted sometime after 1905 that special relativity treated constant speeds v of an observer correctly, but constant angular speeds were not included. Ehrenfest noted that on a rotating disk the flat-space geometry is deformed, and so Einstein eventually was led to view the metric as the gravitational field. As one might expect, there were earlier studies by him which were also based on a scalar gravitational

then he still published his article [164].

field. Einstein studied Nordström's theory [165] and showed that it could be made covariant.[62] Actually, he worked for years on a theory with one scalar field $-g_{00} = c^2(x)$. However, he convinced himself that he needed nonlinear interactions, and this proved very difficult, so this approach was abandoned. His 1915 theory [167] described physical phenomena in a way which does not depend on the actual coordinate system chosen, and he used the notion of a connection for parallel transport of a vector v_ν in curved space: $v_\nu \to v_\nu + \Delta v_\nu$ with $\Delta v_\nu = \Delta x^\mu (\Gamma)_{\mu\nu}{}^\rho v_\rho$. In fact, he used the minimal connection, namely the Riemannian connection which is not an independent field but constructed from the metric

$$(\Gamma)_{\mu\nu}{}^\rho = \left\{ \begin{array}{c} \rho \\ \mu\nu \end{array} \right\} = \frac{1}{2} g^{\rho\sigma}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}) \tag{1.4.2}$$

The history of general relativity has so much been discussed that it need not be repeated here. [168] Let us only mention here that Hilbert and Einstein both separately derived the field equations $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa T_{\mu\nu}$ from an action at the end of 1915.

The success of the geometrical approach of Einstein to gravity led others after him to try to explain other physical theories also by geometrical concepts. One of the first was Weyl. In 1918 he proposed [169] a geometrical theory for gravity and electromagnetism together, by using a more general connection than the Christoffel symbol (1.4.2) which Einstein had used. He added a term involving the vector

[62] It might seem natural to make Newton gravity relativistic by replacing the Laplace equation $\Delta^2\phi(\vec{x}) = 4\pi G\rho(\vec{x})$ and the Newton equation $\vec{F} = md^2\vec{x}/dt^2 = -m\vec{\nabla}\phi$ by $\Box\phi(\vec{x},t) = 4\pi G\rho(\vec{x},t)$ and $F_\mu = mdu_\mu/d\tau = -m(\partial_\mu\phi + u_\mu u^\nu \partial_\nu\phi)$, where ρ is the mass density, $u^\mu = dx^\mu/d\tau$ and $\tau = \sqrt{1-\beta^2}dt$ is the proper time. (Consistency with $u^\mu du_\mu/d\tau = 0$ requires the $u_\mu u^\nu \partial_\nu \phi$ term). However, Einstein noted that this proposal violates the weak (gravitational) equivalence principle. To see this, consider a mass m that moves at $t = 0$ in a horizontal direction. The acceleration in the vertical (z) direction is then $d^2z/dt^2 = -(1-\beta^2)\partial\phi/\partial z$. Hence, the faster a particle moves in a horizontal direction, the slower it falls vertically in this theory. Later Nordström produced an improved second theory, which corresponds in terms of modern field theory to the action $S = \int[-\frac{1}{2}(\partial_\mu\phi)^2 - \frac{4\pi G}{c^2}\phi T]d^4x$ where T is the trace of the stress tensor. For a point particle $T(x) = -\eta_{\mu\nu}\int m\frac{dX^\mu}{d\tau}\frac{dX^\nu}{d\tau}\delta^4(x - X(\tau))d\tau$. In modern terms, this corresponds to a conformal metric $g_{\mu\nu} = \eta_{\mu\nu}\phi$ [166]. Since the same mass m enters in the field equation for ϕ and for X^μ, the weak equivalence principle is now satisfied. From $S_{\text{int}} = -4\pi Gm\int\phi(X)\sqrt{-(dX^\mu/d\tau)^2}d\tau$ one now obtains $d^2X^\mu/d\tau^2 = -\frac{1}{\phi}(\partial^\mu\phi + u^\mu u^\nu\partial_\nu\phi)$.

potential A_μ of the electromagnetic field

$$(\Gamma^W)_{\mu\nu}{}^\rho = \left\{ \begin{array}{c} \rho \\ \mu\nu \end{array} \right\} + \frac{1}{2} g^{\rho\sigma} \frac{e}{\gamma} (g_{\mu\sigma} A_\nu + g_{\nu\sigma} A_\mu - g_{\mu\nu} A_\sigma) \tag{1.4.3}$$

where e was the charge of an electron and γ a free constant. The relative strength of the A_μ-dependent terms was fixed by requiring that Γ^W be symmetric in its two lower indices, and that null vectors (vectors v_ν with vanishing length) remain null vectors after parallel transport. He was the first to introduce non-Riemannian geometry into physics, and to use this for a geometrical description of electromagnetism. Gravity rotated vectors under parallel transport, while electromagnetism changed their lengths. Thus he tried to derive gauge theories from spacial properties of the connection. Under parallel transport along a distance Δx^μ, the length $(g_{\mu\nu} v^\mu v^\nu)^{1/2}$ of a vector v^μ changed in Weyl's theory by the **real** scale factor $-\frac{1}{2}\frac{e}{\gamma} A_\mu \Delta x^\mu$. Rescaling the metric $g_{\mu\nu}(x)$ to $(\exp \lambda(x)) g_{\mu\nu}(x)$, a compensating change $A_\mu \to A_\mu - \frac{\gamma}{e} \partial_\mu \lambda$ did not change his connection, but it would make lengths depend on the choice for λ. One obtains the same result in (1.4.3) if one uses the original connection in (1.4.2) together with a rescaled metric

$$\hat{g}_{\mu\nu}(x_2) = g_{\mu\nu}(x_1) e^{\frac{e}{\gamma} \int_{x_1}^{x_2} A_\mu dx^\mu} \tag{1.4.4}$$

and uses $\hat{g}_{\mu\nu}$ to define lengths. Then the length of a vector became λ-independent. Weyl called the change in $g_{\mu\nu}$ and A_μ due to λ a change in gauge. This is the origin of the concept of gauge invariance![63] Weyl hoped that electromagnetic current conservation $\partial^\mu j_\mu = 0$ would follow from the local scale invariance due to λ transformations, just as energy momentum conservation $\partial^\mu T_{\mu\nu} = 0$ follows from invariance under general coordinate transformations.[64] So he went from 4 to 5 local invariances, just as Nordström had gone from 4 to 5 dimensions. He constructed an action for

[63] Eich-Transformation. The word Eich means measure in German. It comes from the latin verb aequare, to make equal, namely to make the length equal to a standard one. [14]

[64] This argument is incorrect. The usual Noether current is due to rigid symmetries of matter actions, but for matter actions coupled to gravity such that the matter action is invariant under general coordinate transformations with parameter ξ^μ, the invariance of the matter action under this local symmetry implies

gravity which was invariant under local scale transformations, namely the Weyl action $\mathcal{L}_{\mathcal{W}} = \mathcal{C}_{\mu\nu\rho\sigma}\mathcal{C}^{\mu\nu\rho\sigma}$ which can be written as $\mathcal{L}_W = R_{\mu\nu}(\Gamma^W)^2 - \frac{1}{3}R(\Gamma^W)^2$ in 4 dimensions using the Gausz-Bonnet theorem. Upon expanding in terms of A_μ he found that the terms with A_μ canceled, $\mathcal{L}_W(\Gamma^W) = \mathcal{L}_W(\Gamma)$. He therefore added the Maxwell action by hand and obtained in this way an action that was both locally scale invariant, and contained electromagnetism

$$\mathcal{L}_W(\Gamma^W) = \mathcal{L}_W(\Gamma) - \frac{1}{4}F^2_{\mu\nu} \tag{1.4.5}$$

Here definitely was a geometrical origin of electromagnetism. However, his scale transformations were not integrable: under parallel transport from a point x_1 to a point x_2 the length would scale by the factor $\exp(\int_{x_1}^{x_2} A_\mu dx^\mu)$. For a closed loop, the change in scale would be $\exp(\oint F_{\mu\nu}dx^\mu dx^\nu)$ which is gauge-invariant, but in general does not vanish. Thus the change in length would depend on the path taken.

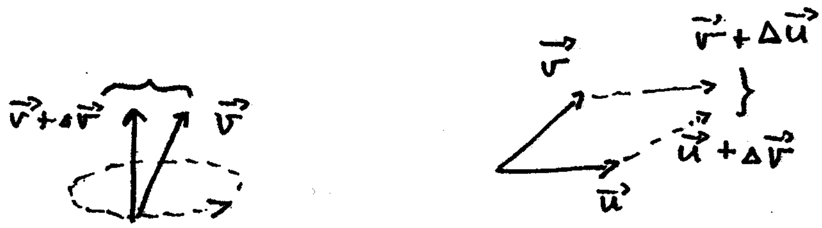
Einstein objected in a postscript to Weyl's article that a change in length under parallel transport would imply that the notion of standard clocks and standard times could not be maintained in such a theory. Chemical elements with definite spectral lines could not exist, contrary to observation. Weyl disagreed, and tried to save his theory, but it did not survive in this form. (See however below). Still, Weyl supported the idea to use geometrical concepts of general relativity to unify gravity and electromagnetism, and he identified the connection as the essential object to describe electromagnetism as well as gravity.

Another generalization of the Riemannian gravitational connection was proposed by Cartan [170], who added an antisymmetric part called torsion tensor to the connection for parallel transport

$$(\Gamma_\mu)_\nu{}^\rho = \left\{ \begin{array}{c} \rho \\ \mu\nu \end{array} \right\} + T^\rho_{\mu\nu} \text{ with } T^\rho_{\mu\nu} = -T^\rho_{\nu\mu} \tag{1.4.6}$$

$\delta S(g_{\mu\nu}, \varphi) = 2\frac{\delta S}{\delta g_{\mu\nu}}D_\mu\xi_\nu + \frac{\delta S}{\delta\varphi}\xi^\mu\partial_\mu\varphi = 0$. If $\frac{\delta S}{\delta\varphi} = 0$ one obtains $D^\mu T_{\mu\nu} = 0$ where $T_{\mu\nu} \sim \delta S/\delta g^{\mu\nu}$, and in flat space this becomes $\partial^\mu T_{\mu\nu} = 0$. Local scale invariance with $\delta g_{\mu\nu} = \lambda g_{\mu\nu}$ implies instead that $T_{\mu\nu}$ is traceless, not that the electromagnetic current is conserved.

Geometrically the meaning of this addition is that parallelograms no longer close: transporting a small vector u^μ along another small vector v^μ does not give the same result as transporting v^μ along u^μ. (In Einstein's general relativity parallelograms did close, but a vector did not return to its original orientation after round-transport along a small circle).



(1.4.7)

Figure 24: Parallel transport of a vector yields rotation proportional to the curvature. Parallelograms due to parallel transport do not close if there is torsion.

The torsion of Cartan never really became a crucial part of classical general relativity, but in 1976 torsion found its true meaning: the theory of supergravity is a theory of torsion induced by gravitinos (the fermionic spin 3/2 partners of the bosonic spin 2 gravitons). Thus the antisymmetric extra term in the connection introduced by Cartan has found physical applications, whereas the symmetric extra term introduced by Weyl has not survived in time. Cartan was also the first to introduce vielbein fields (square roots of the metric, called "repères mobiles" by Cartan, and Vierbeine in German (vier=four)[65]), and as Schrödinger, Weyl and Fock later noted, one needs vielbein fields to describe fermions in curved space. Cartan came to the conclusion that vierbein fields are needed because he discovered that the group $Gl(n, R)$ has no spinor representations. (In the tangent planes the symmetry group is $SO(n)$ which does have spinor representations, hence spinors are Einstein-scalars in general relativity). As far as we know, Cartan was not aware of the close connection between torsion and fermions. For him vierbein fields $e_a^\mu(x)$ gave the orientation of the axes of inertial frames (freely falling lifts) w.r.t. the coordinates of the curved manifold.

[65]Vierbeine are called tetrads in English. The term vielbein is due to M. Gell-Mann (viel=many in German).

These frames could be chosen arbitrarily at each point, and the theory did not depend on this choice.

In 1918 Emmy Noether [171] showed that symmetries are related to currents which are conserved if the field equations are satisfied and conserved currents generate symmetries of the action. Others had already found some relations between symmetries, currents and identities such as the Bianchi identity $D^\mu G_{\mu\nu} = 0$, but she gave a completely general account.[66] In modern field theory, the Noether theorem is of fundamental significance. Any **rigid** symmetry of the action leads to a current, the Noether current, which is conserved if the fields on which the current depends, satisfy their equations of motion. Integrating the time-component of the Noether current over space yields the symmetry charge Q. The commutation relation $[H, Q] = 0$ between the Hamiltonian and a charge Q can be interpreted in two ways: either H is invariant under the action of Q, or Q is conserved. For the construction of gauge theories the Noether current also plays a central role: coupling the Noether current to a gauge field associated to the symmetry, and, if necessary, adding further terms to the action and transformation laws to achieve full invariance of the action, the rigid symmetry becomes promoted to a local (gauge) symmetry. Noether considered not only rigid symmetries also local symmetries. Weyl (1929) generalized the relations between symmetries and conserved currents to gauge theories.

In 1921 Kaluza [172], a docent in mathematics in Königsburg (now Kaliningrad) sent an article to Einstein which was in a way the reverse of what Nordström had tried to do in 1912. Kaluza proposed to start with a purely gravitational theory in 5 dimensions, with a 5×5 gravitational tensor which contained the usual gravitational field $g_{\mu\nu}$, and the Maxwell field of ordinary space, and further a mysterious real scalar

[66] If an action S has a rigid continuous symmetry with constant parameter λ, and one lets λ become local, then the action varies into $\delta S = \int \partial_\mu \lambda j^\mu d^4x$, where j^μ is the Noether current.

field (called graviscalar in supergravity theories, and dilaton in string theories)

$$g_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} g_{\mu\nu} & A_\nu \\ A_\mu & \varphi \end{pmatrix} . \tag{1.4.8}$$

The five coordinates $x^{\hat{\mu}}$ decomposed into the usual coordinates x^μ of 4-dimensional Minkowski space, and a fifth coordinate x^5 which Kaluza set to zero. Setting $\partial/\partial x^5 g_{\hat{\mu}\hat{\nu}} = 0$, the 5-dimensional Einstein action $\sqrt{-\hat{g}}\hat{R}$ decomposed into a sum of the 4-dimensional Einstein action, the Maxwell action and the Klein-Gordon action for φ. General coordinate transformations with ξ^5 (where $\xi^{\hat{\mu}} = (\xi^\mu, \xi^5)$) turned into gauge transformations for A_μ (see below), so here was a spacetime origin of the internal symmetries! Note that in Weyl's 1918 approach A_μ was part of the connection $\Gamma_{\mu\nu}{}^\rho$ but in Kaluza's approach A_μ was part of the metric.

The work of Kaluza was expanded by Klein [173] and Fock [174] who elevated his classical ideas to quantum mechanics. They proposed that the fifth dimension was a small circle, and that one should not set $x^5 = 0$, but expand $g_{\hat{\mu}\hat{\nu}}(x, x^5)$ into a basis of periodic functions

$$\begin{aligned} g_{\mu\nu}(x, x^5) &= \sum_m g^m_{\mu\nu}(x) e^{imx^5} \\ g_{\mu 5}(x, x^5) &= \sum_m A^m_\mu(x) e^{imx^5} \\ g_{55}(x, x^5) &= \sum_m \varphi^m(x) e^{imx^5} \end{aligned} \tag{1.4.9}$$

The 5-dimensional Dalembertian $\pentagon = \sum(\partial/\partial x^\mu)^2 + (\partial/\partial x^5)^2$ gave then a mass to the fields, except when the integer m was zero. For example

$$\pentagon g_{55}(x, x^5) = \sum_m (\Box - m^2)\varphi^m(x) e^{imx^5} \tag{1.4.10}$$

Hence, the masses of fields were due to the compactness of the internal space and proportional to the inverse of the size of that space, but there were always a finite number of x-space fields which were massless. Much work continues to be done in supergravity and string theory based on these ideas, but the massless fields in nongravitational gauge theories acquire masses by another mechanism, namely by spontaneous symmetry breaking.

In the 1930's and 1940's others tried to find a similar spacetime origin for the weak and strong interactions. Klein tried to extend Kaluza's ideas of unification by introducing a 5×5 metric in which the entry $g_{\mu 5}$ was itself a 2×2 **matrix** [14]. Expanding this matrix in terms of Pauli matrices he found what we would now call $SU(2)$ gauge fields B_μ^a. The field B_μ^3 was identified with the electromagnetic field A_μ field, but the relativist C. Mǿller noticed that rigid $SU(2)$ symmetry (charge symmetry of the nuclear forces) required that there be separate fields B_μ^3 and A_μ. [14] Klein then added a term proportional to the unit 2×2 matrix, thereby constructing an early precursor of the $SU(2) \times U(1)$ theory. He even noticed that one could couple either a proton-neutron pair or an electron-neutrino pair to these gauge fields using the Dirac equation, but it was too soon to recognize this as a description of the weak instead of the strong interactions. In fact, he was not even clear about whether these ideas should be applied to the strong or the weak interactions.

Pauli took the next step of considering a 6×6 metric in 1953. However, like Weyl and instead of Kaluza and Klein, he identified the new gauge fields as part of the connection and not of $g_{\mu\alpha}$. Furthermore he struggled with the problem whether these gauge fields should be massive or massless and did not write down an action. He did not publish his results but handwritten letters to A. Pais exist [14]. The construction of an action and transformation laws for nonabelian gauge symmetries was finally solved by Yang and Mills [107] who did not use any higher dimensions. In the 1970's, renewed attempts based on the ideas of Kaluza and Klein were made to derive all nongravitational interactions from a higher-dimensional supergravitational theory. Since there were many more gauge fields, one needed to go to much higher dimensions. Let us only mention here that 11-dimensional supergravity can be dimensionally reduced to 4 dimensions by compactifying 7 dimensions to a compact space. By taking this space to be the coset space $[SU(3)/SU(2) \times U(1)] \times [SU(2) \times U(1)/(U(1))_{\text{diagonal}}]$, one obtains a spacetime origin of the internal symmetry group $SU(3) \times SU(2) \times U(1)$. The smallest coset which leads to $SU(3) \times SU(2) \times U(1)$ as gauge group is this one, and the highest dimension where supergravity exists is 11: it just fits the physics. The problem is that in this way one cannot obtain chiral spinors in 4 dimensions. Later work using orbifolds in string theory resolved this problem. Another problem concerned the cosmological constant: it was huge, proportional to the square of the inverse size of the compact 7-dimensional internal manifold.[67] This problem remains with us today.

The modern extension of the Kaluza-Klein interpretation of gauge symmetries as spacetime symmetries of a suitable extra internal space is as follows [176]. Consider dimensional reduction from D dimensions down to d dimensions. Let the coordinates of the d-dimensional spacetime (our world) be x^μ and those in the internal $(D-d)$ dimensional space y^α. The metric is assumed to decompose into $g_{\mu\nu}(x)$ for our world and $g_{\alpha\beta}(y)$ for the internal space. As internal space one chooses a compact manifold (for example a sphere) with a certain number of space symmetries (generalized translations or rotations) labeled by I. The latter are described by Killing vectors (a particular set of spherical harmonics) ${K_I}^\alpha(y)$ which satisfy $D_\alpha K_{I\beta} + D_\beta K_{I\alpha} = 0$ where $K_{I\alpha} = g_{\alpha\beta}(y) K_I^\beta(y)$. [177] Since the composition rule (=commutator) of two space symmetries is another space symmetry, the Killing vectors satisfy a group property $[K_I, K_J] = {f_{IJ}}^K K_K$ with $K_I = {K_I}^\alpha \frac{\partial}{\partial y^\alpha}$ and ${f_{IJ}}^K$ constants. One next chooses a local Lorentz gauge in D dimensions which casts the vielbein field into triangular form[68]

$$\begin{pmatrix} {e_\mu}^m(x)\Delta^{-\frac{1}{d-2}} & {B_\mu}^I(x){K_I}^\alpha(y)e_\alpha^a(x,y) \\ 0 & e_\alpha^a(x,y) \end{pmatrix} \tag{1.4.11}$$

where $\Delta = \det {e_\alpha}^a$ and $e_\alpha^a(x,y)$ are scalars. (More precisely, e_α^a is decomposed into

[67]One cannot add a compensating cosmological constant to eleven-dimensional supergravity because this violates local supersymmetry.

[68]The Ansatz in (1.4.11) holds in some cases, but in more general cases the precise form of the Ansatz is a very complicated matter. It is correct for supergravities if the compact spaces are S_7, S_5, S_4 [178] or S_2, but it already fails for pure gravity [179].

a sum of terms which are a product of a scalar field in spacetime times a spherical harmonic on the interior space). These scalars are the generalization of the single scalar field Kaluza had found. If one then makes a general coordinate transformation in D dimensions with the special parameter $\xi^\beta(x,y) = \Lambda^I(x)K_I{}^\beta(y)$ and $\xi^\mu = 0$, one finds that the d-dimensional vector fields $B_\mu{}^I(x)$ transform as gauge fields of a Yang-Mills gauge group with structure constants $f_{IJ}{}^K$, namely $\delta B_\mu{}^I = \partial_\mu \Lambda^I + f_{JK}{}^I B_\mu{}^J \Lambda^K$. After integration over y^α the Einstein action in D dimensions becomes the sum of the Einstein action for e_μ^m, the Yang-Mills action for $B_\mu{}^I$, and Klein-Gordon actions for scalars in d dimensions. These ideas can be extended to fermions. One starts with a supergravity theory in D dimensions. The fermionic symmetries of the D- dimensional vacuum solution correspond to Killing spinors $\epsilon_I^A(y)$ (another set of spherical harmonics which are square roots of Killing vectors) with spinor index A. They satisfy the Killing spinor equation

$$D_\alpha \epsilon_I^A + m(\gamma_\alpha)^A{}_B \epsilon_I^B = 0 \tag{1.4.12}$$

where m is proportional to the size of the internal manifold and γ_α are Dirac matrices for the internal space. Integration of the gravitino action over y^α now produces the gravitino (spin 3/2) action and Dirac actions in d dimensions. So all of Dirac theory, Klein-Gordon theory, and Yang-Mills theory can be obtained from the spacetime geometry in higher space dimensions.

The Kaluza-Klein-Fock program was never worked out in the 1920's to the point where it would yield the Yang-Mills action with the explicit form of the nonabelian gauge transformations. Another approach which was not based on the use of higher dimensions, was more successful and led to the concept of gauge theory. In 1922 Schrödinger wrote a remarkable paper [181] in which he observed that in several examples the Bohr-Sommerfeld quantization rule led to a quantization condition on Weyl's connection. Weyl had not specified the constant γ in his connection in (1.4.3). Schrödinger noted that it had the dimension of an action, hence the two natural choices were that γ was proportional to $\hbar$ or to e^2/c ($e^2/\hbar c$ is of course dimensionless). The latter gave huge Weyl factors, whereas the former seemed nicely related to quantum effects. In fact, Schrödinger observed that if one sets $1/\gamma = i/\hbar$, then the Weyl factor would reduce to unity for a closed Bohr-Sommerfeld orbit. Strangely enough, in Schrödinger's 1926 paper [182] where he discovers quantum mechanics, this observation seems forgotton.

London picked up the trail in 1927. [183] He noted that the wave function in de Broglie's theory

$$\psi(x) = e^{\frac{i}{\hbar}(W(x)-m\tau)} \tag{1.4.13}$$

where $W(x)$ is a solution of the relativistic Hamilton-Jacobi equation (with the integration constant chosen such that $\psi(x)$ becomes single-valued), changes along an orbit just such that it produces the Weyl factor. In other words, the wave function acquires in an electromagnetic field a phase factor $\exp \frac{ie}{\hbar} \int A_\mu dx^\mu$. This same phase factor would result if one introduced the notion of minimal covariant derivatives $\partial_\mu - ieA_\mu$ in the Schrödinger equation. This was stressed by Fock [174] (also in 1927) who derived the Schrödinger equation with minimal coupling from a 5-dimensional Laplacian with metric $(ds)^2 = (dx^\mu)^2 + (dw)^2$ where $dw = \frac{e}{m} A dx^\mu + \frac{1}{m} dx^5$.

Thus Schrödinger, London and Fock had shown that Einstein's objection to Weyl's nonintegrable factor could be removed by introducing the factor i of quantum mechanics. Only in 1980 Yang noted that one still could ask whether Einstein's observation about the nonintegrability of phases would have physical consequences. It was then realized that the Aharonov-Bohm effect [184] could be interpreted as due to Weyl's phase factors. Since this effect has been demonstrated to occur in Nature, one may say that the troublesome beginning of Weyl's idea of gauge transformations got a happy ending thanks to quantum mechanics.

We now reach the year 1929 when Weyl published his seminal paper "Elektron und Gravitation" [185] in which he firmly established gauge theory in general. His paper contains the following discoveries:

(i) it studies 2-component spinors in gravity. Spinors (relativistic 4-component spinors) appeared in Dirac's 1928 equation for the electron, and nonrelativistic 2-component spinors had been introduced by Heisenberg and Jordan in 1926, and used by Pauli, to describe the spin of the electron [186]. In mathematics E. Cartan had found the spinorial representations of the orthogonal groups for any dimension as early as 1913, although his well-known book appeared only in 1937. [187] Relativistic 2-component spinors were first discussed by Weyl in a full-fledged and systematic way in his 1929 article. (Later a much simpler presentation of spinor theory was constructed based on Clifford algebras [188]). Lorentz transformations of these spinors are represented by the matrices of $Sl(2, C)$. He noted that Lorentz invariance forbids a mass term for two-component spinors.[69] Because $(\psi)^*$ transforms in a different representation of the Lorentz group as ψ (the $(0, \frac{1}{2})$ representation instead of the $(\frac{1}{2}, 0)$ representation) **parity is violated** in his two-component fermion theory. (In terms of 4-component spinors, his action contained the projection operator $\frac{1}{2}(1 + \gamma_5)$). Years later, two-component spinors found their final destiny in the $V - A$ theory of the weak interactions, as we have discussed. However, Weyl considered this violation of parity undesirable and proposed to use 4-component spinors, unifying the electron with the proton. The mass term $\bar{\psi}\psi$ would require that the mass of the electron and proton were the same, but he expected that this mass problem would be solved by gravity. In his book "Raum, Zeit, Materie" he added shortly after 1929 this stunning observation: "The problem of the proton and electron will be mixed with the symmetry properties of the quantum theory with respect to left and right, past and future, and positive and negative charge." So he anticipated the CPT theorem, even before

[69] Note that $v^\mu = \psi^\dagger \sigma^\mu \psi$ with $\sigma^\mu = (I, \vec{\sigma})$ and ψ a two-component spinor, transforms as a 4-vector, hence $\psi^\dagger\psi$ is not a scalar. Weyl noted that v^0 transforms under Lorentz transformations into $|S\psi|^2$ where S is an $Sl(2, C)$ matrix. Thus $(v^0)'$ is always positive, and time-reversal of v^0 was excluded in his theory. He considered this encouraging. Incidentally, one can construct a mass term for chiral spinors which is Lorentz invariant, namely $\psi^\alpha \psi^\beta C_{\alpha\beta}$ where $C_{\alpha\beta} = \epsilon_{\alpha\beta}$ is the charge conjugation matrix, but such a mass term violates fermion number and was not considered by Weyl.

antiparticles had been put forward, or been discovered.

(ii) He gave a complete treatment of Vierbein fields in physics. Wigner had already observed [189] that in curved space the Dirac matrices should satisfy

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}(x) \tag{1.4.14}$$

Thus the Dirac matrices should depend on x, and he introduced vielbein's ${e_a}^\mu$ by $\gamma^\mu(x) = e^\mu_a(x)\gamma^a$. Then using the Dirac anticommutation relations $\{\gamma^a, \gamma^b\} = 2\eta^{ab}$ the vielbeins were seen to be square roots of the metric, $g^{\mu\nu} = \eta^{ab}{e_a}^\mu {e_b}^\nu$. Recall that Cartan had already introduced the notion of local coordinate frames (repères mobiles, Einstein's freely falling lifts) and interpreted the e^μ_a as the orientation of the local coordinates w.r.t. the coordinates of curved manifolds. However, Weyl noted that the solution of the equation $e^\mu_a {e_b}^\nu \eta^{ab} = g^{\mu\nu}$ is gauge-invariant under local Lorentz transformations ${e_a}^\mu(x) \to {L_a}^b(x) e^\mu_b(x)$. He worked out the relation between the two local symmetries which act on ${e_a}^\mu$: local Lorentz invariance and general coordinate invariance.

(iii) He wrote the Dirac equation in curved space by introducing a local Lorentz gauge field[70] ${\omega_\mu}^a{}_b = {e_\sigma}^a D_\mu(\Gamma){e_b}^\sigma$ (with $D_\mu(\Gamma){e_b}^\sigma = \partial_\mu {e_b}^\sigma + \Gamma_{\mu\ \tau}^{\ \sigma}{e_b}^\tau$)

$$\mathcal{L} = -(\det e^a_\mu)\bar{\psi}\gamma^a e^\mu_a \left(\partial_\mu + \frac{1}{4}{\omega_\mu}^{ab}\gamma_a\gamma_b\right)\psi \tag{1.4.15}$$

The factor $\det e^a_\mu = \sqrt{-\det g_{\mu\nu}}$ was known from Einstein's work, and the combination $\gamma^a e^\mu_a$ was already found by Wigner, but Weyl introduced the covariant derivative $D_\mu\psi = \partial_\mu\psi + \frac{1}{4}{\omega_\mu}^{ab}\gamma_{ab}\psi$ (where $\gamma_{ab} = \frac{1}{2}(\gamma_a\gamma_b - \gamma_b\gamma_a)$).

(iv) He derived conservation laws for the stress tensor $T_{\mu\nu}$ and angular momentum

[70] This $\omega_{\mu\ b}^{\ a} = (e\partial_\mu e^{-1})^a{}_b + \Gamma_{\mu\ b}^{\ a}$ with $e = e^a{}_\nu, e^{-1} = e^\nu{}_a$, and $\Gamma_{\mu\ b}^{\ a} = \Gamma_{\mu\ \tau}^{\ \sigma}{e_b}^\tau {e_\sigma}^a$ is called spin connection, and transforms like a gauge field under local Lorentz transformations. It has the form $g^{-1}D_\mu g$ with $g = e^a{}_\nu$ an element of the gauge $Gl(4,R)$. So Weyl used two connections: ${\Gamma_{\mu\nu}}^\rho$ for general coordinate transformations and the spin connection for local Lorentz symmetry, but they were related by the vielbein "postulate" $\omega_\mu = eD_\mu e^{-1}$. (Often one rewrites this equation as $\mathcal{D}_\mu \rho^a{}_\nu = 0$ where $\mathcal{D}_\mu$ contains both connections). These ideas are also present in Cartan's work. Of course Cartan could not have used (1.4.14) because the Dirac equation was only discovered in 1928.

from general coordinate and local Lorentz invariance. Hilbert had already derived the conservation of $T_{\mu\nu}$ from translational invariance in flat space [192], and Noether had already established the existence of conserved current for any rigid symmetry [171]. Weyl extended these results to local symmetries and curved space. He also proved that the gravitational stress tensor density $\sqrt{-g}T_a{}^\mu = \delta/\delta e_\mu^a\, S$ (or, rather, $T^{\mu\nu} \equiv e^{a\nu}T_a{}^\mu$) is symmetric on-shell, and hence angular momentum is conserved, another standard result of modern gravity theories.

(v) He rewrote the Hilbert-Einstein action in the spin-connection formalism

$$R = e_a^\nu e_b^\mu R_{\mu\nu}{}^{ab}; R_{\mu\nu}{}^{ab} = \partial_\mu \omega_\nu{}^{ab} + \omega_\mu{}^a{}_c \omega_\nu{}^{cb} - \mu \leftrightarrow \nu \tag{1.4.16}$$

Here the spacetime curvature appears as the Yang-Mills field strength for the non-compact Lorentz group.

(vi) Weyl considered his best result his (again unsuccessful) unification of electromagnetism and gravity. The 2-component spinors transformed under the Lorentz group as representations of $Sl(2,C)$. Representations of $Sl(2,C)$ can be extended to $Gl(2,C)$, and, in fact, his gravitational action had a rigid $Sl(2,C) \times U(1) \times R = Gl(2,C)$ symmetry, where the $U(1)$ extension corresponds to a phase factor $\exp i\varphi$. Just as Weyl had generalized Minkowski space to curved space by introducing vielbeins, he now generalized the rigid phase symmetry to a local symmetry, as $\psi'(x) = e^{ie\alpha(x)}\psi(x)$. He introduced the electromagnetic potential A_μ as the connection for this local $U(1)$ symmetry, as $D_\mu\psi = \partial_\mu\psi - ieA_\mu\psi$. What results is the modern approach with minimal gauge couplings. If one defined suitable rescalings of the vielbeins, the massless Dirac action even became locally scale (= Weyl) invariant, which corresponds to the R in the decomposition of $Gl(2,C)$. Together with his description of gravity as a gauge theory with the connection $\omega_\mu{}^{ab}$ for local Lorentz symmetry, this established gauge theory. However, the proposal that $Gl(2,C)$ unifies Lorentz symmetry and electromagnetism cannot be considered to be correct because some spinors (neutrons for example) do not transform under the $U(1)$ of electromagnetism. [14] Worse, this

$U(1)$ symmetry violates parity because left-handed and right-handed spinors acquire complex-conjugated phases instead of the same phase.

We end this section with a few reactions from physicists on Weyl's work, see [14] for more details. First of all Einstein. He wrote to Weyl the following letter after having received a copy of the proofs of Weyl's 1919 book "Raum, Zeit, Materie" (Space, Time and Matter)

"Dear Colleague, I am reading the proofs of your book with admiration. It is like a master-symphony. Each little word is related to the whole and the structure is grandiose. The splendid method of deriving the Riemann tensor from parallel-transfer. How natural everything is!"

This is not the kind of reaction one nowadays gets from one's colleagues. Next Pauli. Pauli who had a sound distaste for too much mathematics without physics, had been critical at first about Weyl's 1929 paper. After reading a short announcement of the paper to appear, he wrote this sarcastic letter:

"Before me lies the April edition of the Proc. Nat. Acad. (U.S.). Not only does it contain an article from you under "Physics" but shows that you are now in a "Physical Laboratory": from what I hear you have even been given a chair in "Physics" in America. I admire your courage, since the conclusion is inevitable that you wish to be judged, not for successes in pure mathematics, but for your true but unhappy love for physics." Note the sarcastic quotation marks. However, upon closer reading, Pauli realized that Weyl had written a very important paper, and he wrote a second letter:

"In contrast to the nasty things I said, the essential part of my last letter has since been overtaken ..... Here I must admit your ability in Physics. Your earlier theory with $g'_{ik} = \lambda g_{ik}$ was pure mathematics and unphysical. Einstein was justified in criticizing and scolding. Now the hour of your revenge has arrived".

A nice letter, although it did not come easily, see the word "must". The word "revenge" says more about Pauli, than about Physics. For an excellent biography of Pauli with interesting discussions of his discoveries see [190].

Let us end with a short anecdote [191], once again involving Pauli. When Yang gave a seminar at Princeton in 1954 about his work with Mills, Pauli was in the audience, and Oppenheimer was in charge. As we have seen, Pauli had been struggling with the question whether the gauge fields for the nuclear interactions were massive or massless. Soon after Yang wrote the equation $(\partial_\mu - i\epsilon B_\mu)\psi$ on the blackboard, Pauli asked "What is the mass of this field B_μ?" Yang replied that this was a good question, and that he and Mills had studied it, but had reached no conclusions. Pauli got annoyed, and repeated his question. Yang answered, and said it was a complicated problem. Pauli said "That is not sufficient excuse". Yang sat down, and Oppenheimer tried to defuse the situation by saying "we should let Frank proceed". So Yang went on, but the following day he found the following short note in his mailbox "Dear Yang, I regret that you made it almost impossible for me to talk with you after the seminar. All good wishes, Sincerely yours, W. Pauli."

Fifty years after studies in gravity had led to the notion of gauge theories for the nongravitational interactions, gravity and particle physics were finally unified. It started with the construction of a new symmetry called supersymmetry (initially called supergauge symmetry). This was a rigid symmetry of certain nongravitational models which transformed bosons into fermions, and fermions into bosons. In rigidly supersymmetric models, the quadratic ultraviolet divergences in the self-energy of scalars (such as Higgs particles) due to fermion loops are canceled by self-energy graphs of the form of the letter Ω with a bosonic loop at a $\lambda\varphi^4$ vertex. Absence of quadratic divergences is necessary in order that successive loop corrections in the renormalized theory remain small (the "hierarchy problem"). Subsequently, a new gauge theory called supergravity, was developed which was based on local supersymmetry. It could be reformulated as a gauge theory in higher (up to eleven) dimensions. Local supersymmetry transformations of fields turned out to be general coordinate transformations of the coordinates θ^α in superspace (a space with the ordinary space-time coordinates x^m and additional anticommuting spinorial coordinates θ^α). Thus supergravity extended Einstein's theory of general relativity to a fermion-boson symmetric theory. The Standard Model was extended to a supersymmetric Standard Model, in which for each particle there is a supersymmetric partner. Even in the minimal supersymmetric Standard Model (MSSM model) one needs two instead of one complex scalar Higgs doublets. In order that supersymmetry breaking can occur in a phenomenologically acceptable way (namely without bringing the hierarchy problem back), one can only add soft-breaking terms by hand. These soft-breaking terms are the same as the terms obtained by coupling the MSSM to supergravity and taking a certain limit. In the present decade experiments at CERN will look for supersymmetric particles, and if found, a third revolution in physics will occur, comparable to relativity and quantum mechanics.

Parallel to these developments, a string model for the strong interactions was constructed, but when it was found to contain a massless spin 2 boson, the latter was identified as the graviton, and string theory became a theory of both gravity and matter. Supersymmetry was found to be needed for the consistency of string theory, leading to superstrings, and supergravity turned out to be the "low-energy" limit of superstring theory. String theory was found to be consistent only in 10 space-time dimensions, and the Kaluza-Klein-Fock program was applied to string theory, and in some cases one came close to a realistic four-dimensional theory. As an ordinary quantum field theory of gravity, supergravity was not finite nor renormalizable, but dualities related properties of classical supergravity to nonperturbative quantum properties of non-gravitational matter. This unification of gravity with nongravitational quantum field theory via strings and supergravity has only just begun. String theory and supergravity have its dedicated advocates and its equally fierce opponents, so it may be wise at this moment to close with a noncontroversial statement: the future lies ahead.

5 Quantization, unitarity and renormalizability

By the mid 1970's it had become clear that all fundamental nongravitational interactions except QED are based on nonabelian gauge theories of the kind discovered by Yang and Mills in 1954 [107]. The question how to quantize and renormalize nonabelian gauge fields has also an interesting history. The answer to this question involves the ghost action and unitarity. For nonabelian gauge theories based on a semisimple gauge group, the need for a ghost action becomes already clear at the one-loop level because without a ghost action, both renormalizability and unitarity are violated. As we shall see in later chapters, without ghosts the divergences in the self-energy of gauge fields are no longer transversal, while the divergences in the 4-point vertex corrections have a different functional form from the terms in the classical action and cannot be absorbed by multiplicative renormalization (or removed by additive renormalization). We shall also demonstrate that unitarity requires that the contributions in the unitarity equation from the longitudinal and timelike polarizations of gauge fields cancel against those of the ghosts and antighosts, hence absence of the latter also violates unitarity.

The concept of unitarity in field theory was conceived and developed by Heisenberg[71] [195]. He had been studying the multiple-particle showers which are produced by a single collision in cosmic rays, and believed that field theory could not provide an adequate description. He concluded that at very short distances other, as yet unknown, physical laws were operative[72]. Pauli, with whom he had collaborated since the early developments of QED, suggested that he should focus only on observables, and forget about particular Hamiltonians which would describe physics at very short

[71] There is actually an earlier paper on the S matrix by J.A. Wheeler [194] who used it as a framework to describe few-nucleon problems in nuclear physics.

[72] As models for new physics that could not be described by the field theory of that time he considered nonlocal interactions, and he introduced the concept of a minimal length (10^{-13} cm) beyond which field theory would break down. He even speculated that in the future theory a universal minimum length would appear as a new quantum number.

distances. To be able to parametrize these unknown theories, Heisenberg proposed to describe incoming and outgoing particles by asymptotic states (plane waves), but not to try to describe what went on at distance scales less than 10^{-13} cm. These incoming and outgoing states could still be described by quantum mechanics, in particular the superposition principle should still hold. The amplitude for the transition from a set of incoming particles to a set of outgoing particles was given by a matrix which he called S for Streuung [195]. (Streuung means scattering in German). Conservation of probability required that the total sum of probabilities that a given in-state $|n>$ decays into out-states $|m>$ be unity

$$\sum_m \mid S_{mn} \mid^2 = 1 \tag{1.5.1}$$

This meant that S had to be unitary

$$SS^\dagger = S^\dagger S = I \tag{1.5.2}$$

Heisenberg tried to determine the S matrix further by imposing other conditions. It should be relativistically invariant, satisfy the spin-statistics relation, and should also be invariant under other symmetries which follow from conservation laws. Kramers noted that S should be analytic in the momenta, except for singularities. Heisenberg noted that bound states appeared as poles of the S matrix along the positive imaginary axis of the complex k plane. However, determining the details of dynamics only from symmetry considerations seemed a doubtful proposition. Pauli called in 1943 this S matrix program an empty concept, and one of his students (Ma) discovered further zeros of the S matrix which did not correspond to bound states. When at the end of the 1940's the renormalization of QED was achieved, interest in the S matrix program faded. However, when in the late 1950's no progress was made in the construction of a field theory for the strong or weak interactions, a second era of interest in S matrix physics occurred. Since the coupling constant for the strong interactions was estimated to be of order unity, perturbation theory seemed out of the question, and some physicists set out to develop dispersion relations for the S-matrix amplitude of the strong interactions, with singularities in the Mandelstam variables s, t and u which, although discovered in perturbation theory, were assumed also to occur at the nonperturbative level. In the mean time Heisenberg had discovered that a minimum length violated causality at the microscopic level, and therefore he changed strategy and tried to construct a new kind of local but nonlinear field theory (it contained products of four spinor fields at one point). With Pauli initially as an enthusiastic supporter but later (after a visit to Columbia and Caltech) as a caustic critic, Heisenberg developed a unified field theory: a completely nonlinear and non-renormalizable field theory in terms of only fermionic fields. Also this second radical proposal led to nothing, and when nonabelian gauge theories were renormalized, this unified field theory, and with it the whole S matrix program, disappeared once again from sight.

The modern point of view is that field theory can describe physics at any short distance scale, and the S matrix can be computed, at least to any order in perturbation theory, by ordinary local renormalizable field theory, once the action is given. However, the S matrix thus computed should be unitary, and in this sense something is still left of the original ambitious program of Heisenberg. For further reading, see [196].

The issue thus arose of determining under what conditions a field theory is unitary. In 1965 a student at Utrecht, M. Veltman, wrote a Ph.D in which he developed cutting rules for field theories with scalar particles, and determined under which conditions these theories preserved unitarity and causality at the perturbative level. [197] His proof, which we will discuss in a later chapter on unitarity, was based only on combinatorics and did not need analytic continuation of momenta and multi-variable complex function theory, unlike earlier work by Cutkowsky. [198] He included in his studies of unitarity the case of unstable particles. The next problem obviously was to extend these considerations to gauge theories, but here the problem of renormalizability complicated matters.

Before discussing the developments which led to the renormalization of nonabelian gauge theories, we go back in time, and discuss the issue of zero point energies in quantum field theory, and the problem with early loop calculations in QED. Afterwards, we shall return to the renormalization and unitarity of nonabelian gauge theories.

In 1900 Planck made a fit to experimental data on blackbody radiation [199] which agreed with the Wien displacement law ($\lambda_{\max}T =$ constant) for large frequencies ν and low temperatures T, while also agreeing with the experimental data for small ν and high T (the Rayleigh-Jeans law $\mathcal{E}(\nu) = \frac{8\pi\nu^2}{c^3}kT$ which states that each oscillator of the electromagnetic field has energy $\frac{1}{2}kT$ as a consequence of the classical equipartition theorem of energy). His interpolation formula became the celebrated "Planck distribution"

$$U(\nu) = \frac{h\nu}{\mathrm{e}^{h\nu/kT} - 1}; \quad \mathcal{E}(\nu) = \frac{8\pi h\nu^3/c^3}{\mathrm{e}^{h\nu/kT} - 1} \tag{1.5.3}$$

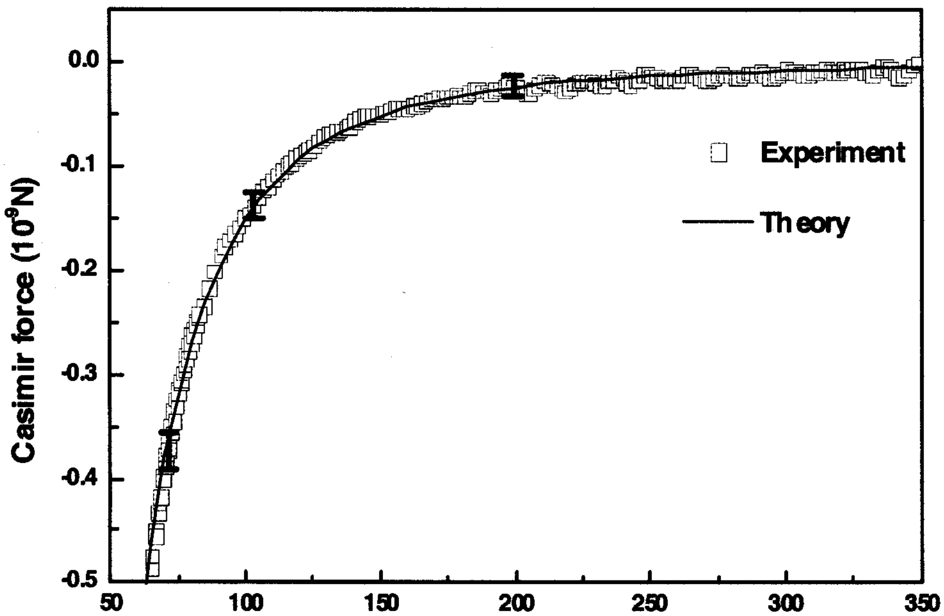
for the average energy U of one harmonic oscillator of frequency $\omega = 2\pi\nu$ at temperature T, and the corresponding energy density $\mathcal{E}(\nu)$ of blackbody radiation of the electromagnetic field. One can derive this formula by assuming that the energy of photons with frequency ν is quantized as $E_n = nh\nu$, and then applying the Boltzmann law for the probabilities of these states to occur, summing over the two polarizations of a photon. We discussed this earlier, see (1.1.3). (Incidentally, it was Einstein who in 1906 asserted that the energy of material oscillators is quantized; Planck never made this claim). In this formula no zero-point energy appears. In 1911 he published "a second theory", in which he replaced quantal absorption by continuous absorption but kept quantal emission, and found as a direct result an extra term $\frac{1}{2}h\nu$ in U, the zero-point energy. [1] In 1916, Einstein derived his famous A and B coefficients for spontaneous and stimulated emission, and rederived the Planck spectrum in (1.5.3), but zero point energies played no role in his derivation. The zero-point energy of the harmonic oscillator was derived by Heisenberg in his treatment of quantum mechanics in 1925 but a year later Schrödinger showed that there is no zero-point energy for a rotating hydrogen molecule [200] (the energy levels are proportional to $l(l+1)\hbar^2$). More generally, the rotational spectra of di-atomic molecules do not have zero-point energies, but the vibrational spectra can be described by a harmonic oscillator and contain zero-point energies [201]. The zero-point energy came back in electromagnetic field theory in the paper by Born, Heisenberg and Jordan of 1926. In the work of Fermi and Dirac the zero-point energy of electrons was also found, but most physicists came to believe that the zero-point energy was a constant without physical meaning which could be eliminated by a suitable shift in the definition of energy. However, it was also noted in the 1930's that the zero-point oscillations of the electromagnetic field could explain stimulated emission in atoms. According to this point of view, spontaneous emission was stimulated emission by the zero-point fluctuations of the electromagnetic field. (This is not quite correct: stimulated emision by the zero-point fluctuations gives 1/2 of the formula for spontaneous emission. Another factor 1/2 is due to radiation reaction, based on Lorentz' self-accelerating electrons. [202] We do not pursue these ideas here any further.) In 1950 Wick introduced "Wick ordering" in interactions, putting all creation operators to the left of annihilation operators, and this removed zero-point energy from the Wick-ordered energy operator [180]. Thus the status of the infinite zero-point energy of the electromagnetic field remained unclear: it might have something to do with spontaneous emission, but on the other hand one could also get rid of it by redefining the energy of the vacuum (at least in flat space; in curved space it should couple to gravity and be physical, yielding perhaps a cosmological constant). Pauli summerized the situation as follows [203]: 'At this point, it should be mentioned that it is more consistent not to introduce here a zero-point energy of $\frac{1}{2}h\nu$ per degree of freedom, in contrast to the material oscillator. Because, on the one hand, this would lead to an infinitely large energy per volume unit due to the infinite number of degrees of freedom; on the other hand, this infinite energy cannot be observed in principle, since it may be emitted, absorbed or diffracted - hence cannot be enclosed by walls - and since it does not create any gravitational field - as is known from experience.' Pauli argued, in connection with the latter observation, that else 'the radius of the world would not extend beyond the moon'.

The situation changed dramatically in 1948 when Casimir showed that one could explain the force between two perfectly conducting parallel plates by assuming that the vacuum contained the zero-point energies of all harmonic oscillators of the electromagnetic field. [204] The zero-point energy of each harmonic oscillator would increase if one brought the plates together, but if all frequencies above a particular frequency (the plasma frequency) would leak through the plates, there would effectively be less vacuum energy between the plates if they approached each other. The result was an attractive force exerted by one plate on one cm^2 of the other plate, given by $F(d) = -\frac{\pi^2 \hbar c}{240 d^4}$ where d is the distance in cm between the two plates. Thus Casimir shifted the emphasis from "the action at a distance" between the molecules in the plates to the local energy density of quantum fields[73], even though no plasma frequency is taken into account. Recent experiments [206] have confirmed[74] this force to an accuracy of 2% (or 5% if all the uncertainty in the position is taken into account at only one point). Much work continues to be done on the "Casimir effect", and it

[73]One can derive the Casimir energy density $\mathcal{E}$ using dimensional regularization. Taking a real scalar field φ, and multiplying the final result by 2 to account for the two polarizations of photons, one finds with the boundary conditions $\varphi(x, y, z = 0) = 0$ and $\varphi(x, y, z = d) = 0$ that the frequencies are $\omega/c = \left[k_x^2 + k_y^2 + \left(\frac{m\pi}{d}\right)^2\right]^{1/2}$. Straightforward calculation yields then for the energy density in $n > 2$ instead of $n = 2$ transversal dimensions $\epsilon = \frac{1}{2}\sum_{m=1}^{\infty}\int \frac{d^n k}{(2\pi)^n}\omega = -\pi^{-\frac{n}{2}-\frac{1}{2}} 4^{-\frac{n}{2}-1}\left(\frac{\pi}{d}\right)^{n+1}\Gamma\left(-\frac{n+1}{2}\right)\zeta(-n-1)$. Substituting $n = 2$ yields the formula for the Casimir energy [205].

[74]In these experiments one hangs a conducting ball above a conducting plate. The force can then be calculated in the approximation that the lower half of the sphere consists of a stacking of rings of diameters ΔR which one assumes to be parallel to the plate. Using the Casimir force for each ring and summing over these rings, one finds $F = \frac{2\pi R d}{3} F(d)$, where R is radius of the sphere, and d the distance between the ball and the plane. For a distance $d = 0.06\mu m$ the force on the ball is $0.48\ 10^{-9}$ Newton, which is many orders of magnitude larger than the gravitational force between the ball and the plate.

has found various applications. For further reading see [202, 205, 207].



(1.5.4)

Figure 25: the Casimir force as a function of the plate-sphere separation in nm. (Harris, Chen and Mohideen [206]). The squares denote data points (the size of the squares has no meaning) and the three error bars denote standard deviation.

At around the same time as Casimir published his result, the Shelter Island conference took place where the issues concerning the renormalization of QED got renewed attention. Stimulated by these renewed activities in QED, Welton [74] gave an explanation of the Lamb shift in terms of zero-point fluctuations of the electromagnetic field in an atom. (The electron oscillates due to the fluctuating electromagnetic field, and experiences as a result of these oscillations a modified Coulomb force which leads to tiny shifts in the energy spectrum of the atom). It seems that Casimir was unaware of these developments in QED. Historically, Casimir first studied with Polder the attractive force between colloids in a fluid due to van der Waals interactions. They found that one needed to include retardation effects which change the interaction from $1/R^6$ to $1/R^7$. When Casimir told Bohr of these results, the latter "mumbled" [208] that van der Waals forces "must have something to do with zero-point energy", and this led Casimir to his discovery. The conclusion is that zero-point energies are real physical phenomena which may very well play a role in the understanding of the cosmological constant and "dark energy".

The existence of physical effects which could be attributed to zero point energies should have cast doubt on the normal-ordering procedures widely adopted at that time in quantum field theory. However, only after regularization schemes (in particular dimensional regularization) had been developed, it was noted that normal-ordering is inconsistent with some basic properties one would like to be satisfied. Namely, normal ordering excludes equal-time contractions, and hence closed loops with only one vertex were set to zero. This violated the mass-independence of Z-factors in dimensional regularization [209], and it also violated the Ward identities in scalar QED, as noted by Kriplovich [158].

Aside from the issue of zero point-energies (which was anyhow not considered to be a real issue by many physicists), the problem of divergences in higher-order corrections was already encountered in the late 1920's and was considered an enormous problem. The problem with infinities in loop calculations was first noticed by Jordan and Klein [210], and then by Heisenberg and Pauli [211] when they started working on their second paper. Pauli suggested to Oppenheimer, who had come to Zürich for a stay, to start working on the electromagnetic selfenergy of bound electrons. This piece of work was initially planned to be part of the second paper of Heisenberg and Pauli, with Oppenheimer as coauthor, but it was published separately [212]. Oppenheimer's article observed that spectral lines will be shifted by self-energy corrections of the electron. Simultaneously, Waller studied the selfenergy of a free electron [213], and Rosenfeld the self-energy of an electron bound in a harmonic oscilator [214]. Waller observed that the self-energy of a free electron is quadratically divergent (see below). All of them used Dirac's second-order perturbation theory and the Coulomb gauge to calculate the energy shift of an electron in a state $|m\rangle$

$$\Delta E_m = \sum_{n,\vec{k},\lambda} \frac{\langle m|H^{(1)}|n,\vec{k},\lambda\rangle\langle\lambda,\vec{k},n|H^{(1)}|m\rangle}{E_m - E_n - |k|c} \tag{1.5.5}$$

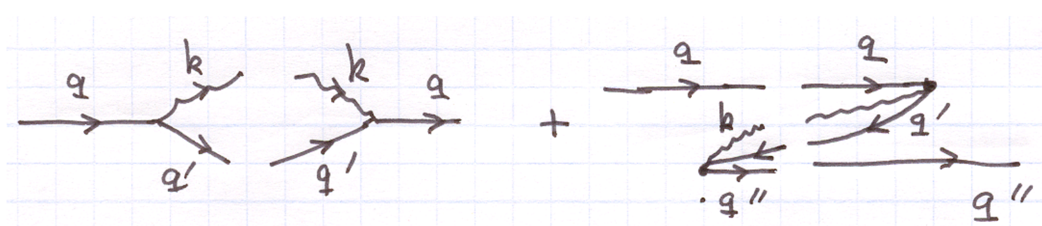
where $H^{(1)} = ie(\bar{\psi}\vec{\gamma}\psi) \cdot \vec{A}^{tr}$ with $\vec{A}^{tr}$ the transverse radiation field. The Dirac sea had not yet been filled, so they summed only over positive energy states ("pre-hole theory"). The selfenergy of the electron diverged quadratically[75]! However, they noticed that the quadratic divergences in the selfenergy canceled if one took the difference between the shifts of two levels. Some years later, Weisskopf repeated this calculation in the theory with the Dirac sea filled and the positron present as a new particle [215]. First he considered the electrostatic selfenergy operator for the Coulomb interactions

$$\hat{E}_{\text{stat}} = \frac{1}{2} \int \int \frac{(\rho(\vec{x}) - \rho_0(\vec{x}))(\rho(\vec{y}) - \rho_0(\vec{y}))}{|\vec{x} - \vec{y}|} d^3x d^3y \tag{1.5.6}$$

Here $\rho(\vec{x}) = e\psi^\dagger\psi$ is the charge density, and $\rho_0(\vec{x})$ is the **vacuum** expectation value $\langle 0|\rho(\vec{x})|0\rangle$ to lowest order. Dirac had already proposed to define the physical charge density by $\rho(\vec{x}) - \rho_0(\vec{x})$, but since $\langle 0|\rho(\vec{x})\rho(\vec{y})|0\rangle$ is not equal to $\langle 0|\rho(\vec{x})|0\rangle\ \langle 0|\rho(\vec{x})|0\rangle$, the vacuum expectation value of $\hat{E}_{\text{stat}}$ does not vanish. Weisskopf calculated the expectation value for a state with one electron at rest, and subtracted the vacuum expectation value

$$\Delta E = \langle e^-, \vec{p} = 0|\hat{E}_{\text{stat}}|e^-, \vec{p} = 0\rangle - \langle 0|\hat{E}_{\text{stat}}|0\rangle \tag{1.5.7}$$

The result was only logarithmically divergent! This was due to a second term involving positrons, as indicated in the following figure



(1.5.8)

Figure 26: In the second diagram, the intermediate states contain two electrons, a positron and a photon. The quadratic divergences in both diagrams cancel in the sum.

[75]Classically, the selfenergy of an electron with radius R is $E = \frac{e^2}{2R}$, hence classically one finds only a linear divergence for $R \to 0$.

He then repeated Oppenheimer's calculation for the selfenergy due to the radiation field, using again the Dirac theory with positrons. The result was disappointing: ΔE seemed to diverge quadratically [215]. However, Furry noticed an algebraic error in the calculations, and allowed Weisskopf to publish a correction [216], according to which the quadratic divergence again canceled! Thus relativity (which implies the existence of positrons) improved the situation of divergences in quantum field theory.

One can understand the reason for the quantum improvement in the relativistic theory most easily if one uses modern covariant Feynman rules, and identifies which contributions come from electrons and which from positrons. The fermion selfenergy in QED is given by

$$\bar{u}(q) \int d^4k \frac{(-i\not{k} + m)}{k^2 + m^2 - i\epsilon} \quad \frac{1}{(k+q)^2 - i\epsilon} u(q) \tag{1.5.9}$$

The factor $(k^2 + m^2 - i\epsilon)$ can be written as $[(k_0 + \omega)^{-1} - (k_0 - \omega)^{-1}]\frac{1}{2\omega}$ where $\omega = (\vec{k}^2 + m^2)^{1/2}$, and the $\int dk_0$ contour integral around the pole at $k_0 + \omega = 0$ yields the contributions of electrons. (For negative times, one must close the k_0 contour in the lower half plane, and then the pole at $k_0 - \omega = 0$ gives the contributions from the positrons. Note that $-k_0 = mk^0 = E$ is positive for electrons). The electron selfenergy in QED in (1.5.9) is logarithmically divergent due to symmetric integration, but replacing the electron propagator $\int d^4k/k^2 + m^2 - i\epsilon$ by $2\pi i \int d^3k/2\omega$ leads to **two** sources for worse divergences: first of all, the photon propagator $(k+q)^2$ becomes $2k\cdot q - m^2$ due to $q^2 + m^2 = 0$, and secondly in the numerator the term with $\not{k}$ no longer cancels due to symmetric integration because k_0 is now replaced by $-\sqrt{\vec{k}^2 + m^2}$. The result is that there are quadratic divergences in the electron selfenergy without taking positrons into account, as Weisskopf indeed found.

The separate evaluation of contributions from Coulomb interactions and contributions from the radiation field leads to complicated algebra. One can write the Coulomb contributions as covariant Feynman graphs with longitudinal and timelike photons, and the contributions from the transversal radiation field as covariant Feynman graphs with transversal photons. For the direct-scattering contributions this follows from inserting unity into the expression for the Coulomb interactions

$$[(\bar{u}_1'\gamma^0 u_1)(\bar{u}_2'\gamma^0 u_2)]\frac{-1}{(\vec{q})^2} = [\text{same}]\frac{-(\vec{q})^2 + q_0^2 + i\epsilon}{(\vec{q})^2}\frac{1}{q^2 - i\epsilon} \tag{1.5.10}$$

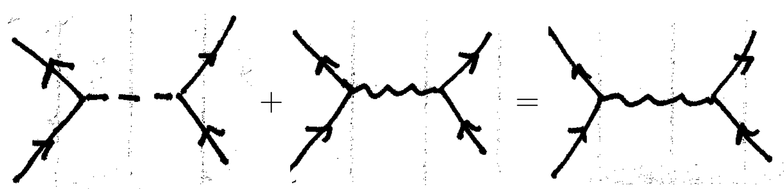
while the transversal radiation field gives

$$[(\bar{u}_1'\gamma^i u_1)(\bar{u}_2'\gamma^j u_2)](\delta_{ij} - q_i q_j/(\vec{q})^2)\frac{1}{q^2 - i\epsilon} = \sum_{m=1}^{2}[\text{same}]\frac{\epsilon_i^m \epsilon_j^m}{q^2 - i\epsilon} \tag{1.5.11}$$

In the sum of the Coulomb contributions and the contributions from the radiation field in the Coulomb gauge, the noncovariant terms $(q_i\bar{u}\gamma^i u)^2/\vec{q}^{\,2}$ cancel if one uses current conservation, $q_0\bar{u}_1'\gamma^0 u_1 = -\bar{u}_1'\vec{\gamma}\cdot\vec{q}u_1$, and one is left with the covariant Feynman propagator for photons

$$\bar{u}_1'\gamma^\mu u_1 \bar{u}_2'\gamma^\mu u_2 \frac{1}{q^2 - i\epsilon}\ . \tag{1.5.12}$$

A similar result holds for the exchange contributions with $(\bar{u}_2'\gamma u_1)(\bar{u}_1'\gamma u_2)$, and one may depict this combination of noncovariant contributions into a covariant result as follows



Coulomb transverse covariant (1.5.13)

Figure 27: The instantaneous Coulomb interaction exchanges timelike and longitudinal photons. The radiation field in the Coulomb exchanges two transversely polarized photons. These contributions sum up to the exchange of four types of photons, and produce the covariant Feynman propagator $\eta^{\mu\nu}/k^2 - i\epsilon$ in the Lorentz gauge with $\xi = 1$.

For fermions one finds in a similar way that the noncovariant contribution to the propagator from electrons and the noncovariant contribution from positrons sum up to the covariant Feynman propagator for fermions. In fact, the contribution to the covariant fermion propagator from electrons comes from a term $< 0|\psi(x_2)\bar{\psi}(x_1)|0 >$ $\theta(x_2^0 - x_1^0)$, and the contribution from positrons involves the term $- < 0|\bar{\psi}(x_1)\psi(x_2)|0 >$

$\theta(x_1^0 - x_2^0)$. The latter contribution can be interpreted as describing electrons running backward in time (from x_2^0 to x_1^0 instead of from x_1^0 to x_2^0). This idea already appears in Stuckelberg's work [218], and became one of the cornerstones of Feynman's approach. In this way the old Hamiltonian approach with ordinary perturbation theory involving energy denominators became replaced by the covariant approach with Feynman rules. For a clear and well-written account of these ideas, see [219]).

Pauli and Weisskopf [220] went further and studied "a toy model": scalar QED. Several one-loop processes came out only logarithmically divergent, but the selfenergy of the scalar field was later found to be quadratically divergent [221]. (When a spinless meson, the pion, was discovered in 1947, this model became of course more then a toy model). In addition to studies of selfenergies of electrons and scalars, of course also the selfenergy of photons was studied. Initial studies showed that it was logarithmically divergent, but Serber noted that it should vanish due to gauge invariance (he coined the expression to "renormalize" to indicate the removal of infinities from the polarization of the vacuum. [222])

One of the consequences of "the Dirac theory of the positron" (the Dirac equation coupled to the electromagnetic field) is that matter can transform into radiation, and radiation into matter. In particular, two colliding photons can produce a virtual electron-positron pair, which in turn can annihilate into two or more photons. This suggests that QED produces nonlinear correction terms to the Maxwell action. Heisenberg and collaborators studied this problem by calculating (in modern terms) the one-loop determinent for an electron loop in a constant external electromagnetic field [223]. This problem can be reduced to a problem with two harmonic oscillators, and can be solved exactly.[76]

[76] A constant field strength $F_{\mu\nu}$ can be cast into 2×2 block form. One can then choose the "Fock-Schwinger gauge" for A_μ, namely $A_1 = 0, A_2 = F_{12}x^1 = ax^1$ and $A_3 = 0, A_0 = F_{30}x^3 = bx^3$. Then $(\not{D} + m)(-\not{D} + m) = -D_\mu^2 + m^2 + \frac{1}{2}ie\gamma^\mu\gamma^\nu F_{\mu\nu}$, with $D_\mu = \partial_\mu - ieA_\mu$ and $\hbar/i\partial_\mu = p_\mu$, becomes

$$p_1^2 + p_3^2 + (p_2 - eax^1)^2 - (p_0 - ebx^3)^2 + m^2 + ie\gamma^1\gamma^2 a + ie\gamma^3\gamma^0 b$$

They found the following result for the one-loop corrections to the Maxwell action in spinor QED with electron mass m

$$\mathcal{L}^{(1)}_{\text{spinor}} = -\frac{1}{hc}\int_0^\infty \frac{d\eta}{\eta^3} e^{-\eta e E_c}\left\{\frac{e^2 ab\eta^2}{\tanh eb\eta \ \tan ea\eta} - 1 - \frac{e^2\eta^2}{3}(b^2-a^2)\right\} \quad (1.5.14)$$

where $E_c = \frac{m^2c^3}{e\hbar}$, $a^2 - b^2 = \vec{E}^2 - \vec{B}^2$ and $ab = \vec{E}\cdot\vec{B}$. They subtracted the term of zeroth and quadratic order in η^2 in order to remove singularities for small η. In modern terms this amounts to on-shell renormalization conditions for QED: the term -1 subtracts the free field case, while the term with $b^2 - a^2$ is proportional to the Maxwell action, and accounts for charge renormalization. They also noted that one should deform the integration contour away from the singularities at $\tan ea\eta = 0$, and this yields an imaginary part to $\mathcal{L}^{(1)}$ which they identified as ($\frac{1}{2}$ times) the pair production rate. Expanding $\mathcal{L}^{(1)}$ to quartic order yielded Euler's correction to the Maxwell action [224]

$$\mathcal{L}^{(1)} = \frac{e^4}{360\pi^2 m^2}[(E^2-B^2)^2 + 7(\vec{E}\cdot\vec{B})^2] \quad (1.5.15)$$

These nonlinear corrections were interpreted as describing the quantum vacuum as a polarizable medium [225]. Only with the modern formulation of QED was this process calculated for arbitrary (non-constant) electromagnetic fields [226] using Pauli-Villars regularization.

The quantization of scalar fields in the early 1930's posed problems not present in the Dirac theory (for example, one could not fill a Dirac sea for scalars due to Bose-Einstein statistics), hence it was very relevant to redo the Heisenberg-Euler calculation for scalar QED (with a complex scalar field in the loop). Weisskopf tackled this problem, and found that the scalars modified the Maxwell action as follows [227]

$$\mathcal{L}^{(1)}_{\text{scalar}} = \frac{1}{2hc}\int_0^\infty \frac{d\eta}{\eta^3} e^{-\eta e E_c}\left\{\frac{e^2 ab\eta^2}{\sinh(eb\eta)\sin(ea\eta)} - 1 + \frac{e^2\eta^2}{6}(b^2-a^2)\right\} \quad (1.5.16)$$

A similarity transformations with $\exp ip_1p_2/(ea)$ removes p_2, and another similarity transformation with $ip_3p_0/(eb)$ removes p_0. One is then left with two harmonic oscillators, (p_1, x^1) and (p_3, x^3).

No new conceptual problems were encountered; as expected the overall sign was opposite to the electron case.

These corrections to Maxwell theory resembled modifications Born and Infeld had proposed at the classical level for Maxwell theory[77] (they tried to construct a consistent classical theory in which the electron had a finite radius and finite energy [228]). Born intended this modified Maxwell theory as the starting point of a new quantum theory, while Heisenberg viewed his own results as only some of the radiative corrections which are produced by Dirac theory. Thus Heisenberg did not question the fundamental theory itself, but he realized that quantum field theory was far from established:
" ...It is at this time hardly possible to make definite predictions about the final form of the Maxwell equations in a future Quantum Field Theory, because that would require entering into the details of all processes with high-energy particles ...".

In 1951 Schwinger gave a formal treatment of pair production [230] which extended the leading-order result of Heisenberg and Euler to all orders. Later these calculations were extended to nonabelian gauge theories (QCD) for the case of covariantly constant field strengths ($D_\mu F_{\nu\rho} = 0$). These days the program Heisenberg and Euler started in the 1930's has blossomed into a wide range of applications in particle physics and quantum field theory [229].

The state of affairs of loop corrections in quantum field theory remained till 1947 as follows: quadratic divergences canceled in all processes of QED [221] but not in scalar QED. Various physicists started developing renormalization programs. For example, as we already mentioned, Dirac proposed to subtract the background charge density $\rho_0(\vec{x})$ as in (1.5.6), and several others tried to generalize these ideas. How-

[77]The Born-Infeld action was $\det(\eta_{\mu\nu} + F_{\mu\nu})$, and has no problems with causality, whereas the Euler-Heisenberg theory violates causality (the energy contains a term - B^4, see (1.5.15) if one adds the one-loop corrections to the action (and presumably also if one adds all loop corrections, due to the so-called Landau ghost in QED).

ever, rather than make an all-out frontal and systematic attack on this fundamental problem, many physicists felt that the time was not yet ripe because they believed that field theory itself was inconsistent. Some felt that QED would break down at energies of the order of 137 $m_e c^2$. We already mentioned Heisenberg's doubts which led him to propose a minimal length beyond which field theory would break down. Dirac, confronted with the apparent breakdown of energy conservation in neutron decay (resolved by Pauli's neutrino hypothesis) had this to say "... The only important fact that we have to give up is quantum electrodynamics ... we may give it up without regrets – in fact because of its extreme complexity, most physicists will be glad to see the end of it." Heisenberg wrote to Pauli in 1935: ... "In regard to quantum electrodynamics we are still at the stage in which we were in 1922 with regard to quantum mechanics. We know that everything is wrong ... ". This lack of confidence in the internal consistency of QED may explain a curious absence in the literature, noticed by S. Weinberg [231]. Oppenheimer, Waller and Rosenfeld had noted that differences between energy levels were better convergent than the energies themselves, and Weisskopf had found that in the relativistic theory again cancellations occurred due to extra processes involving positrons. It may nowadays seem obvious that one should try to obtain finite results by considering differences in energy levels in the theory with positrons and electrons but there exists no record of such a calculation. Apparently, nobody had enough confidence in the consistency of quantum field theory at that time.

It was the experimental discovery of the Lamb shift and the other shifts in atomic energy levels, which totally transformed the physical landscape. In a few years, the renormalization of the QED was achieved, as we have described earlier, and "field theory was back". However, the euphoria did not last very long, and the Dark Ages of mesotron theories, Regge poles, etc. brought new despair. We now make a leap to the late 1960's when the problem of the renormalization of nonabelian gauge theories entered a decisive stage.

A direct way to demonstrate that the S matrix for QCD differs in a significant way from QED already at the tree level is to compute the Born cross section for quark-antiquark annihilation into either two photons or into two gluons[78]. In the former case one can perform the sum over the two photon polarizations by replacing $\sum_{m=1}^{2} \epsilon_m^\mu \epsilon_m^\nu$ by $\eta^{\mu\nu} = \sum_{m=1}^{4} e_m^\mu e_m^\nu$. In QED one obtains in both cases the same answer, and tree unitarity (the optical theorem) is satisfied as Feynman showed in his 1949 article [232].

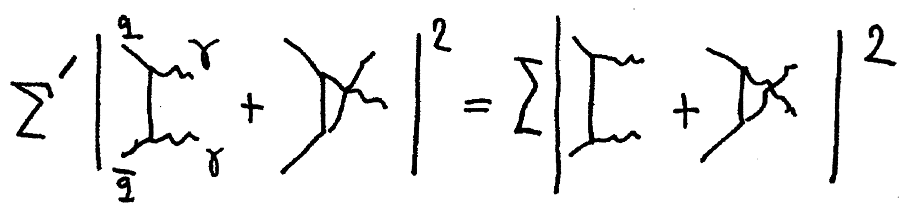


Figure 28: the longitudinal and timelike photons decouple from the S matrix in QED.

The prime on the summation symbol on the left-hand side indicates that one should sum only over photon states with transverse polarizations on the left-hand side, while on the right-hand side one should sum over all four polarizations.

In QCD the same simple-minded procedure to perform the sum over the two gluon polarizations in the square of the tree graphs yields a different result when one uses the explicit expressions for the two polarization vectors, or when one replaces $\sum_{m=1}^{2} \epsilon_m^\mu \epsilon_m^\nu$ by $\eta^{\mu\nu}$.

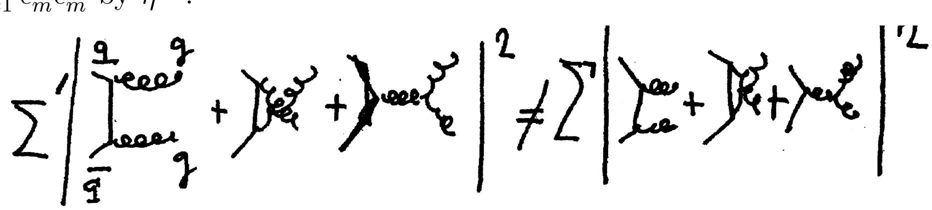


Figure 29: the longitudinal and timelike gluons do not decouple from the S matrix in QCD.

Thus an indication for the need for ghosts in QCD can already be found at the level of tree graphs, without studying loop graphs. The correct way to evaluate the process

[78]Due to confinement, no S-matrix exists for QCD, but in this section we ignore this fact.

of quark-antiquark annihilation into two gluons with fixed initial and final energies is of course to use transverse polarization vectors for the gluons, corresponding to the set of diagrams on the left-hand side. Using $\eta^{\mu\nu}$ for the polarization sum violates unitarity. The reason is that gluons can couple to each other (as indicated in the figure), and this produces a gluon loop. Unitarity requires also to add a ghost loop. This means that one should add a further term on the right-hand side in the case of QCD which corresponds to the square of the tree graph for two quarks decaying into a ghost-antighost pair. With this extra term the inequality turns into an equality.

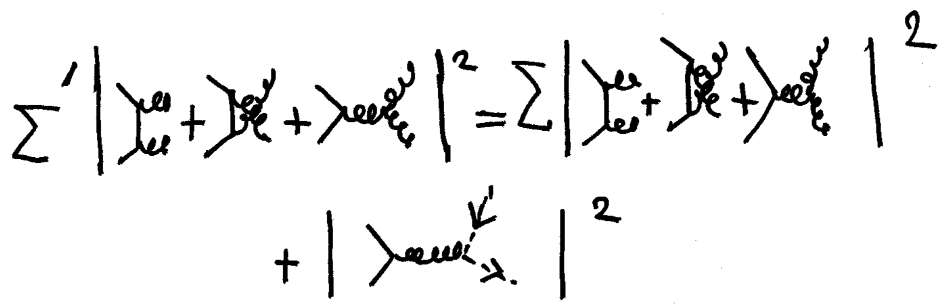


Figure 30: ghosts are needed for tree unitarity of QCD.

One can use gauges for QCD which do not lead to interacting ghosts. These are cumbersome for detailed loop calculations, but they have the advantage that physical properties can be clearly exhibited. One such gauge is the Coulomb gauge $\partial^k A_k^a = 0$. A particularly important and interesting physical phenomenon is asymptotic freedom. It is due to antiscreening of Coulomb gluons. Consider the following two Feynman graphs, depicting two selfenergy loops of a Coulomb gluon exchanged between two heavy quarks.

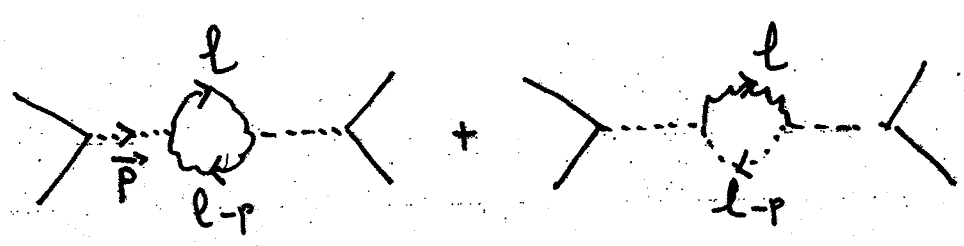


Figure 31: The two selfenergy graphs in the Coulomb gauge which lead to asymptotic freedom.

The first loop contains two transverse gluons, and this is the spin 1 equivalent of a matter loop. The second loop contains one transversal and one Coulomb gluon. This graph has no counterpart in the matter sector, it only exists in nonabelian gauge theories. Unitarity arguments show that in QED matter loops lead to screening, and a decreasing strength of the effective coupling constant for increasing distance. One expects therefore that also the loop due to transversal gluons leads to screening and a negative one-loop β function. This is indeed the case, but the second loop leads to antiscreening, and is larger resulting in the well-known factor 11 in the one-loop β function of QCD. Together these graphs produce a potential $\frac{1}{r} + \beta\frac{1}{r}\ln r/r_0$ which becomes stronger than $\frac{1}{r}$ for larger r. Qualitatively one may understand this difference in signs by opening up ("cutting") one transversal gluon in each graph.

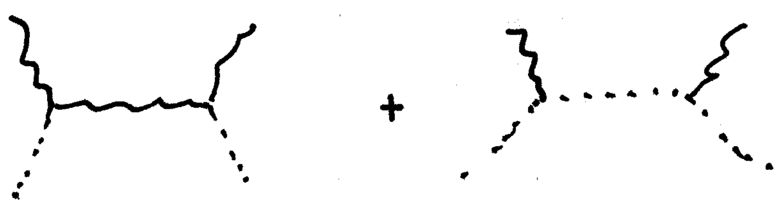


Figure 32: The graph on the left corresponds to an interaction of parallel currents which is attractive according to the law of Biot and Savard and leads to screening, while the graph on the right corresponds to a charge-charge interaction which is repulsive according to the law of Coulomb and leads to antiscreening. [217]

There are no contributions to this effect from the box and crossed box graphs because these are not ultraviolet divergent by power counting. The vertex corrections do not produce a $\ln r/r$ potential either because ghosts decouple in the Coulomb gauge, as a result of which a $Z_1 = Z_2$ type of Ward identity makes the vertex corrections finite. As expected there are no infrared divergences. The quadratic ultraviolet divergence is cancelled by the seagull graph with a 4-point coupling. The logarithmic divergences $\ln(|\vec{p}|/\Lambda)$ are proportional to the expected factor 11, due to a contribution with $+5$ for the first loop and -16 for the second loop. Subtracting at momentum $|\vec{p}_0|$, one finds a factor $\ln|\vec{p}|/|\vec{p}_0|$, and Fourier transforming yields a one-loop correction to the potential in x-space proportional to $\beta \ln r/r$. (Other corrections without logarithm require the calculation of the vertex and box graphs). We discuss this in more detail in section (III.7), where also other explanations of the physical reason for asymptotic freedom are discussed.

In QED there are no ghosts (or, rather, the ghosts are free fields[79] because the ghost couplings are proportional to the structure constants of the gauge group, which vanish for an abelian group.) Hence, for QED Heisenberg and Pauli's "formaler Kunstgriff" of just adding a gauge fixing term without also adding a ghost action, was correct, after all. However, for nonabelian gauge theories with covariant gauges, ghosts do couple and play an important role.

One can determine the ghost action by analyzing Feynman diagrams and trying to add suitable terms to the action involving new fields which restore unitarity and/or renormalizability; this was the path followed by Feynman [233] at the one-loop level and continued at the two-loop level by Veltman [234] who generalized the Stückelberg method [235] to the nonabelian case. Both started from Yang-Mills theory with an explicit mass term and showed that at the one-loop level one may replace the nonrenormalizable propagator $\eta_{\mu\nu} + k_\mu k_\nu/m^2$ by $\eta_{\mu\nu}$ and still preserve unitarity if at the same time one **subtracts** the contribution of one real scalar particle. (This real scalar corresponds at the one-loop level to the two anticommuting Faddeev-Popov ghosts and a real commuting Stückelberg scalar as we shall explain in the next chapter). Since this part of the history of the path to the renormalization of the gauge fields is not well-known, we give some details.

In the mid-1960's current algebra was invented to derive sum rules for various processes. Gell-Mann's current algebra [236] was a set of algebraic relations between vector and axial vector currents, abstracted from a constituent quark model for hadrons, with the aim of making possible calculations of electromagnetic and weak processes without requiring details of the then unknown dynamics of quarks. The first result

[79]More precisely, ghosts are free in QED if one uses gauges linear in A_μ, namely gauge fixing terms quadratic in A_μ. If one uses as gauge fixing term $-\frac{1}{2}\xi(\partial^\mu A_\mu + \alpha A^\mu A_\mu)^2$ with ξ and α constants, the ghosts of QED are no longer free.

of current algebra depended only on the integrated axial vector charge commutator, and led to the Adler-Weisberger relation between the axial vector coupling constant for β-decay of nucleons and the strong coupling constant of pions to nucleons [237]. It agreed well with the experimental data, and encouraged many people to enter the program of current algebra. The current algebra program led in a few years to two great discoveries: scaling and the renormalization of nonabelian gauge theories.

First scaling. Requiring that one could insert a set of elementary constituents between two currents and saturate the current relations gave indications that new elementary constituents were present inside nucleous. In 1969, Bjorken clarified the precise saturation mechanism [238] and identified new particles which Feynman called partons, and which turned out to be the quarks of 1964 but with colour. Experiments at SLAC gave experimental proof of the existence of these quarks [239].

The other development of current algebra brought field theory back in particle physics. For reasons to be shortly given, at the end of the 1960's and beginning of the 1970's currents were replaced by a renormalizable quantum field theory in which these currents appeared as elementary vertices instead of composite operators. The evaluation of matrix elements of currents was reduced to a calculation of Feynman graphs with well-defined propagators and vertices.

Current algebra provided the transition from the S-matrix approach to field theory. The current commutation relations involved vector and axial vector currents, and could be also be written as divergence relations of the form $\partial_\mu j^\mu = \ldots$, or to include QED, as $D_\mu j^\mu = \ldots$ where $D_\mu = \partial_\mu - ieA_\mu$ is the electromagnetic covariant derivative. Veltman studied application of these current relations to weak interactions, and for consistency he needed to add terms to the charged weak current relation $D^\mu j^+_\mu + \ldots = 0$. Bell noted that $(D^\mu j^+_\mu + \ldots)$ looked like a generalized covariant derivative, and this suggested to Veltman that the current algebra formalism was based on a new kind of gauge field theory with nonabelian gauge fields. Since the weak interactions were of short range, he decided to take massive gauge fields, and as action he took, after some searching around, the theory Yang and Mills had proposed in 1954, augmented by a mass term $M^2(A^a_\mu)^2$. Here a serious problem arose: the propagator contained $\eta_{\mu\nu} - k_\mu k_\nu/M^2$ in the numerator. In QED in the Landau gauge one finds $\eta_{\mu\nu} - k_\mu k_\nu/k^2$ which did not seem to present major problems, but the term $k_\mu k_\nu/M^2$ definitely upset power-counting renormalizability. At this point Veltman made a crucial observation: on-shell at the one loop level, the nonrenormalizable divergences which were induced by the $k_\mu k_\nu/M^2$ term, canceled. He decided to introduce a free scalar field φ, and make a field redefinition of the fields A^a_μ and φ such that the $k_\mu k_\nu/M^2$ term in the propagator would cancel. Because φ was a free scalar, one was still dealing with the original theory, but after the transformation nothing remained of φ if one considered Green functions with external A^a_μ but no φ. The field redefinition had as far as A^a_μ was concerned, the form of a gauge transformation, but note that the field A^a_μ was massive to begin with. In QED at one-loop level this program worked fine, but in Yang-Mills theory dropping the terms $k_\mu k_\nu/M^2$ in the propagator required to add by hand new vertices. At the 2-loop level, nonrenormalizable divergences in five-point Green functions remained. Here the program got stuck. Later it was noted that by adding a Higgs particle, these nonrenormalizable divergences could have been removed, but further vertices would have needed to be added, and in this way one might have discovered spontaneously broken gauge theories.

However, in 1970 he and van Dam, and independently Zacharov, discovered that the massive theory does not limit smoothly to the massless theory already at the one-loop level [87], and thereafter most physicists focused on the massless theory.[80] We shall give a simpler derivation of these results based on the Stückelberg formalism in the next chapter, and discuss there an even simpler approach to obtain the ghost

[80] In the massless theory unitarity is satisfied at the one-loop level if one subtracts the contributions of two real scalar particles, which is equivalent to adding a loop with ghosts. In the massive case one must add a ghost loop and a loop with a real scalar particle, which is equivalent to the subtraction of one real scalar loop.

action which works at the tree level.

Another way to find the form of the ghost action is to use path integrals [88, 240, 241]. Path integrals yield a third approach to quantum physics, in addition to Heisenberg's operator and Schrödinger's wave function approach. They are due to Feynman [88], who developed in the 1940's an approach Dirac had briefly considered in 1932 [241]. Dirac and Feynman derived path integrals with $\frac{i}{\hbar}$ times the action in Minkowski space in the exponent from quantum mechanics. In mathematics Wiener had already studied path integrals in the 1920's but these path integrals contained (-1) times the free Euclidian action for a point particle in the exponent. Wiener's path integrals were Euclidean path integrals which are mathematically well-defined but Feynman's path integrals do not have a similarly solid mathematical foundation. Nevertheless, path integrals have been successfully used in almost all branches of physics: particle physics, atomic and nuclear physics, optics, and statistical mechanics.

In many applications one uses path integrals for perturbation theory, in particular for semiclassical approximations, and in these cases there are no serious mathematical problems. In other applications one uses Euclidean path integrals, and in these cases they coincide with Wiener's path integrals. However, for the nonperturbative evaluations of path integrals in Minkowski space a completely rigorous mathematical foundation is lacking. The problems increase in dimensions higher than four [242]. Feynman was well aware of this problem, but the physical ideas which stem from path integrals are so convincing that he (and other researchers) considered this not worrisome.

The first steps in the direction of path integrals began with Dirac who wrote in 1932 an article in a USSR physics journal [241] in which he tried to find a description of quantum mechanics which was based on the Lagrangian instead of the Hamiltonian approach. (It was published in a Soviet journal because Dirac was making with

Heisenberg a trip around the world, and took the trans-Siberian railway to arrive in Moscow). In those days all work in quantum mechanics (including the work on quantum field theory) started with the Schrödinger equation or operator methods in both of which the Hamiltonian played a central role. For quantum mechanics this was fine, but for relativistic field theories an approach based on the Hamiltonian had the drawback that manifest Lorentz invariance was lost (although for QED it had been shown that physical results were nevertheless relativistically invariant). Dirac considered the transition element.

$$\langle x_2, t_2 | x_1, t_1 \rangle = K(x_2, t_2 | x_1, t_1) = \langle x_2 | e^{-\frac{i}{\hbar}\hat{H}(t_2 - t_1)} | x_1 \rangle \tag{1.5.17}$$

(for time-independent H), and asked whether one could find an expression for this matrix element in which the action was used instead of the Hamiltonian. (The notation $\langle x_2, t_2 | x_1, t_1 \rangle$ is due to Dirac who called this element a transformation function. Feynman introduced the notation $K(x_2, t_2 | x_1, t_1)$ because he used it as the kernel in an integral equation which solved the Schrödinger equation.) Dirac knew that in classical mechanics the time evolution of a system can be written as a canonical transformation, with Hamilton's principal function $S(x_2, t_2 | x_1, t_1)$ as generating functional [243]. This function $S(x_2, t_2 | x_1, t_1)$ is the classical action evaluated along the classical path that begins at the point x_1 at time t_1 and ends at the point x_2 at time t_2. In his 1932 article Dirac wrote that $\langle x_2, t_2 | x_1, t_1 \rangle$ *corresponds to* $\exp \frac{i}{\hbar} S(x_2, t_2 | x_1, t_1)$. He used the words "corresponds to" to express that at the quantum level there were presumably corrections so that the exact result for $\langle x_2, t_2 | x_1, t_1 \rangle$ was different from $\exp \frac{i}{\hbar}(x_2, t_2 | x_1, t_1)$. Although Dirac wrote these ideas down in 1932, they were largely ignored until Feynman started his studies on the role of the action in quantum mechanics.

In the last years of the 1930's Feynman started studying how to formulate an approach to quantum mechanics based on the action. As we already mentioned in section 1, the reason he tackled this problem was that with Wheeler he had developed a theory of quantum electrodynamics from which the electromagnetic field had been eliminated. In this way they hoped to avoid the problems of the self-acceleration and infinite self-energy of an electron which are due to the interactions of an electron with the electromagnetic field and which Lienard, Wiechert, Abraham and Lorentz had in vain tried to solve. The resulting "Wheeler-Feynman theory" arrived at a description of the interactions between two electrons in which no reference was made to any field. These theories were nonlocal in space and time. (In modern terminology one might say that the field A_μ had been integrated out from the path integral by completing squares). Wheeler and Feynman set out to quantize this system, but a Hamiltonian treatment turned out to be hopelessly complicated[81]. Thus Feynman was looking for an approach to quantum mechanics in which he could avoid the Hamiltonian. The natural object to use was the action.

At this moment in time, an interesting discussion helped him further. A physicist from Europe, Herbert Jehle, who was visiting Princeton, mentioned to Feynman (spring 1941) that Dirac had already in 1932 studied the problem how to use the action in quantum mechanics. Together they looked up Dirac's paper, and of course Feynman was puzzled by the ambiguous phrase "corresponds to" in it. He asked Jehle whether Dirac meant that they were equal or not. Jehle did not know, and Feynman decided to take a very simple example and to check. He considered the case $t_2 - t_1 = \epsilon$ very small, and wrote the time evolution of the Schrödinger wave function $\psi(x,t)$ as follows

$$\psi(x, t+\epsilon) = \frac{1}{\mathcal{N}} \int \exp\left(\frac{i}{\hbar}\epsilon L(x, t+\epsilon; y, t)\right) \psi(y,t) dy. \tag{1.5.18}$$

With $L = \frac{1}{2}m\dot{x}^2 - V(x)$ one obtains, as we now know very well, the Schrödinger

[81]By expanding expressions such as $\frac{1}{\partial_x^2 + \partial_t^2 - m^2}$ in a power series in ∂_t, and using Ostrogradsky's approach to a canonical formulation of systems with higher order ∂_t derivatives, one can give a Hamiltonian treatment, but one must introduce infinitely many new fields $B, C, \ldots$ of the form $\partial_t A = B, \partial_t B = C, \ldots$. All these new fields are, of course, equivalent to the oscillators of the original electromagnetic field.

equation, by expanding to the first order in ϵ, provided the constant $\mathcal{N}$ is given by

$$\mathcal{N} = \left(\frac{2\pi i\hbar\epsilon}{m}\right)^{\frac{1}{2}} \tag{1.5.19}$$

(The factor $\frac{dy}{\mathcal{N}}$ is nowadays called the Feynman measure). Thus, as Dirac correctly guessed, $\langle x_2, t_2|x_1, t_1\rangle$ was analogous to $\exp\frac{i}{\hbar}\epsilon L$ for small $\epsilon = t_2 - t_1$; however they were not equal but rather proportional.

Feynman then asked himself how to treat the case that $t_2 - t_1$ is not small. This Dirac had already discussed in his paper: by inserting complete set of x-eigenstates one obtains

$$\begin{aligned} \langle x_f, t_f|x_i, t_i\rangle &= \int \langle x_f, t_f|x_{N-1}, t_{N-1}\rangle\langle x_{N-1}, t_{N-1}|x_{N-2}, t_{N-2}\rangle \ldots \\ &\quad \ldots \langle x_1, t_1|x_i, t_i\rangle dx_{N-1}\ldots dx_1 \, . \end{aligned} \tag{1.5.20}$$

Taking $t_j - t_{j-1}$ small and using that for small $t_j - t_{j-1}$ one can use $\mathcal{N}^{-1}\exp\frac{i}{\hbar}(t_j - t_{j-1})L$ for the transformation function, Feynman arrived at

$$\langle x_f, t_f|x_i, t_i\rangle = \int \exp\left[\frac{i}{\hbar}\sum_{j=0}^{N-1}(t_{j+1} - t_j)L(x_{j+1}, t_{j+1}; x_j, t_j)\right]\frac{dx_{N-1}\ldots dx_1}{\mathcal{N}^N} \; . \tag{1.5.21}$$

All this point Feynman recognized that for large $\mathcal{N}$ one obtains the action in the exponent and that by first summing over j and then integrating over x one is summing over paths. Hence $\langle x_f, t_f|x_i, t_i\rangle$ is equal to a sum over all paths of $\exp\frac{i}{\hbar}S$ with each path beginning at x_i, t_i and ending at x_f, t_f.

Of course one of these paths is the classical path, but by summing over all other paths (arbitrary paths not satisfying the classical equation of motion) quantum mechanical corrections are introduced. The tremendous result was that all, nonperturbative as well as perturbative, quantum corrections were included if one summed over all paths. Dirac had entertained the possibility that in addition to summing over paths one would have to replace the action S by a generalization which contained terms with higher powers in $\hbar$.

Reviewing this development more than half a century later, when path integrals have largely superseded operators methods and the Schrödinger equation for relativistic field theories, one notices how close Dirac came to the solution of using the action in quantum mechanics, and how different Feynman's approach was to solving the problem. Dirac anticipated that the action had to play a role, and by inserting complete set of states he did obtain (1.5.21). However, he did not pursue the observation that the sum of terms in (1.5.21) is the action because he anticipated for large $t_2 - t_1$ a more complicated expression. Feynman, on the other hand, started by working out a few simple examples, curious to see whether Dirac was correct that the complete result would need a more complicated expression than the action, and found in this way that the truth lies in between: Dirac's transformation functions (Feynman's transition kernel K) is equal to the exponent of the action up to a constant. This constant diverges as ϵ tends to zero, but for $N \to \infty$ the result for K (and other quantities) is finite.

Feynman initially believed that in his path integral approach to quantum mechanics ordering ambiguities of the p and x operators of the operator approach would be absent (as he wrote in his PhD thesis of May 1942). However, later in his fundamental 1948 paper in Review of Modern Physics [88] he realized that the same ambiguities would be present. Schrödinger [244] had already noticed that ordering ambiguities occur if one tries to promote a classical function $F(p, x)$ to an operator $\hat{F}(\hat{p}, \hat{x})$. Furthermore, one can in principle add further terms linear and of higher order in $\hbar$ to such operators $\hat{F}$. These are further ambiguities which have to be fixed before one can make definite predictions.

Path integrals for gauge theories are divergent because in the sum over all paths there exist for each path infinitely many gauge-equivalent paths. Clearly one had to fix a gauge which removed this degeneracy. As shown by Faddeev and Popov [240], in order to factorize out an overall infinite factor (which is the product of the finite group volume of the semisimple group at all points in spacetime), one may fix the gauge by adding a Dirac delta function to the path integral measure at each point in spacetime. This Dirac delta function by itself is not gauge-invariant, and to restore gauge invariance of the path integral measure, one should also add a determinant at each point in spacetime which measures how far two hypersurfaces corresponding to two infinitesimally different gauge parameters are separated from each other at different points in spacetime. Exponentiating this Faddeev-Popov determinant using ghost and antighost fields, one obtains the ghost action [245].

At this point G. 't Hooft decided to study first massless gauge theories, and to focus on the renormalizability of Green functions instead of the S-matrix. As all students of Veltman he had studied the paper by Gell-Mann and Levy on the linear σ-model, [104] and he had followed lectures by Ben Lee of Stony Brook at the Cargèse summer school on the renormalization of this spontaneously broken model [246]. He developed further the diagrammatic methods of Veltman, and his Ph.D consisted of one of the fundamental discoveries in gauge theory: the renormalizability of non-abelian gauge fields. He began with massless Yang-Mills fields at the one-loop level. The Feynman rules had been derived before by Mandelstam, DeWitt, Faddeev and Popov, and Fradkin and Tyutin [240]. 't Hooft observed that since the S-matrix should be independent of the gauge choice, one may average over the argument of the Dirac delta function which fixes the gauge. Using an exponential function for this averaging yields

$$\int \left\{ \prod_x \delta\left[F(x) - f(x)\right] \right\} e^{-\frac{1}{2\xi}\int f^2(y) d^4 y} [df] = e^{-\frac{1}{2\xi}\int F^2(x) d^4 x} \tag{1.5.22}$$

Adding this gauge fixing term to the classical action and the ghost action, one obtains the quantum action in the form it is commonly used nowadays. He then gave a proof of the Ward identities which must be satisfied for renormalizability of the massless theory and proved unitarity using diagrammatic methods developed by Veltman for the massive case. As he wrote ... "An auxiliary "ghost particle" appears. In fact, it will be seen to cancel the third polarization direction of the W particle"... [247].

Whether this "ghost particle" was an anticommuting scalar or a scalar with an extra minus sign in front of its action was not explicitly stated at this point, but Faddeev and Popov had proposed the former. 't Hooft first proved one loop renormalizability for the massless theory in 1971 [247]. The spontaneously broken case was solved half a year later [248]. By then the ghost particle was identified as an anticommuting scalar.... "Furthermore an extra factor -1 must be inserted for each closed loop ..." [248]. This article contains the proof of renormalizability and unitarity to all orders in the number of loops, based on diagrammatic methods.

Having obtained the Feynman rules for nonabelian gauge theories, the issue of a suitable regularization scheme that maintained the diagrammatic identities (equivalent to the Ward identities we shall derive from BRST symmetry) became crucial. It was soon realized that crude schemes like cutting off the momenta at some upper limit did not preserve the Ward identities. Also Pauli-Villars regularization was not suitable because it requires a mass for the gauge fields. An old idea brought a solution: adding extra dimensions. 't Hooft introduced a fifth coordinate for one-loop graphs, and stated.... "By introducing more dimensions one can give a consistency proof for all orders, but we shall not present it here".

With Veltman he constructed the dimensional regularization scheme that works at any loop level for theories without γ_5 or $\epsilon^{\mu\nu\rho\sigma}$ symbols that manifestly preserved the Ward identities of gauge symmetry [249]. (The use of dimensional regularization or dimensional reduction in theories with γ_5 and $\epsilon^{\mu\nu\rho\sigma}$ tensors requires a detailed discussion which we will give later). They applied it to the renormalization of spontaneously broken gauge theories with Higgs-Englert-Brout "Higgs bosons", and they wrote a beautiful article in which they worked out an example in detail [250]. The importance of their work was not only that it solved the problem of the renormalizability of the weak interaction, but rather that it finally gave the rules according to which any nonabelian gauge theory can be renormalized.

Modern nonabelian quantum gauge field theory is based on the quantization, unitarity and renormalizability of 't Hooft and Veltman (with important contributions from Faddeev and Popov, B. Lee and J. Zinn-Justin, and others) and on the classical work of Yang and Mills; it has become in the 20th century what Maxwell theory was in the 19th century.

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A Relativistic corrections to the spectrum of hydrogen.

Since the Dirac equation is even today the final word on quantum mechanics for one-electron atoms with infinitely heavy pointlike nuclei, we shall discuss the various relativistic corrections to the nonrelativistic Schrödinger equation by starting from the Dirac equation itself, and identifying the various terms which are additions to the Schrödinger equation. These corrections all belong to the domain of quantum mechanics and by themselves they do not belong in a book on quantum field theory. However, in later chapters we discuss field-theoretical corrections to the spectrum of hydrogen (due to the anomalous magnetic moment of the electron and the Lamb shift), and then one should distinguish these two classes of relativistic corrections. We begin with the work of Balmer, Bohr and Sommerfeld, leading to the relativistic mass corrections of the hydrogen spectrum (the p^4 terms). A detailed account of the history of quantum mechanics is given in the 9-volume opus by Mehra and Rechenberg [252].

"Hydrogen, the atomic weight of which is by far the smallest of all substances known to date ... seems more qualified than any other body to open new vistas in the investigation about the nature and properties of matter. In particular, the wavelengths of the first four hydrogen lines excite and arrest attention ...". These were prophetic words by Balmer in 1885 [1, 253]. He was docent in mathematics at the university of Basel in Switzerland and also a high-school teacher, and a bit of a number freak. He had discovered that the formula $\lambda_m = \text{constant} \times \frac{m^2}{m^2-n^2}$ with $m = 3, 4, 5, 6$ and $n = 2$, did reproduce the wavelengths of the first four spectral lines ($H_\alpha, H_\beta, H_\gamma$ and H_δ) up to the experimental accuracy at that time, which was about 0.5 Angström[82]. The way he arrived at this formula is miraculous. He started from the observation that the ratios of the wavelengths formed simple fractions

$$\begin{aligned} H_\alpha/H_\beta &= 27/20; H_\alpha/H_\gamma = 189/125; H_\alpha/H_\delta = 8/5 \\ H_\beta/H_\gamma &= 28/25; H_\beta/H_\delta = 32/27; H_\gamma/H_\delta = 200/189 \end{aligned} \tag{1.A.1}$$

Then he noted that this implied that the wavelengths themselves could be written as a common constant k times even simpler fractions

$$H_\alpha = 9/5k; H_\beta = 4/3k = 16/12k; H_\gamma = 25/21k; H_\delta = 9/8k = 36/32k \tag{1.A.2}$$

Finally he noted that after replacing 4/3 by 16/12 and 9/8 by 36/32 as indicated, the

[82]Balmer first used Angströms results of 1868 for H_α, H_β, H_γ and H_δ, namely $H_\alpha = 6562.1$ (modern value 6562.8), $H_\beta = 4860.7$ (modern value 4861.3), $H_\gamma = 4340.1$ (modern value 4340.5) and $H_\delta = 4101.2$ (modern value 4101.7). Of course, the Balmer lines are nowadays known to higher precision and are split due to the fine structure, but this does not affect the first five digits of the modern values quoted. Later he found also agreement for the cases $n = 2$, $m = 7, \ldots, 16$. Balmer was interested in numerology and in such things as the number of steps of the Pyramid. One day he complained to a friend that he "had run out of things to do". The friend replied: "Well, you are interested in numbers, why don't you see what you can make of ... [the wavelengths of the first few lines of the hydrogen spectrum]", see volume 1 of [252].

numerators were given by $3^2, 4^2, 5^2$ and 6^2, while the denominators were $3^2 - 2^2, 4^2 - 2^2, 5^2 - 2^2$ and $6^2 - 2^2$. His publication list consisted of only 3 articles [1], but his discovery made him world famous.

During the sixty years following Balmer's discovery experimentalists would time and again find deviations from theory, and theorists would rise to the challenge and produce new terms to be added to the formula for the spectral lines.

Thirty years after it was discovered the Balmer formula got a theoretical foundation. Based on his studies of radioactivity, Rutherford had proposed in 1912 a dynamical model of the atom, consisting of a small positively-charged heavy nucleus surrounded by a swarm of electrons which circle around the nucleus like planets. Bohr quantized the motion of the electrons in this model. The Bohr model of 1913 of an electron in a circular orbit around the nucleus [60] with angular velocity ω and a quantized kinetic energy $E_{\text{kin}} = \frac{1}{2}|E_{\text{pot}}| = \frac{1}{2}n\hbar\omega$ explained the Balmer formula and gave an expression for the constant k (hence for R).[83] (Later, following Wilson and Sommerfeld, the angular momentum l instead of the kinetic energy was quantized, $l = n\hbar$, but for circular motion these quantization conditions are equivalent. We discuss this below.)

However, already at the time when Bohr was constructing his model, experiments indicated a splitting of the lines of the Balmer series into doublets. For the first 38 lines of hydrogen ($n = 2, m = 3, \cdots, 40$) splits were found of the order of 0.1 Angström (corresponding to 40 μeV) or less, which was about 50 times larger than the experimental accuracy [256].

Since special relativity was well-known by then, it was natural to try to use it to explain these splits. Bohr substituted the relativistic expressions for the energy

[83]For a while a problem arose with the Bohr model: experimentally one found that for helium $R(He^+)/R(H)$ was equal to 4.0016 instead of exactly 4. Bohr realized that this was due to the reduced mass of the electron, and the correction factor $4\left(\frac{m_e+m_p}{m_e+4m_p}\right) = 4.00163$ resolved this problem [255].

and momentum into his formula but this did not give sensible results.[84] Wilson and Sommerfeld proposed instead to quantize the closed phase space integrals $\oint pdq$. [258] In ordinary nonrelativistic classical mechanics one finds elliptic orbits given by $r = \frac{b^2}{a+c\cos\varphi}$ with $b^2/a = l^2/me^2$ and $(c/a)^2 = 1 + \frac{2l^2}{me^4}W$ where l is the angular momentum and W the (negative) binding energy. (For a circular orbit, $a = b$ and $c = 0$, one recovers Bohr's formulas $r = l^2/me^2$ and $W = -\frac{me^4}{2l^2}$). Phase space quantization gave $l = n_\varphi \hbar$ and $W = -me^4/2n^2\hbar^2$ where $n = n_\varphi + n_r$, with n_φ and n_r integers. Thus there was degeneracy: the energy only depended $n_\varphi + n_r$, so only on the major semiaxis a of the ellipse, and Bohr's nonrelativistic results were reobtained. (One easily derives $a = 2e^2/W$. The excentricity $\left(\frac{c}{a}\right)^2$ becomes equal to $\left(\frac{c}{a}\right)^2 = 1 - \frac{n_\varphi^2}{(n_\varphi+n_r)^2}$ so $\frac{b}{a} = \frac{n_\varphi}{n_\varphi+n_r}$).

Sommerfeld used this approach to compute relativistic corrections to Bohr's model [258], and derived what we shall call below the p^4 terms in the fine structure.[85] He found these to be proportional to $\frac{1}{l+1} - \frac{3}{4n}$ where $0 \le l \le n-1$, instead

[84]Bohr expanded $H = m_e c^2/\sqrt{1-\beta^2} - \frac{e^2}{r}$, and using the equation of motion for circular motion $m_e r\dot{\varphi}^2/\sqrt{1-\beta^2} = e^2/r^2$ and the quantization condition $l = m_e r^2\dot{\varphi}/\sqrt{1-\beta^2} = n\hbar$, he obtained for the binding energy $W_n = -m_e c^2 \frac{\alpha^2}{2n^2}(1 + \frac{1}{4}\alpha^2\frac{1}{n^2})$ where m_e is the reduced mass of the electron. [257]

[85]For interested readers we present a complete derivation of these corrections. In modern terms, Sommerfeld [258] took the Lagrangian for a point particle in special relativity, $L = -m_e c^2\sqrt{1-\beta^2} + \frac{e^2}{r}$ with $\beta^2 c^2 = \dot{r}^2 + r^2\dot{\varphi}^2$. The Hamiltonian is then given by $H = E = m_e c^2/\sqrt{1-\beta^2} - e^2/r$ which becomes $m_e c^2\sqrt{1 + p_r^2/m_e^2c^2 + p_\varphi^2/m_e^2 r^2 c^2} - e^2/r$ in terms of the conjugate momenta $p_r = m_e\dot{r}/\sqrt{1-\beta^2}$ and $p_\varphi = m_e r^2\dot{\varphi}/\sqrt{1-\beta^2}$. Squaring $E + \frac{e^2}{r} = m_e c^2/\sqrt{1-\beta^2}$ and using $\frac{1}{1-\beta^2} = 1 + \frac{\beta^2}{1-\beta^2}$ where $\frac{\beta^2}{1-\beta^2} = (p_r^2 + \frac{1}{r^2}p_\varphi^2)/m_e^2c^2$, he found an equation for $(ds/d\varphi)^2$ upon division by p_φ^2 and use of $\frac{p_r}{p_\varphi} = -\frac{ds}{d\varphi}$ with $s = 1/r$. Differentiation w.r.t. φ led to the equation $\frac{d^2s}{d\varphi^2} + \gamma^2 s = e^2E/(l^2c^2)$, describing elliptic orbits with precession ($r = \frac{b^2}{a+\hat{c}\cos\gamma\varphi}$ as in general relativity, but larger than the perihelium precession of $0.1''$ per revolution of Mercury, namely $1-\gamma^2 = \frac{e^4}{l^2c^2} = 6''$. The excentricity is $a - \hat{c} = \epsilon$ and $b^2 = a^2 - \hat{c}^2$. The relativistic energy E and the relaticistic angular momentum $l = p_\varphi$ can be obtained by evaluating the expression for the Hamiltonian at the extrema where $\dot{r} = 0$ and $r = a \pm \hat{c}$. Nonrelativistically, one finds in this way $l^2 = me^2b^2/a$ and $E = -\frac{1}{2}e^2/a$). He required that the momenta $p_\varphi = mr^2\dot{\varphi}$ and $p_r = m\dot{r}$ (with $m = m_e/\sqrt{1-\beta^2}$) satisfied the quantization rules $\int_0^{2\pi/\gamma} p_\varphi d\varphi = n_\varphi h$ and $\oint p_r dr = n_r h$ for closed periodic orbits in phase space. Since $p_\varphi =$ constant, he obtained $p_\varphi = n_\varphi\hbar\gamma$. The case $n_r = 0$ corresponded to circular orbits, but the case $n_\varphi = 0$ was excluded since it corresponded to linear motion of the electron but then the electron would collide with the nucleus. The integral $\oint p_r dr$ ran from $r_{\min}$ to

of the $\frac{1}{4n}$ which Bohr had found. For circular orbits ($l+1=n$) both results agreed, but Sommerfeld's result lifted the degeneracy and gave excellent agreement with the values of the doublet splittings. Quantum mechanics, but still without spin, corrected this in 1926 to $\frac{1}{l+1/2}-\frac{3}{4n}$, see footnote 86.

A new set of problems were already encountered in 1898 when Zeeman put atoms in an external magnetic field. The normal Zeeman effect (which splits spectral lines into doublets or triplets, see below) was explained by Lorentz, but the anomalous Zeeman effect (which leads to more complicated splittings) posed an enormous problem. Further splitting of spectral lines beyond the doublet splittings mentioned before, the anomalous Zeeman effect, and the need for selection rules which could explain why certain spectral lines were not seen, led to the concept of spin in 1925 [19] (the same year as quantum mechanics was discovered) and added the spin-orbit coupling in the fine structure formula. The Dirac theory of 1928 explained all these terms in the fine structure formula, and added a further term (the Darwin term). It yielded a factor $\frac{1}{j+\frac{1}{2}}-\frac{3}{4n}$ for the α^4 corrections, which gave the same energy levels as Sommerfeld had obtained, even though the latter had not taken spin into account. The doublet-structure of the Balmer lines was explained as being due to the fine structure, according to which the $2^2p_{3/2}$ level was lying higher than the $2^2p_{1/2}$ and $2^2s_{1/2}$ levels, but the $2^2s_{1/2}$ and $2^2p_{1/2}$ levels remained degenerate. (Other levels at higher n were

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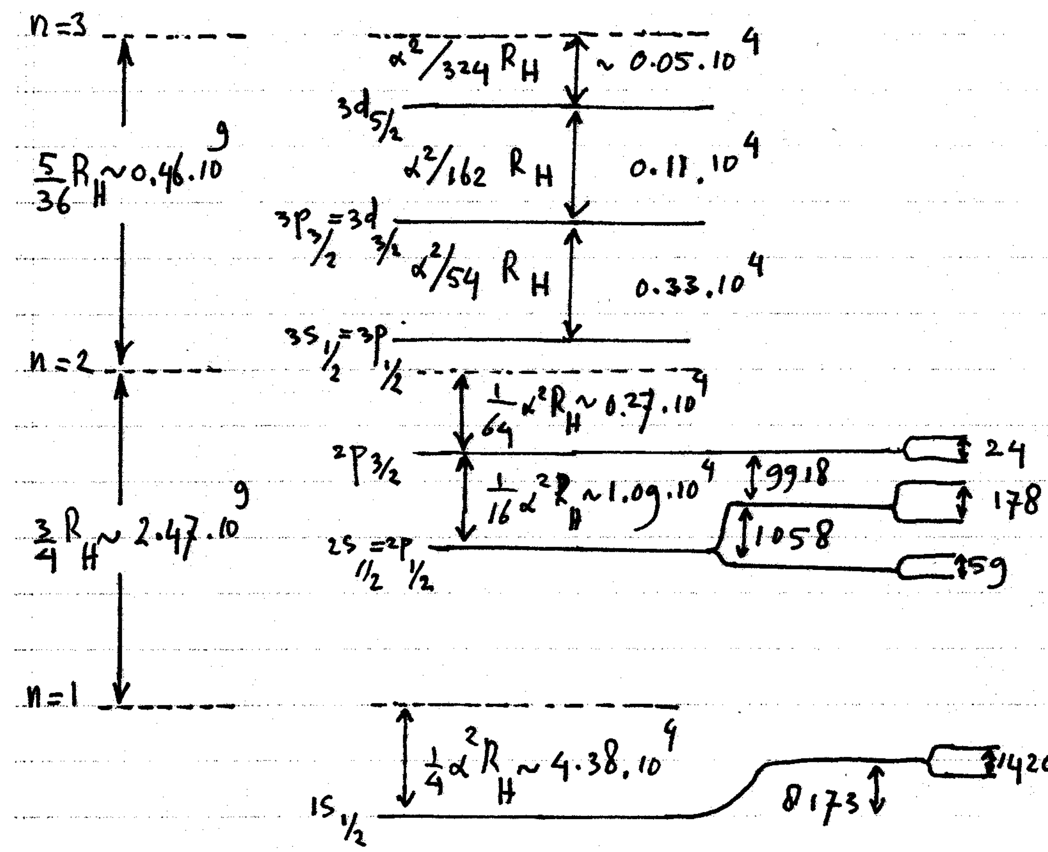
$r_{\max}$ and back. Solving for p_r from $(E+\frac{e^2}{r})^2 = m_e^2c^4(1+\frac{\beta^2}{1-\beta^2})$ where $E-m_ec^2 \equiv W$ is the relativistic binding energy, he found an expression for p_r in terms of p_φ and W, namely $p_r=\sqrt{A+2\frac{B}{r}+\frac{C}{r^2}}$, where $Ac^2=W^2+2Wm_ec^2$, $Bc^2=(W+m_ec^2)e^2$ and $Cc^2=e^4-n_\varphi^2\hbar^2\gamma^2c^2$. In the complex r-plane, p_r has a cut from $r_{\min}$ to $r_{\max}$ (both positive), and $\oint p_r dr$ can be written as an anti-clockwise contour integral around the cut. Using complex function theory to move the contour from around the cut in the complex plane to around the poles at $r=0$ and $r=\infty$, he obtained $\oint p_r dr = -2\pi i(\sqrt{C}-B/\sqrt{A})$. The quantization of p_r and p_φ leads then to a quantized expression for W, namely $1+W/(m_ec^2)=[1+\alpha^2/\{n_r+\sqrt{n_\varphi^2-\alpha^2}\}^2]^{-1/2}$. Expansion to order α^4 gave Sommerfeld's relativistic correction $W=-m_ec^2[\frac{\alpha^2}{2n^2}+\frac{\alpha^4}{2n^3}(\frac{1}{n_\varphi}-\frac{3}{4n})]$ where $n_r+n_\varphi=n$ and $n_\varphi=l+1$ in modern terms. This is the formula mentioned in the text. Quantum mechanics without spin gives the same result but with $n_\varphi=l+1/2$ instead of $l+1$. This would correspond to Bohr-Sommerfeld quantization rules with $n_\varphi-1/2$ instead of n_φ, but this modification seems never to have been considered because in the mean time quantum mechanics became the superior theory.

split into triplets etc., but these splittings were smaller and initially not observed.)

New splittings of spectral lines at the time of Dirac's formula led to the notion that the nucleus has a magnetic moment [263] that can couple to the orbital magnetic moment of the electron ($\vec{\mu}_p \cdot \vec{L}$ terms) [264]. Finally Fermi closed the circle in 1930 and also took the coupling of the nuclear magnetic moment to the electronic magnetic moment into account ($\vec{\mu}_p \cdot \vec{\mu}_e$ terms) [73]. The latter two corrections yield the hyperfine structure which contains a factor m_e/m_p and other factors, and is about a factor 100 smaller than the fine structure for hydrogen. After 1930 the spectra of atoms were believed to be completely understood. However in the 1940's new splittings (for example, the Lamb shift between the $2^2s_{1/2}$ and the $2^2p_{1/2}$ lines in hydrogen which are degenerate in Dirac's theory as we already mentioned) could only be explained by using quantum field theory, and from then on quantum mechanics had to be supplemented with quantum field theory. We now give some details.



(1.A.3)

Figure caption: On the left: the Bohr levels corresponding to $W_n = -R_H \frac{1}{n^2}$. The frequencies are given in MHz. ($1\mu eV = 8.065\ 10^{-3} cm^{-1} = 242$ MHz). In the middle: the fine structure of the first three Bohr levels of hydrogen (not to scale) according to the Dirac formula $W_{nlj} = W_n \left[1 + \frac{\alpha^2}{n}\left(\frac{1}{j+\frac{1}{2}} - \frac{3}{4n}\right)\right]$ where $j = 1/2$ if $l = 0$ and $j = l \pm 1/2$ if $l \neq 0$. (The dotted lines are the Bohr levels.) The levels are the same as in Sommerfeld's relativistic theory without spin, and the observed doublet structure of the transitions to $n = 2$ levels is well explained by the $2p_{3/2}$ and $2p_{1/2} = 2s_{1/2}$ splitting. Further to the right: the Lamb shift for the first three levels. It lifts the $2s_{1/2}$ level above the $2p_{1/2}$ level by 1058 MHz which is much larger than the hyperfine splitting of these two levels. On the far right: the hyperfine interactions split these levels further; for example, for $l = 0$ the spin-triplets lie above the spin-singlets by $1420/n^3$ MHz (= 21 cm for $n = 1$). In general the spin triplets move up by 1/4 of the hyperfine splitting, while the spin singlets move down by 3/4 of the hyperfine splitting.

The Dirac equation for an electron in an external electromagnetic field reads

$$\left[\gamma^\mu(\partial_\mu - \frac{ie}{\hbar c}A_\mu) + \frac{mc}{\hbar}\right]\psi(\vec{x},t) = 0, \quad e < 0$$
$$\gamma^k = \begin{pmatrix} 0 & -i\sigma^k \\ i\sigma^k & 0 \end{pmatrix} \text{ for } k = 1,2,3; \quad \gamma^0 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \tag{1.A.4}$$

where ψ is a 4-component complex spinor, $A^k = A_k$ is the vector potential, and $A^0 = -A_0 = \phi$ the scalar potential, while $e = -|e|$ is the charge of an electron. The Dirac matrices γ^μ with $\mu = 0,1,2,3$ satisfy $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$ with $\eta^{\mu\nu} = (-1,1,1,1)$, and the particular representation in (1.A.4) is useful for taking the nonrelativistic limit.

To exhibit the various relativistic corrections, we first turn the Dirac equation into an equivalent equation which is quadratic in derivatives, by acting with $\gamma^\nu(\partial_\nu - \frac{ie}{\hbar c}A_\nu) - \frac{mc}{\hbar}$ on the linear Dirac equation. This yields

$$\left[\left(D^\mu D_\mu - \left(\frac{mc}{\hbar}\right)^2\right) - \frac{ie}{2\hbar c}\gamma^\mu\gamma^\nu F_{\mu\nu}\right]\psi = 0 \tag{1.A.5}$$

with $D_\mu = \partial_\mu - \frac{ie}{\hbar c}A_\mu$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The operator $D^\mu D_\mu - \left(\frac{mc}{\hbar}\right)^2$ is the relativistic field operator for a one-component wave function of a spin-zero particle, which Schrödinger and others had already proposed in 1926 to describe the hydrogen atom [18]. Apparently, Schrödinger first tried the relativistic equation $\left[D^\mu D_\mu - \left(\frac{mc}{\hbar}\right)^2\right]\psi(\vec{x},t) = 0$ for the hydrogen atom [251], but getting wrong results for the spectrum[86] he then took the nonrelativistic limit $E = mc^2 + W$ and $(E - e\phi)^2 \to mc^2(mc^2 + 2W - 2e\phi)$ thus obtaining for $A_k = 0$ his famous $\left(-\frac{\hbar^2}{2m}\partial^k\partial_k + e\phi\right)\psi = i\hbar\frac{\partial\psi}{\partial t}$ for $\psi(\vec{x}, t) = \psi(\vec{x})\exp\left(-\frac{i}{\hbar}Wt\right)$.

One way to generalize the nonrelativistic Schrödinger equation to a relativistic equation is to take both quadratic space derivatives and quadratic time derivatives, and this yields "the relativistic Schrödinger equation", while another way is to allow only linear space and time derivatives, and this yields the Dirac equation. The difference is the term with $\gamma^\mu\gamma^\nu F_{\mu\nu}$ in (1.A.5) which can be decomposed as follows

$$\gamma^\mu\gamma^\nu F_{\mu\nu} = 2i\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}\cdot\vec{B} + 2\begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}\cdot\vec{E} \tag{1.A.6}$$

where $\vec{\sigma}$ are the Pauli matrices, $F_{ij} = \epsilon_{ijk}B^k$ and $F_{i0} = E_i$. We now go back to the time when there was not yet a Dirac equation.

When in 1925 Goudsmit and Uhlenbeck [19] proposed that an electron has twice the magnetic moment of a classical spin 1/2 particle[87], this not only explained both the normal and the anomalous Zeeman splittings in the spectrum, but it also gave a theoretical basis for Pauli's exclusion principle (also from 1925) which stated that one can put two (instead of one) electrons in a given electron orbit. Since the electron had a magnetic moment which could take on two values, Pauli asked himself the question: which 2×2 matrices satisfy the commutation relation of angular momentum? The answer were the three matrices σ^k $(k = 1, 2, 3)$ now known as Pauli matrices. Pauli proposed to describe the hydrogen atom by a two-component spinor $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ on which these matrices σ^k act [41]. Before discussing this Pauli spin theory, we first discuss how spin resolved problems with the Zeeman effect.

[86] For the relativistic scalar field equation the energy levels in a Coulomb field to order α^4 read $E = mc^2\left[1 - \frac{\alpha^2}{2n^2} - \frac{\alpha^4}{2n^4}\left(\frac{n}{l+\frac{1}{2}} - \frac{3}{4}\right)\right]$ with $l = 0, \ldots, n-1$ [261]. The Dirac equation yields $E = mc^2\left[1 - \frac{\alpha^2}{2n^2} - \frac{\alpha^4}{2n^4}\left(\frac{n}{j+\frac{1}{2}} - \frac{3}{4}\right)\right]$ with where $j = 1/2$ if $l = 0$ and $j = l \pm 1/2$ if $l \neq 0$. [261]. The nonrelativistic Schrödinger equation gives $W = -mc^2\frac{\alpha^2}{2n^2}$. The fine structure for a given n as predicted by the relativistic scalar field equation is much too large: (for $n = 1$ it is a factor 5 too large).

[87] We discuss the amusing history of the magnetic moment of the electron in the beginning of the chapter on the anomalous magnetic moment of the electron and muon.

The Zeeman effect led to tremendous confusion from the moment it was observed until its resolution by the discovery of spin. In 1896 Zeeman noticed that the two narrow yellow sodium (Na) lines seemed to become broader in an external magnetic field of 10 kGausz. (The lines are separated by 6 Angström, and the broadening was 1/40 of the line separation). This seemed to suggest that each line splits into two or more other lines. He consulted Lorentz who gave an explanation based on his theory of classical electromagnetism and the Lorentz force. In an external field a charged particle can perform linear oscillations along the magnetic field, and clockwise or anticlockwise circular motions in a plane orthogonal to the magnetic field. The frequencies of the latter two modes are higher and lower by an amount $\Delta\omega = \pm\frac{e}{2mc}B$ (about 50 μeV for 10 kGausz) than the frequency of the linear oscillation, which is not changed by the magnetic field. Viewed from a direction orthogonal to the magnetic field one should see three evenly split linearly polarized spectral lines, but viewed from a direction parallel to the magnetic field one should only see two spectral lines with opposite circular polarization. (A linearly oscillating electron does not radiate in the direction of its motion). Furthermore, the orientation of the polarization would reveal whether the oscillating charged particles were positively or negatively charged, and the magnitude of the splits would reveal the value of e/m (the ratio of the electric charge and the mass of the oscillating particles. In 1896 it was not yet known that these particles ere electrons; for example Lorentz called them ions). Initially Zeeman did not actually observe the splitting of spectral lines, only their broadening, but soon more refined experiments showed that various spectral lines did indeed split. He found that oscillating particles had negative charge, but for e/m he found a result which was more than 1000 times larger than the result for the Sommerfeld ion, the hydrogen. The next year J.J. Thomson also determined the value e/m from experiments on cathode rays, and found the same value. (In 1899, Thomson determined the value of e separately, and this often considered to be the discovery of the electron [4]).

A year later Zeeman reported that the blue line of Cadmium (4800 Å) indeed splits into a doublet or triplet, as Lorentz had predicted. However, soon afterwards others found that the two yellow sodium lines (6 Å apart from each other, which is about 5000 μeV) split into 4 and 6 lines, respectively, and this could not be explained by the Lorentz model. The structure of the alkali atoms (Na, K, Rb, Cs, Fr) is similar to the structure of the H and Li atoms, yet it seemed that the spectral lines of H and Li did split into triplets. It became clear that the anomalous Zeeman effect is the generic case, and the normal Zeeman effect is the exception. A time of confusion followed, which we do not describe in detail; we only mention that Landé at some moment proposed that some angular momenta (of the electrons in inner orbits in an atom) have half-integer instead of the integer values of the Bohr-Sommerfeld model. The discovery by Goudsmit and Uhlenbeck in 1925 that all electrons have a fourth degree of freedom, spin, resolved all confusion. To explain how complicated "the Zeeman effect" of spectral lines in fact was, we will briefly summerize the situation from the modern point of view based on quantum mechanics and spin.

There are actually three effects: the normal Zeeman effect, the anomalous Zeeman effect, and the Paschen-Back effect. A Zeeman effect occurs if the splitting of levels due to the spin-orbit coupling (or due to Coulomb screening for states with different l) is larger than the Zeeman energies $-\mu_e B_z(L_z + 2S_z)$, while a Paschen-Back effect operates when the Zeeman energies dominate. (If the Zeeman energies of both energy levels yielding a given transition are dominant, one has a so-called complete Paschen-Back effect, whereas if the Zeeman energy of only one of the two energy levels dominates, one speaks of a partial Paschen-Back effect.) If a Zeeman effect leads to a splitting into 3 or 2 spectral lines according to Lorentz, one calls it a normal Zeeman effect, whereas any other splitting yields by definition an anomalous Zeeman effect. For a Zeeman effect one uses j eigenstates for the wave functions of the electrons of each level, while for Paschen-Back effect one uses Russell-Saunders (L-S) coupling[88]. We shall now explain the early results on Na, Cd, Li and H, using

[88]Nowdays one can make such strong magnetic fields (of 5 Tesla or more) that the diamagnetic term

elementary quantum mechanics and spin.

(i) Consider the red 6438.47 Å Cadmium line for the transition from the $5s5d\,^1D_2$ spin-singlet state to the $5s5p\,^1P_1$ spin-singlet state (there are also three nearby ^3D_J spin-triplet states with $J = 1, 2, 3$, and three nearby ^3P_J spin-triplet states with $J = 2, 1, 0$, but we consider here the spin-singlet states.) Because the spins of the two electrons couple to a singlet, spin plays no role in this transition. All transitions satisfy the selection rule $\Delta L = 0, \pm 1$, and since all levels are split by the same amount $\Delta E = -\frac{e\hbar}{2mc}(\Delta L_z)B$, one observes a splitting into triplets or doublets, as predicted by Lorentz. Thus this is a genuine "normal" Zeeman effect, and classical physics and quantum mechanics give the same results.

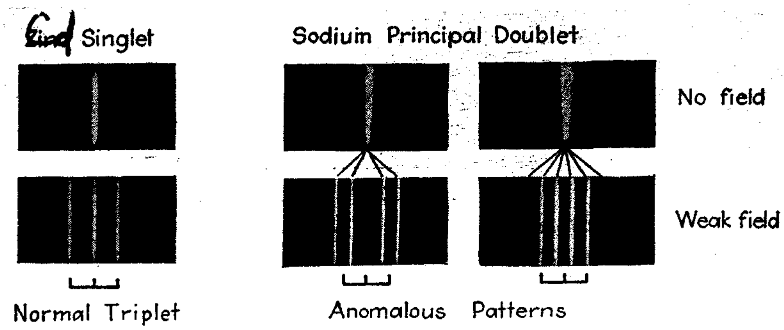
(ii) Consider next the two yellow sodium lines D_1 and D_2; the D_1 line corresponds to the $2\,^2p_{1/2} \rightarrow 2\,^2s_{1/2}$ transition, while the D_2 line (above the D_1 line) is due to the transition $2\,^2p_{3/2} \rightarrow 2\,^2s_{1/2}$. For a magnetic field of 10 kGauss, the spin-orbit energies are much larger than the Zeeman energies and the spins do not couple to yield a spin singlet, hence we are dealing with an anomalous Zeeman effect. The selection rules are $\Delta m_j = 0, \pm 1$, and one indeed finds that the D_1 line splits into 4 lines while the D_2 line splits into 6 lines.

(iii) Consider finally the $2p - 1s$ transition of Li or H in a strong magnetic field (40 kGausz). Now the spin-orbit energies are smaller than the Zeeman energies $\Delta E = -\frac{e\hbar}{2mc}(L_z + 2S_z)B$ so we are dealing with a complete Paschen-Back effect. To describe it we use the eigenstates with $|LSm_em_s>$ where $L = 1, m_e = \pm 1, 0$ and $m_s = \pm 1/2$, or $L = 0, m_e = 0$ and $m_s = \pm 1/2$. The selection rules are $\Delta m_s = 0, \Delta m_l = 0, \pm 1$. One obtains a splitting into six lines, but they occur as 3 narrow doublets[89], and in earlier experiments it seemed that here one was dealing

(proportional to $(\vec{A})^2$) dominates. In our historical account this term plays no role.

[89]Neglecting spin-orbit couplings, one finds exactly a triplet, but using perturbation theory, the spin-orbit interaction splits these lines into narrow doublets.

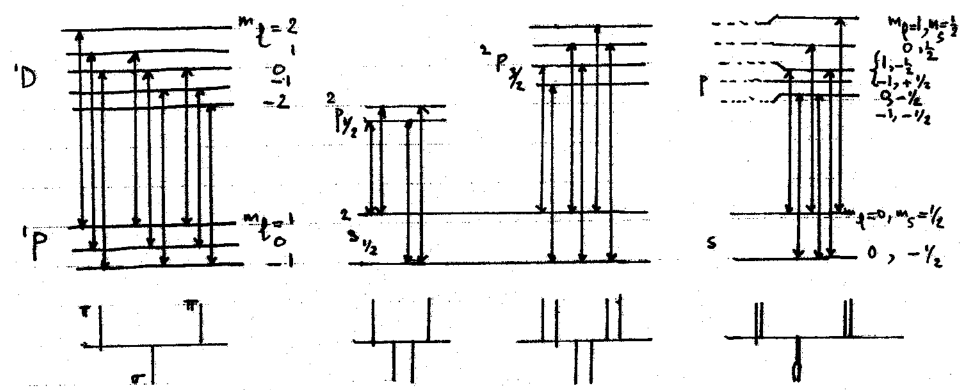
with a normal Zeeman effect.



(1.A.7)

Figure caption: The normal Zeeman effect in the red Cadmium line, and the anomalous Zeeman effect in the two D-lines of Sodium (Na), both viewed perpendicular to the magnetic field.

The introduction of spin not only correctly explained these patterns of splitting, but also their polarizations and intensities. For further reading see [1], or the books by Bethe and Salpeter [259], and by Condon and Shortley [260], or the textbooks by Schiff [261], and Powell and Craseman [262].



(1.A.8)

Figure caption: Zeeman splittings observed from a direction orthogonal to the magnetic field B. Polarizations parallel to B (denoted by π for parallel) are indicated by vertical bars under the transition schemes along the $+z$ direction, while polarizations orthogonal to B (denoted by σ, for senkrecht = perpendicular in German) are along the $-z$ axis.

On the left: the normal Zeeman effect in the red line of Cadmium. Due to screening of the Coulomb interactions, the D level lies for above the P level, and since both states are spin-singlets, spin plays no role in this transition. The Zeeman splittings are small and equal for both levels, so that one observes a triplet splitting.

In the middle: the anomalous Zeeman effect in the yellow doublet of sodium (Na). The values of the interaction $\frac{e\hbar}{2mc}\vec{B}(\vec{L}+2\vec{S})$ on j-eigenstates are given by $\frac{e\hbar}{2mc}Bgm$ where $-J \leq m \leq J$ and g

is called the Landé factor[90]. One finds $g = \frac{4}{3}$ for the $p_{3/2}$-levels, and $g = 2/3$ for the $p_{1/2}$ levels, but for the s-levels is $g = 2$. Clearly the D_2 line splits into $2 \times 4 - 2 = 6$ equally-spaced lines while the D_1 line splits into $2 \times 2 = 4$ not equally-spaced lines. (with energies proportional to $\pm\frac{1}{3} \pm 1$). Because the spin-orbit coupling is much larger than the Zeeman energies, the quartet and the sextet are clearly separated from each other.

On the right: the complete Paschen-Back effect in Lithium. The dotted lines indicate the energy levels with the Zeeman energies $\frac{e\hbar}{2mc}B(m_l + 2m_s)$ before taking the small spin-orbit corrections into account. On $L-S$ eigenstates the Zeeman energies are equally spaced for the p levels, and the s levels have twice the spacing of the p levels. The spin-orbit corrections are diagonal on all nondegenerate and degenerate states in the $L - S$ coupling scheme, and to first-order perturbation theory they are proportional to $< l, s; m_l, m_s|\vec{L} \cdot \vec{S}|\ l, s; m_l, m_s >\sim m_l m_s$. This explains the splittings of the two lines on the left and the two lines on the right. The doublet in the middle is then degenerate, but higher-order perturbation theory also splits this doublet.

To discuss the two-component spinor theory of Pauli, we first expand the exact quadratic Dirac equation by setting $E = mc^2 + W$. This yields in the Coulomb gauge $\partial_k A^k = 0$

$$\Big[W - e\phi + \tfrac{\hbar^2}{2m}\vec{\nabla}^2 + \tfrac{1}{2mc^2}(W - e\phi)^2 - \tfrac{ie\hbar}{mc}\vec{A}\cdot\vec{\nabla} \\ - \tfrac{e^2}{2mc^2}\vec{A}\cdot\vec{A} + \tfrac{e\hbar}{2mc}\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}\cdot\vec{B} - \tfrac{ie\hbar}{2mc}\begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}\cdot\vec{E}\Big]\psi = 0. \tag{1.A.9}$$

The first 3 terms yield the nonrelativistic Schrödinger equation, while the next 3 terms are due to the relativistic Schrödinger equation. Only the last two terms depend on spin. The fourth term can be written to leading order by using $(W - e\phi)\psi \sim -\frac{\hbar^2}{2m}\vec{\nabla}^2\psi$ as follows

$$H_{Som} = -\frac{1}{2mc^2}(W - e\phi)^2 \sim -\frac{p^4}{8m^3c^2} \tag{1.A.10}$$

and accounts for the relativistic kinetic correction $\sqrt{m^2c^4 + (pc)^2} - mc^2 = \frac{p^2}{2m} - \frac{1}{8}\frac{p^4}{(m^3c^2)} + \ldots$ (One obtains this result if one lets factor p^2 act to the right in the expectation value and the other factor p^2 to the left). This term corresponds to the relativistic corrections introduced by Sommerfeld [258] but the expectation value of the operator p^4 in quantum mechanics differs from the value Sommerfeld obtained,

[90]The Landé factor is in general $g = 1 + \frac{J(J+1) - L(L+1) + S(S+1)}{2J(J+1)}$ and for $L = 1, J = 3/2$ one finds $g = 4/3$, while $L = 1, J = 1/2$ yields $g = 2/3$.

as we have discussed before. The two terms with $\vec{A}$ describe the effect of an external vector potential on the electron. We already discussed in the main text that the first term had been used by Dirac to compute Einstein's A-coefficient for spontaneous emission, and that he had used the term with $\vec{A}^2$ to describe the scattering of an electron by an atom. Pauli introduced the coupling $\frac{e\hbar}{2mc}\vec{\sigma}\cdot\vec{B}\ \psi$ for a two component spinor; this correctly described the interaction Goudsmit and Uhlenbeck had proposed [19] and, as we now discuss, it finally gave an explanation of the confusing Zeeman splittings of spectral lines in an external magnetic field.

The difference between the exact quadratic Dirac equation with a 4-component spinor and the Pauli model with a 2-component spinor is, apart from the relativistic Sommerfeld corrections, due to the last term in (1.A.9). Writing the Dirac spinor as $\psi = \begin{pmatrix} u_A \\ u_B \end{pmatrix}$ with u_A the Pauli spinor, the linear Dirac equation yields

$$\begin{aligned} (E - mc^2 - e\phi)u_A &= \vec{\sigma}\cdot(c\vec{p} - e\vec{A})u_B \\ (E + mc^2 - e\phi)u_B &= \vec{\sigma}\cdot(c\vec{p} - e\vec{A})u_A \end{aligned} \tag{1.A.11}$$

To evaluate the last term in (1.A.9), we use that the Pauli matrices satisfy not only the commutation relations of angular momentum, but also, as pointed out to Pauli by Jordan (footnote 2 of [41]), the anticommutation relations $\{\sigma^i, \sigma^j\} = 2\delta^{ij}$. Approximating $E + mc^2 - e\phi$ in the linear Dirac equation by $2mc^2$ and omitting the term with $\vec{A}$, one finds

$$\vec{\sigma}\cdot\vec{E}\ u_B = \frac{1}{2mc^2}\left[\vec{E}\cdot c\vec{p} + i\vec{\sigma}\cdot(\vec{E}\times c\vec{p})\right]u_A \tag{1.A.12}$$

The term with $\vec{E}\times c\vec{p}$ yields the spin-orbit coupling,[91] including the Thomas factor 2 [267]. It yields for the hydrogen atom[92] with $\phi = |e|/r = -e/r$

$$H_{spin-orbit} = \frac{ie\hbar}{2mc}\frac{1}{2mc^2}i\vec{\sigma}\cdot\left(-\vec{\nabla}\phi\times c\vec{p}\right)$$

[91] This interaction was first obtained by R. de Laer Kronig, but he had the unlucky idea of discussing it with Pauli, who convinced him it had nothing to do with reality [190]. Thus Kronig never published his result.

[92] For S-states, the radial integral seems to diverge as $\int_0^\infty dr\frac{1}{r}$, but this is due to the approximation $E + mc^2 - e\phi \simeq 2mc^2$ used in (1.A.12). Keeping $E + mc^2 - e\phi$ near $r = 0$, one finds that the radial

$$= \frac{e\hbar}{4m^2c^2}(\vec{\sigma}\cdot\vec{L})\frac{1}{r}\frac{d\phi}{dr} = \frac{e^2}{2m^2c^2r^3}(\vec{s}\cdot\vec{L}) \quad \text{with} \quad \vec{s} = \frac{\hbar}{2}\vec{\sigma} \quad (1.A.13)$$

and shifts states with $j = l + 1/2$ above states with $j = l - 1/2$. Pauli also included this spin-orbit coupling in his theory. The Pauli theory with a two-component wave function for the electron [41] contained thus the following terms: the nonrelativistic terms of the Schrödinger equation, Sommerfeld's p^4 relativistic correction, the $\vec{\sigma}\cdot\vec{B}$ magnetic correction of Goudsmit and Uhlenbeck with $g_e = 2$ inserted by hand, and the spin-orbit coupling with Thomas factor 2 included. The term with $\vec{E}\cdot c\vec{p}$, on the other hand, present in Dirac theory, was not included in the Pauli approximation. For a central electric field $\vec{E} = -\vec{\nabla}\phi$ with $\phi = -e/r$ the expectation value of the term with $\vec{E}\cdot c\vec{p}$ was evaluated by Darwin and Gordon [48]

$$\begin{aligned} H_{Darwin} &= \frac{ie\hbar}{2m_ec}\frac{1}{2m_ec^2}\left\langle\left(-\frac{\partial\phi}{\partial r}\right)c\frac{\hbar}{i}\frac{\partial}{\partial r}\right\rangle \\ &= -\frac{e^2\hbar^2}{4m_e^2c^2}\int_0^\infty R(r)\frac{1}{r^2}\left(\frac{d}{dr}R(r)\right)r^2dr \\ &= \frac{e^2\hbar^2}{8m_e^2c^2}\{R(r=0)\}^2 \qquad (1.A.14) \end{aligned}$$

where $R(r)$ is the radial part of the electron wave function. Since $R(r)$ vanishes at $r = 0$ for states with $l > 0$, the Darwin term only contributes to S waves. In fact, it contributes the same amount as one would get from the spin-orbit coupling if one were to take the formula for $j = l + 1/2$ and substitute $l = 0$ into this result[93]. (As we explained before, the spin-orbit coupling is proportional to $\vec{L}\cdot\vec{s}$ and hence does not contribute to S states). In physical terms the Darwin term describes the coupling of the "small spinors" (u_B) to the "large spinors" (u_A) at the position of the nucleus where the singular potential of order $\frac{1}{r}$ becomes competitive with $2mc^2$.

integral is finite, and hence S states do not contribute to the term with $\vec{s}\cdot\vec{L}$. However the Darwin term which we discuss next gives precisely the same contribution as one would obtain if one assumed that the formula for the energy due to spin-orbit coupling (which holds only for $l \neq 0$) also would hold for $l = 0$.

[93]For $l \neq 0$ the numerator of the spin-orbit contribution is proportional to $j(j+1) - l(l+1) - s(s+1)$. For $j = l + 1/2$ this becomes l, but the denominator of the matrix element contains a term $1/l$ [260]. Hence, setting $l = 0$ yields a nonvanishing result.

The exact solutions of the Dirac equation for the bound state energies depend only on j but not on l. For example, the $2\,^2s_{1/2}$ and the $2\,^2p_{1/2}$ states have the same energies, and only the radiative corrections of quantum field theory lift this degeneracy. It is of some interest to see how the l-dependence of the various terms in the Pauli approximation to Dirac theory cancels in the sum. There are three terms which give corrections of order α^4: the relativistic p^4 Sommerfeld correction, the spin-orbit coupling due to $\vec{E} \times \vec{p} \cdot \vec{\sigma}$, and the Darwin term due to $\vec{E} \cdot \vec{p}$

$$E = \left\langle -\frac{1}{2mc^2}(W - e\phi)^2 + \frac{e}{2m^2c^2}\frac{1}{r}\frac{d\phi}{dr}(\vec{L} \cdot \vec{s}) - \frac{e\hbar^2}{4m^2c^2}\frac{d\phi}{dr}\frac{\partial}{\partial r} \right\rangle \tag{1.A.15}$$

The symbol $\langle \quad \rangle$ denotes the expectation value w.r.t. the solutions of the nonrelativistic Schrödinger equation with quantum numbers n and l. Substituting $\phi = -\frac{e}{r} = \frac{|e|}{r}$, one obtains for the α^4 corrections

$$E(\alpha^4) \;\; = \int drr^2 R_{n,l}(r) \left[-\tfrac{1}{2mc^2} \left\{ \left(\tfrac{\alpha^2}{2n^2} mc^2 \right)^2 + 2 \tfrac{\alpha^2}{2n^2} mc^2 \left(-\tfrac{e^2}{r} \right) + \tfrac{e^4}{r^2} \right\} \right. \tag{1.A.16}$$
$$\left. + \tfrac{e^2\hbar^2}{2m^2c^2} \tfrac{1}{r^3} \left(\tfrac{1}{2} l \;\text{ or }\; -\tfrac{1}{2}(l+1) \right) - \tfrac{e^2\hbar^2}{4m^2c^2} \tfrac{1}{r^2} \tfrac{d}{dr} \right] R_{n,l}(r) \tag{1.A.17}$$

where the radial eigenfunctions are normalized to $\int_0^\infty R_{n,l}^2(r) r^2 dr = 1$. Substituting the expectation values $\left\langle \frac{1}{r} \right\rangle = \frac{1}{a}\frac{1}{n^2}$ and $\left\langle \frac{1}{r^2} \right\rangle = \frac{1}{a^2}\frac{1}{n^3(l+1/2)}$ where $a = \frac{\hbar^2}{me^2}$ is the Bohr radius, the relativistic p^4 corrections yield

$$E(p^4) = -\frac{\alpha^4 mc^2}{2n^3} \left(\frac{1}{l+1/2} - \frac{3}{4n} \right) \tag{1.A.18}$$

The spin-orbit correction yields, using $\left\langle \frac{1}{r^3} \right\rangle = \frac{1}{a^3}\frac{1}{n^3(l+1)(l+\frac{1}{2})l}$ for $l \neq 0$,

$$E_{\text{spin-orbit}} = \frac{\alpha^4 mc^2}{2n^3} \frac{\left(\frac{1}{2} l \;\text{ or }\; -\frac{1}{2}(l+1) \right)}{(l+1)(l+\frac{1}{2})l} \quad \text{for} \quad l \neq 0 \tag{1.A.19}$$

for the cases $j = l + 1/2$ or $j = l - 1/2$. The Darwin term contributes only to $l = 0$ and yields, upon using $R_{n,l=0}^2(r=0) = \frac{4}{n^3}\frac{1}{a^3}$,

$$E_{(Darwin)} = \frac{\alpha^4 mc^2}{8} \left(\frac{4}{n^3} \right) \quad \text{for} \quad l = 0 \tag{1.A.20}$$

First we notice that setting $l = 0$ in the first term of $E_{spin-orbit}$, yields $E_{(Darwin)}$, as we already mentioned. Hence, for all l the l-dependent terms are given by the expression in (1.A.19), but for $l = 0$ we should only take the first term inside the parentheses. In the total result the terms depending on l for the case $j = l + 1/2$ are proportional to $\frac{1}{l+1} = \frac{1}{j+1/2}$, while for the case $j = l - 1/2$ the result is proportional to $\frac{1}{l} = \frac{1}{j+1/2}$. Hence, in terms of j the result is the same for both cases: there is degeneracy between $j = l + 1/2$ and $j = l - 1/2$.

The total result for the α^4 corrections of Pauli theory reads, also including the Darwin term for $l = 0$,

$$E(\alpha^4; n, l) = -\frac{\alpha^4 mc^2}{2n^3}\left(\frac{1}{j+1/2} - \frac{3}{4n}\right) \tag{1.A.21}$$

and this agrees with the exact result of Dirac theory up to order α^4. Levels with the same n and l such as the $2\,^2p_{1/2}$ and $2\,^2p_{3/2}$ levels are only split by the spin-orbit coupling corrections, but not by the p^4 corrections. On the other hand, for transitions between levels with different l the p^4 corrections by themselves as derived from quantum mechanics are far too large. For example for the $2\,^2p_{1/2} \to 1\,^2s_{1/2}$ transition, the relativistic scalar equation yields a result for ΔE which is five times larger than the data. This was for Schrödinger the reason to abandon the relativistic scalar equation, as we already discussed.

Increased experimental accuracy led within a year to new disagreements in the spectrum of the hydrogen atom [263]. These were resolved by taking into account [263, 264] that the proton has a finite mass m_p and a magnetic momentum μ_p given by

$$\vec{\mu}_p = \frac{|e|\hbar}{2m_p c} g_p \vec{\sigma}, \qquad g_p = 2.79 \tag{1.A.22}$$

Strictly speaking, one should now use a relativistic Dirac equation for 2 particles, the electron and the proton, and this is a complicated problem which was tackled by Breit. [265] Moreover, relativity requires the presence of antiparticles, and then pair creation becomes possible, which requires field theory instead of quantum mechanics. However a full quantum field theory for QED had not been yet developed by 1930, and approximate methods were used which gave satisfactory results.

The finite mass of the proton can be taken into account by replacing $1/m_e$ in $p^2/2m_e$ by the reduced mass $1/m = 1/m_e + 1/m_p$. The magnetic moment of the proton is taken into account by coupling the magnetic moment of the electron to the vector potential $\vec{A}$ generated by the nuclear magnetic moment. This yields the hyperfine coupling[94]

$$H_{hf} = -\vec{\mu}_e \cdot \vec{B} + \frac{ie\hbar}{m_e c}\vec{A} \cdot \vec{\nabla}; \qquad \vec{A} = \frac{\vec{\mu}_p \times \vec{r}}{r^3} \tag{1.A.23}$$

where $\vec{B} = curl\vec{A}$ and $\vec{\mu}_e = \frac{e\hbar}{2m_e c}\vec{\sigma}_e$, while $\vec{\mu}_p = \frac{|e|\hbar}{2m_p c} g_p \vec{\sigma}_p$ where $g_p = 2.79$. (So $g_p = 1$ would correspond to a Dirac magnetic moment). The first term yields the spin-spin coupling, while the second term yields the spin-orbit coupling. Evaluation of $\vec{B} = curl\vec{A}$ using $\vec{B} = -\vec{\nabla} \times \left[\vec{\mu}_p \times \vec{\nabla}\frac{1}{r}\right] = -\vec{\mu}_p \nabla^2 \frac{1}{r} + \mu_p^l \partial_l \vec{\nabla}\frac{1}{r}$ yields[95]

$$\vec{B} = 3\vec{r}(\vec{r} \cdot \vec{\mu}_p)/r^5 - \vec{\mu}_p/r^3 + (8\pi/3)\vec{\mu}_p \delta^3(\vec{r}) \tag{1.A.24}$$

and thus

$$H_{hf} = -3(\vec{\mu}_e \cdot \vec{r})(\vec{\mu}_p \cdot \vec{r})/r^5 + (\vec{\mu}_e \cdot \vec{\mu}_p)/r^3 - \frac{8\pi}{3}(\vec{\mu}_e \cdot \vec{\mu}_p)\delta^3(\vec{r}) - \frac{e}{m_e c r^3}(\vec{\mu}_p \cdot \vec{L}) \tag{1.A.25}$$

The expectation value of the hyperfine interaction for S states is only due to the

[94] The idea that the nucleus has a nonvanishing total angular momentum which interacts with the electrons was first proposed by Pauli [266] in 1924. However Pauli considered orbital angular momentum, not spin.

[95] Use $-\nabla^2 \frac{1}{r} = 4\pi\delta^3(\vec{x})$ and $\partial_l \partial_i \frac{1}{r} = 3x_l x_i/r^5 - \delta_{li}/r^3 + \alpha_{li}\delta^3(\vec{x})$ with α_{li} a constant. To fix α_{li} either take the trace over l and i, or apply Gauss theorem to $\int \partial_l \partial_i \frac{1}{r} d^3r = \int \partial_k \left[\partial_l \delta_i^k \frac{1}{r}\right] d^3x = \oint (x_k/r)\left(-x_l \delta_i^k/r^3\right) dO = -(4\pi/3)\delta_{li}$. Hence $\alpha_{li} = -(4\pi/3)\delta_{li}$, and this explains the term $(8\pi/3)\vec{\mu}_p \delta^3(\vec{x})$ in $\vec{B}$.

term with $\delta^3(\vec{r})$ and it yields[96]

$$H_{hf} = \frac{e\hbar}{2m_e c}\frac{e\hbar}{2m_p c} g_p \frac{8\pi}{3}(\vec{\sigma}_e \cdot \vec{\sigma}_p)|\psi(\vec{r}=0)|^2 \quad \text{for} \quad l=0 \tag{1.A.26}$$

where $|\psi(\vec{r}=0)|^2 = |Y_{l=0,m=0}R_{n,l=0}|^2 = \frac{1}{4\pi}\frac{4}{n^3}\left(\frac{m_e e^2}{\hbar^2}\right)^3$. This result was first obtained by Fermi [73], and it shows that singlet states (for which $\vec{\sigma}_e \cdot \vec{\sigma}_p = -3$) lie below triplet states (for which $\vec{\sigma}_e \cdot \vec{\sigma}_p = 1$). For states with $l \neq 0$ the term with $\delta^3(\vec{r})$ does not contribute but the evaluation of the remaining terms requires some tricks [259]. The result is proportional to

$$\left[\frac{f(f+1) - i(i+1) - j(j+1)}{j(j+1)(2l+1)}\right] \tag{1.A.27}$$

where $\vec{i}$ is the spin of the nucleus ($i = 1/2$ for the proton) and $\vec{f} = \vec{i} + \vec{j}$ is the total angular momentum. In fact, this same formula holds also for the case $l = 0$. Hence, for the hydrogen atom with $f = j \pm 1/2$

$$\begin{aligned} H_{hf} &= \frac{m_e c^2 \alpha^4}{n^3}\left(g_p \frac{m_e}{m_p}\right)\left[\frac{f(f+1) - 3/4 - j(j+1)}{j(j+1)(2l+1)}\right] \quad \text{any } l \\ &= \frac{m_e c^2 \alpha^4}{n^3} g_p \frac{m_e}{m_p}\frac{[j \quad \text{or} \quad -j-1]}{j(j+1)(2l+1)} \quad \text{any } l \\ &= \frac{m_e c^2 \alpha^4}{n^3} g_p \frac{m_e}{m_p}\frac{4}{3}\left[\frac{1}{2} \quad \text{or} \quad -\frac{3}{2}\right] \quad \text{for } l=0 \text{ and } S=1,0 \end{aligned} \tag{1.A.28}$$

So spin singlets lie below spin triplets.

Since the Dirac equation was consistent with relativity, it was natural to ask whether a "spinor" calculus for ψ existed, similar to the tensor calculus of special relativity. Ehrenfest put this question to van der Waerden in Göttingen, introducing the name spinor for the wave functions. The answer [268] introduced the "dotted and undotted spinors", widely used in supersymmetry, superspace and supergravity.

[96] The expectation value of $\frac{1}{r^3}$ in S states must be regularized. First evaluating all integrals for $r \geq \epsilon$ and then sending $\epsilon \to 0$ yields a vanishing result for the first two terms in (1.A.25) because even for $r \geq \epsilon$ the result is proportional to $l(l+1)$ which vanishes when $l = 0$. The last term in (1.A.25) clearly also vanishes for $l = 0$.

The problems of the fine structure and hyperfine structure of the hydrogen atom were successfully solved in the 1920's and early 1930's by quantum mechanics, but in the 1940's new descrepancies were found in the spectrum of hydrogen, and these were solved by taking field theory into account. We summarize the relativistic corrections for the $n = 1$ and $n = 2$ levels of the Bohr hydrogen atom. The Bohr values are $W(n = 1) = -13.60$ eV and $W(n = 2) = -3.45$ eV. The corrections in μeV due to the p^4 term, the s-o (spin-orbit) coupling (combined with the Darwin term), the hf (hyperfine) interaction for spin-triplets or spin-singlets, and the radiative Lamb shift, as well as the experimental value of the difference of the total energy of an excited state and the ground state energy in cm^{-1}, are as follows

	p^4	s-o	hf	Lamb	exp
$n = 1,\ s_{1/2}$	-910μeV	720μeV	$5.72\left(\frac{1}{4} \text{ or } -\frac{3}{4}\right)\mu$eV	34μeV	0
$n = 2,\ s_{1/2}$	-147μeV	90μeV	$0.71\left(\frac{1}{4} \text{ or } -\frac{3}{4}\right)\mu$eV	4.3μeV	82258.907
$n = 2,\ p_{1/2}$	-26μeV	-30μeV	$0.24\left(\frac{1}{4} \text{ or } -\frac{3}{4}\right)\mu$eV	$-\frac{1}{20}\mu$eV	82258.942
$n = 2,\ p_{3/2}$	-26μeV	15μeV	$0.095\left(\frac{3}{4} \text{ or } -\frac{5}{4}\right)\mu$eV	small	82259.272

The value of the Lamb shift of the $p_{3/2}$ level is much smaller than other effects such as the finite size of the nucleus [269], so we do not quote it. These numbers clearly illustrate that in the 1940's a quantum field theory for electromagnetic processes had become a necessity.

B Anomalous magnetic moment

The anomalous magnetic moment $a = \frac{1}{2}(g - 2)$ of the electron (a_e) and muon (a_μ) have been measured and calculated over the years to extremely high precision. In early 1947, Nafe, Nelson and Rabi found that the hyperfine structure of the ground state of hydrogen and deuterium[97] deviated by 0.26% from theory [4]. Breit suggested that the electron might possess an anomalous contribution to its magnetic moment of

[97]The magnetic moment of the proton was known at that time with 0.03% accuracy and that of the deuteron with 0.04% accuracy. The hyperfine structure was according to Pauli [1] due to the interaction of the nuclear and electronic magnetic moments. The theoretical result for the hyperfine splitting of S states

the order of α times the value of the magnetic moment in Dirac theory. [3] Instigated by Rabi, Foley and Kusch [5] looked for similar effects in more complicated atoms and found a discrepancy of 0.1% between the measured value of the of the g factor in Na and Ga atoms and the theoretical value $g = 2$ as predicted by Dirac theory. A correction of 0.1% to the magnetic moment would explain both the deviations in the Na and Ga atoms, and also the hyperfine discrepancies because the electron and the nucleus contribute each a 0.1% correction. Furthermore, at about the same time Lamb and Retherford [6] found shifts in energy levels which should be degenerate according to Dirac theory. This started the modern era of quantum electrodynamics, where field quantization of the electrons supplants quantum mechanics based on the Dirac equation. In a $g - 2$ experiment for muons at Brookhaven, the calculated one-loop **electroweak** corrections to a_μ are four times the expected experimental uncertainty, and as a consequence this $g - 2$ experiment leads to another test of the electroweak sector of the Standard Model. It might even lead to a breakdown of the Standard Model and be an indication for supersymmetry [7]. We discuss the supersymmetric contributions to $g - 2$ in Appendix C.

Before we begin our discussions of the field theoretical contributions to the anomalous magnetic moment, we recall that in 1928 the Dirac equation had given a firm theoretical derivation that the magnetic moment corresponds to $g = 2$ for an electron. Classical electrodynamics predicted, of course, $g = 1$. As one might expect, there was a time before 1928 when experiments yielded puzzling discrepancies between the measured value of the magnetic moment and the theoretical value with $g = 1$. A little anecdote illustrates this confusion. (I thank E. Remiddi and V. Telegdi for providing me with this anecdote. See also A. Pais "Subtle is the Lord, The science and life of Albert Einstein", section 14b, page 245. A detailed account of the early experiments measuring g is given in Peter Gallison, "How experiments end", Univ. Chicago Press

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was given by E. Fermi [2]. G. Breit calculated the corrections to Fermi's result due to nuclear motion [3]. The total uncertainty in the calculated values of the hyperfine splitting was 0.05%.

1987, chapter 2.)

According to classical electrodynamics, a charged particle with angular momentum $\vec{M}$, charge e, and mass m carries a magnetic dipole moment $\vec{\mu}$ given by

$$\vec{\mu} = g \frac{e}{2mc} \vec{M} \tag{1.B.1}$$

with $g = 1$. When it was discovered in the beginning of the 1900's that matter consists of charged particles with very different mass to charge ratio (the positive components have a charge whose absolute value is a small integer multiple of the electron[98] charge, while their masses are larger by a factor 2000 or more), it was realized that a change in the magnetization of a bar should induce a change in its angular momentum and *vice versa.*[99] That fact became known as the Einstein-de Haas effect after the paper by A. Einstein and W.J. de Haas, "Experimenteller Nachweis der Ampèreschen Molekularströme", *Verh. d. Deutsch. Phys. Ges.* **17** (1915) 152 ("Nachweis" means proof in German.) This is perhaps the only experimental paper written by Einstein. Eq. (9.0.1) above, with $g = 1$, was written as

$$\vec{M} = \frac{2mc}{e} \vec{\mu} = \lambda \vec{\mu} \tag{1.B.2}$$

and an experiment was proposed to measure λ in order to obtain a new value for the charge to mass ratio of the lightest particle, the electron.

The theoretical value for λ which follows from (9.0.2) is $\lambda = 1.13 \times 10^{-7}$ in Gaussian units (the effect is small). They found $\lambda = 1.11 \times 10^{-7}$, with an agreement which was almost embarassing. Indeed the authors observed that even if the agreement was

[98]The discovery of the electron is sometimes attributed, in addition to J.J. Thomson, to H.A. Lorentz and P. Zeeman who received the second Nobel prize in physics in 1902 "for their researches into the influence of magnetism upon radiation phenomena". Thomson received the 1906 Nobel prize "for his theoretical and experimental investigations on the conduction of electricity by gases".

[99]Applying a magnetic field to a bar, the small magnetic moments in the bar due to electrons in their orbits and (although unknown at that time) also due to the electron spins, become aligned, yielding a net, nonvanishing angular momentum. The bar must then counter-rotate to preserve angular momentum.

due to chance ("*auf Zufall beruhen*"), nevertheless even with a 10% uncertainty the effect was quantitatively established.

A related experiment on "Magnetization by Rotation" was carried out almost at the same time by S.J. Barnett, *Phys. Rev.* **6** (1915) 239 and **10** (1917) 7. His aim was to show that rotation can induce magnetization for explaining the magnetization of the earth in terms of its daily rotation[100]. When he became aware of the work of Einstein and de Haas, he presented his results as a measurement of the gyromagnetic ratio of the electron. For "electrons in slow motion" he expected from the theory 7.1×10^{-7} in the proper units, and obtained 3.1×10^{-7} in the first experiment, and results ranging from 5.1×10^{-7} to 6.5×10^{-7} in the second experiment. He claimed satisfactory agreement with the Einstein model (but was not satisfied in other respects: "... Their paper contains no reference to the previous work of Maxwell, Schuster, Richardson, or myself").

Later on, the experiment was repeated by Emil Beck, "Zum experimentellen Nachweis der Ampèreschen Molekularströme", *Ann. d. Physik* **60** (1919) 109. He carried out three series of measurements, with final results $\lambda = 0.57 \times 10^{-7}, \lambda = 0.60 \times 10^{-7}, \lambda = 0.64 \times 10^{-7}$ "*sehr genau die Hälfte des zu erwartenden Wertes*" 1.13×10^{-7} (very precisely half of the expected value). He could not explain the disagreement with the Einstein-de Haas result, despite "*eine persönliche Unterredung mit Prof. Einstein*" (a personal discussion with Prof. Einstein), which was for him "*noch ganz besonders wertvoll*" (quite valuable).

We now know that Beck was right - the factor g in eq. (9.0.1) is equal to 2 for the electron, and slightly larger than 2 when QED radiative corrections are accounted for. (Only the spins of electrons contribute to (9.0.2), the orbital angular momenta cancel each other). This factor 2 should be in the denominator of the r.h.s. of eq. (9.0.2), implying a theoretical value of λ equal to $(1.13/2) \times 10^{-7} = 0.565 \times 10^{-7}$, very

[100]If one gives an unmagnetized piece of iron an angular acceleration, the little permanent magnets inside it experience a torque that alignes them: rotation produces magnetism.

close to the values found by Beck. But at that time g was still equal to 1, and Beck could only get a job as high school teacher (de Haas continued his scientific career in Leiden).

When in 1925 Goudsmit and Uhlenbeck (both at Leiden, but unaware of de Haas's work) proposed $g = 2$ [8] to fit the experimental data on the anomalous Zeeman splitting of spectral lines[101], and the Dirac equation of 1928 gave a theoretical explanation, theory and experiment seemed for almost two decades in agreement as far as g was concerned. However, studies of radiative corrections in the 1930's and 1940's for various processes in QED gave infinities and the problem of eliminating these became a central issue. So when in 1947 Rabi and coworkers reported that experiments saw small but definitely nonvanishing departures of the value of the magnetic moment from Dirac theory, and Lamb and coworkers simultaneously reported similar deviations in the spectral lines for certain atomic energy transitions, theorists had their work cut out. In a few years, the full renormalizable theory of QED was established by Schwinger, Feynman, Dyson, Tomonaga, Kramers, Bethe, Breit, French, Weisskopf and others.

We present in what follows the theory of radiative corrections to the anomalous magnetic moment of the electron and muon. This is an excellent exercise in what is technically called on-shell renormalization of QED. The value of $g - 2$ is a relatively simple S-matrix element. One must deal simultaneously with ultraviolet and infrared divergences, and also take into account that the external fermions are on-shell. In Appendix A we discuss these issues further. We calculate the one-loop correction of Schwinger in Appendix B, but also discuss in detail the two-loop corrections. Next we discuss 3-loop and higher-loop corrections. Then we discuss a recent experiment on $g - 2$ for the muon, and calculate the contributions due to the weak interactions.

[101] A few months earlier, Pauli had put forward his exclusion principle [9], but he found that one can put **two** electrons in each state. The discovery of spin by Goudsmit and Uhlenbeck also explained this puzzling factor 2.

In Appendix C, we discuss the predictions of the minimally susy Standard Model (the MSSM) for $g-2$. For a review of the status of QED, see [10].

At the one-loop level, Schwinger's famous result from end 1947 [11] states that the magnetic moment of charged leptons (electron, muon, and since the late 1970's also the tau lepton) is related to its spin by $\vec{\mu} = g\frac{e\hbar}{2mc}\frac{\vec{\sigma}}{2} = (1+a)\frac{e}{mc}\vec{s}$ where $\vec{s} = \frac{\hbar}{2}\vec{\sigma}$ and a at the one-loop level is given by

$$\begin{aligned} &g = 2(1+a) \\ &a_e = a_\mu = a_\tau = \tfrac{\alpha}{2\pi} = 0.001\ 161\ 409\ 7 \end{aligned} \tag{1.B.3}$$

We shall repeat this calculation below; only the one-loop vertex corrections contribute, and the radiative corrections to a are both ultraviolet (UV) and infrared (IR) finite, as well as independent of the gauge chosen. The physical picture behind this calculation is quite simple[102]: an electron dissociates part of the time into an electron and a photon, during which time the electron has a different four-momentum and during this time it couples differently to the magnetic field. This resolved the problem Rabi and coworkers had found, but it also raised the question whether theory and experiment agree at the two-loop level.

Since a_μ and a_e should be dimensionless, and UV and IR convergent, they can only contain mass-independent terms and terms proportional to the ratios of masses, but they cannot depend on the renormalization scale. In the QED sector with electrons (e), muons (μ) and tau-leptons (τ) one has thus [10]

$$\begin{aligned} a_e &= a_e(\text{no } m) + a_e\left(\frac{m_e}{m_\mu}\right) + a_e\left(\frac{m_e}{m_\tau}\right) + a_e\left(\frac{m_e}{m_\mu}, \frac{m_e}{m_\tau}\right) \\ a_\mu &= a_\mu(\text{no } m) + a_\mu\left(\frac{m_\mu}{m_e}\right) + a_\mu\left(\frac{m_\mu}{m_\tau}\right) + a_\mu\left(\frac{m_\mu}{m_e}, \frac{m_\mu}{m_\tau}\right) \end{aligned} \tag{1.B.4}$$

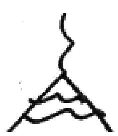
The 2-loop m-independent terms due to QED were first calculated by R. Karplus and N.M. Kroll in 1949 [13], and a small error was corrected by A. Petermann [14] and

[102] This idea is due to Wick who explained in this way why the magnetic moment of the proton is very different from the value Dirac theory predicts. [12]

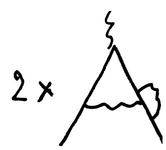
C. Sommerfeld [15]. The result is

$$a_e^{QED}(\text{no m}) = a_\mu{}^{QED}\ (\text{no m}) = \left(\frac{\alpha}{\pi}\right)^2 \left[\frac{197}{144} + \frac{\pi^2}{12} + \frac{3}{4}\zeta(3) - \frac{1}{2}\pi^2 \ln 2\right]$$
$$= -0.328\,478\,965\ldots \left(\frac{\alpha}{\pi}\right)^2 = -.000\,001\,772 \qquad (1.B.5)$$

There are five graphs which contribute to the anomalous magnetic moment of the electron: a ladder and a crossed box graph with two virtual photons, and further 1-loop vertex corrections and 1-loop fermion selfenergy corrections and a 1-loop photon selfenergy correction inserted into Schwinger's one-loop graph. Furthermore, there are 1-loop graphs with an insertion of a 1-loop counter term. We display the graphs, and quote below in each column the contribution to (9.0.5), as obtained from dimensional regularization, omitting an overall factor $(\frac{\alpha}{\pi})^2$.

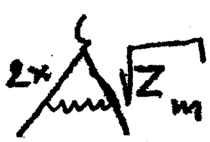


$\left(-\frac{3}{4}\frac{1}{d-4} + \frac{107}{48} + \frac{1}{18}\pi^2\right)$



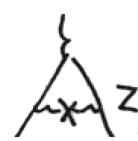
$\left(-\frac{1}{2}\frac{1}{d-4} - \frac{19}{24} + \frac{1}{18}\pi^2 + \frac{1}{3}\pi^2 \ln 2 - \frac{1}{2}\zeta(3)\right)$

$\left(-\frac{1}{d-4} + \frac{5}{24} - \frac{1}{18}\pi^2\right)$



$\left(\frac{3}{2}\frac{1}{d-4} - 1\right) \times \frac{1}{2}(1 - 2(d-4))$

$\left(\frac{3}{2}\frac{1}{d-4} - \frac{7}{4}\right)$



(1.B.6)

$$\underbrace{}_{\left(\frac{11}{48} + \frac{1}{18}\pi^2\right) = 0.778} \quad \underbrace{}_{\left(\frac{1}{6} + \frac{13}{36}\pi^2 - \frac{5}{6}\pi^2 \ln 2 + \frac{5}{4}\zeta(3)\right) = -0.467} \quad \underbrace{}_{\left(-\frac{7}{3} + \frac{1}{3}\pi^2 \ln 2 - \frac{1}{2}\zeta(3)\right) = -0.654} \quad \underbrace{}_{\left(\frac{119}{36} - \frac{1}{3}\pi^2\right) = 0.016}$$

As one can see, the contributions from different graphs cancel each other a good deal, and as a consequence the two-loop corrections are a factor 1000 smaller than the one-loop correction.

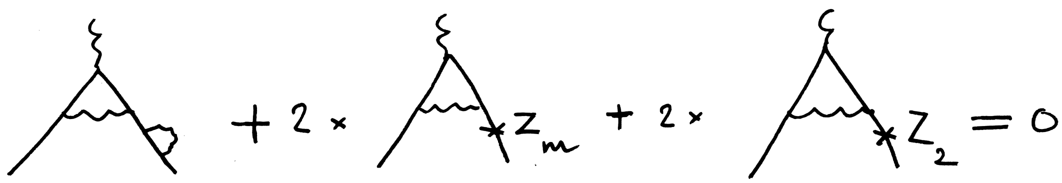
These results were obtained using dimensional regularization. [16] In the works of Petermann and Sommerfeld, the IR divergences of individual graphs were regulated by giving the photon a small mass[103] λ, and one finds then that $\frac{1}{d-4}$ is replaced by $\ln \lambda^2/m_e^2$. For higher-loop calculations (3 loops and 4 loops), dimensional regularization is far simpler than any other scheme, and hence we shall base our discussion of the two-loop corrections on ordinary 't Hooft-Veltman dimensional regularization.

Before dimensional regularization became the universally preferred regularization scheme, the method of dispersion relations was widely used for higher-loop calculations. Using dispersion relations and a particular regularization scheme (Pauli-Villars for example), the subtraction procedure leads to finite dispersion integrals, and the subtraction constants are fixed by the renormalization procedure. In this way one obtains renormalized quantities in terms of subtracted dispersion relations without the need for any explicit knowledge of counter terms (Z factors). In the dispersion approach, one first treats selfenergies and vertex corrections with the dispersion method, and then one uses the results as building blocks in larger diagrams. On the other hand, in dimensional regularization one evaluates separately the diagrams with and without counter term insertions, and only at the end one adds their contributions.

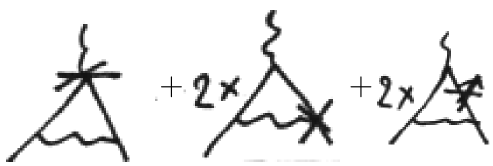
Other approaches that have been used include a partial wave expansion in 4-dimensional Euclidean space, and, of course, various numerical methods. We do not discuss there approaches but refer to the article by Kinoshita in [10].

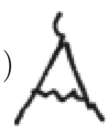
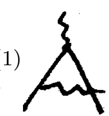
[103] If one regulates QED by giving the photon a small mass, one should sum over 3 rather than 2 polarizations. This is crucial for the Lamb shift [17]. Schwinger and Feynman who initially overlooked this subtlety got an incorrect result. French and Weisskopf who got the correct result for the Lamb shift by using noncovariant methods, delayed publication because their calculations gave a result which differed from Schwinger's and Feynman's. For the anomalous magnetic moment one can safely ignore the subtleties introduced by a longitudinal polarization of the massive photon because $k_\mu k_\nu/m^2$ terms in the photon propagator cancel due gauge invariance.

Because the residue of the renormalized (finite) fermion propagator is unity according to on-shell renormalization, the usual correction factors $(\sqrt{\text{residue}})^{-1}$ for external lines in the definition of the S-matrix are just unity, and all corrections on external fermion lines cancel. For example

2×


Because $Z_1 = Z_2$ in QED, only one factor Z_1 contributes to the 2-loop graphs



$= (3Z_1^{(1)} - 2Z_2^{(1)})$

$= Z_1^{(1)}$

(1.B.7)

(The minus sign in front of $-Z_2^{(1)}$ can only be understood by an explicit calculation of the third graph, keeping track of all factors i.) However, since $Z_1^{(1)} = Z_m^{(1)}$[104] one can also write the contribution with $Z_1^{(1)}$ as $Z_m^{(1)}$ times the one-loop graph, and this has been done in the first column of the 2-loop graphs. As a result, one only finds counter terms with Z_m and Z_3, but none with Z_1 and Z_2. Also at higher loop it is believed that all contributions to $g-2$ only need the counter terms Z_m and Z_3, but not Z_1 or Z_2.

The product of the one-loop graph and $Z_m^{(1)}$ does not only yield a contribution $2(\sqrt{Z_m} - 1)\frac{\alpha}{2\pi}$, where $\frac{\alpha}{2\pi}$ is the one-loop correction to $\frac{1}{2}(g-2)$, but because there is

[104] At the one-loop level, in the Lorentz gauge and using dimensional regularization, $Z_m = 1 - \frac{\alpha}{4\pi}(\frac{3}{\epsilon} + 3\{\frac{4}{3} - \gamma_E + \ln\frac{4\pi\mu^2}{m^2}\})$ and $Z_1 = 1 - \frac{\alpha}{4\pi}(\frac{1}{\epsilon} - 2(-\frac{1}{\epsilon}) + 4 - 3\gamma_E + 3\ln\frac{4\pi\mu^2}{m^2})$ where $\epsilon = \frac{1}{2}(4-d)$ and the pole $\frac{1}{\epsilon}$ in Z_1 is due to an ultraviolet divergence but the pole indicated by $(-\frac{1}{\epsilon})$ in Z_1 is due to an infrared divergence [19]. Clearly the total Z_1 and Z_m satisfy $Z_1^{(1)} = Z_m^{(1)}$. The equality of the γ_E and $\ln(4\pi\mu^2/m^2)$ terms is not surprising because one can multiply diagrams by overall factors with Γ functions, which lead to the γ_E and logarithms, but the equality of the pole terms and the finite terms is surprising. The 3-loop on-shell Z factors can be found in [18], and there one can see that Z_m is no longer equal to Z_1 at higher loop levels.

a pole in Z_m, one must also calculate the 1-loop correction to order $d-4$.[105] The on-shell mass renormalization correction can be found in textbooks [19], and reads

$$Z_m = 1 + \frac{\alpha}{\pi}\left(\frac{3}{2}\frac{1}{d-4} - 1 + \frac{3}{4}\gamma_E - \frac{3}{4}\ln\frac{4\pi\mu^2}{m^2}\right) + \mathcal{O}\left(\frac{\alpha}{\pi}\right)^2 \tag{1.B.8}$$

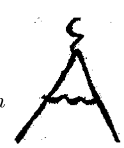
On the other hand, the one-loop correction to the magnetic moment in d dimensions to order $d-4$ will be derived in Appendix B. Dropping the terms with the Euler constant γ_E, with $\ln 4\pi$ and with $\ln(\mu^2/m_e^2)$ one obtains [20]

$$a^{(1)}(d \text{ dims}) = \frac{\alpha}{2\pi}(1 - 2(d-4)) \tag{1.B.9}$$

The product of (9.0.8) and (9.0.9) yields the result for the second graph in the first column of (9.0.6).

The crossed box is UV and IR finite by itself. There are no IR and UV subdivergences as one easily checks either by power counting or by letting the momenta of one or both photons tend to zero, and the overall divergence of this graph contributes only to the charge renormalization.

The "corner graphs" with vertex corrections on the side and the "selfenergy graphs" with an electron selfenergy are each divergent. It is natural to combine the selfenergy graph with the graph with a mass renormalization counter term for the internal electrons, but there remain divergences as one can see. However, the sum of the corner graphs, selfenergy graphs, and the graph with an internal mass renormalization is UV and IR finite. (The contributions from the counter terms with Z_1 and Z_2 to these vertex corrections and fermion selfenergy cancel separately). The contribution from the Z_m counter term is not simply Z_m times the one-loop result

 $Z_m \neq Z_m$  (1.B.10)

[105]The need of terms of higher order in $d-4$ is perhaps the only disadvantage of dimensional regularization. The continuous dimensional regularization method is much more convenient than the Pauli-Villars method together with a small photon mass.

Rather, the graph with a Z_m counter term insertion contains 3 instead of 2 fermion propagators, and must be calculated separately.

Finally, the vacuum polarization graph and the counter term with Z_3 produce together an IR and UV finite part which is also gauge-choice independent. This contribution should be UV and IR finite and gauge-choice independent because the imaginary part of the vacuum polarization graph yields the cross section for e^+e^- annihilation into an e^+e^- pair. We calculate this contribution in Appendix B by using a dispersion integral.

Chapter 2

BRST symmetry

In the previous chapter we discussed the historical development of quantum gauge field theory. In this chapter we shall follow a more recent approach to determine the quantum action and its properties, namely by requiring that the it have a certain symmetry. (By quantum action we mean the action which appears in the path integral, so the sum of the classical action, gauge fixing term and ghost action.) This symmetry is BRST symmetry, where the letters B,R,S and T stand for its inventors Becchi, Rouet, Stora and Tyutin [1]. It is a residual **rigid** symmetry with an anticommuting Lorentz-scalar parameter which remains after the classical gauge symmetry has been broken by adding a gauge fixing term. (By a rigid symmetry we mean a symmetry whose parameter is constant, i.e., spacetime independent). Also supersymmetric theories have a constant anticommuting parameter, but in this case the parameter is a Lorentz-spinor. Nevertheless, there is a connection between BRST symmetry and supersymmetry as we show in section 9.

The importance of BRST symmetry is, of course, not that it gives one more derivation of the quantum action, but it allows to derive Ward identities for proper graphs (and connected graphs) which simplify the proofs of renormalizability (and unitarity) enormously. In the work of 't Hooft and Veltman these identities can already be found, but they were derived by a diagrammatic method, whereas the BRST method applied to path integrals allows a far simpler derivation. The BRST method is a general method which can be applied to any gauge theory. BRST symmetry of a given model does not always imply that such a model is unitary or renormalizable. For example, models with higher covariant derivatives in the classical gauge action, or ordinary gauge actions but with an opposite overall sign are not unitary, but are BRST invariant.[1] However, such actions are pathological and one would not use them. One can also find models without such pathologies where BRST symmetry and unitarity do not hold simultaneously. We give an example of a model which is BRST invariant but not unitary [2] in section 5, and an example of a model which is unitary but not BRST invariant appears in [3]. These models are, however, peculiar exceptions, and in general unitarity and BRST symmetry imply each other.

The relation between renormalizability and BRST symmetry is of a similar nature: in general they imply each other although there are exceptions. For example, adding a so-called Pauli coupling $\frac{1}{m}g\bar{\psi}\gamma^\mu\gamma^\nu\psi(\partial_\mu A_\nu - \partial_\nu A_\mu)$ to QED preserves BRST symmetry because it is gauge invariant, but destroys renormalizability. Also the gauge-invariant higher-derivative gauge theories we mentioned before are BRST invariant but not renormalizable. In both examples coupling constants with a negative dimension appear (g/m), which in general violates renormalizability (the prime example being gravitation). One can also construct examples of theories with dimensionless coupling constants which are BRST invariant but not renormalizable. For example, dropping the $\lambda\varphi^4$ coupling of a scalar field theory coupled to gauge fields, the quantum action remains BRST invariant but renormalizability is lost.[2] (The reason is that proper box diagrams with four external φ fields and two internal gauge fields lead to divergences proportional to φ^4.)

[1] As we shall discuss in the next section, for classical fields the BRST transformations are equal to gauge transformations with parameter $\lambda^a = c^a\Lambda$ where c^a are the ghost fields and Λ is an anticommuting constant parameter. Hence classical gauge invariant actions are automatically BRST invariant.

[2] More precisely, multiplicative renormalizability is lost but the model without classical $\lambda\varphi^4$ term is still additively renormalizable.

In a path integral approach it is not sufficient that the quantum action be invariant under a symmetry transformation, also the measure should be invariant. If the measure is not BRST invariant, there could be BRST anomalies, and these lead in general to a violation of renormalizability and unitarity. In section 3 we shall therefore analyze whether the Jacobian for BRST transformations is unity. This is usually checked without regularization, but such an approach is ill-defined[3], and we shall regularize the BRST Jacobian with heat kernel methods. We use heat kernel regularization because it is particularly well suited for the regularization of Jacobians, but other schemes could equally well be used. The conclusion will be that pure Yang-Mills theory is free from BRST anomalies. For matter-coupled gauge theories, the BRST Jacobian is unity if and only if the quantum gauge field theory does not contain triangle anomalies in the chiral gauge symmetries.

Having shown that there are no anomalies in one regularization scheme is sufficient to conclude that there are also no genuine anomalies in any other regularization scheme. It can be shown using the action principle that the results for the effective action obtained from different regularization schemes differ only by local finite counter terms ΔS [4]. Thus, if in one regularization scheme there are no anomalies, one can also in any other scheme make the theory anomaly free by adding a suitable local finite counter term to the effective action. Adding a finite local counter term to the quantum action does not violate additive renormalizability. However, it would be more satisfactory if there were a direct method to study anomalies in Ward identities without relying on even one particular regularization scheme. Such a method exists, and is called the cohomology approach. As we already mentioned, the conclusion is as follows: when there are no chiral anomalies there are no BRST anomalies (so then the measure is BRST invariant), and when there are no BRST anomalies, there are

[3]For chiral symmetry of models with fermions the need for regularization is well appreciated. The Jacobian is in this case proportional to the trace of γ_5, and would vanish without regularization. With regularization one obtains a nonvanishing result proportional to $\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}$.

no chiral anomalies.

In section 4 we discuss anti-BRST symmetry. It plays an important role in string theory, but for field theory its relevance is limited because the only known power-counting renormalizable model which is both BRST and anti-BRST invariant, is the Curci-Ferrari model, as we shall show, and this model is renormalizable but not unitary.

In section 5 we consider massive Yang-Mills theory, i.e. Yang-Mills theory with a mass term $m^2 A_\mu^2$ added by hand instead of generating a mass term by spontaneous symmetry breaking. This theory has no gauge invariance and is manifestly unitary, but the question we study is whether it is renormalizable or not. This question is of more than academic interest because, if searches for the Higgs boson lead to a negative result, it could offer an alternative to the Higgs mechanism if it were renormalizable. Unfortunately, as we shall prove, it is nonrenormalizable.

In section 6 we discuss alternatives to BRST quantization. In particular, we compare the Faddeev-Popov method based on choosing a gauge choice such as $\partial^\mu A_\mu^a = 0$ with the quantization methods developed in string theory where one makes an orthogonal decomposition of a gauge field A_μ^a into a pure gauge part $D_\mu \omega^a$ and its physical part. The string quantization method is formulated in terms of an operator P, defined by $P A_\mu^a = D_\mu \omega^a$, and its hermitian conjugate $P^\dagger$, and requires a study of zero modes of $PP^\dagger$ and $P^\dagger P$, but the final results for gauge theories are the same as obtained from Faddeev-Popov quantization.

In the remainder of this chapter we discuss some aspects of classical gauge field theory, which are not directly relevant for BRST symmetry but which deal with the basic structure of gauge theories. We derive in section 7 the full nonlinear structure of classical and quantum Yang-Mills theory by using the Noether method. This method uses as input the free action and its symmetries, and derives order-by-order in the coupling constant g the nonlinear terms. Some people motivate their interest in gauge theory by pointing to its esthetic beauty, but as Boltzmann has said, beauty is only good for tailors.[4] We show in section 8 that gauge invariance follows from unitarity, because the only theories for vector fields which are unitary are gauge theories. Section 9 contains some historical comments and elaborates on results obtained in this chapter.

For a first reading only sections 1 and 2 are indispensable.

1 Invariance of the quantum action for gauge fields

Consider the Yang-Mills action for nonabelian gauge fields, for example the QCD action, with the usual relativistically invariant gauge fixing term and ghost action

$$\begin{aligned} \mathcal{L} &= -\frac{1}{4}(F_{\mu\nu}{}^a)^2 - \frac{1}{2\xi}(\partial^\mu A_\mu{}^a)^2 - (\partial^\mu b_a)(D_\mu c)^a \\ F_{\mu\nu}{}^a &= \partial_\mu A_\nu{}^a - \partial_\nu A_\mu{}^a + g f^a{}_{bc} A_\mu{}^b A_\nu{}^c; (D_\mu c)^a = \partial_\mu c^a + g f^a{}_{bc} A_\mu{}^b c^c \end{aligned} \tag{2.1.1}$$

where ξ is a real parameter, called the gauge-fixing parameter. We use the Minkowski metric $\eta_{\mu\nu} = \{-1, +1, +1, +1\}$, so $\partial^\mu A_\mu{}^a = \sum_j \partial_j A_j{}^a - \partial_0 A_0{}^a$ with $j = 1, 2, 3$.

Although in the developments of QED untill 1950 the Coulomb gauge was dominant, work after 1950 saw the ascent of the relativistically invariant Lorentz gauge, and nowadays this gauge is almost always used. The two unphysical polarizations of gauge bosons are cancelled by the two unphysical ghost and antighost particles. More precisely, the contributions of these fields cancel in the unitarity equations, as we shall discuss at length in the chapter on unitarity. The extra minus sign needed for this cancellation is obtained by requiring that b and c anticommute. More general gauge fixing terms than the one in (2.1.1) will be discussed later. The ghost c^a and antighost b_a in the action are Grassmann variables: $b_a c^b = -c^b b_a, c^a c^b = -c^b c^a$ and $b_a b_b = -b_b b_a$. At the operator level, there exist corresponding Heisenberg operators, whose anticommutators need not vanish, but in the action the ghost fields

[4] “Die Eleganz die überlass ich den Schneidern”, in L. Boltzmann, Vorlesungen über Gastheorie.

always satisfy a simple Grassmann algebra. Following Kugo and Ojima [5] **we take $\boldsymbol{b_a}$ imaginary and $\boldsymbol{c^a}$ real**, in order that $\mathcal{L}$ (ghost) is real (hermitian). Hermiticity of the quantum action allows a straightforward proof of unitarity as we shall see.[5] We could, of course, have taken both b_a and c^a real, but then we would have needed a factor i in front of the ghost action and we prefer not to have to deal with such factors of i. In some of the literature the notation $\bar{c}$ is used for the antighost, and sometimes $\bar{c}$ is viewed as the complex conjugate of c. Then the ghost action is not hermitian, and we prefer to use a different symbol (b_a) for the antighost to stress that the antighost is not the complex conjugate of the ghost. In the BRST approach it becomes particularly clear that the ghost c^a has a definite reality (purely real or purely imaginary) because as we shall see $c^a\Lambda$ replaces the real gauge parameter λ^a, where Λ is an anticommuting constant. Clearly both c^a and Λ must have definite reality properties in order that $c^a\Lambda$ be real. We choose c^a to be real and Λ to be purely imaginary; then $c^a\Lambda$ is real.

Another property we shall need is the dimension of fields. Since the action is dimensionless in units with $\hbar = c = 1$, and coordinates have dimension -1 in four dimensions, the gauge fields have dimension $+1$ and the sum of the dimensions of the ghost and antighost field is $+2$. Without loss of generality (because all terms in the quantum action conserve ghost number) we may take the dimension of ghosts and antighosts to be equal, hence $+1$. In the literature one sometimes chooses c^a to have dimension zero and b_a dimension 2, but we prefer to treat c^a and b_a on the same footing as ordinary scalar fields, and so assign dimension 1 to both. This is of course also the choice which is made in most textbooks.

[5] Actually, hermiticity of the gauge fixing term and consequently of the corresponding quantum action allows a relatively simple proof of unitarity, but it is not strictly necessary as we show in the chapter on unitarity. Because the S-matrix is independent of the choice of gauge fixing term, one can choose complex gauge fixing terms and corresponding nonhermitian ghost actions, and still satisfy unitarity. For a suitable choice of complex gauge fixing terms one can greatly simplify the interactions in the quantum action.

The classical action is, of course, invariant under infinitesimal gauge transformations[6]

$$\delta_{\rm gauge} A_\mu{}^a = (D_\mu \lambda)^a \equiv \partial_\mu \lambda^a + g f^a{}_{bc} A_\mu{}^b \lambda^c \tag{2.1.2}$$

The basic idea of BRST symmetry is to replace the classical gauge parameter λ^a by $c^a\Lambda$ where c^a is the ghost field and Λ is a constant, anticommuting and imaginary parameter with ghost number -1 (ghosts having by definition ghost number $+1$) and dimension -1 (since λ^a is dimensionless while ghosts have dimensions $+1$). Taking the ghost field to be real (or hermitian in an operator approach), it is clear that $c^a\Lambda$ is again real, just as λ^a is real. Hence the combination $c^a\Lambda$ can be viewed as a particular choice of λ^a, and thus the classical action is invariant under the following BRST transformations of the gauge fields

$$\delta_B A_\mu{}^a = (D_\mu c)^a \Lambda \tag{2.1.3}$$

If there are scalars φ^i transforming as $\delta\varphi^i = -g(T_a)^i{}_j \varphi^j \lambda^a$ under classical gauge transformations (with $[T_a, T_b] = f_{ab}{}^c T_c$), then the corresponding BRST transformations read $\delta_B \varphi^i = -g(T_a)^i{}_j \varphi^j c^a \Lambda$. The classical gauge invariant action $-\frac{1}{2}(D_\mu \varphi^i)^2$ with $D_\mu \varphi^i = \partial_\mu \varphi^i + g(T_a)^i{}_j \varphi^j A_\mu{}^a$ is then also BRST invariant. Similarly, for fermions, BRST transformations are gauge transformations with λ^a replaced by $c^a\Lambda$.

We shall now derive the remaining BRST transformation rules from the requirement that the quantum action be BRST invariant. This is thus a dynamical approach.

[6]We recall that the gauge action can also be written in terms of Lie-algebra valued fields $A_\mu \equiv A_\mu^a T_a$ as $S = \frac{1}{2}\int Tr F_{\mu\nu} F^{\mu\nu}$ with the antihermitian generators for the fundamental representation T_a normalized to $Tr T_a T_b = -\frac{1}{2}\delta_{ab}$. This action is clearly invariant under the infinitesimal gauge transformation $\delta_{\rm gauge} F_{\mu\nu} = [F_{\mu\nu}, \lambda]$ where $\lambda = g\lambda^a T_a$. Since $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + g[A_\mu, A_\nu]$, we can introduce the one-form $A \equiv A_\mu dx^\mu$ and the two-form $F = \frac{1}{2} F_{\mu\nu} dx^\mu dx^\nu$ and find then $F = dA + gAA$ and $\delta_{gauge} F = [F, \lambda]$. Finite gauge transformations are given by $F' = U^{-1} F U$ where $U = \exp\lambda$. It is then relatively easy to find the finite gauge transformation of A which leads to this expression for F', namely the covariant derivative transforms as a vector in the adjoint representation: $(d + gA)' = d + gA' = U^{-1}(d + gA)U$. Indeed, the curvature is the commutator of two covariant derivatives, $F = \frac{1}{g}(d + gA)(d + gA)$, and thus $F' = U^{-1} F U$. For A_μ^a expansion to first order in λ yields the result (2.1.2). The finite gauge transformation of A_μ^a reads explicitly $g(A_\mu^a)' T_a = e^{-\lambda}(\partial_\mu e^\lambda) + e^{-\lambda} g A_\mu e^\lambda$.

Afterwards we shall check the nilpotency of these transformation rules. We could instead have started with a kinematical approach, namely by requiring that the BRST transformation rules be nilpotent, and then afterwards construct a quantum action that is invariant under these rules. The results of both approaches are the same.

To make the sum of the gauge fixing term and the ghost action in (2.1.1) invariant, it is clearly sufficient that

1. the BRST variation of $\partial^\mu(D_\mu c)$ vanishes separately and

2. the variation of the gauge fixing term be canceled by that variation of the ghost action which is due to a suitable variation of the antighost.

$$\underbrace{\mathcal{L}(\text{gauge})}_{\text{invariant}} \underbrace{-\frac{1}{2\xi}(\partial^\mu A^a_\mu)^2 + b_a}_{\text{invariant}} \underbrace{\partial^\mu D_\mu c^a}_{\text{invariant}} \tag{2.1.4}$$

In fact, this is the **only** way the quantum action can be BRST invariant once one has imposed (2.1.3), because variation of the term in $\partial^\mu D_\mu c$ still leaves the field b in the variations, and $\partial^\mu A_\mu$ does not vary into b. So, the variation of b in $\mathcal{L}(\mathit{ghost})$ must necessarily cancel against the variation of the gauge fixing term. Then one is left with the variation of the fields in $\partial^\mu D_\mu c$.

The BRST variation of the gauge fixing term is according to (2.1.3) given by

$$\delta_B \mathcal{L}(\text{fix}) = -\frac{1}{\xi}(\partial \cdot A_a)\partial^\mu (D_\mu c)^a \Lambda \tag{2.1.5}$$

Variation of the antighost field by a transformation law $\delta_B b_a$ which we do not yet know, yields after partial integration (we discuss boundary terms later)

$$\delta_B \mathcal{L}(\text{ghost}) = (\delta_B b_a)\partial^\mu (D_\mu c)^a \tag{2.1.6}$$

Clearly, the sum of the variation of the gauge fixing term and the variation due to $\delta_B b_a$ in $\mathcal{L}$ (ghost) cancels for the following transformation law of the antighost

$$\delta_B b_a = -\frac{1}{\xi}(\partial^\mu A_\mu)_a \Lambda \tag{2.1.7}$$

(Moving Λ past $\partial^\mu (D_\mu c)^a$ yields an extra minus sign).

All that is left, is to make sure that $\delta_B \partial^\mu (D_\mu c)^a = 0$. Let us first make the slightly stronger requirement that $\delta_B D_\mu c = 0$. Since we already have fixed $\delta_B A_\mu^a$, we must now see whether we can fix $\delta_B c^a$ such that $\delta_B D_\mu c^a = 0$. Evaluating the BRST variation of $D_\mu c$ is straightforward

$$\begin{aligned} \delta_B (D_\mu c)^a &= \delta_B(\partial_\mu c^a) + \delta_B(g f^a{}_{bc} A_\mu{}^b c^c) \\ &= \partial_\mu(\delta_B c^a) + g f^a{}_{bc} (D_\mu c)^b \Lambda c^c + g f^a{}_{bc} A_\mu{}^b (\delta_B c^c) \end{aligned} \tag{2.1.8}$$

One can solve this equation in one step by rewriting it as

$$D_\mu \delta_B c^a + \frac{1}{2} D_\mu (g f^a{}_{bc} c^b \Lambda c^c) = 0 \tag{2.1.9}$$

We used that $f^a{}_{bc}$ is an invariant tensor of the gauge group.[7] The solution is then (2.1.11). However, for readers who prefer more details, we now give an explicit elementary derivation. Assuming no knowledge about $\delta_B c^a$ at all, it could in principle contain terms with A_μ^a and terms without A_μ^a. Let us first assume that $\delta_B c^a$ does not contain terms depending on A_μ^a. The terms in (2.1.8) without $A_\mu{}^b$ yield the condition

$$\partial_\mu(\delta_B c^a) - g f^a{}_{bc} (\partial_\mu c^b) c^c \Lambda = 0 \tag{2.1.10}$$

Since $f^a{}_{bc}(\partial_\mu c^b) c^c = \partial_\mu (\frac{1}{2} f^a{}_{bc} c^b c^c)$ due to the antisymmetry of the structure constants, one finds a solution for $\delta_B c^a$

$$\delta_B c^a = \frac{1}{2} g f^a{}_{bc} c^b c^c \Lambda \tag{2.1.11}$$

[7] To prove that (2.1.8) and (2.1.9) are equivalent note that both ghosts contribute equally in (2.1.9). In (2.1.9) one finds a term $\frac{1}{2} f^a{}_{pq} A_\mu^p (g f^q{}_{bc} c^b \Lambda c^c)$, while in (2.1.8) one finds $g f^a{}_{bc} (g f^b{}_{pq} A_\mu^p c^q) \Lambda c^c$. The difference is proportional to $(\frac{1}{2} f^a{}_{pq} f^q{}_{bc} - f^a{}_{qc} f^q{}_{pb}) c^b \Lambda c^c$. Writing the second term as two terms, each with a factor 1/2, by antisymmetrization in bc, one obtains 3 terms which yield the Jacobi identity, and thus vanish. The Jacobi identity reads $f^a{}_{pq} f^q{}_{bc} + f^a{}_{bq} f^q{}_{cp} + f^a{}_{cq} f^q{}_{pb} = 0$. It can be written as $M^a{}_q f^q{}_{bc} - f^a{}_{bq} M^q{}_c - f^a{}_{qc} M^q{}_b = 0$ with $M^a{}_q = f^a{}_{pq}$, and states that if one transforms all indices of $f^a{}_{bc}$ with a matrix M in the adjoint representation, it is invariant: the structure constants are invariant tensors of the gauge group. For x-independent invariant tensors, the covariant derivative vanishes, $D_\mu f^a{}_{bc} = 0$, and this we used to obtain (2.1.9).

It remains to prove that the remaining terms in (2.1.8) (the terms depending on ${A_\mu}^a$) cancel also. As we already mentioned, it could have happened that one needs ${A_\mu}^c$ dependent terms in $\delta_B c^a$ (just like in $\delta_B b_a$), but the result for $\delta_B c^a$ in (2.1.11) is already the complete answer, as inspection of the A_μ dependent variations in (2.1.8) shows

$$\begin{aligned} & g {f^a}_{bc}(g {f^b}_{pq} {A_\mu}^p c^q \Lambda) c^c + g {f^a}_{bc} {A_\mu}^b \left(\frac{1}{2} g {f^c}_{pq} c^p c^q \Lambda\right) = \\ & -g^2 \left({f^a}_{bs} {f^b}_{pq} - \frac{1}{2} {f^a}_{pb} {f^b}_{qs}\right) {A_\mu}^p c^q c^s \Lambda \end{aligned} \tag{2.1.12}$$

We claim that these terms vanish as a consequence of the Jacobi identities for the structure constants. After using the antisymmetry in c^q and c^s to write the first term as two terms, each with a factor 1/2, one obtains in (2.1.12) a factor

$$\begin{aligned} & {f^a}_{bs} {f^b}_{pq} - {f^a}_{bq} {f^b}_{ps} - {f^a}_{pb} {f^b}_{qs} = \\ & {f^a}_{bs} {f^b}_{pq} + {f^a}_{bq} {f^b}_{sp} + {f^a}_{bp} {f^b}_{qs} = 0 \end{aligned} \tag{2.1.13}$$

The last line is cyclic in the indices s, p, q and is the usual form of the Jacobi identities for the structure constants.

The most general solution for $\delta_B c^a$ is a sum of a particular solution of the inhomogeneous solution (which we found) and the most general solution of the homogeneous equation $\partial_\mu \delta_B c^a = 0$. Since there is no homogeneous solution which is a polynomial in fields and their derivatives, the solution for $\delta_B c^a$ is unique.

Coming back to the requirement that only $\delta_B \partial^\mu (D_\mu c)^a$ need vanish, instead of $\delta_B (D_\mu c)^a$, we find from the ${A_\mu}^a$-independent variations the condition $\partial^\mu \partial_\mu \delta_B c^a + \partial^\mu (g {f^a}_{bc} \partial_\mu c^b \Lambda c^c) = 0$. Since the solution of the homogeneous equation $\partial^\mu \partial_\mu \delta_B c^a = 0$ for general off-shell fields c^a is only $\delta_B c^a = 0$,[8] the solution is still only (2.1.11). Hence, the vanishing of $D_\mu c^a$ is equivalent to the vanishing of $\partial^\mu D_\mu c^a$. We conclude that the

[8]The solution for $\delta_B c^a$ should be a polynomial in the fields, and after a Wick rotation the operator $\partial^\mu \partial_\mu$ in Euclidean space has no eigenfunctions with vanishing eigenvalue. (The integral $\int (\delta_B c^a) \partial^\mu \partial_\mu (\delta_B c^a) = -\int (\partial^\mu \delta_B c^a)^2$ is negative definite, and only vanishes for $\delta_B c^a = 0$.)

quantum action has a rigid symmetry (with constant parameter Λ) given by

$$\delta A_\mu{}^a = (D_\mu c)^a \Lambda, \;\; \delta c^a = \frac{1}{2} g f^a{}_{bc} c^b c^c \Lambda, \;\; \delta b_a = -\frac{1}{\xi} \partial^\mu A_{\mu a} \Lambda \tag{2.1.14}$$

These rules preserve the reality properties of the fields provided one declares that Λ is purely imaginary.

It is now clear that BRST symmetry is also present if one uses other gauge fixing terms. In general, $\delta_B A_\mu{}^a$ and $\delta_B c^c$ are the same, but if (using a notation in the next equation which will be explained in the next paragraph)

$$\mathcal{L}(\text{fix}) = -\frac{1}{2} \gamma_{ab} F^b F^a, \;\; \mathcal{L}(\text{ghost}) = b_a \delta_B F^a / \Lambda \tag{2.1.15}$$

where F^a is **any** gauge fixing term and γ_{ab} any field-independent matrix, then

$$\delta_B b_a = -\gamma_{ab} F^b \Lambda \tag{2.1.16}$$

Indeed $\mathcal{L}(\text{fix}) + \mathcal{L}(\text{ghost})$ is still BRST invariant

$$\delta_B \left(-\frac{1}{2} \gamma_{ab} F^b F^a \right) + (\delta_B b_a) \delta_B F^a / \Lambda = -\gamma_{ab} F^b \delta_B F^a + (-\gamma_{ab} F^b \Lambda) \delta_B F^a / \Lambda = 0 \tag{2.1.17}$$

The relativistic gauge fixing term corresponds to $F^a = \partial^\mu A_\mu{}^a$ and $\gamma_{ab} = \frac{1}{\xi} \delta_{ab}$. For field-dependent γ_{ab}, see section 9, eq. (2.9.4). We shall, however, mostly use the most used gauge fixing term in (2.1.1).

A few words about notation. We have explicitly written the parameter Λ in all variations. To discard Λ in $\delta_B F^a$ we use the notation $\delta_B F^a / \Lambda$ which indicates that one should move Λ to the far right and then discard it. One sometimes introduces a symbol s by $\delta_B A = (sA)\Lambda$ (another convention is $\delta_B A = \Lambda s A$) for any object A, so that in sA one omits the Λ (and calls the resulting rules "antiderivations"), but then one should specify the rules how to take variations of products of fields. With Λ present, no such additional information is needed. For paedagogical reasons we shall use the perhaps more cumbersome notation with Λ present. Readers who are more used to the notation with s should have no problem converting our notation to theirs because they only must delete Λ at various places. In terms of s the BRST rules read

$$sA_\mu{}^a = (D_\mu c)^a,\ sc^a = \frac{1}{2} g f^a{}_{bc} c^b c^c,\ sb_a = -\frac{1}{\xi}\partial^\mu A^a_\mu\,. \tag{2.1.18}$$

Before we proceed to study the BRST transformation laws further, we should point out that they are infinitesimal transformation rules. Ordinary symmetry transformation rules can be exponentiated to yield finite transformation rules, but this does not yield a more general result for the transformation rules in (2.1.14) because if Λ is in the exponent, expansion of the exponent would yield terms with Λ^2, Λ^3, etc. which all vanish.

Note that the position of indices on b_a, c^a and F^a is such that one does not need a metric to raise or lower group indices in $\delta_B A_\mu{}^a$ or $\delta_B c^a$. Hence one can extend these BRST rules to the case of non-semisimple groups (for which the group metric $g_{ab} \sim TrT_aT_b$ has no inverse by definition). However, in the gauge fixing terms in the action in (2.1.15) (and thus also in the ghost action) a symbol γ_{ab} appears which plays the role of a metric. In practical applications, this γ_{ab} is often the Killing metric, and can be taken to be equal to δ_{ab} (see the next paragraph). In the next section we shall see that one can remove this γ_{ab} also from $\delta_B b_a$ by introducing an auxiliary field. Then γ_{ab} is absent from all transformation rules (i.e., γ_{ab} is absent at the kinematical level). Thus γ_{ab} is then entirely a dynamical object (an object appearing only in the action).

In the classical action one should contract the group indices of the two Yang-Mills curvatures with a metric γ_{ab} which is proportional to the Killing metric $g_{ab} = -f_{ap}{}^q f_{bq}{}^p$. Then the classical action is gauge invariant[9]. For $SU(n), \gamma_{ab} = \delta_{ab}$ if one normalizes the generators T_a in the defining representation of $SU(n)$ by $TrT_aT_b =$

[9]The gauge invariant of $\gamma_{ab}F^a_{\mu\nu}F^{b,\mu\nu}$ follows from the fact that γ_{ab} is an invariant tensor. Namely, transforming each index of γ_{ab} with a matrix in the adjoint representation, the result vanishes as a consequence of the total antisymmetry of the structure constants: $\delta\gamma_{ab} = f_{ac}{}^{a'}\gamma_{a'b} + f_{bc}{}^{b'}\gamma_{ab'} = f_{acb} + f_{bca} = 0$.

$-\frac{1}{2}\delta_{ab}$. (For $SU(2)$, this normalization corresponds to $f^a{}_{bc} = \epsilon^{abc}$ and $T_a = -\frac{i}{2}\tau_a$ with τ_a the Pauli matrices.) We have implicitly assumed this normalization in (2.1.1). Thus group indices are raised and lowered by the Kronecker delta δ^{ab} and δ_{ab}, respectively.

Because the quantum action has a rigid BRST symmetry, there is a Noether current for BRST symmetry and a Noether charge Q. This BRST charge is nilpotent, $Q^2 = 0$, and plays a crucial role in string theory, and is also used in gauge theories to define physical states. In the next chapter we shall construct a differential operator which is also nilpotent and follows from the BRST symmetry of the effective action. It is sometimes called the Slavnov-Taylor operator and it plays a crucial role in the proof of renormalization of non-abelian gauge theories. We denote it by $\mathcal{S}$.

By using forms, the structure of the BRST transformations comes out more clearly. Define $A = T_a A^a_\mu dx^\mu$ and $c = T_a c^a$, then $sA = dc + \{A, c\}$ and $sc = cc$, where d is the exterior derivative, $d = dx^\mu \partial/\partial x^\mu$, and we assumed that ghosts anticommute with dx^μ. (This is natural if one views ghosts as one-forms, $c^a = c^a_b d\varphi^b$ where φ^b are the group coordinates. We discuss this further in the chapter on anomalies.) The antighost and auxiliary field form “a contractible pair”, by which in general one means a pair A and B such that $sA = B$ and $sB = 0$. The “geometric sector” with A and c is completely decoupled from the sector with a contractible pair.

Having derived the BRST symmetry of the quantum action for gauge theories, the reader may have noticed a tacit assumption which we made right at the beginning. Namely, we began by deriving the BRST symmetry of the quantum action by requiring that the classical action $\mathcal{L}(\text{class})$ and the sum $\mathcal{L}(\text{fix}) + \mathcal{L}(\text{ghost})$ are each separately invariant.

Is it possible to find other BRST-like transformations which leave the whole quantum action invariant, but not $\mathcal{L}(\text{class})$ by itself? Indeed, this is possible, but only in very particular models, an example being ordinary (i.e., non-supersymmetric) 3-dimensional Chern-Simons theory in the Landau gauge (but not in any other gauge).

It has a symmetry which has been called vector supersymmetry,[10] a somewhat misleading term as it has nothing to do with supersymmetry (although also supersymmetric Chern-Simons theory in the Landau gauge has this symmetry). Another question one might already raise at this moment is the following: can one interchange the role of ghost and antighost in the BRST formalism, and begin with $\delta A^a = D_\mu(b^a\Lambda)$ instead of $\delta A^a_\mu = D_\mu(c^a\Lambda)$? This is indeed possible as we shall discuss in section 4, and leads to another symmetry which has been called anti-BRST symmetry.

2 Nilpotency and auxiliary field

The BRST transformation laws of ${A_\mu}^a$ and c^a are nilpotent. For ${A_\mu}^a$ one BRST variation yields $(D_\mu c)^a$ and we already showed that the BRST variation of $(D_\mu c)^a$ vanishes. Hence, the BRST transformations are nilpotent on ${A_\mu}^a$. Note that we consider here the product of two transformations, not a commutator. However, since $\Lambda_1\Lambda_2 = -\Lambda_2\Lambda_1$, the product $\delta(\Lambda_1)\delta(\Lambda_2)$ is equal to the commutator $\frac{1}{2}[\delta(\Lambda_1), \delta(\Lambda_2)]$.

On c^a we find for the product of two BRST transformations

$$\begin{aligned}
&\delta(\Lambda_1)\delta(\Lambda_2)c^a = \delta(\Lambda_1)\frac{1}{2}g{f^a}_{bc}c^b c^c\Lambda_2 = \\
&g{f^a}_{bc}(\delta(\Lambda_1)c^b)c^c\Lambda_2 = g{f^a}_{bc}\left(\frac{1}{2}g{f^b}_{pq}c^p c^q\Lambda_1\right)c^c\Lambda_2 \\
&= \frac{1}{2}g^2({f^a}_{bc}{f^b}_{pq})c^p c^q c^c\Lambda_2\Lambda_1
\end{aligned} \tag{2.2.1}$$

[10]The action and vector supersymmetry transformation rules are [6]

$$\begin{aligned}
&\mathcal{L}_{CS} = \frac{1}{4}\epsilon^{\mu\nu\rho}\left(F^a_{\mu\nu}A^a_\rho - \frac{1}{3}gf_{abc}A^a_\mu A^b_\nu A^c_\rho\right),\ \delta A^a_\mu = -\epsilon_{\mu\nu\rho}\partial^\nu b^a\epsilon^\rho,\ \delta d_a = (\partial_\nu b_a)\epsilon^\nu \\
&\mathcal{L}_{\text{fix}} + \mathcal{L}_{\text{ghost}} = d_a\partial^\mu A^a_\mu - (\partial^\mu b_a)(D_\mu c^a),\ \delta c^a = A^a_\mu\epsilon^\mu,\ \delta b_a = 0
\end{aligned}$$

where ϵ^μ is anticommuting and $\epsilon^{\mu\nu\rho}\epsilon_{\mu\sigma\tau} = -2\delta^\nu_{[\sigma}\delta^\rho_{\tau]}$ in Minkowski space. One can also interchange the role of b_a and c^a to obtain "anti-vector supersymmetry". The classical Chern-Simons action can be made supersymmetric by adding only a mass term $\bar\lambda\lambda$ for Majorana fermions λ. The classical action has then both ordinary and vector supersymmetry. (As an aside we mention that at the quantum level it is better to use superfields because then the gauge-fixing term and the ghost action are also supersymmetric. However, this introduces more ordinary fields into the action).

In the second line we used that variation of c^c gives the same result as variation of c^b. Since $c^p c^q c^c$ is totally antisymmetric in p, q, c, the Jacobi identities can be used to show that this expression vanishes. Hence, BRST transformations are also nilpotent on the ghost fields.

On the antighost one finds no nilpotency but, rather, as we shall show, the product of two BRST transformations of the antighost is proportional to its field equation.

This situation is the same as in supersymmetry, where the commutator of two supersymmetry transformations of the fermion fields contains a term proportional to their field equation. The procedure to remove the field equation in the BRST algebra is the same as in supersymmetry: one adds an auxiliary field to restore nilpotency of the BRST transformations (which corresponds to closure of the supersymmetry algebra).

First, let us demonstrate that the product of two BRST transformations of the antighost does not vanish. Using (2.1.5) one finds

$$\delta(\Lambda_1)\delta(\Lambda_2)b_a = \delta(\Lambda_1)\left[-\frac{1}{\xi}(\partial^\mu A_\mu{}^a)\Lambda_2\right] = -\frac{1}{\xi}\partial^\mu(D_\mu c)^a\Lambda_1\Lambda_2 \tag{2.2.2}$$

(We used here two different parameters Λ_1 and Λ_2 instead of only one Λ because Λ^2 would vanish. Using antiderivations (BRST variations $sA_\mu = D_\mu c$ without any Λ) would avoid this complication).

This expression is not zero, but it is proportional to the antighost field equation $\frac{\delta}{\delta b_a}S(\text{ghost}) = \partial^\mu(D_\mu c)^a$. (The letter S denotes actions and by $\frac{\delta}{\delta b_a}$ we mean the functional derivative with respect to the field $b_a(x)$.) We now observe that if one replaces the gauge fixing term in (2.1.1) by

$$\mathcal{L}(\text{fix}, \text{aux}) = \frac{1}{2}\xi(d_a)^2 + d_a(\partial \cdot A)^a \tag{2.2.3}$$

then the quantum action is again BRST invariant under

$$\delta_B b_a = d_a\Lambda \quad , \delta_B d_a = 0 \tag{2.2.4}$$

Indeed, the term $\frac{1}{2}\xi(d^a)^2$ is invariant by itself, while the variations

$$d_a\delta_B(\partial \cdot A)^a + (\delta_B b_a)\partial^\mu (D_\mu c)^a \tag{2.2.5}$$

cancel each other as before. Eliminating d^a by its algebraic field equation yields $d^a = -\frac{1}{\xi}\partial \cdot A^a$, and substituting the result back into $\delta_B b_a = d_a\Lambda$, one regains the previous result $\delta_B b^a = -\frac{1}{\xi}\partial^\mu A_\mu{}^a\Lambda$. Similarly, substitution of the d^a field equation into the action, one recovers (2.1.1). This substitution of this field equation into an action does not mean that one is on-shell; rather, it amounts to a Gaussian integration over the nonpropagating field d^a in the path integral and does not imply that any of the remaining fields are on-shell.

In supersymmetry, auxiliary fields play an important role, because they close the supersymmetry algebra. Here the auxiliary field makes BRST transformations nilpotent, which one can write algebraically as $Q^2 = 0$. Thus also here "'the algebra closes". The analogy goes further because just as supersymmetry can be reformulated in superspace with superfields, one can also reformulate BRST symmetry in a similar superspace [7].

Using the new fields d_a to linearize the action in terms of the gauge fixing term F^a is an example of first order formalism. Another example is writing $\frac{1}{2}\dot{q}^2$ as $p\dot{q} - \frac{1}{2}p^2$ in classical mechanics.

The formulation with d_a present has the advantage that the BRST laws are also nilpotent on b_a (and also on d_a). Nilpotency of the BRST laws is a useful property in the proofs of renormalizability and unitarity. In fact, we have arrived at transformation rules which are purely kinematical, in the sense that they no longer depend on the particular form of the gauge fixing term. (In the action there is still a dependence on a metric γ^{ab} in the term $\frac{1}{2}\gamma^{ba}d_a d_b$ where γ^{ab} is the inverse of γ_{ab}. We recall that in (2.1.1) γ^{ab} is equal to $\xi\delta^{ab}$).

Another advantage of the BRST formalism with auxiliary field is that if one partially integrates to write the gauge fixing term as

$$\mathcal{L}(\text{fix}, \text{aux}) = \frac{1}{2}\xi(d_a)^2 - (\partial^\mu d_a)A_\mu{}^a \tag{2.2.6}$$

then the Lagrangian density, not only the action, is BRST invariant. Without auxiliary field the same is true if one writes the ghost action as $b_a \partial^\mu D_\mu c^a$, but this is less desirable for canonical quantization since the ghost field then carries a double time derivative. Thus, in the formulation with auxiliary fields and using the gauge fixing term in (2.2.6) there are no boundary terms in the BRST variation of the quantum action.[11]

We have derived the BRST laws from requiring that the action be invariant, and subsequently discovered that these laws are nilpotent. One can also start from the requirement of BRST nilpotency, and then try to construct invariant actions afterwards.

We summarize the BRST transformation rules with auxiliary field

$$\delta_B A_\mu{}^a = (D_\mu c)^a \Lambda, \delta_B c^a = \frac{1}{2} g f^a{}_{bc} c^b c^c \Lambda, \delta_B b_a = d_a \Lambda, \delta_B d_a = 0 \tag{2.2.7}$$

We can write the complete quantum action as follows

$$\begin{aligned} \mathcal{L}(qu) &= \mathcal{L}(cl) + \delta_B \left(b_a \left(F^a + \frac{1}{2}\xi d^a \right) \right) /\Lambda \\ &= \mathcal{L}(cl) + s(b_a(F^a + \frac{1}{2}\xi d^a)) \end{aligned} \tag{2.2.8}$$

where $F^a = \partial \cdot A^a$. This is also the quantum action for more general gauge-fixing terms, see section 9. In this form the BRST invariance is manifest: the classical action is BRST invariant because it is gauge invariant whereas the gauge artefacts are BRST invariant because the BRST transformations are nilpotent. In some modern developments one views δ_B as a kind of exterior derivative, satisfying $\delta_B^2 = 0$, and then the classical action is closed but not exact, whereas the gauge artefacts are exact.

[11] More precisely, there are no boundary terms due to varying the gauge fixing term and the ghost action. There can be boundary terms from varying the classical action. For example in general relativity there are such boundary terms, but not in Yang-Mills theory.

3 The BRST Jacobian

From the path integral for gauge theories we shall obtain Ward identities by making a change of integration variables which corresponds to an infinitesimal BRST transformation.[12] We must then evaluate the Jacobian. For pure gauge theories this Jacobian is given by

$$\delta(x-y) + Tr\left(\partial\delta_B A_\mu{}^a(x)/\partial A_\nu{}^b(y) - \partial\delta_B c^a(x)/\partial c^b(y) - \partial\delta_B b_a(x)/\partial b_b(y)\right) \quad (2.3.1)$$

where the trace Tr includes a summation over the indices $a = b$ and $\mu = \nu$ and an integral over the spacetime points $x = y$. (For anticommuting fields one gets an extra minus sign because the Jacobian is in general the superdeterminant, which reduces for infinitesimal variations to unity plus a supertrace [8]. This minus sign has the same origin as the minus sign for a fermion loop in a Feynman graph).

Formally each term in the trace vanishes since the structure constants of semisimple Lie algebras are traceless. For example

$$\partial\delta_B A_\mu{}^a(x)/\partial A_\nu{}^b(y) = \partial(D_\mu c^a)(x)/\partial A_\nu{}^b(y)\Lambda = gf^a{}_{bc}c^c(x)\delta_\mu{}^\nu\delta(x-y)\Lambda \qquad (2.3.2)$$

and the trace over a, b yields $f^a{}_{ac} = 0$. Similarly the contribution of the ghost fields is proportional to $f^a{}_{ac}$ and vanishes. For the term with the antighosts there is even no b-dependent term in $\delta_B b_c(y)$, hence this contribution to the trace vanishes even more clearly. With auxiliary field, one should add a term $\partial\delta_B d_a(x)/\partial d_b(y)$ to (2.3.1), but since $\delta_B d_a = 0$, also this term vanishes. Hence, the Jacobian seems unity. (For gravity one finds similar results.[13])

[12] The BRST transformation is infinitesimal in the sense that it is linear in Λ. However, one cannot claim that Λ is small because it is Grassmann parameter. Fortunately, the Jacobian can be defined without Λ, by just dropping Λ. No nontrivial definition of a finite BRST transformation is known.

[13] The vielbein field e_μ^m transforms classically as $\delta e_\mu^m = \xi^\nu \delta_\nu e_\mu^m + (\partial_\mu \xi^\nu) e_\nu^m + \lambda^m{}_n e_\mu^n$. The parameter ξ^μ for general coordinate transformations becomes $\xi^\mu = c^\mu \Lambda$ and the parameter $\lambda^a{}_b$ for local Lorentz transformations is replaced by $\lambda^a{}_b = c^a{}_b \Lambda$. Nilpotency of the BRST transformations on $e_\mu{}^m$ determines the transfomation rule $\delta c^\mu = c^\nu \Lambda \partial_\nu c^\mu$ for the coordinate ghosts and

However, this argument is incomplete since one should regulate the trace because it contains a sum over all spacetime points which itself is infinite and "zero times infinity" can be a finite but nonvanishing number. This is the origin of anomalies. For example, for chiral transformations $\delta\psi = i\alpha\gamma_5\psi$, the Jacobian is proportional to $Tr\gamma_5$ and formally this trace vanishes. However, after regularization with for example the regulator $\exp(-\not{D}\not{D}/M^2)$ with $\not{D} = \gamma^\mu D_\mu$, the trace no longer vanishes, as we shall discuss in the chapter on anomalies. Likewise, regularization of the trace in (2.3.1) might yield a nontrivial result because off-diagonal terms in the Jacobian might combine with off-diagonal terms in the regulator to produce a nonvanishing trace.

As with all situations involving regularization, there are two ways to proceed. One may pick a particular regularization scheme and explicitly compute the regularized Jacobian. Or one might first write down the most general expression for the Jacobian which any possible regularization scheme could ever produce, only restricted by consistency conditions which we discuss below. Then one should study whether or not this most general expression can always be canceled by adding a suitable local finite counter term to the effective action whose BRST variation equals minus the contribution from the regularized Jacobian. The latter approach is based on the mathematical theory of cohomology. We shall discuss this approach in the next chapter when we analyze the Zinn-Justin equation $\{\Gamma, \Gamma\} = \Delta$. In this section we shall follow [9] and compute the BRST Jacobian with a particular regularization scheme, namely the heat kernel method. The calculations are straightforward if somewhat tedious, but since heat kernel methods are widely used, a complete calculation which uses them is of interest for its own sake. The impatient reader may skip all details and only read the last paragraph of this section. We shall see that the BRST Jacobian

$\delta c^m{}_n = c^m{}_t \Lambda c^t{}_n + c^\nu \Lambda \partial_\nu c^m{}_n$ for the local Lorentz ghosts. These transformation rules are again nilpotent. Furthermore, now the contributions to the Jacobian from the vielbein and ghosts do not cancel separately, but their sum still cancels naively (i.e., before regularization).

for pure Yang-Mills theory is nonvanishing, but it is the BRST variation of a local finite counter term. Hence, by starting with the effective action minus this counter term, the BRST anomalies are canceled, and BRST Ward identities (to be derived in the next chapter) hold without extra terms.

Before performing the actual calculation, we should for completeness discuss what can be said about taking any other particular regularization scheme. It can be shown that the results for the Jacobian (and also for the effective action) obtained by using different regularization schemes are all equivalent modulo local finite counter terms in the action **provided the anomalies satisfy certain consistency conditions**. An anomaly is the response of the effective action under a symmetry transformation of the quantum action. Hence, if $\delta_\lambda S(qu) = 0$ defines a symmetry transformation of the quantum action with parameter λ, then the one-loop anomaly An is given by $An(\lambda) = \delta_\lambda \Gamma$ where Γ is the effective action.[14] When the symmetries of the quantum action form a closed algebra (when the commutator of two symmetries is again a symmetry), one has $[\delta_{\lambda_1}, \delta_{\lambda_2}]\Gamma = \delta_{\lambda_1 \times \lambda_2}\Gamma$, where the notation $(\lambda_1 \times \lambda_2)^a$ denotes $gf^{abc}\lambda_1^b\lambda_2^c$. This yields the consistency condition $\delta_{\lambda_1} An(\lambda_2) - 1 \leftrightarrow 2 = An(\lambda_1 \times \lambda_2)$. For BRST transformations the product of two BRST transformations, one with parameter Λ_1, and the other with parameter Λ_2, vanishes (if one wishes one can rewrite this product as a commutator because Λ_1 and Λ_2 anticommute). Thus $\delta(\Lambda_1)[\delta(\Lambda_2)\Gamma] = 0$. Then the consistency condition for BRST anomalies states that they are themselves BRST invariant. The crucial theorem is then: when two particular regularization schemes produce each a BRST anomaly which is BRST invariant, then the difference of these anomalies is equal to the BRST variation of a local counter term in the action. We

[14] At higher loops, the effective action is invariant under BRST transformation laws which themselves receive quantum corrections. Namely, as we show in the next chapter, for $\hbar \neq 0$, the BRST invariance can be written as $\delta\hat{\Gamma}/\delta A_\mu{}^a \frac{\partial}{\partial K_a^\mu}\hat{\Gamma} + \partial\hat{\Gamma}/\partial c^a \frac{\partial}{\partial L_a}\hat{\Gamma} = 0$ where $\hat{\Gamma}$ is the effective action Γ minus the gauge fixing term $S(\text{fix})$, and $\frac{\partial}{\partial K_a^\mu}\hat{\Gamma}$ is equal to the classical transformation law $\delta_B A_\mu{}^a = D_\mu c^a \Lambda$ together with all one-particle irreducible diagrams with one vertex given by $\delta_B A_\mu{}^a$. Similarly for $\delta_B c^a$. We discuss this in the next chapter.

shall not prove this theorem but refer to the literature [4].

We call a regularization scheme which produces anomalies which satisfy their consistency conditions a consistent regularization scheme. An anomaly which is BRST exact will be called a trivial anomaly. We can then conclude that if one consistent regularization scheme only produces a trivial anomaly, then any other consistent regularization scheme will also only produce trivial anomalies. So if one wants to prove that there are no anomalies, it is sufficient to check that one particular consistent regularization scheme only produces a trivial anomaly. We shall now take for this particular consistent regularization scheme heat kernel regularization. For simplicity we work without auxiliary fields.

We regulate the Jacobian J for an infinitesimal BRST transformation with a regulator R whose matrix elements we compute by using heat kernel methods. Denoting the result for the regulated Jacobian by $1 + An$ (where An stands for anomaly) we must calculate

$$An = \lim_{M^2 \to \infty} Tr J e^{R/M^2} \tag{2.3.3}$$

where Tr denotes the supertrace in (2.3.1). As regulator we choose the operator $R^i{}_j = (T^{-1})^{ik} S_{kj}$, where $\phi^i T_{ij} \phi^j$ may be any nonsingular mass matrix for the fields $\phi^i = \{b_a, A_\mu{}^a, c^a\}$ and S_{kj} is the kinetic operator $\partial/\partial\varphi^k S(qu) \overleftarrow{\partial} /\partial\varphi^j$. (The notation indicates that one should differentiate w.r.t. φ^k from the left and w.r.t. φ^j from the right. Note that for anticommuting fields left- and right-derivatives of the action differ by a sign). There is a good reason for picking this regulator, and not, for example S_{kj} alone, but we will not go into the reasons for this choice.[15] There are, of course, other regulators one might take, but as explained before, if there is no anomaly for one regulator, there is none for any other regulator. So it is sufficient to pick one

[15]We follow here the theory of [10] for consistent regulators. The regulators are obtained by comparison with Pauli-Villars regularization, where the Pauli-Villars fields χ^j have kinetic terms $\chi^k S_{kj} \chi^j$ and mass terms $M^2 \chi^k T_{kj} \chi^j$. One can show in general that these regulators yield consistent anomalies. We shall explicitly check that the result for the anomaly is consistent.

regulator (and check that the result for the anomaly is BRST closed).

We choose a nondegenerate mass term which is invariant under rigid Yang-Mills transformations and which has a vanishing ghost number. There is a unique candidate, up to rescalings, namely $\int Tr(A_\mu A^\mu + 2bc)d^4x$. Writing the matrix entries in order of decreasing ghost number, namely as $(b, A, c)T(b, A, c)^T$, we obtain the following 6×6 matrices

$$T = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \eta^{\mu\nu} & 0 \\ -1 & 0 & 0 \end{pmatrix} \delta(x-y) \; ; \quad T^{-1} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & \eta_{\mu\nu} & 0 \\ 1 & 0 & 0 \end{pmatrix} \delta(x-y) \qquad (2.3.4)$$

The operator S_{kj} follows by differentiating the quantum action

$$\int [-\frac{1}{4}(F_{\mu\nu}{}^a(A))^2 - \frac{1}{2}(\partial \cdot A^a)^2 - (\partial^\mu b_a)(D_\mu c^a)]d^4x \qquad (2.3.5)$$

once from the left and once from the right. One obtains then

$$S_{kl} = \begin{pmatrix} 0 & -\partial^\nu c & \partial^\rho D_\rho(A) \\ c\partial^\mu & R^{\mu\nu} & [\partial^\mu b] \\ -D^\rho(A)\partial_\rho & [\partial^\nu b] & 0 \end{pmatrix} \delta(x-y) \qquad (2.3.6)$$

where

$$R^{\mu\nu}(x)\delta(x-y) = \frac{\partial}{\partial A_\nu(y)} D^\rho F_\rho{}^\mu(x) + \partial_x^\mu \partial_x^\nu \delta(x-y) =$$
$$\Big[2F^{\mu\nu}(A) + \eta^{\mu\nu} D^\rho(A) D_\rho(A) - D^\mu(A) D^\nu(A) + \partial^\mu\partial^\nu\Big]\delta(x-y) \qquad (2.3.7)$$

To obtain this result for $R^{\mu\nu}$, we replaced $-D^\nu(A)D^\mu(A)$ by $-D^\mu(A)D^\nu(A)+F^{\mu\nu}(A)$.

Factors of $\frac{1}{2}$ and 2 are easily checked by noting that $S = \frac{1}{2}\varphi^k S_{kl}\varphi^l$ if S is quadratic in φ. The term $\partial^\mu\partial^\nu$ in $R^{\mu\nu}$ is, of course, due to the gauge fixing term. The square brackets in $[\partial^\mu b]$ indicate that this term contains no free derivatives: the derivative ∂^μ acts on b but not beyond b. Furthermore, all entries lie in the adjoint representation of the Lie algebra, for example $c = c^a{}_b \equiv g f^a{}_{cb} c^c$.

The Jacobian for the infinitesimal BRST transformation $\delta b^a = -\partial \cdot A^a \Lambda, \delta A_\mu{}^a = D_\mu c^a \Lambda$ and $\delta c^a = \frac{1}{2} g f^a{}_{bc} c^b c^c \Lambda$ is obtained by right-differentiation [8] of these expressions with respect to ϕ^j and reads

$$J \equiv j(x)\delta(x-y) = \begin{pmatrix} 0 & -\partial^\nu & 0 \\ 0 & -c\delta^\nu_\mu & -D_\mu(A) \\ 0 & 0 & -c \end{pmatrix}(x)\,\delta(x-y)\Lambda \tag{2.3.8}$$

where we recall that $c = c^a{}_b = gf^a{}_{cb}c^c$.

Having obtained explicit expressions for T^{-1}, S and J we can calculate the anomaly in (2.3.3). This calculation is given in the appendix, and the result reads

$$\begin{aligned} An &= \frac{1}{(4\pi)^2}\int \frac{1}{12}Tr(\partial^\nu c)[4A_\mu A_\nu A^\mu - 4A^\mu(\partial_\mu A_\nu - \partial_\nu A_\mu) \\ &- 4A_\nu\partial_\mu A^\mu + \partial^\mu\partial_\mu A_\nu - 3\partial_\nu\partial_\mu A^\mu]d^4x \end{aligned} \tag{2.3.9}$$

In the appendix we show that it is BRST closed, i.e. it is BRST invariant, which is the consistency condition. So, this is a consistent candidate anomaly. The claim that there is no genuine BRST anomaly in pure Yang-Mills theory boils now down to the statement that the expression in (2.A.17) for the anomaly should be BRST exact, i.e., of the form $An = \delta_B \Delta S$. It indeed is of this form, with [9]

$$\Delta S = \frac{1}{(4\pi)^2}\int \frac{1}{12}Tr\left[(\partial\cdot A)^2 + \frac{3}{2}A_\mu A_\nu A^\mu A^\nu - \frac{1}{2}(A^2)(A^2)\right]d^4x \tag{2.3.10}$$

Hence, for pure Yang-Mills theory there is no genuine BRST anomaly: the candidate An for an anomaly can be removed by subtracting the local finite counter term ΔS from the effective action.

4 Anti-BRST symmetry

In this section we study anti-BRST symmetry. Then we analyze which actions are both BRST and anti-BRST invariant. These so-called anti-BRST transformations [2] do not play a role in the discussion of renormalization and unitarity of gauge theories, but they are used in string theory, and for the sake of interest we discuss them briefly.

The basic idea underlying anti-BRST symmetry is to interchange the ghosts and antighosts in the transformation rules. Thus one starts with

$$\delta_{\bar{B}} A_\mu{}^a = (D_\mu b^a)\zeta \tag{2.4.1}$$

where $\bar{B}$ denotes anti-BRST and ζ is an anticommuting, real parameter (because antighosts are antihermitian as we discussed), again with dimension -1 but with ghost number $+1$ (as opposed to Λ which has ghost number -1). Requiring nilpotency of anti-BRST transformations on $A_\mu{}^a$ leads to

$$\delta_{\bar{B}} b^a = \frac{1}{2} g f^a{}_{bc} b^b b^c \zeta \tag{2.4.2}$$

The proof is the same as for BRST transformations. Although we determined $\delta_B b_a$ previously by requiring invariance of the action, it is illuminating to consider all transformation laws purely from a kinematical point of view, and to begin by writing down the most general transformation laws. The expressions for $\delta_B c^a$ and $\delta_{\bar{B}} b^a$ were already fixed by nilpotency of the BRST and anti-BRST transformation laws on $A_\mu{}^a$. Since we want to extend BRST symmetry to a larger symmetry which includes anti-BRST symmetry, we keep the BRST transformation rules $\delta_B b_a$ and $\delta_B d_a$ unchanged. The most general Ansatz for $\delta_{\bar{B}} c^a$ and $\delta_{\bar{B}} d_a$ which is compatible with dimension, ghost number, Lorentz invariance and reality, is of the form $\delta_{\bar{B}} c \sim d\zeta + \partial \cdot A\zeta + bc\zeta ; \delta_{\bar{B}} d \sim bd\zeta + b\partial \cdot A\zeta$. If one requires that the BRST and anti-BRST transformations are nilpotent and commute

$$\delta_B(\Lambda_1)\delta_B(\Lambda_2) = \delta_{\bar{B}}(\zeta_1)\delta_{\bar{B}}\delta(\zeta_2) = [\delta_B(\Lambda), \delta_{\bar{B}}(\zeta)] = 0 \tag{2.4.3}$$

one finds the following result [7]

$$\delta_{\bar{B}} c^a = -d^a \zeta + g f^a{}_{bc} b^b c^c \zeta ; \delta_{\bar{B}} d^a = -g f^a{}_{bc} b^b d^c \zeta \tag{2.4.4}$$

It is not difficult to check by direct calculation that (2.4.4) satisfies (2.4.3).

The next problem is to construct a quantum action which is both BRST and anti-BRST invariant. We begin with the BRST invariant action

$$\mathcal{L} = \mathcal{L}(\text{class}) + \delta_B \left[b_a \left(F^a + \frac{1}{2}\xi d^a \right) \right] \tag{2.4.5}$$

Anti-BRST invariance is certainly obtained if the term in square brackets is itself anti-BRST exact. If this is the case, then the last term in (2.4.5) can be written as $\delta_{\bar{B}}(\delta_B X))$ where X has dimension 2 and ghost number zero. If one imposes also Lorentz invariance, X can only be a linear combination of $(A_\mu{}^a)^2$ and $b_a c^a$. Thus one tries to solve for F^a from the equation

$$b_a \left(F^a + \frac{1}{2}\xi d^a \right) = \delta_{\bar{B}} \left[\alpha (A_\mu{}^a)^2 + \beta b_a c^a \right] / \zeta \tag{2.4.6}$$

There is no solution for F^a if one assumes that F^a is as usual proportional to $\partial \cdot A^a$. A weaker condition would be to require that $\delta_B[b_a(F^a + \frac{1}{2}\xi d^a)/\Lambda] = \delta_{\bar{B}} Y/\zeta$ for some Y and some F^a. This would still be sufficient for anti-BRST invariance, but again there is no solution. However, one may use the order-by-order in g Noether method (see section 7). In this way, one obtains the Curci-Ferrari model [2][16]

$$\begin{aligned}
\mathcal{L} &= \mathcal{L}(\text{class}) - \frac{1}{2\xi}(\partial \cdot A)^2 + \frac{1}{2} b_a (\partial^\mu D_\mu + D_\mu \partial^\mu) c^a \\
&+ \frac{1}{8} g^2 \xi (b \times c)^2 + \frac{1}{2}\xi \left(d + \frac{1}{\xi}\partial \cdot A - \frac{1}{2} g b \times c \right)^2
\end{aligned} \tag{2.4.7}$$

where $(b \times c)^a \equiv f^a{}_{bc} b^b c^c$. From the d^a field equation we see that this model has a

[16] In fact, this model has a much larger symmetry group: a full superalgebra $Osp(1,1/2)$, containing $Sp(2)$ and $SO(1,1)$ algebras which commute with each other, and four fermionic symmetries [11]. The bosonic symmetries are

$$\begin{array}{ll}
Sp(2): & \delta c = c, \delta b = -b \text{ (ghost } U(1) \text{ symmetry)} \\
& \delta c = b, \delta b = -c, \delta d = \frac{1}{2} g (b \times b - c \times c) \text{ (ghost rotations)} \\
& \delta c = b, \delta b = c, \delta d = \frac{1}{2} g (b \times b + c \times c) \\
SO(1,1) & \delta c = b \times (c \times c), \delta b = c \times (b \times b)
\end{array}$$

The fermionic symmetries are BRST, anti-BRST, and further $\delta b = c \times c$, and $\delta c = b \times b$.

ghost-dependent gauge fixing term

$$F^a = \partial \cdot A^a - \frac{1}{2}\xi g \, (b \times c)^a \tag{2.4.8}$$

It is now clear why the simpler approaches discussed before did not work: the gauge fixing term is not simply proportional to $\partial \cdot A^a$, but it contains in addition terms with ghosts. The final action for the gauge artefacts is equal to $\delta_B \delta_{\bar{B}}(\alpha(A^a_\mu)^2 + \beta b_a c^a)$ with $\alpha = -1$ and $\beta = -\frac{1}{2}\xi$ and can be written in the usual form $\delta_B[b_a(F^a + \frac{1}{2}\xi d^a)]$ with F^a given in(2.4.8).

This model is renormalizable but not unitary. That the model is renormalizable requires detailed study involving Ward identities for one-particle irreducible graphs [12]. Similarly, to prove that unitarity is broken in this model one may either construct the physical states in this model (states which are both BRST and anti-BRST invariant) and show that some of them have negative norms [2,12], or one may study Ward identities for connected graphs. Since all these issues will be discussed at length in future chapters, we do not elaborate further at this point. We only mention that one can add a mass term to the model without breaking BRST and anti-BRST symmetry of the action

$$\mathcal{L}(\text{mass}) = -\frac{1}{2}m^2 {A_\mu}^2 - \xi m^2 b_a c^a \tag{2.4.9}$$

One must then also add m-dependent terms $\delta_B d_a = -m^2 c\Lambda$ and $\delta_{\bar{B}} d_a = -m^2 b_a \zeta$ to the transformation law of d_a. The BRST transformations which leave this action invariant are no longer nilpotent (not even on-shell), and the gauge artefacts are not BRST exact.[17] One might study whether there is an anti-BRST anomaly, or one might study adding mass terms which are only BRST but not also anti-BRST invariant. We leave these issues to the reader.

Because this model would yield a mass for Yang-Mills theory without the need of introducing Higgs fields, this model has received a good deal of attention, but

[17] One can derive the gauge-fixing term in (2.4.8) from $\mathcal{L}$ (mass) by requiring that $\mathcal{L}$ (mass) be BRST invariant and use that the antighost b_a varies into the gauge fixing term.

we repeat that it is not unitary and therefore not a viable alternative to the Higgs mechanism.

5 Nonrenormalizability of massive gauge theory

As we have seen in the previous section, adding a mass term for gauge bosons to the action such that BRST and anti-BRST symmetry are preserved leads to a nonunitary (although still renormalizable) model. However, one might instead consider adding a mass term by hand and abandon all BRST symmetry. In the 1960's the renormalizability of this kind of massive Yang-Mills theory was studied, with strong hints that this theory is nonrenormalizable, but no definitive conclusions were reached, and with the advent in the 1970's of spontaneously-broken renormalizable gauge theories ("Higgs models"), the issue of renormalizability of gauge theories with an explicit mass term was relegated to the background. Since electroweak precision data indicate that the Higgs mass is around 200 GeV, so that Higgs particles should be detected this decade if they exist, it may be the right time to come back to the issue of massive gauge fields and discuss the question whether massive gauge fields are indeed not an alternative to Higgs models.

Adding a mass term $-\frac{1}{2}m^2(A_\mu{}^a)^2$ to the gauge action leads to a propagator with numerator $\eta_{\mu\nu}+k_\mu k_\nu/m^2$ which is unitary (because $\eta_{\mu\nu}+k_\mu k_\nu/m^2 = \sum_{m=1}^3 \epsilon_\mu^m \epsilon_\nu^m$ when $k^2+m^2=0$, with ϵ_μ^m the three polarization vectors for a massive vector boson) but not power-counting renormalizable. In fact, the degree of divergence of a L-loop proper graph with E external massive Yang-Mills vector bosons is $D \le 6L+E-2$.[18] Hence, for a fixed n-point function, each extra loop raises the degree of divergence. However,

[18]The degree of divergence D is the number of vertices V_{3A} with a derivative plus 4 times the number of loops: $D = V_{3A} + 4L$. Propagators tend to $k_\mu k_\nu/k^2$ for large momenta, so they are treated as constants. The topological relation $I - V_{3A} - V_{4A} = L - 1$ relates the number of internal lines I to the number of vertices. Since each internal line ends on two vertices, while an external line ends on one vertex, one has $3V_{3A} + 4V_{4A} = 2I + E$. Eliminating V_{3A} one finds $D = 6L + E - 2$.

renormalizability of Green's functions is a luxury, not a necessity, and the weaker property of renormalizability of the S-matrix is sufficient. In fact, it is known that massive Maxwell theory (Proca theory) is renormalizable (and unitary), precisely because of "miraculous cancellations" of divergences in the S-matrix.[19] (We shall prove its renormalizability in the next chapter). The main approach to analyzing the issue of renormalizability of massive Yang-Mills theory will be to first rewrite the theory such that the term $k_\mu k_\nu/m^2$ in the propagator is replaced by $k_\mu k_\nu/k^2$, at the expense of introducing an extra real scalar field φ^a into the theory. This is the approach taken by Feynman [13], who studied this problem using path integrals, and Veltman [14], who studied this problem using diagrammatic techniques. Their results were confirmed by Boulware [15] who used a path integral approach. At the end we shall compare this model with the corresponding Higgs model. We begin with a path integral derivation, and then use the Stückelberg formalism to rederive the results.

In spontaneously broken gauge theories, the kinetic terms contain both a gauge fixing term $-\frac{1}{2\xi}(\partial^\mu A_\mu{}^a)^2$, and a mass term $-\frac{1}{2}m^2(A_\mu{}^a)^2$ where $m = \frac{1}{2}gv$ with v the vacuum expectation value of the Higgs field. The propagator for these models is obtained by inverting the kinetic operator $(k^2+m^2)(\eta_{\mu\nu}-k_\mu k_\nu/k^2)+(m^2+k^2/\xi)k_\mu k_\nu/k^2$. Because $\eta_{\mu\nu} - k_\mu k_\nu/k^2$ and $k_\mu k_\nu/k^2$ are orthogonal projection operators, the result can immediately be written down:

$$\frac{1}{k^2+m^2}(\eta_{\mu\nu} - k_\mu k_\nu/k^2) + \frac{1}{m^2+k^2/\xi}k_\mu k_\nu/k^2 \tag{2.5.1}$$

We can eliminate the $k_\mu k_\nu/k^2$ in the numerators to obtain

$$\frac{\eta_{\mu\nu} + k_\mu k_\nu/m^2}{k^2+m^2-i\epsilon} - \frac{k_\mu k_\nu/m^2}{k^2+\xi m^2-i\epsilon} = \frac{1}{k^2+m^2-i\epsilon}\left(\eta_{\mu\nu} - \frac{(1-\xi)k_\mu k_\nu}{k^2+\xi m^2-i\epsilon}\right) \tag{2.5.2}$$

This propagator is clearly of the power-counting renormalizable type.

To explain how one can change the term $k_\mu k_\nu/m^2$ in the propagator to a term $k_\mu k_\nu/k^2$ we first recall some properties of Higgs models. The massive theory is not

[19] For example, if a term $k_\mu k_\nu/m^2$ ends on a fermion current $\bar{\psi}\gamma_\nu\psi$, with on-shell fermions, current conservation cancels this $k_\mu k_\nu/m^2$ term. For internal propagators the analysis is more complicated.

gauge invariant, but we can nevertheless follow the same steps as in Higgs models in the hope of also in this case replacing the $k_\mu k_\nu/m^2$ terms by $k_\mu k_\nu/k^2$ terms. We therefore add "unity" to the path integral according to the Faddeev-Popov trick

$$I = \int Dg \prod_x \Delta_F(A)\delta[\partial^\mu A_\mu^g - C] \tag{2.5.3}$$

where A_μ^g is the gauge transform of A_μ and Dg denotes the group-invariant Haar measure (to be discussed below). For later use we recall that $\Delta_F(A)$ is gauge-invariant (for a careful proof, see the chapter on the Gribov problem). Inserting this decomposition of unity into the path integral leads to

$$Z = \int D{A_\mu}^a D\psi Dg \left(\prod_x \Delta_F(A)\delta(\partial^\mu A_\mu^g(x) - C(x))\right) e^{\frac{i}{\hbar}S_{cl} + \frac{i}{\hbar}\int J_a^\mu {A_\mu}^a d^4x} \tag{2.5.4}$$

where ψ denotes any matter fields. Since the measure $D{A_\mu}^a D\psi$ is gauge invariant, we may replace $D{A_\mu}^a D\psi Dg$ by $D{A_\mu}^{a,g} D\psi^g Dg$, and since $\Delta_F(A)$ is gauge invariant, we also may replace $\Delta_F(A)$ by $\Delta_F(A^g)$. In the action we may everywhere replace A_μ by A_μ^g except in the mass term, where we use the identity

$$-\frac{1}{2}m^2 A_\mu^2 = -\frac{1}{2}m^2 \left\{(A_\mu^g)^{g^{-1}}\right\}^2 \tag{2.5.5}$$

Finally we also replace A_μ in $J_a^\mu {A_\mu}^a$ by $(A_\mu^g)^{g^{-1}}$, which is obviously the same.

Before going on we should stress that the gauge parameters which are present in A_μ^g are to be considered as new scalar fields in what follows. For that reason we will denote them by φ instead of the more usual notation ω.

For S-matrix elements it makes no difference whether we couple J_a^μ to $(A_\mu)^{g^{-1}}$ or to A_μ. (In general a field redefinition "$\Phi \to \Phi +$ terms nonlinear in fields" does not change the S-matrix [16]. In our case $A_\mu \to A_\mu + \partial_\mu\varphi$ + nonlinear terms. The term $\partial_\mu\varphi$ is linear in fields, but when one takes J_a^μ to be conserved,[20] we can drop

[20] Connected Green's functions are obtained by replacing J_a^μ by a physical polarization vector ϵ^μ and truncating the propagator. Since $k_\mu\epsilon^\mu = 0$, one may omit the term $\partial_\mu\omega \cdot J^\mu$ if one is only interested in S-matrix elements.

the term $\partial_\mu \varphi \cdot J^\mu$. Therefore we use $J_a^\mu A_\mu^{a,g}$ as source term). We then denote A_μ^g by A'_μ and drop the prime. Finally we raise the arguments $C(x)$ of the delta functions into the exponent by suitable Gaussian integrals, and find then the following action

$$\mathcal{L} = -\frac{1}{4}(G_{\mu\nu}^a)^2 - \frac{1}{2\xi}(\partial^\mu A_\mu{}^a)^2 - (\partial^\mu b_a) D_\mu c^a - \frac{1}{2} m^2 (A_\mu^{a,g^{-1}})^2 + J_a^\mu A_\mu^a \qquad (2.5.6)$$

We stress that the Green functions of this model are very different from those of massive gauge field theory, but the S matrix elements are the same. Hence, if the S matrix elements constructed from (2.5.6) contain nonrenormalizable divergences, then the same holds for the S matrix elements of massive gauge field theory. The great advantage of (2.5.6) is of course that it has the renormalizable propagator of (2.5.2).

One can obtain this action in another way which stresses the role of gauge invariance in the process of creating a mass term. For abelian theories this approach is due to Stückelberg, and is called the Stückelberg formalism [17]. The extension to nonabelian theories is due to Veltman [14]. One begins by constructing a gauge-invariant field $\hat{A}_\mu$ which is a sum of the original gauge field A_μ and further terms which depend on a scalar field φ. The scalar field φ is a Goldstone boson since it transforms as $\delta\varphi = -\omega + \cdots$. The following field combination is invariant under finite nonabelian gauge transformations

$$\begin{aligned} \hat{A}_\mu &= e^{-\varphi} D_\mu e^{\varphi} = A_\mu + \partial_\mu \varphi + [A_\mu, \varphi] + \frac{1}{2}[\partial_\mu \varphi, \varphi] + \cdots \\ D_\mu^g &= \partial_\mu + A_\mu^g = e^{-\omega}(\partial_\mu + A_\mu) e^{\omega} \;;\; (e^{\varphi})^g = e^{-\omega} e^{\varphi} \qquad (2.5.7) \end{aligned}$$

Using this field combination we can add a gauge invariant mass term to the massless action

$$\mathcal{L} = -\frac{1}{4}(G_{\mu\nu}^a)^2 + m^2 Tr(e^{-\varphi} D_\mu e^{\varphi})^2 \qquad (2.5.8)$$

The mass term generates an unpleasant off-diagonal coupling

$$\mathcal{L}(\text{mass}) = -\frac{1}{2} m^2 (A_\mu^a)^2 - m^2 A_\mu^a \partial^\mu \varphi^a - \frac{1}{2} m^2 (\partial_\mu \varphi^a)^2 + \cdots \qquad (2.5.9)$$

where we used that $A_\mu = A_\mu^a T_a$ and $TrT_aT_b = -\frac{1}{2}\delta_{ab}$. However, we can use the fact that the action is still gauge invariant by choosing a suitable gauge fixing term which cancels this off-diagonal kinetic term

$$\mathcal{L}_{\text{fix}} = \frac{-1}{2\xi}(\partial^\mu A_\mu^a + \xi m^2 \varphi^a)^2 \tag{2.5.10}$$

(A similar trick was used by 't Hooft for Higgs models where one also finds an off-diagonal kinetic term in the classical gauge action). Because φ is dimensionless, we added the factor m^2, and in order to obtain a gauge field propagator with a free ξ parameter as in (2.5.2), we used a so-called R_ξ gauge (a renormalizable gauge with a free ξ parameter). From the gauge fixing term $\partial^\mu A_\mu + \xi m^2 \varphi$ we find the ghost action in the usual way (by making an infinitesimal gauge variation, and then replacing the gauge parameter ω by a corresponding ghost field c). This yields

$$\mathcal{L}_{\text{ghost}} = (-\partial^\mu b_a) D_\mu c^a - \xi m^2 b_a (c^a + \cdots) \tag{2.5.11}$$

We used that the gauge variation $\delta\varphi$ of φ is given by

$$e^{\varphi+\delta\varphi} = e^{-\omega} e^{\varphi} = e^{\varphi - \omega + \frac{1}{2}[\varphi,\omega] + \cdots} + \mathcal{O}(\omega^2) \tag{2.5.12}$$

hence $\delta\varphi = -\omega + \cdots$

We finally couple the gauge field A_μ to an external current. We find then the following action

$$\begin{aligned}
\mathcal{L} &= -\frac{1}{4}(G_{\mu\nu}^a)^2 - \frac{1}{2}m^2 (A_\mu^a)^2 - \frac{1}{2\xi}(\partial \cdot A^a)^2 + J_a^\mu A_\mu^a \\
&- \frac{1}{2}m^2 (e^{-\varphi}\partial_\mu e^{\varphi})^a (e^{-\varphi}\partial^\mu e^{\varphi})^a - \frac{1}{2}\xi m^4 (\varphi^a)^2 \\
&+ m^2 (A_a^\mu (e^{+\varphi}\partial_\mu e^{-\varphi})^a + A_a^\mu \partial_\mu \varphi^a) \\
&+ b_a \left[\partial^\mu D_\mu c^a - \xi m^2 c^a + \frac{1}{2}\xi m^2 [\varphi, c]^a + \cdots\right]
\end{aligned} \tag{2.5.13}$$

The term $A_a^\mu \partial_\mu \varphi^a$ will be canceled shortly by expanding the term preceding it. For $\xi = 1$, the propagators for all fields are proportional to $(k^2 + m^2)^{-1}$, and the gauge propagator has as numerator $\eta_{\mu\nu}$. The Feynman rules for this model are thus very simple.

If we compare this action with the action in (2.5.6) we see that the leading terms agree, but at the level of subleading terms there are differences: for example, the ghosts are now massive, and the off-diagonal coupling which was present in $-\frac{1}{2}m^2(A_\mu^{a,g})^2$ is now absent. We could, of course, have used the gauge-fixing term in (2.5.10) also in (2.5.6) and would then have obtained the same result as in (2.5.13). However, if one is not clever enough to use the gauge fixing term in (2.5.10) one can still continue by diagonalizing the kinetic terms by hand. For historical interest, and also to demonstrate some useful manipulations, we discuss this approach now; readers who are only interested in the final result may go directly to below (2.5.18).

The mass term in (2.5.6) can be written more explicitly as follows

$$\begin{aligned}
&-\frac{1}{2}m^2(A_\mu^{a,g^{-1}})^2 = m^2 Tr\left\{e^\omega\left(\partial^\mu + A^\mu\right)e^{-\omega}\right\}\left\{e^\omega\left(\partial_\mu + A_\mu\right)e^{-\omega}\right\}\\
&= -\frac{1}{2}m^2({A_\mu}^a)^2 - 2m^2 Tr A^\mu(e^{-\omega}\partial_\mu e^\omega) + m^2 Tr(e^{-\omega}\partial^\mu e^\omega)(e^{-\omega}\partial_\mu e^\omega)
\end{aligned} \tag{2.5.14}$$

where we set $g = e^\omega$ and used $Tr T_a T_b = -\frac{1}{2}\delta_{ab}$. The kinetic terms which come from the mass term read

$$-\frac{1}{2}m^2({A_\mu}^a)^2 + m{A_\mu}^a\partial^\mu\varphi^a - \frac{1}{2}(\partial_\mu\varphi^a)(\partial^\mu\varphi^a) + \cdots \tag{2.5.15}$$

where we defined $\varphi^a = m\omega^a$. The linearized field operator in (A_μ, φ) space reads

$$\begin{pmatrix} (\Box - m^2){\delta_\mu}^\nu - \left(1 - \frac{1}{\xi}\right)\partial_\mu\partial^\nu & m\partial_\mu \\ -m\partial^\nu & \Box \end{pmatrix} \tag{2.5.16}$$

and by inverting it we find the propagator in (A_μ, φ) space

$$\begin{pmatrix} {\delta_\mu}^\nu/(\Box - m^2) + \left(\frac{-1}{\Box(\Box - m^2)} + \xi/\Box^2\right)\partial_\mu\partial^\nu & -\xi m\partial_\mu/\Box^2 \\ \xi m\partial^\nu/\Box^2 & \frac{1}{\Box} - \xi m^2/\Box^2 \end{pmatrix} \tag{2.5.17}$$

If we go to the Landau gauge $\xi \to 0$,[21] the propagator becomes diagonal in $A_\mu{}^a$ and φ^a, and the gauge field propagator in the massive model coincides then with the gauge field propagator in the Higgs model, also in the Landau gauge, see (2.5.2)

$$\langle A_\mu{}^a A_\nu^b \rangle = \delta^{ab} \frac{-i}{-\Box + m^2 - i\epsilon} (\eta_{\mu\nu} - \partial_\mu \partial_\nu (\Box + i\epsilon)^{-1}) \tag{2.5.18}$$

This propagator has an extra $k_\mu k_\nu / k^2$ term and is not as simple as the propagator $\eta_{\mu\nu}(k^2 + m^2)$ for the gauge field, but one can work with (2.5.18) and compute loops. We now turn back to (2.5.13).

We have thus tamed the propagator of the massive theory at the expense of introducing new real scalar fields φ^a. The degree of divergence of an L-loop proper graph is now $D \leq 2L + 2$, which is much less than the $D \leq 6L + E - 2$ for the massive vector bosons we started out with. We can sharpen this result if we restrict our attention to graphs without external φ fields. One finds then $D \leq 2L + 2 - E$. In particular, for one-loop graphs ($L = 1$) one obtains the same degree of divergence as for standard renormalizable gauge theories.

$$D = 4 - E \quad \text{(one-loop graphs)} \tag{2.5.19}$$

The interactions can be written in terms of group vielbeins $e_{\hat{c}}^b(\varphi)$ defined by

$$e^{-\varphi} \partial_\mu e^\varphi = T_b e_{\hat{c}}^b(\varphi) \partial_\mu \varphi^{\hat{c}} \tag{2.5.20}$$

where in the terminology of coset models we refer to the index b as a flat index and $\hat{c}$ as a curved index. The Lie-algebra valued group vielbein field is $e_{\hat{b}} = e_{\hat{b}}^a T_a$. The group metric $g_{\hat{b}\hat{c}}$ is defined by $g_{\hat{b}\hat{c}} = -2Tr e_{\hat{b}} e_{\hat{c}} = \delta_{bc} e_{\hat{b}}{}^b e_{\hat{c}}{}^c$. For the group $SU(2)$ it is not difficult to find an explicit formula for $e_{\hat{a}}{}^b$, but we shall not need it. The interactions of the φ fields without ghosts can now be written as

$$\mathcal{L}_{int}(\varphi) = -m^2 A_a^\mu \left[e_{\hat{b}}{}^a(-\varphi) - \delta_{\hat{b}}^a \right] \partial_\mu \varphi^{\hat{b}} - \frac{m^2}{2} g_{\hat{a}\hat{b}}(\varphi) \partial_\mu \varphi^{\hat{a}} \partial^\mu \varphi^{\hat{b}} \tag{2.5.21}$$

[21]There are no singularities to be concerned about for the limit $\xi \to 0$, as ξ only appears in the propagators but not in the vertices.

This is the action of a nonlinear sigma model, which in four dimensions is nonrenormalizable by power counting. So one might expect trouble with renormalization. However, we want to study the S matrix of this particular model further, to see whether miraculous cancellations occur.

Let us prove at this point that **massive QED is renormalizable**. The structure constants in the interactions of the group scalars vanish ($e^{-\omega}\partial_\mu e^\omega = \partial_\mu\omega + \frac{1}{2}[\partial_\mu\omega,\omega] + \frac{1}{3!}[[\partial_\mu\omega,\omega],\omega] + \cdots$), hence only the kinetic terms in (2.5.21) survive, and the ghosts are free fields. One can also obtain directly this abelian action from the Stückelberg model: $\mathcal{L} = -\frac{1}{4}(F_{\mu\nu})^2 - \frac{m^2}{2}(A_\mu + \partial_\mu\varphi)^2$. It is invariant under the gauge transformations $\delta A_\mu = \partial_\mu\omega$ and $\delta\varphi = -\omega$. Choosing the gauge $\varphi = 0$ one obtains massive QED (also called Proca theory), but using $-\frac{1}{2\xi}(\partial \cdot A + \xi m^2\varphi)^2$ as gauge fixing term we obtain the previous results with a renormalizable propagator for the photon and a free φ field. This model is thus renormalizable.

We return to nonabelian gauge field theory. In the path integral one integrates over $\varphi^{\hat{a}}$ with the Haar measure in (2.5.3)

$$Dg = \prod_x (\det e_{\hat{b}}{}^b) \prod_{\hat{a}} d\varphi^{\hat{a}} \tag{2.5.22}$$

However, going through the path integral quantization procedure in phase space, one must at some point integrate out the canonical momenta of $\varphi^{\hat{a}}$, and this leads to another factor $(\det g_{\hat{a}\hat{b}})^{-1/2}$ in the measure which clearly cancels the factor $\det e_{\hat{b}}{}^b$ in the Haar measure. The Feynman rules of a nonlinear sigma model in phase space (with momenta present) involve the interaction Hamiltonian, which differs from minus the interaction Lagrangian in configuration space. However, according to Matthew's theorem, one may work in configuration space with the interaction Lagrangian if one uses the naive propagators given above and derivatives thereof.[22]

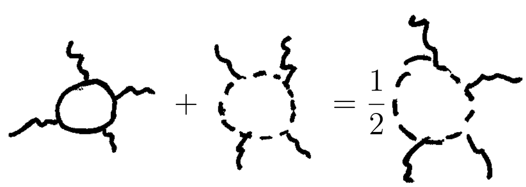
[22]One may introduce the notion of a T^* product in configuration space instead of the T product, and then the only relevant result of this formalism is that the propagator for $\dot{A}_j(x)\dot{A}_k(y)$ is the $\frac{\partial}{\partial x^0}\frac{\partial}{\partial y^0}$ derivative of the Feynman propagator. In phase space there is also a delta function $\delta(x^0 - y^0)$

The domain of the group coordinates φ^a is a compact space because the gauge group is compact, but in perturbation theory one usually ignores this fact and integrates the φ^a from $-\infty$ to $+\infty$, which leads to a field-independent overall constant in the path integral ($\int d\varphi^a_{\text{all}} = N \int d\varphi^a_{\text{group}}$), which we drop.[23]

Let us now make a more detailed study of one-loop proper graphs. The interactions of φ for one-loop graphs follow from (2.5.13) by expanding to second order in φ. This is the part of the action one needs if one considers one-loop graphs without external φ fields. The interactions to second order in φ are contained in

$$\mathcal{L}^{\text{int}} = -\frac{1}{4}(G^a_{\mu\nu})^2 - \frac{1}{2}(\partial^\mu \varphi_a) D_\mu \varphi^a - (\partial^\mu b_a) D_\mu c^a \tag{2.5.23}$$

(We rescaled φ to obtain standard dimensions). Using (2.5.13) at $\xi = 1$ we get standard propagators for A_μ, b and c, and φ but with $k^2 + m^2$ instead of k^2. Note that this action for φ has the same form as the ghost action,[24] but φ is real and commuting. Hence, to take φ-loops into account, one only needs to multiply the contributions from ghost loops by a factor $-1/2$. This is the origin of the van Dam-Veltman-Zacharov mass discontinuity between massless and massive gauge theory at the one-loop level. [18] In the massless case unitarity requires only the presence of a ghost loop, but in the massive case one gets $1 - \frac{1}{2} = \frac{1}{2}$ times the massless result.


for $m \neq 0$ (2.5.24)

In the massive case one needs only half as many ghosts as in the massless case, because there are only timelike but not longitudinal unphysical polarizations. The limit $m \to 0$

which results from differentiating the $\theta(x^0 - y^0)$ in the propagator, but these terms may be dropped according to Matthews' theorem if at the same time one uses the interaction Lagrangian instead of minus the interaction Hamiltonian. We discussed this in the previous chapter.

[23] This issue has been studied by G. Moore and P. Nelson, C.M.P. **100** (1985) 83, in the context of σ-model anomalies.

[24] More precisely, ghost loops are due to the vertices $-g\partial^\mu b_a (f^a{}_{bc} A^b_\mu c^c)$ and loops with φ fields are due to the vertices $-\frac{1}{2} g (\partial^\mu \varphi_a)(f^a{}_{bc} A^b_\mu \varphi^c)$. If one partially integrates the latter vertex, terms with $\partial^\mu A^a_\mu$ vanish. So the contribution from φ loops is $-\frac{1}{2}$ times the contribution of ghost loops.

in the massive theory thus exists, because the term $k_\mu k_\nu/m^2$ in the massive propagator has been eliminated by introducing a new field φ_a, but the massless limit does not yield the massless theory.

Let us now draw conclusions about the renormalizability of the model. For the gauge field selfenergy the combination of a gauge loop and a ghost loop is no longer transversal, but this merely means that the gauge fixing term must be renormalized by a Z_ξ which is no longer equal to a Z_3 (see the next chapter). The form of the divergences in the gauge 3-point function is unique, namely it can always be written as proportional to $g(\partial^\mu A^{a,\nu} - \partial^\nu A^{a,\mu}) f_{abc} A^b_\mu A^c_\nu$, and hence multiplication of the ghost loop by a factor 1/2 merely modifies the one-loop contribution to $Z^{(1)}$. However, for the 4-point function the results are disastrous: multiplying the ghost loop by a factor 1/2 leads to divergences with 4 external A fields whose functional form is different from the terms in the action. These terms are nonvanishing on-shell, even with physical polarizations, and this proves that massive Yang-Mills theory is not multiplicatively renormalizable even at the one-loop level.[25] This rules out the models for non-abelian gauge fields with an explicit mass term.

One can actually make field redefinitions such that the Stückelberg model becomes a polynomial model, still with well-behaved propagators, but then vertices with up to 4 derivatives are present. [19] Also in this form the model is nonrenormalizable.

[25]The same conclusions are reached if one uses additive renormalization. Dropping the divergences whose functional form is different from the terms in the action, yields one-loop finite Green functions which do not satisfy the Ward identities. This in turn leads to violation of unitarity at the two-loop level.

6 BRST, Faddeev-Popov and string-like quantization

Quantization of gauge theories has been called an art: one begins with some intuition and often some ill-defined procedures. Then, as one goes along, there comes a point where one can define the results more precisely, and this end product is then called the quantization procedure.

The BRST quantization procedure can be defined without introducing path integrals, as an invariance of quantum actions and transformation rules. However, having obtained the quantum action one still needs to use it in path integrals, and all subsequent work on BRST cohomology for anomalies and Ward identities is formulated in terms of path integrals. Without further discussion these path integrals are ill-defined.

The Faddeev-Popov quantization procedure starts out with path integrals, and divides out at each point in spacetime the finite volume of the compact gauge group. Since there are infinitely many points in spacetime, this dividing-out process is by itself ill-defined, but afterwards one arrives at a formulation which makes sense.

A third quantization procedure which is widely used is the procedure used in string theory. One can use the same ideas in gauge field theory, and we call this quantization procedure string-like quantization.

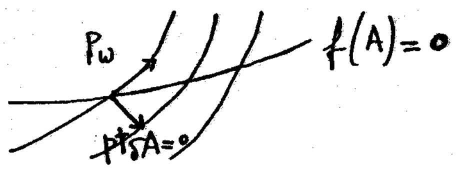
In this section we want to compare these three quantization methods. There are other quantization procedures but for our purposes these three methods are all we need. We shall mostly discuss the relation between the Faddeev-Popov and string-like quantization, but before reaching that point we want to make a short comment on the relation between BRST and Faddeev-Popov quantization.

In the Faddeev-Popov procedure one encounters the absolute value of the Faddeev-Popov determinant Δ_F. Furthermore, since there are in general Gribov copies if one uses relativistic gauges (more than one solution to the equation $\partial^\mu A^a_\mu = F^a(x)$ for given $F^a(x)$), one is supposed to integrate in functional space up to some boundary, in order not to integrate over Gribov copies. The precise form of this boundary is unknown; much has been written about it and interesting speculations have been made, but the matter cannot be considered as settled. In the BRST approach, one encounters the determinant when one integrates out the ghosts and antighosts, but now the determinant appears without absolute value signs. Furthermore, it is tacitly assumed that one integrates in the BRST approach over all field configurations, so without a boundary which excludes Gribov copies. A boundary would (hopelessly?) complicate many of the formal elegant derivations of Ward identities. Clearly, there seems at first sight some disagreement between BRST and Faddeev-Popov quantization. Since both are much used, it would be unsettling if they were not equivalent. We mention here one speculation which is so simple and elegant that it would be a great disappointment if it were not valid. Namely, integrating over all field configurations in the BRST approach, it might be that Gribov copies cancel in pairs in such a way that only configurations remain for which Δ_F is positive. The region which remains in configuration space after all these cancellations have taken place should then be bounded by the boundary which one would need in the Faddeev-Popov approach. We discuss this further in the chapter in the Gribov problem, and now turn to the relation between Faddeev-Popov and string-like quantization.

The Faddeev-Popov quantization method uses a gauge fixing condition $f(A) = 0$ which picks, assuming there are no Gribov copies, precisely one representative from each gauge orbit $U^{-1}(\partial_\mu + A_\mu)U$. As a result the condition $f(A) = 0$ determines a surface in configuration space which in general is not orthogonal to the gauge orbits. In string theory one decomposes the fluctuations around a given configuration into pure gauge parts (along the gauge orbits) and fluctuations perpendicular to the gauge orbits. We apply in this section the ideas of string quantization to gauge theories following [20], and show that the final quantization results are equivalent to

Faddeev-Popov quantization.

We begin by introducing a linear operator P which maps from gauge parameter space $\omega = \omega^a T_a$ (with small ω) to configuration space $A_\mu + \delta A_\mu = ({A_\mu}^a + \delta {A_\mu}^a) T_a$ where the fields $A + \delta A$ are near a given reference field A_μ



$$P : \omega \to \delta A_\mu \equiv -D_\mu(A)\omega \tag{2.6.1}$$

So the fields $P\omega$ lie along the gauge orbits (they are pure gauge). We introduce positive-definite inner products in δA space and ω space as follows

$$(\delta A, \delta B) = -tr \int (\delta A^\dagger \delta B) d^4x; (\omega, \omega') = -tr \int \omega^\dagger \omega' d^4x \tag{2.6.2}$$

where we assumed that $tr T_a T_b = -\delta_{ab}$. (One usually defines $tr T_a T_b = -\frac{1}{2}\delta_{ab}$, but for simplicity of notation we use in this section the normalization without factor 1/2). We work in Euclidean space, so the inner products are really positive definite.

We can then define the hermitian conjugate of P by

$$(P\omega, \delta A) = (\omega, P^\dagger \delta A) \tag{2.6.3}$$

Clearly $P^\dagger$ maps fluctuations δA back into the Lie algebra

$$P^\dagger : \delta A_\mu \to D^\mu(A)\delta A_\mu \equiv \omega \tag{2.6.4}$$

The fluctuations in the kernel of $P^\dagger$ (those δA for which $P^\dagger \delta A = 0$) are orthogonal to the gauge orbits because $(\omega, P^\dagger \delta A) = (P\omega, \delta A) = 0$ for all ω.

The Laplacians $PP^\dagger$ acting in δA space and $P^\dagger P$ acting in ω space have the same nonzero eigenvalues with the same degeneracy because if $PP^\dagger \delta A = \lambda \delta A$ then $(P^\dagger P)(P^\dagger \delta A)$ is equal to $\lambda(P^\dagger \delta A)$, while if $P^\dagger P\omega = \lambda\omega$ then $(PP^\dagger)P\omega = \lambda P\omega$. Hence for every eigenfunction δA of $PP^\dagger$, there is an eigenfunction ω of $P^\dagger P$, with the same eigenvalue, and vice-versa. To show that the degeneracies are the same, assume the opposite. Then there is an eigenfunction δA of $PP^\dagger$ which is mapped by $P^\dagger$ into zero: $\omega = P^\dagger \delta A = 0$. This is not possible if δA has nonzero eigenvalue. The same argument holds for the eigenfunctions of $P^\dagger P$. Thus the range of $P^\dagger$ and the range of P are mapped onto each other by P and $P^\dagger$. The explicit forms for these Laplacians are

$$PP^\dagger = -D_\nu D^\mu, P^\dagger P = -D^\mu D_\mu \tag{2.6.5}$$

(Thus $PP^\dagger$ acts in the space of δA_μ, and $P^\dagger P$ in the space of ω). The eigenvalues of $PP^\dagger$ and $P^\dagger P$ are real and nonnegative. If $P^\dagger P\omega = 0$, then ω lies in the kernel of $P(P\omega = 0)$ because $(\omega, P^\dagger P\omega) = 0 =\parallel P\omega \parallel^2$. Similarly, if $PP^\dagger \delta A = 0$ then δA lies in the kernel of $P^\dagger$. Using the norms defined in (2.6.2), we can define a nonsingular (finite and well-defined) Gaussian measure for path integrals

$$1 = \int D(\delta A_\mu) e^{-\pi(\delta A, \delta A)}; 1 = \int D\omega e^{-\pi(\omega,\omega)} \tag{2.6.6}$$

Expanding δA_μ into a complete orthornormal set of eigenfunctions of the Laplacian $PP^\dagger$

$$\delta A_\mu(x) = \sum_{\lambda \geq 0} c_\lambda \phi_\mu^\lambda(x), PP^\dagger \phi_\mu^\lambda(x) = \lambda \phi_\mu^\lambda(x) \tag{2.6.7}$$

we see from $(\delta A, \delta A) = \sum_{\lambda\geq 0} c_\lambda^2$ that the measure $D\delta A_\mu$ is (up to a constant) equal to

$$D\delta A_\mu = \prod_{\lambda \geq 0} dc_\lambda \tag{2.6.8}$$

In a similar way we can expand $\omega(x) = \sum_{\lambda\geq 0} e_\lambda \chi^\lambda(x)$ and find then that (up to another constant) the measure $D\omega$ is equal to

$$D\omega = \prod_{\lambda\geq 0} de_\lambda \tag{2.6.9}$$

Of course, the path integral in (2.6.6) consists of infinitely many integrals over the coefficients c_λ and e_λ, and for the inner products to be well-defined we assume that $\sum c_\lambda^2$ and $\sum e_\lambda^2$ are finite.

The measure for the path integral over δA which is used for gauge theories is singular since it contains the usual infinite gauge factor. Dividing out the latter we must determine the explicit form of the remaining measure $D\delta A/D\omega$. The space δA splits into the kernel of $P^\dagger$ and the orthogonal complement $(\ker P^\dagger)_\perp =$ range P. Similarly, ω space consists of the kernel of P and the orthogonal complement $(\ker P)_\perp =$ range $P^\dagger$.

$$\begin{array}{ccc} \omega & P & \delta A \\ \begin{array}{|c|}\hline \text{range } P^\dagger \\ \hline \ker\ P \\ \hline \end{array} & \begin{array}{c} \xrightarrow{\hspace{6em}} \\ \xleftarrow{\hspace{6em}} \\ P^\dagger \end{array} & \begin{array}{|c|}\hline \text{range } P \\ \hline \ker\ P^\dagger \\ \\ \hline \end{array} \end{array}$$

Hence

$$D\delta A/D\omega = \frac{D(\ker P^\dagger)_\perp}{D(\ker P)_\perp}\,\frac{D(\ker P^\dagger)}{D(\ker P)} = \frac{D(\text{range}P)}{D(\text{range}P^\dagger)}\frac{D(\ker P^\dagger)}{D(\ker P)} \tag{2.6.10}$$

Any δA in the range of P can be written as (suppressing the index μ)

$$\delta A(\text{range } P) = \sum_{\lambda>0} c_\lambda \phi^\lambda = P\left(\sum_{\lambda>0} e_\lambda \chi^\lambda\right) = \sum_{\lambda>0} e_\lambda P\chi^\lambda \tag{2.6.11}$$

As we have seen before, $P\chi^\lambda$ is proportional to ϕ^λ, so χ^λ and ϕ^λ have the same eigenvalue under $P^\dagger P$ or $PP^\dagger$, but since

$$\parallel P\chi^\lambda \parallel^2 = \left(\chi^\lambda, P^\dagger P\chi^\lambda\right) = \lambda \parallel \chi_\lambda \parallel^2 = \lambda \tag{2.6.12}$$

it follows that $P\chi^\lambda = \sqrt{\lambda}\phi^\lambda$. (By choosing the phases of ϕ^λ suitably, there are no minus signs in this relation). Hence $\delta A(\text{range}P) = \sum_{\lambda>0} e_\lambda\sqrt{\lambda}\phi^\lambda$, and $c_\lambda = \sqrt{\lambda}e_\lambda$. This determines the first ratio in the measure in (2.6.10)

$$\frac{D(\text{range } P)}{D(\text{range } P^\dagger)} = \frac{\prod dc_{\lambda>0}}{\prod de_{\lambda>0}} = \prod_{\lambda>0}\sqrt{\lambda} = \sqrt{\det(P^\dagger P)'} \tag{2.6.13}$$

The prime indicates that one should only take the determinant of $P^\dagger P$ in the space orthogonal to the zero modes (orthogonal to $\ker P$, so in the range of $P^\dagger$).

Next we must determine the ratio of $D(\ker P^\dagger)$ and $D(\ker P)$. The kernel of P consists of all those ω that satisfy $D_\mu(A)\omega = 0$ at all x for given (fixed) reference configuration A. These ω satisfy $[F_{\mu\nu}, \omega] = 0$, and also $[[F_{\mu\nu}, F_{\rho\sigma}], \omega] = 0$, etc. If at a fixed point x the $F_{\mu\nu}(x)$ for all μ, ν span the whole Lie algebra, then ω vanishes according to Schur's lemma at that point x. But since $D_\mu(A)\omega = 0$ at all x, it follows that then $\omega(x) = 0$ at all points x. This is the generic case. The set of reference A's for which $\ker P$ is nonvanishing has measure zero in the space of all configurations, and for each such reference A the measure $D(\ker P)$ is finite-dimensional. Therefore, in the field theory case which we are considering, we set $D(\ker P) = 1$. (This is not possible in string theory, see the comment at the end of this section).

Next we consider the measure $D(\ker P^\dagger)$. The space $\ker P^\dagger$ is the linear vector space of fluctuations $\delta A^\perp$ with $D^\mu(A)\delta A^\perp_\mu = 0$. These $\delta A^\perp$ are orthogonal to the gauge orbits because $(\omega, D^\mu \delta A^\perp_\mu) = 0 = (\delta A^\perp_\mu, D^\mu \omega)$ where $D^\mu\omega$ is a general element in the gauge orbit. Let a general δA_μ be expanded into eigenfunctions of $PP^\dagger$

$$\delta A_\mu = \sum_{\lambda=0} a_m \phi^m_\mu + \sum_{\lambda>0} b_\lambda \phi^\lambda_\mu \tag{2.6.14}$$

where the ϕ^m_μ form a basis in $\ker P^\dagger$. Then

$$D(\ker P^\dagger) = \prod_m da_m \tag{2.6.15}$$

We thus obtain for the total path integral measure in the string-like quantization procedure

$$D\delta A_\mu / D\omega = [\det(P^\dagger P)']^{1/2} \prod_m da_m \tag{2.6.16}$$

The path integral itself is then given by a

$$\int [\det(P^\dagger P)']^{1/2} \prod_m da_m e^{-S[A+\delta A]} \tag{2.6.17}$$

The classical action $S[A + \delta A]$ is gauge invariant, so we may drop the pure gauge parts of δA in $S[A + \delta A]$ (the δA which lie in the range of P) and are then left with $S[A + \sum a_m \phi^m]$. Hence only an integration da_m over the orthogonal variations (orthogonal to the gauge orbits) is left.[26] This path integral is not of the Faddeev-Popov form because $\left[\det(P^\dagger P)'\right]^{\frac{1}{2}}$ is not equal to the Faddeev-Popov determinant, while there is also no gauge fixing term in the action.

Let us now establish the connection with Faddeev-Popov quantization. We begin with the δA satisfying $f(A+\delta A)=0$. Hence

$$\int \partial f(A(y))/\partial A_\mu(x)\delta A_\mu(x)d^4x = 0 \tag{2.6.18}$$

(Usually one extracts a factor $\delta(y-x)$ from $\partial f(A(y))/\partial A_\mu(x)$ but it is useful to think of $\partial f(A)/\partial A_\mu$ as a matrix in what follows). We assume that this condition precisely picks out one point on each gauge orbit, so we exclude those A's for which Gribov copies exist ($f(A+D\omega)=0$). We want to find the relation between the integration $\prod da_m$ which was left in the string-like path integral, and the Faddeev-Popov integration. So we introduce a Jacobian $\tilde{J}(A)$ which accounts for the difference

$$D(\ker P^\dagger) = \prod_m da_m = D(\delta A_\mu)\prod_x \delta\left[\int \partial f(A(x))/\partial A(y)\delta A(y)dy\right]\tilde{J}(A) \tag{2.6.19}$$

Since $D\delta A_\mu = \prod_m da_m \prod_{\lambda>0} db_\lambda$, see (2.6.14), we find after dividing by $\prod_m da_m$

$$\begin{aligned} 1 &= \prod_{\lambda>0} db_\lambda \tilde{J}(A)\prod_x \delta\left[\int \partial f(A(x))/\partial A(y)\left(\sum_m a_m\phi^m + \sum_{\lambda>0} b_\lambda \phi^\lambda\right)(y)dy\right] \\ &= \prod_{\lambda>0} db_\lambda \tilde{J}(A)\prod_x \delta\left[\sum_m < x|\partial f/\partial A|m > a_m + \sum_{\lambda>0} < x|\partial f/\partial A|\lambda > b_{\ \lambda}\right] \\ &= \tilde{J}(A)|\det < x|\partial f/\partial A|\lambda > |^{-1} \end{aligned} \tag{2.6.20}$$

The values of b_λ for which the delta function vanishes, and which we must substitute wherever b_λ appears after integrating over b_λ, are given by

$$b_\lambda^{(0)} = - < \lambda|(\partial f/\partial A)^{-1}|x >< x|\partial f/\partial A|m > a_m \tag{2.6.21}$$

The Jacobian $\tilde{J}(A)$ is thus given by

$$\tilde{J}(A) = |\det < x|\partial f/\partial A|\lambda > | \tag{2.6.22}$$

[26]In string theory one replaces at this point $\prod da_m$ by the Weyl-Peterson measure.

which we can also write as

$$\tilde{J}(A) = \det{}^{1/2}\left[\sum_{\lambda>0} < x|\partial f/\partial A|\lambda >< \lambda|\partial f/\partial A|y >\right] \tag{2.6.23}$$

assuming that $\partial f/\partial A$ is real (hermitian). Next we write the sum $\sum\limits_{\lambda>0} |\lambda >< \lambda|$ over the nonzero modes as a projection operator onto the space orthogonal to $\ker P^\dagger$

$$\sum_{\lambda>0} |\lambda >< \lambda| = P\frac{1}{P^\dagger P}P^\dagger \tag{2.6.24}$$

Since $P^\dagger$ removes the zero modes of $P^\dagger$, the operator $(P^\dagger P)^{-1} = P^{-1}P^{\dagger -1}$ never becomes singular. Thus

$$\begin{aligned} \tilde{J}(A) &= \det{}^{1/2} < x|\partial f/\partial A \;\; P\frac{1}{P^\dagger P}P^\dagger \;\; \partial f/\partial A|y > \\ &= |\det \frac{\partial f}{\partial A_\mu}D_\mu(A)|\frac{1}{[\det(P^\dagger P)']^{1/2}} \end{aligned} \tag{2.6.25}$$

Substituting these results into the string-like path integral with the measure in (2.6.10) we find that the factors with $[\det(P^\dagger P)']^{1/2}$ cancel, and we are left with

$$\begin{aligned} &\int D\delta A/D\omega e^{-S[A+\delta A]} = \int [\det(P^\dagger P)']^{1/2}\prod_m da_m e^{-S[A+\delta A]} \\ &= \int D\delta A\prod_x \delta[f(A+\delta A)]\,|\det\frac{\partial f}{\partial A_\mu}D_\mu(A)| \end{aligned} \tag{2.6.26}$$

Since the eigenvalues of $(\partial f/\partial A_\mu)D_\mu(A)$ are positive inside the Gribov horizon, we may take away the absolute value signs, and we obtain then the usual Faddeev-Popov determinant $\Delta_F = \det(\partial f/\partial A_\mu)D_\mu(A)$.

Thus string-like quantization is equivalent to Faddeev-Popov quantization, but we had to argue that one may neglect the kernel of P, which is the space of gauge parameters ω satisfying $D_\mu(A)\omega = 0$. For closed strings, $\ker P = 0$ defines the conformal Killing vectors which exist only for the sphere and the torus but one should be careful to include them in the measure when one performs the discrete sum over surfaces of all genera.

7 Classical and quantum Yang-Mills theory from the Noether method

The Noether method has played a major role in the construction of new gauge theories, such as supergravity, W gravity and string theory, and also Yang-Mills theory and Einstein gravity can be derived by this method in a few steps. For the interested reader we give a derivation of Yang-Mills theory from scratch, using this method. For a detailed discussion of the Noether method in supergravity, see [33].

This method is called "Noether method" because for gauge symmetries the first nonlinear terms in the action (trilinear in fields) are a product of a gauge field times the Noether current which follows from the rigid part of the local gauge symmetry. The higher order terms are in general not proportional to the Noether current, but are nonetheless constructed in an unambiguous way.

To derive classical Yang-Mills theory one must begin by specifying the free action and its symmetries; this is the input. The rest is output. For Yang-Mills theory, the free action is a set of Maxwell actions

$$\mathcal{L}_{cl}^{(0)} = -\frac{1}{4}(\partial_\mu A_\nu{}^a - \partial_\nu A_\mu{}^a)^2 \tag{2.7.1}$$

(In the next section we show that even this input follows from unitarity). It has clearly the local gauge invariance

$$\delta_g^{\rm lin} A_\mu{}^a = \partial_\mu \lambda^a(x) \tag{2.7.2}$$

However, it also has a rigid symmetry under homogeneous adjoint transformations

$$\delta_r A_\mu{}^a = g f^a{}_{bc} A_\mu{}^b \eta^c \ , \ \eta^c \text{ constant} \tag{2.7.3}$$

We have introduced a coupling constant g in (2.7.3) because we anticipate that interactions will appear shortly. (One could have written (2.7.3) with a parameter η' without g, but then one would find at a later stage that $\eta' = \eta g$.) The transformation rules in (2.7.3) are a symmetry of (2.7.1) if the Yang-Mills curvatures have been contracted with a metric proportional to the Killing metric $g_{ab} = f_{ap}{}^q f_{bq}{}^p$ because then $g_{ad} f^d{}_{bc} \equiv f_{abc}$ is totally antisymmetric. We use a normalization of the generators of the Lie algebra such that $g_{ab} = \delta_{ab}$. The equations (2.7.1), (2.7.2), and (2.7.3) constitute the input for the Noether method.

We now try to make the parameter η^c local. Then the action in (2.7.1) is no longer invariant under (2.7.3), but it transforms into $\partial_\mu \eta^c$ times the Noether current. This Noether current should be conserved if one uses the equations of motion. We find for the variation of (2.7.1) under (2.7.3)

$$\delta S_{cl}^{(0)} = \int \left[-(F_{(0)}^{\mu\nu,a}) g f^a{}_{bc} A_\nu{}^b \partial_\mu \eta^c(x) \right] d^4x \tag{2.7.4}$$

and hence the Noether current is in this case

$$j_c^\mu = -F_{(0)}^{\mu\nu,a} g f^a{}_{bc} A_\nu{}^b \tag{2.7.5}$$

where $F_{\mu\nu}^{(0)a} = \partial_\mu A_\nu{}^a - \partial_\nu A_\mu{}^a$. The conservation of $j^\mu{}_c$ follows from the field equation $\partial_\mu F_{(0)}{}^{\mu\nu} = 0$ and the antisymmetry of the structure constants.

The Noether method tries now to add a term either to $\mathcal{L}_{cl}^{(0)}$ or to $\delta_r A_\mu{}^a$ or to both, such that the contributions from these extra terms produce a second variation which cancels (2.7.4). It is clear from (2.7.4) that the following extra term in the action achieves this

$$\mathcal{L}_{cl}{}^{(1)}(\text{extra}) = -\frac{1}{2} F_{(0)}^{\mu\nu,a} g f^a{}_{bc} A_\mu{}^b A_\nu{}^c \tag{2.7.6}$$

provided one also adds an extra term to the transformation law in (2.7.3)

$$\delta_r(\text{extra}) A_\mu{}^a = \partial_\mu \eta^a(x) \tag{2.7.7}$$

Since both fields $A_\mu{}^b$ and $A_\nu{}^c$ appear symmetrically in (2.7.6), one only needs to vary one of them and multiply the result by a factor 2, and using (2.7.7) for this variation, one produces minus the variation in (2.7.4). Thus the action is now invariant up to order g.

Note that the laws in (2.7.7) and in (2.7.2) are the same. We thus see that requiring invariance under local η transformations fuses the field-independent local symmetry and the field-dependent rigid symmetry into one local nonlinear symmetry

$$\delta A_\mu{}^a = \partial_\mu \eta^a(x) + g f^a{}_{bc} A_\mu{}^b \eta^c(x) \tag{2.7.8}$$

We are not yet done, however, because after having added $\mathcal{L}_{cl}^{(1)}$(extra) in (2.7.6) to the action, we should also subject this term to the variations in (2.7.3). (The variation in (2.7.7) was already taken into account).

It may clarify to count powers of g. The extra term in (2.7.6) is of order g, and the variation in (2.7.3) is also of order g. Hence we obtain a variation of order g^2. In general we can try to cancel such an order g^2 variation by a combination of two modifications:

(1) by adding a term of order g^2 to the transformation laws such that this new variation acting on $\mathcal{L}^{(0)}$ produces new order g^2 variations or

(2) by adding a new term to the action of order g^2, such that the g-independent variation (2.7.7) produces a new order g^2 variation. In the former case one finds terms proportional to the $\mathcal{L}^{(0)}$ field equations (the Maxwell equations). Hence we require

$$\delta(\text{order } g^2)S(\text{order } g^0) + \delta(\text{order } g^0)S(\text{order } g^2) + \delta(\text{order } g)S(\text{order } g) = 0 \tag{2.7.9}$$

The last term in (2.7.9) is the variation we discussed below (2.7.8). If the parameter in (2.7.3) were constant, this variation would vanish since all indices a, b, c etc. are contracted in a group invariant way. Hence one only gets a contributions from the variation of the factor $F_{\mu\nu(0)a} = \partial_\mu A_\nu{}^a - \partial_\nu A_\mu{}^a$ in (2.7.6) under (2.7.3) if the derivatives in $F_{\mu\nu(0)}$ hit $\eta^c(x)$. We find

$$\delta(\text{order } g)S(\text{order } g) = -\int g f^a{}_{pq} A^{\nu,p}(\partial^\mu \eta^q)(g f^a{}_{bc} A_\mu{}^b A_\nu{}^c) d^4x \tag{2.7.10}$$

Since this result does not have enough derivatives to contain the Maxwell field equation, it cannot be canceled by the first term in (2.7.9). That leaves the second term.

From (2.7.10) we can determine what $S(\text{order } g^2)$ is: by replacing $\partial_\mu \eta^a$ by $-A_\mu{}^a$ in (2.7.10), we find that the new term is

$$\mathcal{L}_{cl}{}^{(2)}(\text{extra}) = -\frac{1}{4}(gf^a{}_{pq}A_\mu{}^p A_\nu{}^q)(gf^a{}_{bc}A_\mu{}^b A_\nu{}^c) \tag{2.7.11}$$

The factor $\frac{1}{4}$ is needed because there are four gauge fields which appear symmetrically and hence to produce the result in (2.7.10) with an overall factor unity, we need a factor $\frac{1}{4}$ in (2.7.11).[27]

At this point we have found the following action which is invariant through order g^2

$$\begin{aligned} \mathcal{L} \;\; = \;\; & -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 - \frac{1}{2}(\partial_\mu A_\nu{}^c - \partial_\nu A_\mu{}^a)(gf^a{}_{bc}A_\mu{}^b A_\nu{}^c) \\ & -\frac{1}{4}(gf^a{}_{bc}A_\mu{}^b A_\nu{}^c)^2 \end{aligned} \tag{2.7.12}$$

This expression factorizes into the product of two Yang-Mills curvatures

$$\mathcal{L} = -\frac{1}{4}(F_{\mu\nu}{}^a)^2, F_{\mu\nu}{}^a = \partial_\mu A_\nu{}^a - \partial_\nu A_\mu{}^a + gf^a{}_{bc}A_\mu{}^b A_\nu{}^c \tag{2.7.13}$$

The transformation rules obtained so far read

$$\delta A_\mu{}^a = \partial_\mu \eta^a + gf^a{}_{bc}A_\mu{}^b \eta^c \equiv (D_\mu \eta)^a \tag{2.7.14}$$

It is too early to conclude that we have reproduced standard Yang-Mills theory because there are still the variations of order g^3 to consider. They only come from varying (2.7.11) under (2.7.3) (the order g^2 terms in the action under the order g transformation laws). However, since in (2.7.11) no derivatives appear, it makes in the calculation no difference whether we take constant or local η's, and since in (2.7.11) all indices are contracted, it is invariant under the "rotation" in (2.7.3). Hence, we have indeed reproduced classical Yang-Mills theory from the Noether method.

A few general features of the Noether method become clear from this simple calculation. At each step in the procedure one loses one derivative, and if the original

[27] This is really an integrability condition. In supergravity models one finds even fermionic integrability conditions.

action has only a finite number of derivatives, one is guaranteed that only a finite number of steps is needed to arrive at the final result. However, not always success is guaranteed: there exist cases where the Noether method proceeds a few steps and then stops because no solution at that point can be constructed. One needs to satisfy bosonic (or fermionic!) integrability conditions, and sometimes there exists no solution for them. (An example is the cosmological constant in 11-dimensional supergravity). Even in our derivation of Yang-Mills theory we found such an integrability condition: to solve (2.7.4) we needed a solution which was symmetric in both gauge fields. Fortunately, such a solution did exist, see (2.7.6).

A simplification of this derivation of Yang-Mills theory can be obtained by using a first-order formalism in which ${F_{\mu\nu}}^a$ and ${A_\mu}^a$ are both independent fields. We look for a free field action which is linear in derivatives and equivalent to the linear approximation of the full action. This leads to

$$\mathcal{L}^{(0)} = -\frac{1}{2}{F^{\mu\nu}}_a(\partial_\mu {A_\nu}^a - \partial_\nu {A_\mu}^a) + \frac{1}{4}{F^{\mu\nu}}_a {F_{\mu\nu}}^a \tag{2.7.15}$$

Indeed, solving ${F_{\mu\nu}}^a$ from its own algebraic field equation yields ${F_{\mu\nu}}^a = (\partial_\mu {A_\nu}^a - \partial_\nu {A_\mu}^a)$, and substituting this result back into the action one retrieves the Maxwell action. The first-order action has again a local symmetry.

$$\delta_g {A_\mu}^a = \partial_\mu \eta^a(x), \delta_g {F_{\mu\nu}}^a = 0 \tag{2.7.16}$$

and a rigid symmetry

$$\delta_r {A_\mu}^\alpha = g{f^a}_{bc}{A_\mu}^b\lambda^c, \delta_r {F_{\mu\nu}}^a = g{f^a}_{bc}{F_{\mu\nu}}^b\lambda^c \tag{2.7.17}$$

Letting λ^c become spacetime dependent, the variation of $\mathcal{L}^{(0)}$ becomes

$$\delta\mathcal{L}^{(0)} = -{F^{\mu\nu}}_a(\partial_\mu\lambda^c)g{f^a}_{bc}{A_\nu}^b \tag{2.7.18}$$

To cancel this variation we identify $\eta^a(x) = \lambda^a(x)$. We can then cancel this variation by adding a new term to the action

$$\mathcal{L}^{(1)} = -\frac{1}{2}{F^{\mu\nu}}_a g{f^a}_{bc}{A_\mu}^b{A_\nu}^c \tag{2.7.19}$$

because $\delta A_\mu{}^a = \partial_\mu \lambda^a$ applied to $\mathcal{L}^{(1)}$ cancels (2.7.18). The factor $\frac{1}{2}$ is again needed since both A_μ fields yield the same variation. No further variations need be studied since $\mathcal{L}^{(1)}$ contains no derivatives and all indices are contracted, so that it is invariant under (2.7.17) whether λ is constant or local. The advantage of using a first-order formalism is that one already gets rid of one derivative in the action by using an independent field $F_{\mu\nu}{}^a$; as a result one needs one step less than in a second-order approach. The final action, $\mathcal{L}^{(0)} + \mathcal{L}^{(1)}$, is the Yang-Mills action in first-order form

$$\mathcal{L}^{(0)} + \mathcal{L}^{(1)} = \frac{1}{4} F^{\mu\nu}_a F_{\mu\nu}{}^a - \frac{1}{2} F^{\mu\nu}{}_a (\partial_\mu A_\nu{}^a - \partial_\nu A_\mu{}^a + g f^a{}_{bc} A_\mu{}^b A_\nu{}^c) \tag{2.7.20}$$

The transformation rules are

$$\delta A_\mu{}^a = D_\mu \lambda^a, \delta F_{\mu\nu}{}^a = g f^a{}_{bc} F_{\mu\nu}{}^b \lambda^c \tag{2.7.21}$$

and we have recovered Yang-Mills theory in first-order form.

One can also apply the Noether method to the S matrix to deduce which ghosts one needs in Yang-Mills theory [21]. This is similar to Feynman's original approach, but in this case one only needs tree graphs and no loops. One can also apply the Noether method to gravity. The free field action for massless spin 2 fields (which itself is the linearized limit of the nonlinear Hilbert-Einstein action) is input [23], and output is the Hilbert-Einstein action. Again first-order formalism simplifies the analysis [24].

8 Gauge invariance from tree unitarity

In the derivation of Yang-Mills theory given in the previous section, we needed the free part of the gauge action as input. Even this input can be derived from a more fundamental property, namely unitarity, as we now show. So, gauge invariance follows from unitarity.

To determine the free field part of a gauge theory, one may add the coupling to an external source and require that the residues of the propagator at its poles are positive definite. This is necessary (but not sufficient) for unitarity; it is sometimes called tree unitarity. For Yang-Mills theory and Maxwell theory, the most general Lorentz-invariant free field action is

$$\mathcal{L} = -\frac{1}{2}\left[\alpha(\partial_\mu A_\nu)^2 + \beta(\partial^\mu A_\mu)^2 - \gamma m^2(A^\mu A_\mu)\right] - J^\mu A_\mu \tag{2.8.1}$$

where for completeness we consider both massless and massive theories. To facilitate the analysis, we introduce a complete orthonormal set of projection operators

$$\begin{aligned} P^1 + P^0 &= I, (P^1)^2 = P^1, (P^0)^2 = P^0, P^1P^0 = P^0P^1 = 0 \\ (P^1)_\mu{}^\nu &= \delta_\mu{}^\nu - \partial_\mu\partial^\nu/\Box; (P^0)_\mu{}^\nu = \partial_\mu\partial^\nu/\Box; \Box = \partial^\lambda\partial_\lambda \end{aligned} \tag{2.8.2}$$

Then the field equation is given by $\mathcal{O}A = J$ where the kinetic matrix $\mathcal{O}$ is given by

$$\mathcal{O} = [\alpha(P^1 + P^0) + \beta P^0]\Box + \gamma(P^1 + P^0)m^2 \tag{2.8.3}$$

The propagator $\prod$ is the inverse of the field operator (if this inverse exists), $\mathcal{O}\prod = I$, and writing $\prod$ as $\prod = aP^1 + bP^0$, one finds a set of linear relations for a and b

$$\begin{aligned} &\left\{\alpha\Box + \gamma m^2\right\} aP^1 + \left\{(\alpha+\beta)\Box + \gamma m^2\right\} bP^0 = I \\ &a = (\alpha\Box + \gamma m^2)^{-1}, b = [(\alpha+\beta)\Box + \gamma m^2]^{-1} \end{aligned} \tag{2.8.4}$$

We first consider general values of α, β and γ; the special cases (which are the interesting cases) will be discussed later. Completing squares in the action we find

$$\frac{1}{2}A\mathcal{O}A - JA = \frac{1}{2}(A - J\mathcal{O}^{-1})\mathcal{O}(A - \mathcal{O}^{-1}J) - \frac{1}{2}J\mathcal{O}^{-1}J \tag{2.8.5}$$

where

$$-\frac{1}{2}J\mathcal{O}^{-1}J = -\frac{1}{2}J\Pi J = \frac{1}{2}J\left(\frac{1}{\alpha k^2 - \gamma m^2}P^1 + \frac{1}{(\alpha+\beta)k^2 - \gamma m^2}P^0\right)J \tag{2.8.6}$$

If $\alpha > 0$, we must require that $\gamma < 0$ to exclude a tachion. (In our conventions, $k^2 = \vec{k}^2 - k_0^2$). Then the residue JP^1J is indeed positive since $P^1{}_\mu{}^\nu$ projects onto the three orthonormal polarizations of a massive spin 1 particle

$$P^1{}_\mu{}^\nu = \sum_{m=1}^{3} \epsilon_\mu{}^m \epsilon_m{}^\nu ; \epsilon^m{}_\mu k^\mu = 0 \tag{2.8.7}$$

The other pole must then have $\alpha + \beta > 0$ to exclude a tachyon, but now JP^0J is negative, since $P^0{}_\mu{}^\nu$ projects on the timelike polarization vector proportional to k_μ. (Recall that $J(P^1 + P^0)J = J^\mu J_\mu = \sum_{m=1}^{3} |\epsilon \cdot J|^2 - |k \cdot J|^2/m^2$ and this minus sign makes JP_0J negative definite). Hence, for generic values of α, β, γ no free field action exists without tachyons and ghosts.

We now study the special cases. The most interesting case is $\alpha + \beta = \gamma = 0$. Then $\mathcal{O} = \alpha P^1 \Box$, and $\mathcal{O}A = J$ implies that the source J must satisfy the constraint $\Pi^0 J = 0$, i.e., J is conserved. The propagator $\Pi = \mathcal{O}^{-1}$ does not exist because the operator P^1 is singular (it has eigenvectors with vanishing eigenvalue, namely the vectors k_μ). The usual approach to deal with this situation is to add a gauge fixing term to the action, but if one is only interested in $J\Pi^{-1}J$, one does not need to add a gauge fixing term as we now explain. One can write $\Pi = aP^1 + bP^0$, and the ambiguity in the value of b cancels in $J\Pi J$. One can understand the emergence of this constraint on J as a consequence of the presence of a gauge invariance as follows. Since $\mathcal{O}A = 0$ when $A = P^0\lambda$, the field equation and the action with $J = 0$ have a local gauge invariance $\delta A = P^0\lambda$. Local gauge invariance means in general that certain field components are not present in the action, and this is quite clear in the present case: P^0A is not present. The local gauge invariance leads then to a constraint in J which eliminates the ambiguities in the propagator.

We can now again study the residue of the propagator. Since

$$-J\Pi J = J \left(\frac{1}{\alpha k^2} \right) P^1 J \tag{2.8.8}$$

we see that for $\alpha > 0$ the residue is positive[28]

$$JP^1J = \sum_{m=1}^{3} (\epsilon^m{}_\mu J^\mu)^2 \geq 0 \tag{2.8.9}$$

The result is precisely Maxwell theory. Note that unitarity has led to gauge invariance: gauge invariance is derived, not imposed on esthetic grounds.

The reader may check that for $\gamma \neq 0$ one finds a massive Maxwell theory (Proca theory) which is free from ghosts and tachyons provided $\gamma < 0$.

$$\mathcal{L}(\text{Proca}) = -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 - \frac{1}{2}m^2 A_\mu{}^2 \tag{2.8.10}$$

Another special solution is the case $\alpha = 0, \beta \neq 0$, i.e., the theory with

$$\mathcal{L} = \frac{1}{2}(\partial \cdot A)^2 + \frac{m^2}{2} A_\mu{}^2 + J^\mu A_\mu \tag{2.8.11}$$

The field equation reads $\partial_\mu \partial \cdot A - m^2 A_\mu = J_\mu$, i.e. $\{(\Box - m^2)P^0 - m^2 P^1\}A = J$. One finds then

$$-J\Pi J = -J\left(\frac{1}{m^2}P^1 + \frac{1}{k^2 + m^2}P^0\right)J \tag{2.8.12}$$

This theory has no ghosts or tachyons, but it propagates only scalar fields and is equivalent to a scalar field theory, as one may note by putting $A_\mu = \partial_\mu \sqrt{\frac{1}{\Box}}\varphi$.

The approach to derive gauge invariance from unitarity can also be applied to gravitation [22] and one obtains then the linearized Einstein equations (the Fierz-Pauli action [23].) Applying the Noether method to derive the interactions, one can obtain the full Einstein action as discussed in the previous section. Hence, in the case of gravitation unitarity yields the same end product as derived by Einstein from geometry.

[28] Of course, there are two physical polarizations in Maxwell theory, not three. For massless fields one may use the decomposition $\eta_{\mu\nu} = \sum_{i=1}^{2} \epsilon_\mu \epsilon_\nu + (k_\mu \bar{k}_\nu + k_\nu \bar{k}_\mu)/k \cdot \bar{k}$ where $k^\mu \epsilon_\mu = \bar{k}^\mu \epsilon_\mu = 0$, and if $k^\mu = (\vec{k}, k^0)$ then $\bar{k}^\mu \equiv (\vec{k}, -k_0)$. Using that the current is conserved, $k \cdot J = 0$, it is clear that the residue contains only two squares instead of three. The reason is that **two** terms in $J\Pi J$ now cancel, namely $(\bar{k} \cdot J)(k \cdot J)$ and $(k \cdot J)(\bar{k} \cdot J)$.

9 Historical and other comments

1. A few historical comments. Feynman noted in 1963 that both in gravity and in Yang-Mills theory unitarity seems violated [13]: summing squares of tree graphs over intermediary gauge fields with only physical polarizations did not reproduce the imaginary part of the forward scattering amplitude if one used propagators with $\eta_{\mu\nu}$. (As we now know, the Faddeev-Popov ghosts were lacking). For Yang-Mills theory he added an explicit mass term $-\frac{1}{2}m^2 A_\mu^2$ (to avoid a divergence in the path integral due to gauge invariance; at the end this mass was understood to be sent to zero). In addition he added another term $-\frac{1}{2}(\partial^\mu B_\mu - D^\mu D_\mu \omega + m^2\omega)^2$ where ω lies in the Lie algebra and he integrated over the group. By a suitable shift of ω the term $\partial^\mu B_\mu$ can be transformed away so one is still dealing with the original massive gauge theory. He used a background field formalism with $A_\mu = A_\mu$ (back) $+B_\mu$ and worked to second order in B_μ (the one-loop approximation). He found in this way Yang-Mills theory for B_μ with mass term **and** with a term $-\frac{1}{2}(\partial^\mu B_\mu)^2$, and further one real scalar field with action $-\frac{1}{2}\partial^\mu\omega D_\mu\omega - \frac{1}{2}m^2\omega^2$.

In 1972 he came back to this problem [13], and showed that at the one-loop level adding a mass term $-\frac{1}{2}m^2 B_\mu^2$ to YM theory[29] is equivalent to (i) adding both a mass term $-\frac{1}{2}m^2 B_\mu^2$ and a Lorentz gauge fixing term $-\frac{1}{2}(\partial^\mu B_\mu)^2$ (yielding propagators with $\eta_{\mu\nu}$) and (ii) **subtracting** a loop due to a real massive scalar field with action $-\frac{1}{2}\partial^\mu\omega D_\mu\omega - \frac{1}{2}m^2\omega^2$. Subtracting a real scalar loop is at the one-loop level the same as adding a loop of anticommuting ghosts and a loop of a real commuting scalar: $-1 = -2 + 1$. The latter is the correct approach at higher loops, as we have shown in section 5. Feynman did not note that the limit $m \to 0$ of the massive theory did not agree with the massless theory at 1-loop for Yang-Mills theory; this was discovered

[29] Feynman also discussed gravity with a linear (!) mass term $m^2 g_{\sigma\sigma}$, and showed that it was equal to gravity with this mass term and with a de Donder gauge fixing term $g^{\alpha\beta}\bar{H}_{\alpha\nu,\nu}\bar{H}_{\beta\sigma,\sigma}$ with $\bar{H}_{\mu\nu} = H_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}H_{\sigma\sigma}$ divided (in the path integral sense) by the action of a real massive vector field with $\mathcal{L} = -\partial^\nu\eta^\sigma\overline{D_\nu\eta_\sigma} - M^2\eta^\sigma\eta_\sigma$. The propagator of the graviton again involved only $\eta_{\mu\nu}$.

by van Dam and Veltman, and Zacharov already in 1970 [18]. Thereafter, all efforts were concentrated on the massless theory.

At the end of the 1960's, stimulated by Feynman's results of 1963, DeWitt, Mandelstam, Fradkin and Tyutin constructed path integrals for massless nonabelian gauge theories, and derived Feynman rules for these theories [25]. These constructions were formal in the sense that possible infrared divergences were ignored, but their work went beyond loop approximations and was crucial in the further developments of the theory, in particular it opened the door to studies of the renormalizability of nonabelian gauge theories. 't Hooft [26] first derived some Ward identities in Yang-Mills theory for Green's functions involving $\partial^\mu A_\mu{}^a$, using diagrammatic techniques developed for massive Yang-Mills fields by Veltman. An excellent review is [27]. He also gave a path integral derivation of the Ward identities for these Green functions. Shortly afterwards Slavnov derived Ward identities for more general Green functions using a path integral approach for connected graphs in which the Faddeev-Popov determinant $\Delta(A_\mu{}^a)$ was part of the measure [28]. He made a nonlocal gauge transformation with parameter $[\partial^\mu D_\mu]^{-1}\eta$ where η is an arbitrary local function, and showed that the product $\Delta(A_\mu{}^a) dA_\mu{}^a$ was invariant. This nonlinear transformation already appeared in an early article by Fradkin and Tyutin [29] to whom he refers. (More precisely, the article appears in the list of references, but in the text no reference is made to it). A more expanded version of this formalism to derive Ward identities was given in a series of papers in 1973 by B.W. Lee and J. Zinn- Justin [30].

In order to recast this complicated nonlinear transformation into something simpler, Itzykson and Stora studied Slavnov's paper and found some formulas which are now part of the BRST equations. Stora and his postdoc Rouet then wrote some lecture notes for a course given at Lausanne. Becchi, upon visiting Stora and Rouet in Marseille, recognized these equations as symmetries of the quantum action in the spring of 1974.

Faddeev and Popov had already observed that one can expand their determinant $\Delta(A_\mu{}^a)$ diagrammatically by introducing fictitious scalars [31] and Slavnov followed up on this idea and even explicitly wrote down the ghost action as part of the action in the path integral [28]. Finally, BRS noted in the spring of 1974 that Slavnov's Ward identities could be derived from a symmetry principle of the quantum action (so with ghost action). Soon afterwards, Zinn-Justin applied them to the problem of renormalization of gauge theories in lectures which he gave in the summer of 1974 at Bonn [32]. In these lectures the $\Gamma\Gamma$ Ward identity which plays a crucial role in the proof of renormalizability of gauge theories (see the next chapter) appears for the first time.

BRS wrote down transformation rules for the Abelian Higgs model (to avoid infrared divergences) which they characterized as "a type of supergauge transformations" (referring to the supersymmetry which Wess and Zumino had introduced in the west), and called them Slavnov transformations [1]. Their paper was finished at the end of the summer of 1974, but appeared only in print in 1975. They also made an analysis of possible BRST anomalies (violations of the Ward identities, sometimes called obstructions) by using cohomology. For a while the name supergauge transformations stuck. Somewhat later, in February 1975, Tyutin wrote an article in which he arrived at conclusions similar to the BRS article [1]. In those days, one needed in the Soviet Union permission to publish articles, and Tyutin's article remained an internal publication. The author was shown this article during a visit at Lebedev Institute in 1975, and upon return to the west showed this to colleagues in the supergravity community (which early on recognized the importance of BRST symmetry). For that reason, the name was changed into BRST symmetry, and many physicists use this name nowadays. We now quote a letter R. Stora wrote upon request about the history of BRST symmetry.

> Alain Rouet was one of my first doctoral students in Marseille in 1970. We obtained a doctoral fellowship from CEA Saclay for him, to be used in

Marseille.

After some exercises we wanted to use BPHZ on gauge theories, but did not have the right Ward identity. When I went on sabbatical to CERN in 1972-1973 I did lecture at Lausanne (spring '73), and Rouet and I produced some notes. We decided to include what we had understood about gauge theories including an exercise done in collaboration with Claude Itzykson in CERN: redo Slavnov's paper (I was unaware of Taylor's paper for quite some time) for the full Green functional, including sources for the ghost and antighost. $\frac{1}{2}c \times c$ came out, as well as an s antighost gauge function. I met Carlo Becchi in CERN at the end of '72, and being both interested in BPHZ, were able to arrange a visit in Marseille for '73-'74. After a while Carlo read these notes, jumped to Alain's office and remarked that the Slavnov identity was linear in the sources, hence it expressed a symmetry. I was busy with some administration. When I arrived, the formulae were on the blackboard. They worked very fast, **did** introduce the sources coupled to the Slavnov's variations, as we all were followers of Schwinger [and] Symanzik.

At the time, the abelian gauge theories were thought to be easier, so we started with the abelian Higgs-Kibble [model], and as it turned out, were protected by God (in this case C conjugation invariance). The 't Hooft gauge had almost all the nonabelian difficulties and furthermore avoided the zero-mass ghosts of the Landau gauge, i.e. IR troubles.

The abelian Higgs [model] was finished in April '74, including gauge independence of the physics (S-matrix and some gauge invariant local operators), following Lowenstein Schröer for massive QED, but excluding unitarity. I gave a seminar in Hamburg (to Haag, Lehmann [and] Symanzik). When I returned the unitarity proof was sitting on the blackboard. Some algebraic steps involving $\frac{1}{n!} \to \frac{1}{(n-1)!}$ suggested the published proof (PRL).

We spent two weeks all three of us in Saclay and reassured ourselves that the Wess Zumino consistency condition coming from $s^2 = 0$ delivered the ABBJ anomaly. We did that for an arbitrary structure group. When the d symbol came out, I was convinced we had something. Then Carlo went back to Geneva and Alain went to MPI Munich as a postdoc. So, it took a long time to write up the long *Ann. Phys.* paper (the abelian Higgs was written before they left).

There is a amusing story with *Ann. Phys.* paper. It was sent to the journal at the end of July '75, and 'lost'. Roman Jackiw and Arthur Jaffe can tell you the story, if they remember.

J. Zinn-Justin understood very fast and produced the Bonn '74 notes. We had been using the connected Green's functional which lead to the discovery of the symmetry. He introduced the 1 PI Γ and wrote down $\Gamma * \Gamma$ instead of BV's $[\Gamma, \Gamma]$.

I lectured in Hercegnovi, and we lectured in Erice '75.

T: I met him in Moskow in '76. He showed me his paper and when I asked him why he did not publish it, he replied: 'Your papers had already come out'.

To conclude, we were lucky and surprised that BPHZ had been firmly and sufficiently developed to be able to do such complicated models as gauge theories! (End of letter by R. Stora)

In the BRST formalism external currents (called K_a^μ and L_a in this book) appeared which multiplied the BRST variations of the gauge field A_μ^a and the ghost c^a. Somewhat later, problems were encountered in the quantization of antisymmetric tensor fields and spin 5/2 fermionic gauge fields. The resolution was that ghost actions themselves were gauge actions, so that ghosts-for-ghosts and antighosts-for-antighosts were needed. In supergravity with a background-gauge-invariant quantum gauge fixing term of the form $\bar{\psi}\cdot\gamma \not{D}(A_\mu)\gamma\cdot\psi$, a third kind of ghost was also needed, and the quantization rules of more general gauge theories entered a phase of complications. At this time Batalin and Vilkovisky came with a very general quantization method which was conceptually very simple, and contained all previous results as special cases [34]. The BRST sources of Zinn-Justin were called antifields, and we shall call this formalism the antifield formalism (it is also sometimes called the BV formalism). We discuss it later in a separate chapter.

2. The most general approach to BRST symmetry for gauge theories with a closed gauge algebra and which are "irreducible" (i.e., do not contain ghosts-for-ghosts) is as follows [35]. The classical fields are denoted by ϕ^I and transform as $\delta\phi^I = R^I{}_\alpha(\phi)\lambda^\alpha$ under classical gauge transformations where the summation over α contains both a summation over internal and spacetime points. (This notation is due to B.S. DeWitt and is very useful when one analyzes problems from a formal point of view.) For example,

$$\delta_{\text{gauge}} A_\mu{}^a(x) = \int \left[(D_\mu)^a{}_b \delta(x-y)\right] \lambda^b(y) dy \tag{2.9.1}$$

and $I = \{a, \mu, x\}$ while $\alpha = \{b, y\}$. The most general set of transformation rules which is allowed by dimension, statistics and which is nilpotent, reads

$$\begin{aligned}
\delta_B \phi^I &= R^I{}_\alpha(\phi) c^\alpha \Lambda \\
\delta_B c^\alpha &= \frac{1}{2} f^\alpha{}_{\beta\gamma} c^\gamma \Lambda c^\beta \\
\delta_B b_a &= d_a \Lambda, \delta_B d_a = 0
\end{aligned} \tag{2.9.2}$$

(The position of Λ between two ghosts in $\delta_B c^a$ ensures that these rules are valid both for anticommuting ghosts as in Yang-Mills theory, and for commuting ghosts as in supergravity and string theory). We restrict ourselves to transformation laws in which $R^I{}_\alpha$ contains only a finite number of derivatives. We contract all indices in the same way (from south-west to north-east), and these results are generally valid (for example also for supergravity with a closed gauge algebra). Note that the index of

b_a and d_a need not be the same as the superscripts of c^α and $f^\alpha{}_{\beta\gamma}$. (In Yang-Mills theory they are the same, but in string theory they are different. For example, the ghost action in bosonic string theory is given by $b_{++}\partial_- c^+ + b_{--}\partial_+ c^-$.). The structure constants follow from closure of the classical gauge algebra

$$\begin{aligned} &[\delta(\lambda), \delta(\eta)]\, \phi^I = \delta(\lambda) R^I{}_\alpha \eta^\alpha - \lambda \leftrightarrow \eta = \\ &R^I{}_{\alpha,J} R^J{}_\beta \lambda^\beta \eta^\alpha - \lambda \leftrightarrow \eta = R^I{}_\gamma f^\gamma{}_{\alpha\beta} \lambda^\beta \eta^\alpha \end{aligned} \tag{2.9.3}$$

where $R^I{}_{\alpha,J} = \partial R^I{}_\alpha / \partial \phi^J$. Whenever the left-hand side can be written as the right-hand side for some $f^\alpha{}_{\beta\gamma}$, we call the gauge algebra closed. This equation **defines** then the structure "constants". (In more complicated gauge theories such as supergravity and certain string models, the structure "constants" depend actually on gauge fields. The results in (2.9.2) remain valid, but the Jacobi identities now also involve derivatives of $f^\alpha{}_{\beta\gamma}$ w.r.t. gauge fields [7]). The quantum action is given by

$$\mathcal{L} = \mathcal{L}(cl) + \delta_B \left[b_a \left(F^a + \frac{1}{2} d_b \gamma^{ba} \right) \right] / \Lambda \tag{2.9.4}$$

Note that γ^{ba} may be field dependent, and the gauge fixing terms F^a may even contain ghosts as in (2.4.8).

If the gauge algebra does not close ("open gauge algebras") one needs four-ghost terms (and sometimes further ghosts). An example is supergravity without auxiliary fields [33]. New ghosts-for-ghosts are needed if the gauge algebra is "reducible" which means that for suitable $M^\alpha{}_A$, the expressions $R^I{}_\alpha \lambda^\alpha$ with $\lambda^\alpha = M^\alpha{}_A \zeta^A$ vanish if one uses the classical field equations [34]. We discuss these issues further in a separate chapter. Note that Yang-Mills theory and ordinary gravity are as simple as possible: they have a closed gauge algebra which is irreducible.

3. BRST symmetry and supersymmetry. Both BRST symmetry and rigid supersymmetry have a constant anticommuting parameter, but in supersymmetry this parameter is ϵ_α with $\alpha = 1, 4$ and is a spinor under Lorenz transformations, whereas the BRST parameter Λ is a Lorentz scalar. (For what follows we need two-component spinors ϵ_α with $\alpha = 1, 2$ and $\epsilon_{\dot{\alpha}}$ with also $\dot{\alpha} =, 2)$. However, there is a connection in Euclidean space [36]. In Euclidean space the Lorentz group is a direct product of two $SU(2)$ group one of which acts on the indices α, and the other on the indices $\dot{\alpha}$. In the so-called $N = 2$ supersymmetric models, the susy parameters consist of the undotted ϵ^i_α and the dotted $\epsilon^i_{\dot{\alpha}}$ with internal index $i = 1, 2$ and spinor indices $\alpha = 1, 2$ and $\dot{\alpha} = 1, 2$. There is a third $SU(2)$ group which acts on the indices i. One can "twist" these theories by identifying the $SU(2)$ transformations which act on the index i with an $SU(2)$ subgroup of the Lorentz group. Then $\epsilon^1_1 + \epsilon^2_2$ becomes a scalar of the new Lorentz group and is actually the BRST parameter of the twisted theory, and the supersymmetry transformations with parameter $\epsilon^1_1 + \epsilon^2_2$ (whose anticommutator in the superalgebra vanishes) become nilpotent BRST transformations. (Of course, the Lorentz properties of the fields after twisting are very different from those before twisting; for example a spinor can become a scalar).

For the interested reader we give some details. The supersymmetry algebra for two left-handed charges $Q^i_\alpha (\alpha = 1, 2$ and $i = 1, 2)$ in Minkowski spacetime can be written as

$$\{Q_\alpha{}^i, Q_\beta{}^j\} = \epsilon^{ij}\epsilon_{\alpha\beta} Z \tag{2.9.5}$$

where Z is a (possibly complex) central charge. The spinor indices α and $\dot{\alpha}$ are indices for the chiral and antichiral parts $\frac{1}{2}(1 \pm \gamma_5)\lambda$ of four-component spinors λ. In a representation with γ_5 block-diagonal with I and $-I$ along the diagonal, it is clear that $\frac{1}{2}(1 \pm \gamma_5)\lambda$ become two-component spinors. The $N = 2$ supersymmetric Yang-Mills theory in Minkowski spacetime contains one gauge field $A_{\alpha\dot{\alpha}} = A_\mu \sigma^\mu{}_{\alpha\dot{\alpha}}$, two Majorana spinor $\lambda^{i\alpha}$ (with conjugates $(\lambda^{i\alpha})^\dagger = \lambda^{\dot{\alpha}}_i$), two real scalars (one complex scalar ϕ), and a triplet of auxiliary fields $D^{ij} = D^{ji} = D^m \sigma_m{}^{ij}$ with $m = 1, 2, 3$. The transformation rules read

$$\delta A_{\alpha\dot{\alpha}} \;=\; \epsilon^i{}_\alpha \lambda_{i\dot{\alpha}} + h.c.$$

$$\begin{aligned} \delta\lambda^i{}_\alpha &= \epsilon^{i\beta} f_{\alpha\beta} + \epsilon_{j\alpha} D^{ij} + D_{\alpha\dot\alpha}\phi\epsilon^{i\dot\alpha} \\ \delta\phi &= \epsilon^{i\alpha}\lambda_{i\alpha} \\ \delta D^{ij} &= \epsilon^{(i\alpha} D_{\alpha\dot\alpha}\lambda^{\dot\alpha j)} + \epsilon^{(i\alpha}\phi\lambda_\alpha{}^{j)} + h.c. \end{aligned} \tag{2.9.6}$$

where the spinor indices α and $\dot\alpha$ range over the values 1,2. The indices α and $\dot\alpha$ are raised and lowered by the antisymmetric tensors $\epsilon^{\alpha\beta}, \epsilon_{\alpha\beta}, \epsilon^{\dot\alpha\dot\beta}$ and $\epsilon_{\dot\alpha\dot\beta}$ (which are the charge conjugation matrices in these spaces). Similarly, the $SU(2)$ indices $i, j = 1, 2$ are raised and lowered with ϵ^{ij} and ϵ_{ij}. The field D^{ij} is an auxiliary field and $D_{\alpha\dot\alpha}$ denotes the covariant derivative. In this model the central charge Z vanishes, as one may verify from (2.9.6). Thus the Q^i_α anticommute. Furthermore, there is a rigid symmetry group $SU(2)$ which acts on the indices i. (There exist models with a rigid symmetry group $U(2)$ but in this model only the $SU(2)$ is realized). Next one makes a Wick rotation to Euclidean space, which converts the Lorentz group $SO(3,1)$ to $SO(4)$. The SO(4) Euclidean group consists of an SU(2) which acts on $\dot\alpha$, and another $SU(2)$ which acts on α. One can now consider the diagonal subgroup $SU(2)_d$ which acts simultaneously on α and on i (this is called "twisting" [36]). If one defines a new Euclidean Lorentz group by $SU(2) \otimes SU(2)_d$ (where the first $SU(2)$ acts on $\dot\alpha$ as before), then the supersymmetry parameter $\epsilon^{\alpha i} - \epsilon^{i\alpha}$ is Lorentz invariant. This $\epsilon^{\alpha i} - \epsilon^{i\alpha}$ is proportional to $\epsilon^1_1 + \epsilon^2_2$, and becomes the BRST parameter Λ, and $Q^{i\alpha} - Q^{\alpha i}$ becomes the nilpotent BRST charge. The spinor $\lambda^i{}_{\dot\alpha}$ becomes after twisting a vector, and D^{ij} becomes the selfdual part of an antisymmetric tensor.

4. Gauging BRST symmetry. One can gauge BRST and anti-BRST symmetry [7]. The gauge fields are anticommuting vector fields. For a single real anticommuting vector field, the Maxwell action is a total derivative, but if one combines the two anticommuting gauge fields for BRST and anti-BRST into one complex anticommuting vector field, the Maxwell action is not a total derivative. As one might expect from the relation between supersymmetry and supergravity discussed above, there also exists a superspace formalism for BRST symmetry [7].

5. In the spontaneously broken gauge theories for the electroweak interactions, which we discuss later, the ghost and antighost fields become massive. One uses gauge-fixing terms which diagonalize the kinetic terms of the gauge fields. These gauge-fixing terms (introduced by 't Hooft) are of the form

$$\mathcal{L}(\text{fix}) = -\frac{1}{2\xi}\left(\partial^\mu A_\mu{}^a + \frac{1}{2}\xi g v \chi^a\right)^2 \tag{2.9.7}$$

where v is the vacuum expectation value of the Higgs scalar, while χ^a are certain scalar fields called the would-be Goldstone fields. Since in these models classically $\delta\chi^a = gv\lambda^a +$ field-dependent terms, the ghost action, obtained as before by a BRST variation of the gauge fixing term, contains a mass term for the ghosts: $-\frac{1}{2}\xi b_a (gv)^2 c^a$. In fact, the unphysical part of $A_\mu{}^a$, the would-be Goldstone bosons χ^a and the ghost and antighost c^a and b_a all have the same mass (which for $\xi \neq 1$ is different from the mass of the physical part of $A_\mu{}^a$) and form a quartet [5]. The proof of BRST invariance of these electroweak models is step-by-step the same as for QCD. In the first articles on BRST symmetry the abelian Higgs model [37] was considered in order to avoid problems with infrared divergences.

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A Heat kernel regularization of the BRST Jacobian.

We present here details of the computation of the anomaly $Tr\, J \exp T^{-1}S/M^2$ where the 6×6 matrices T, S and J were given in section 3.

The regulator $T^{-1}S$ contains terms with two free derivatives, one free derivative and no free derivatives. The terms with two free derivatives are proportional to the Dalembertian $\partial^\mu\partial_\mu$ because we work in the gauge with $\xi = 1$

$$T^{-1}S = \begin{pmatrix} D^\rho\partial_\rho & -[\partial^\nu b] & 0 \\ c\partial_\mu & R_\mu{}^\nu & [\partial_\mu b] \\ 0 & -\partial^\nu c & \partial^\rho D_\rho \end{pmatrix} \tag{2.A.1}$$

We omit the factors $\delta(x-y)$ which multiply T and S for notational convenience. It can thus be written as follows

$$T^{-1}S = (\partial_\alpha \mathbf{1} + Y_\alpha)\eta^{\alpha\beta}(\partial_\beta \mathbf{1} + Y_\beta) + E \tag{2.A.2}$$

where $\mathbf{1}, Y_\alpha$ and E are 6×6 matrices without any free derivatives and with entries in the adjoint representation. For example, the unit matrix $\mathbf{1}$ has entries $\delta^{ab}, \delta_a{}^b\delta^\mu{}_\nu$

and $\delta_a{}^b$. The calculation of Y_α and E is tedious but straightforward. One finds

$$
\begin{aligned}
Y_\alpha &= \frac{1}{2}\begin{pmatrix} A_\alpha & 0 & 0 \\ c\delta^\mu{}_\alpha & (A_\alpha \delta^\mu_\nu - \frac{1}{2}A^\mu \eta_{\nu\alpha} - \frac{1}{2}A_\nu \delta^\mu_\alpha) & 0 \\ 0 & -c\eta_{\alpha\nu} & A_\alpha \end{pmatrix} \\
E &= \begin{pmatrix} -\frac{1}{2}\partial^\alpha A_\alpha & -\partial^\nu b & 0 \\ -\frac{1}{2}\partial^\mu c & \left\{ \begin{matrix} \frac{3}{2}(\partial^\mu A_\nu - \partial_\nu A_\mu) + \eta^{\mu\nu} A^\rho A_\rho \\ + A^\mu A_\nu - 2A_\nu A^\mu \end{matrix} \right\} & \partial^\mu b \\ 0 & -\frac{1}{2}\partial_\nu c & \frac{1}{2}\partial^\alpha A_\alpha \end{pmatrix} \\
&- \begin{pmatrix} \frac{1}{4}A^\alpha A_\alpha & 0 & 0 \\ (\frac{1}{4}cA^\mu - \frac{3}{4}A^\mu c) & (\frac{1}{4}A^\alpha A_\alpha \delta^\mu_\nu + \frac{1}{2}A^\mu A_\nu - A_\nu A_\mu) & 0 \\ -c^2 & -\frac{1}{4}A_\nu c + \frac{3}{4}cA_\nu & \frac{1}{4}A^\alpha A_\alpha \end{pmatrix}
\end{aligned}
\tag{2.A.3}
$$

The last matrix contains all contributions from $-Y^\alpha Y_\alpha$, and we replaced $F^{\mu\nu}(A)$ by $\partial^\mu A^\nu - \partial^\nu A^\mu + A^\mu A^\nu - A^\nu A^\mu$.

It will simplify the computation of the regularized Jacobian considerably if we can get rid of the free derivatives in J. This can be achieved by symmetrizing J, using that the supertrace in (2.3.1) is invariant under a simultaneous supertransposition of the 6×6 supermatrices (supertransposition is defined by $\varphi^i M \varphi^j = \varphi^j M^T \varphi^i$), a transposition of the derivative operators (which amounts to an extra sign after partially integrating these derivatives) and a transposition of the matrices of the Lie algebra (which amounts to another sign in our case since the adjoint representation is antisymmetric).

To keep track of signs and fermi/bose (anti)commutation properties, one may use the mass term $\phi^i T_{ij} \phi^j$ and derive the symmetrized Jacobian by varying both fields ϕ^j under BRST transformations. (In the derivation of [31], the Pauli-Villars regularization method is used to construct the regulator R and the Jacobian. The variation of the mass term yields then the symmetrized Jacobian). One finds

$$\delta_B(\phi^i T_{ij} \phi^j) = \phi^i (T_{ij} \delta_B \phi^j) + \left(\delta_B \phi^j\right) T_{jk} \phi^k \tag{2.A.4}$$

The symmetrized Jacobian is then

$$\partial \delta_B \phi^i / \partial \phi^j + (T^{-1})^{ik} \left(\frac{\partial}{\partial \phi^k} \delta_B \phi^l \right) T_{lj} \tag{2.A.5}$$

One can prove that the second term gives the same contribution as the first term in (2.3.3) by using that both T and S are symmetric, and using that $trT^{-1}AT\exp T^{-1}S = trA\exp ST^{-1}$. The first term in (A.5) we already evaluated. The second term is obtained as follows. One starts from the variation of the first field ϕ^i in $\phi^i T_{ij}\phi^j$

$$\int\left[-\frac{1}{2}gf^a{}_{bc}c^b c^c\Lambda b_a + (D^\mu(A)c^a)\Lambda A^a_\mu - \partial\cdot A\Lambda c\right]d^4x \tag{2.A.6}$$

Left-differentiation w.r.t. ϕ^i yields the matrix

$$j^tT = \begin{pmatrix} 0 & 0 & 0 \\ 0 & c\eta^{\mu\nu}\Lambda & \partial^\nu\Lambda \\ -c\Lambda & -D^\nu(A)\Lambda & 0 \end{pmatrix} \tag{2.A.7}$$

If one then evaluates $T^{-1}j^tT$ one finds the matrix

$$\begin{pmatrix} c & D^\nu(A) & 0 \\ 0 & c\delta_\mu{}^\nu & \partial_\mu \\ 0 & 0 & 0 \end{pmatrix}\Lambda \tag{2.A.8}$$

Adding this result to J one finds the symmetrized Jacobian

$$j_{\rm sym} = j + T^{-1}j^tT = \begin{pmatrix} c & A^\nu & 0 \\ 0 & 0 & -A_\mu \\ 0 & 0 & -c \end{pmatrix}\Lambda \tag{2.A.9}$$

This expression is indeed without free derivatives.

We evaluate (2.3.3) by inserting complete sets of coordinate eigenstates

$$An = Tr\; j_{\rm sym}(x)\delta(x-y) < y \mid e^{(D^\alpha D_\alpha + E)/M^2} \mid x > \tag{2.A.10}$$

where $D_\alpha = \partial_\alpha + Y_\alpha$. Since there are no free derivatives in $j_{\rm sym}(x)$, the expression for An contains an undifferentiated Dirac delta function. Using the representation

$$\delta(x-y) = \int < x \mid k >< k \mid y > d^4k = \int \frac{d^4k}{(2\pi)^4}e^{ik(x-y)} \tag{2.A.11}$$

the BRST anomaly can be rewritten as a supertrace involving the heat kernel $h(x,y)$ at $x = y$

$$An = \int d^4x str j_{\rm sym}(x)h(x,x); \quad h(x,x) = \int \frac{d^4k}{(2\pi)^4}\int e^{-ikx}e^{R(x)/M^2}e^{ikx} \tag{2.A.12}$$

where $R(x)$ is the differential operator $D^\alpha D_\alpha + E$. Pulling the plane wave $\exp ikx$ to the left of $R(x)$, each free derivative $\frac{\partial}{\partial x^\alpha}$ is replaced by $\frac{\partial}{\partial x^\alpha} + ik_\alpha$, and the exponent becomes $(-k^2 + 2ik^\alpha D_\alpha + R(x))/M^2)$. Rescaling $k = \kappa M$, one obtains

$$h(x,x) = M^4 \int \frac{d^4\kappa}{(2\pi)^4} e^{-\kappa^2} \exp\left[\frac{2i\kappa^\alpha D_\alpha}{M} + \frac{R(x)}{M^2}\right] \tag{2.A.13}$$

Expanding the second exponent, the anomaly is given by the M-independent terms. There are only a few terms which are M independent, and they are given by a set of Gaussian integrals which are easily evaluated. These M independent terms are denoted in the literature[30] by a_2, hence

$$\begin{aligned} a_2 = M^4 \int \frac{d^4\kappa}{(2\pi)^4} e^{-\kappa^2} &\left[\frac{1}{2!}\left(\frac{D^\alpha D_\alpha + E}{M^2}\right)^2 + \frac{1}{3!}\left\{\frac{(D^\beta D_\beta + E)}{M^2}\frac{2i\kappa^\alpha D_\alpha}{M}\frac{2i\kappa^\gamma D_\gamma}{M}\right.\right. \\ &+ \left.\frac{2i\kappa^\alpha D_\alpha}{M}\frac{(D^\beta D_\beta + E)}{M^2}\frac{2i\kappa^\gamma D_\gamma}{M} + \frac{2i\kappa^\alpha D_\alpha}{M}\frac{2i\kappa^\gamma D_\gamma}{M}\frac{(D^\beta D_\beta + E)}{M^2}\right\} \\ &+ \left.\frac{1}{4!}\frac{(2i\kappa^\alpha D_\alpha)^4}{4!}\right] \end{aligned} \tag{2.A.14}$$

Performing the momentum integrals, all terms combine “miraculously” into nice covariant objects

$$a_2 = \frac{1}{(4\pi)^2}\left(\frac{1}{12}Y_{\alpha\beta}^2 + \frac{1}{2}E^2 + \frac{1}{6}\Box E\right) \tag{2.A.15}$$

where $Y_{\alpha\beta}$ are the Yang-Mills curvature for Y_α, so $Y_{\alpha\beta} = \partial_\alpha Y_\beta - \partial_\beta Y_\alpha + [Y_\alpha, Y_\beta]$, while $\Box E = D^\alpha D_\alpha E$ and $D_\alpha E = \partial_\alpha E + [Y_\alpha, E]$. (If one views the composite objects Y_α and E as Yang-Mills gauge fields and scalars in a 6-dimensional space, R is gauge covariant and thus also a_2 must be gauge covariant. This explains the “miracle”). Even though we started with operators $D_\alpha = \partial_\alpha + Y_\alpha$ where Y_α acted just by matrix multiplication and not via a commutator, in the end result the Y_α appear as part of Yang-Mills curvatures or Yang-Mills covariant derivatives with commutators.

Thus the computation of the BRST anomaly An is reduced to an algebraic trace

$$An = \frac{1}{(4\pi)^2}\int str j_{\text{sym}}\left(\frac{1}{12}Y_{\alpha\beta}Y^{\alpha\beta} + \frac{1}{2}E^2 + \frac{1}{6}\Box E\right) d^4x \tag{2.A.16}$$

[30] One can also derive these results from Lagrangian field theory with dimensional regularization. The Lagrangian $\mathcal{L} = (D_\alpha\varphi^i)^2 + E(\varphi^i)^2$ leads to the following one loop divergences: $\frac{1}{\epsilon}\int d^4x(\frac{1}{12}Y_{\alpha\beta}^2 + \frac{1}{2}E^2)$, see G. ’t Hooft, *Nucl. Phys. B* **62** (1973) 444.

The calculation is tedious, but perfectly straightforward; only matrix multiplication is involved. The result reads [9]

$$\begin{aligned} An &= \frac{1}{(4\pi)^2}\int \frac{1}{12} Tr(\partial^\nu c)[4A_\mu A_\nu A^\mu - 4A^\mu(\partial_\mu A_\nu - \partial_\nu A_\mu) \\ &- 4A_\nu \partial_\mu A^\mu + \partial^\mu \partial_\mu A_\nu - 3\partial_\nu \partial_\mu A^\mu] d^4x \end{aligned} \tag{2.A.17}$$

If this anomaly is indeed equal to the BRST variation of the one-loop effective action, it should be BRST invariant as we discussed. It is indeed BRST invariant. To prove this, we must substitute $\delta_B c = c^2$ and $\delta_B A_\mu = A_\mu c - cA_\mu$, but because c and A_μ are in the adjoint representation, and the adjoint representation is antisymmetric, there are further relations, for example

$$Tr\partial_\mu cc A^\mu = Tr A^\mu c\partial_\mu c = Tr\ c\partial_\mu c A^\mu = \frac{1}{2} Tr\partial_\mu(c^2)A_\mu \tag{2.A.18}$$

Using this relation, one may check that An is indeed BRST invariant (BRST closed). Thus the anomaly satisfies the consistency conditions.

The last, and most crucial, step is to decide whether the anomaly is also BRST exact, namely the BRST variation of a local counter term ΔS. By dimensional arguments it is clear that ΔS should be the integral of a polynomial of dimension 4. One may check that the solution is

$$\Delta S = \frac{1}{4\pi^2}\int \frac{1}{12} Tr\left[(\partial\cdot A)^2 + \frac{3}{2}A_\mu A_\nu A^\mu A^\nu - \frac{1}{2}(A^2)(A^2)\right] d^4x \tag{2.A.19}$$

Thus pure Yang-Mills theory has no BRST anomaly. This is as expected because there are BRST anomalies if and only if there are chiral anomalies, and loops with bosonic fields carry no axial anomalies[31].

[31] An exception is loops with selfdual antisymmetric tensor gauge fields.

Chapter 3

Renormalization of unbroken gauge theories

Renormalization is, with unitarity, one of the central issues of quantum field theory. We study in this chapter multiplicative perturbative renormalization of unbroken nonabelian gauge theories, i.e., gauge theories without or with matter but without spontaneous symmetry breaking. By perturbative renormalization we mean that we consider unrenormalized proper Green functions (one-particle irreducible Feynman graphs without counter terms) and construct corresponding finite renormalized proper Green functions loop-by-loop (one-particle irreducible graphs, some of which have counter terms as vertices). We shall follow induction; assuming that all $(n-1)$-loop proper graphs have been made finite, we shall first determine all divergences in proper graphs with n loops. By multiplicative renormalization we mean that these n-loop divergences can be absorbed by rescaling the $(n-1)$-loop renormalized fields and parameters (masses, coupling constants and the gauge parameter ξ) such that the proper graphs computed in terms of these rescaled $(n-1)$-loop renormalized quantities become also finite at the n-loop level. The rescaled $(n-1)$-loop renormalized quantities are then the n-loop renormalized quantities, see (3.3.16).

A more general approach than multiplicative renormalization is additive renormalization, usually called algebraic renormalization. [1] For theories with chiral fermions or γ_5-matrices, such as the electroweak sector of the Standard Model or supersymmetric models, one cannot use multiplicative renormalization, but one must instead use algebraic renormalization. We shall in a later chapter discuss this further, but in this chapter we discuss multiplicative renormalization; this is sufficient for QCD and QED. We shall not specify how one regulates loop corrections. One only requires that the Green functions satisfy the BRST Ward identities. As we shall derive, these Ward identities have the form $\mathcal{S}\Gamma = 0$ where $\mathcal{S}$ is the Slavnov-Taylor operator, which satisfies $\mathcal{S}^2 = 0$,[1] and Γ is the effective action. If there are terms by which the Ward identities are broken, $\mathcal{S}\Gamma = An$, these An are finite (i.e., non-divergent) terms, which one might call candidate anomalies and which must satisfy the consistency condition that they are annihilated by the Slavnov-Taylor operator S. One must then solve the equation $\mathcal{S}An = 0$, and renormalizability requires that any such An is BRST exact: $An = \ csX$. The An are spacetime integrals over polynomials in the fields and derivatives, with the same dimension and ghost number as $\mathcal{S}\Gamma$, namely the An have dimension 5 and ghost number +1. If there are polynomials An which are BRST closed ($\mathcal{S}An = 0$) but which are not BRST exact (meaning that An cannot be written as $\mathcal{S}X$; such An are called nontrivial cohomology in mathematics) there are genuine anomalies in the theory and these prevent renormalization of the theory. If, on the other hand, the candidate anomalies are BRST exact, one can remove them by adding $-X$ to the action as a counter term, and in this case the candidate anomalies are not genuine anomalies. Of course, it is desirable to know beforehand which theories have genuine BRST anomalies, and in which theories one can always remove the candidate BRST anomalies by counter terms. It can be shown that there are BRST anomalies if and only if there are chiral anomalies. So from now

[1] This operator is nilpotent as a consequence of the BRST symmetry of the quantum action, and is itself sometimes called the BRST charge, although strictly speaking it is not the BRST charge, but rather a consequence of BRST symmetry. We shall follow this usage and call expressions X which are annihilated by $\mathcal{S}$ "BRST-closed", instead of Slavnov-Taylor closed.

on we consider only gauge theories without chiral anomalies.[2]

The proof that nonabelian gauge theories are renormalizable is due to 't Hooft and Veltman [3] who received the Nobel prize for their work in 1999. Their work is based on a careful study of properties of Feynman graphs, in particular relations (Ward identities) between different Feynman graphs. These diagrammatical methods [4] are completely equivalent to functional methods developed somewhat later by J. Zinn-Justin [5,6] and B.W. Lee [6] and others. We shall follow here the approach which uses functional methods and base our entire discussion on BRST symmetry. The physical content is the same, but functional methods allow one to summarize properties of sets of Feynman graphs in a very simple and general manner, and BRST methods [5] have the advantage that they eliminate the need for nonlocal expressions [6] in the study of divergences. We shall not be totally one-sided, though, because we shall use Feynman graphs when they clarify formal issues. Functional methods for generating functionals of Feynman graphs with external sources were proposed by Schwinger in [7]. Further references to original articles on functional methods can be found in chapter 6 of [8].

Renormalization of gauge theories differs from renormalization of generic spin 0 or spin 1/2 field theories in the following way. Certain Green functions must have Z factors which are related to the Z factors of other Green functions if multiplicative renormalization is to hold. For example, one can introduce separate Z factors for the 2, 3 and 4 point functions for gauge fields, but then these Z factors are not independent, but satisfy one relation. These relations follow from the BRST symmetry of the quantum action. We shall use this symmetry to derive Ward identities for the effective action Γ. (The effective action is the sum of all proper Green functions). Because the transformation laws of the fields under BRST symmetry are nonlinear in fields (for example, $\delta_B A_\mu = \cdots + gA_\mu \times c$) these Ward identities are quadratic in

[2]If there are no chiral anomalies at the one-loop level, there are also no chiral anomalies at the higher-loop level. [2] We discuss chiral anomalies in a separate chapter.

Γ. We shall refer to these relations as the $\Gamma\Gamma$ equations. For field theories with linear transformation rules, the Ward identities are linear in Γ.

As we already discussed, we shall assume that all formal manipulations with path integrals remain valid when the theory is properly regularized. In particular, we assume that the $\Gamma\Gamma$ equations are satisfied, so we assume that there are no anomalies. We shall not specify in this chapter which regularization scheme we use; all we need to know is that the divergences at the n-loop level are spacetime integrals of local polynomials in the fields and derivatives thereof, which satisfy the Ward identities. A proof that these divergences (and hence also the counter terms which remove them) are local is given in the chapter on unitarity.

The divergences of Green's functions with generic off-shell momenta consist only of ultraviolet divergences. In dimensional regularization, they show up as poles at $n - 4$, and we must remove them by renormalization. If one uses a gauge where the renormalized parameter ξ_{ren} is not equal to unity, there are $k_\mu k_\nu / k^2$ terms in the propagator of massless vector bosons, but also these do not lead to infrared divergences in Green's functions with generic off-shell momenta.

For the computation of cross sections the situation is different. In QED, there are infrared divergences from soft (i.e., with small momenta) emitted photons (called Bremsstrahlung) and also infrared divergences from soft virtual photons in loops. Both kinds of infrared divergences show up as poles at $n - 4$ in dimensional regularization. In the cross sections of QED these infrared poles cancel by themselves, without having to invoke renormalization. In QCD the situation is more complicated as we shall discuss in a special chapter on infrared divergences.

Having made the proper graphs finite order-by-order in the number of loops, the connected graphs can be made finite as well in the following way. Consider a general connected graph, and draw blobs around all proper 2-point, 3-point or 4-point subgraphs. These are the only potentially-divergent proper graphs as we shall prove using power counting. Make these blobs as large as possible. We shall call the resulting subgraphs maximal potentially-divergent proper subgraphs. For example, in a 2-loop selfenergy graph with overlapping divergences, there are two ways to isolate a proper 3-point graph, but both are part of a larger proper graph (the selfenergy itself).

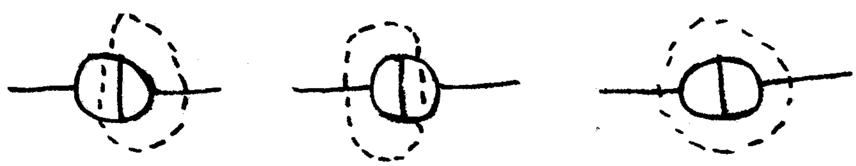


Figure II.1. An example of extending potentially-divergent proper graphs to maximal potentially-divergent proper graphs.

The result of this procedure of identifying a set of maximal potentially-divergent proper subgraphs contained in a given connected graph is actually unique. For example in the following graph, one can draw blobs around the vertex and propagator corrections in various ways, but there is a unique way such that the proper potentially-divergent subgraphs are maximal

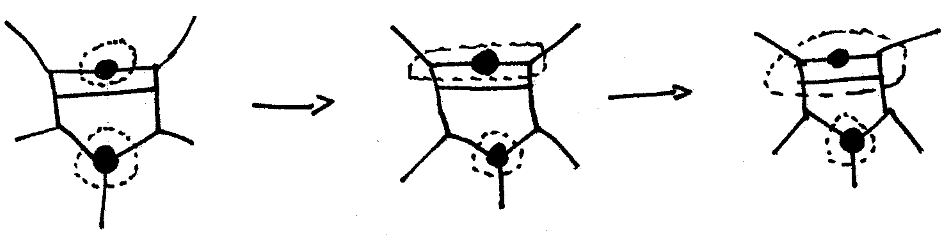
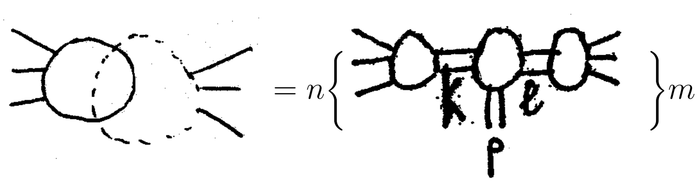


Figure II.2. Maximal potentially-divergent proper subgraphs do not intersect. Note that the blob on top in the second figure is not proper, but the extension in the third graph is proper.

In this example one sees that the maximal potentially-divergent proper subgraphs do not intersect. The claim is that for any graph the blobs around maximal potentially-divergent proper subgraphs are unique and do not intersect. This means that one can make proper subgraphs finite without having to worry about overlapping divergences or about the order in which one makes them finite, and once all divergent proper subgraphs have been made finite by renormalization, the whole connected graph is also finite.

To prove in general that blobs around the maximal potentially-divergent subgraphs do not intersect, assume the contrary.[3] Then there are at least two blobs which are overlapping but neither one is entirely contained in the other. Each is either a 2-point, or a 3-point, or a 4-point function. Furthermore each is maximal: one cannot add a further part of the original graph to a blob such that the result is again proper and has again 2, 3 or 4 external legs. Draw a blob around the vertices in the intersection, and two other blobs around the remaining parts of the original blobs. Let the blob around the intersection have p external lines ($p = 0, 1, 2, \cdots$). Then the intersection-blob must be connected by at least two lines to each of the remaining blobs (since each of the original blobs was proper) and each of the two remaining blobs can have at most $2 - p$ external lines or lines connecting it to the other remaining blob (because the intersection blob and one of the remaining blobs form together one of the original blobs which have at most four external lines).



$$\begin{array}{l} k \geq 2 \\ \ell \geq 2 \\ n + p + l \leq 4 \\ m + p + k \leq 4 \\ \left.\begin{array}{l} n + p \leq 2 \\ m + p \leq 2 \end{array}\right\} \Rightarrow \begin{array}{c} m + n + p \\ \leq 4 \end{array} \end{array} \tag{3.0.1}$$

II.3. Maximal potentially-divergent proper subgraphs are unique. In this graph, $m+n+p \leq 4$ and, as explained in the text, this implies that two maximal potentially-divergent proper graphs do not intersect.

But then the union of the two blobs would be again a blob (since it has at most $m+n+p \leq 4$ external lines). This contradicts the assumption that the original blobs were maximal.

If also five-point functions would be divergent by power counting, one would run into trouble. Suppose five-point functions would be divergent but six-point functions

[3] I thank G. 't Hooft for this proof.

not. One would like also in this case to draw blobs around proper subgraphs with five external lines, but now the identification of proper subgraphs inside a connected graph is not unique as the following example shows

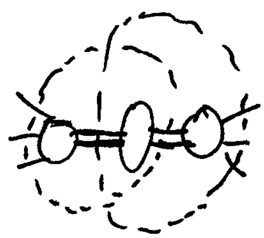


Figure II.4. Example of a proper graph which contains two overlapping 5-point graphs, but the graph itself has more than 5 external legs (namely 6).

Drawing a circle around the two blobs on the left or the two blobs on the right identifies two proper subgraphs with five external lines, but now the whole graph is not a blob by itself, having six external lines. Hence, admitting proper graphs with five external lines, the procedure of identifying maximal potentially-divergent proper subgraphs becomes ambiguous. Fortunately, five-point proper graphs are not divergent in 4 dimensions, as we shall prove by power counting, so we do not need to identify proper subgraphs with five external lines. The topology of proper graphs fits beautifully with the program of renormalization in 4 dimensions.

Having drawn blobs around the maximal proper subgraphs which are potentially divergent, these proper graphs are made finite by the renormalization procedure which is discussed in this chapter. Consider then the set of graphs obtained from the original graph by replacing the subgraphs inside blobs by the sum of subgraphs (including counter terms) which make the blobs finite. All other proper subgraphs are finite by power counting. We can then apply a theorem by Weinberg which states that if all subgraphs of a proper graph are finite according to power counting (or by renormalization), then the graph itself has only local overall divergences, and these occur only in 2, 3 and 4-point Green functions [9]. Removing these by the process of renormalization, one arrives at finite proper Green's functions, and thus also at finite

Green's functions for arbitrary connected n-point functions in gauge theories with or without matter. We now turn to the task of making the divergent proper graphs finite.

1 The Ward identities for divergences in proper graphs

Consider the quantum action for pure Yang-Mills theory

$$\mathcal{L}(qu) = -\frac{1}{4}(F_{\mu\nu}{}^a)^2 - \frac{1}{2\xi}(\partial^\mu A_\mu{}^a)^2 - (\partial^\mu b_a)(D_\mu c)^a \tag{3.1.1}$$

We use the Lorentz-covariant gauge fixing term because this is the one most often used. One can also consider axial-type gauges such as $-\frac{1}{2\xi}(n^\mu A^a_\mu)^2$ where n^μ is a constant vector, in particular the axial gauge $-\frac{1}{2}(A^a_3)^2$ itself, but then there are many more counter terms possible, and the whole analysis becomes much more complicated, see section 6. (For example, there are divergences proportional to $F^2_{\mu\nu}$ and separate divergences proportional to $F^2_{\mu 3}$). In principle one could even use gauge fixing terms which are not even invariant under rigid group transformations such as $(\partial^\mu A^1_\mu)^2 + (\partial^\mu A^2_\mu)^2 + \lambda(\partial^\mu A^3_\mu)^2$, but we prefer not to consider such complications. In fact, only for Lorentz-invariant and rigid-group-invariant gauge fixing terms renormalizability of nonabelian gauge theories has been proven to all orders in loops.

Assuming finiteness of the effective action at the $(n-1)$-loop level, one could compute all proper diagrams at the n -loop level, isolate the divergences, and then multiplicative perturbative renormalizability would mean that one could remove these divergences (and hence render all n-loop 1PI Green's functions finite) by rescaling the objects in the quantum action. These rescalings one would expect to be given by

$$\begin{aligned} A_\mu{}^a &= \sqrt{Z_3}A_\mu{}^{a,\text{ren}}; b_a = \sqrt{Z_{gh}}b_a{}^{\text{ren}}, c^a = \sqrt{Z_{gh}}c_{\text{ren}}{}^a \\ g &= \frac{Z_1}{(Z_3)^{3/2}}\mu^{\frac{1}{2}(4-n)}u\ ,\ \xi = Z_\xi \xi^{\text{ren}} \end{aligned} \tag{3.1.2}$$

Since in the action in (3.1.1) only the product of b_a and c^a appears, the product of their Z factors, $\sqrt{Z_b}\sqrt{Z_c}$, can always be equally distributed over both, hence without loss of generality we can assume that $Z_b = Z_c \equiv Z_{gh}$. New as compared to $\lambda\varphi^4$ theory is, of course, the appearance of the gauge parameter ξ. One might think that one could choose a gauge with $\xi = 1$, and then one would not have to deal with the renormalization of ξ. This is false; even if one would choose the gauge $\xi_{\text{ren}} = 1$ one still would need its Z -factor Z_ξ. This can be understood as follows. A direct computation of the proper one-loop selfenergy for gauge fields (for example with dimensional regularization) yields a transverse result

$$< A_\mu{}^a A_\nu{}^b >= (\eta_{\mu\nu} k^2 - k_\mu k_\nu)\delta^{ab}\Pi(k^2) \tag{3.1.3}$$

This corresponds to a renormalization of the kinetic terms $-\frac{1}{4}(\partial_\mu A_\nu{}^a - \partial_\nu A_\mu{}^a)^2$, hence at one-loop there is no renormalization of the gauge fixing term $-\frac{1}{2\xi}(\partial^\mu A_\mu{}^a)^2$. Renormalization of the kinetic term requires renormalization of A^a_μ. However, renormalization of $A_\mu{}^a$ would lead to a counter term of the form $-\frac{1}{2\xi}Z_3(\partial^\mu A_\mu{}^{a,\text{ren}})^2$, and since the explicit calculation showed that such a term is absent at the one-loop level, one must rescale ξ in the opposite way, such that the total effect of rescaling both ξ and $A_\mu{}^a$ in $\mathcal{L}(\text{fix})$ cancels. Two-loop calculations confirm that the selfenergy is transverse. Hence we conclude that

(i) we need a parameter ξ and

(ii) we must renormalize it as

$$Z_\xi = Z_3 \tag{3.1.4}$$

One can actually prove that at any loop level the complete proper selfenergy of A^a_μ is transversal, see (3.1.40). In this chapter we focus on the divergences and show that the n -loop divergences in the $(n-1)$ loop renormalized selfenergy of the gauge fields are transversal. The proof is given by induction, namely we show that the divergences (and therefore the counter terms) are proportional to the various terms in $S(\text{quantum}) - S(fix)$, so the divergences are transversal (not proportional to $S(fix)$).

In other words, there are no counter terms proportional to $S(fix)$. For renormalization one needs proper graphs, but for unitarity one needs connected graphs. One can also prove that the renormalized connected selfenergy graphs are transversal at all loop levels. We give this proof in the chapter on unitarity.

We shall prove the nonrenormalizability of the gauge fixing term as part of our general proof of renormalizability. In the next chapter we consider spontaneously broken gauge theories. One finds then gauge-fixing terms with several parameters

$$\mathcal{L}(fix) = -\frac{1}{2\xi}(\partial^\mu {A_\mu}^a + \alpha g v \chi^a)^2 \tag{3.1.5}$$

where v is the vacuum expectation value of the Higgs field and χ^a are the would-be Goldstone bosons, and again one begins by restricting the renormalization of these parameters such that $\mathcal{L}(fix)$ does not renormalize: after renormalization $\mathcal{L}(fix)$ has the same form as before renormalization, except that it is written in terms of renormalized quantities (thus: all Z factors in $\mathcal{L}(fix)$ cancel). We shall prove that also in this case the effective action becomes finite after renormalization.

There is a big difference between on the one hand the renormalization of, for example, QED or models with scalars which have a rigid symmetry (for example linear σ models), and on the other hand the renormalization of nonabelian gauge theories. In the latter case, the transformation rules of the symmetry (BRST symmetry) are nonlinear in the fields, and this means that the path integral average of the variation of a field is not equal to the product of the path integral averages of the fields in the variation. For example

$$< \delta_B A_\mu^a > /\Lambda =< g{f^a}_{bc}{A_\mu}^b c^c >\neq g{f^a}_{bc} < {A_\mu}^b > < c^c > \tag{3.1.6}$$

We shall derive Ward identities for generating functionals of connected and proper diagrams using path integrals, and we shall encounter terms like $< \delta_B {A_\mu}^a >$. To still be able to deal with such terms, there is a general method: one adds new terms to the action which are products of external sources and the nonlinear objects. For pure

Yang-Mills theory in the usual relativistic gauge, the nonlinear terms in the BRST transformation rules are only present in $\delta_B A_\mu{}^a$ and $\delta_B c^a$ but not in $\delta_B b_a$. Hence we add to the action the following two terms

$$\mathcal{L}(extra) = K_a{}^\mu (D_\mu c)^a + L_a \frac{1}{2} g f^a{}_{bc} c^b c^c \tag{3.1.7}$$

Clearly, $K_a{}^\mu$ is anticommuting and L_a commuting; moreover both are arbitrary x-dependent fields which only enter into the quantum action in this way, so they have no propagators. In terms of Feynman graphs this means that we only consider proper graphs with external K and/or L lines (in addition to the usual graphs without any external K or L lines). To keep the action real, we declare that $K_a{}^\mu$ and L are purely imaginary. Note that since BRST transformations on A_μ^a and c^a are nilpotent, even without BRST auxiliary field, $\mathcal{L}(extra)$ by itself is BRST invariant (by definition, $K_a{}^\mu$ and L_a do not transform under BRST transformations).

The external sources $K_a{}^\mu$ and L_a were introduced by Zinn-Justin and B. Lee [5,6]. They are called "antifields" in the more recent antifield formalism [10] and are then denoted by $(A_\mu^a)^*$ and $(c^a)^*$. They can be considered as a kind of "covariant momenta" conjugate to $A_\mu{}^a$ and c^a (but with opposite statistics from the usual momenta). In more complicated theories with open gauge algebras, or reducible gauge algebras, this antifield formalism provides a systematic derivation of the correct quantum action.

We are now ready to derive the Ward identities. As in $\lambda\varphi^4$ theory we add the usual external sources which couple to the fields in the quantum action

$$\mathcal{L}(source) = J_a{}^\mu A_\mu{}^a + \beta_a c^a + b_a \gamma^a \tag{3.1.8}$$

The external source β_a is imaginary and γ^a real to make $\mathcal{L}(source)$ real, and both are anticommuting. (Recall that we take c^a to be real and b_a imaginary).

We shall first consider "linear gauges", i.e., gauges in theories like QCD in which the gauge fixing function F^a is linear in fields. For these theories the proof of renormalizability is simplest if one does not introduce the auxiliary fields d_a. Later we shall consider nonlinear gauges where it is simpler to keep d_a as an independent field.

Consider the following path integral for connected and disconnected graphs

$$\begin{aligned} & Z(J_a{}^\mu, \beta_a, \gamma^a; K_a{}^\mu, L_a) = N \int dA_\mu{}^a db_a dc^a \\ & \exp \frac{i}{\hbar} \int [\mathcal{L}(qu) + \mathcal{L}(extra) + \mathcal{L}(source)] d^4x \end{aligned} \tag{3.1.9}$$

The constant N is chosen such that $Z = 1$ when all its arguments vanish. We now make a change of integration variables, from $(A_\mu{}^a, b_a, c^a)$ to $(A_\mu{}^a)' = A_\mu{}^a + \epsilon\delta_B A_\mu{}^a, b'_a = b_a + \epsilon\delta_B b_a, (c^a)' = (c^a + \epsilon\delta_B c^a)$ where ϵ is an infinitesimal commuting constant.[4] We assume that the Jacobian for this infinitesimal BRST transformation is unity, see our discussion in chapter II. Then

$$dA_\mu{}^a db_a dc^a = d(A_\mu{}^a)' db'_a d(c^a)' \tag{3.1.10}$$

Next we use the BRST invariance of the quantum action and $\mathcal{L}$(extra) in (3.1.7) to replace all fields in these actions by BRST-transformed fields

$$\begin{aligned} \mathcal{L}(qu) &= \mathcal{L}\left((A_\mu{}^a)', b'_a, (c^a)'\right) \\ \mathcal{L}(extra) &= K_a{}^\mu (D_\mu c^a)' + L_a \left(\frac{1}{2} g f^a{}_{bc} (c^b)'(c^c)'\right) \end{aligned} \tag{3.1.11}$$

Finally, we replace in $\mathcal{L}$ (source) the fields $A_\mu{}^a$ by $(A_\mu{}^a)' - \epsilon\delta_B A_\mu{}^a$, b_a by $b'_a - \epsilon\delta_B b_a$ and c^a by $(c^a)' - \epsilon\delta_B c^a$. None of these steps changes the value of Z. However, the terms with $\epsilon = 0$ are also equal to Z, since writing Z in terms of primed variables amounts only to a change of name (the Shakespeare theorem [5]). Hence, the expression for

[4] In the literature one usually only works with Λ but one does not introduce a second constant commuting parameter ϵ. One views Λ then as an infinitesimal parameter which is anticommuting. Because the notion of an infinitesimal anticommuting parameter is unclear we prefer for paedagogical reasons to introduce another infinitesimal commuting parameter ϵ and work to first order in ϵ. After having derived the Ward identity, we will no longer need ϵ.

[5] . . . ô be some other name. Whats in a name? that which we call a rose, By any other word would smell as sweete . . . [11].

Z in (3.1.9) equals the expression for Z with extra ϵ-dependent terms. Thus the ϵ dependent terms should cancel by themselves. We shall work to first order in ϵ and find then the following identity for Z

$$\int dA_\mu{}^a db_a dc^a \int \left(J_a{}^\mu(y)\delta_B A_\mu{}^a(y) + \beta_a(y)\delta_B c(y)^a + \delta_B b_a(y)\gamma^a(y)\right) d^4y$$
$$\exp \frac{i}{\hbar}\int \left[\mathcal{L}(qu) + \mathcal{L}(extra) + \mathcal{L}(sources)\right] d^4x = 0. \tag{3.1.12}$$

This identity holds if there are no BRST anomalies (in which case the BRST Jacobian equals unity). Note that this Ward identity would also hold if one had not included the sources K_a^μ and L_a, but they will soon become crucial.

To simplify the notation we write this expression as

$$\int < J_a{}^\mu \delta_B A_\mu{}^a + \beta_a \delta_B c^a + \delta_B b_a \gamma^a > d^4y = 0 \tag{3.1.13}$$

where $<\ >$ denotes the path integral average. We can bring $J_\mu{}^a, \beta_a$ and γ^a outside the brackets and we encounter then the before-mentioned terms $< \delta_B A_\mu{}^a >$ and $< \delta_B c^a >$. Now we see the use of $\mathcal{L}(extra)$, since we can write

$$\frac{i}{\hbar} < \delta_B A_\mu{}^a(y) >= \left(\frac{\partial}{\partial K_a{}^\mu(y)}\Lambda\right) Z; \frac{i}{\hbar} < \delta_B c^a(y) >= \left(\frac{\partial}{\partial L_a(y)}\Lambda\right) Z \tag{3.1.14}$$

In linear gauges (gauges which are linear in quantum fields), $< \delta_B b_a >$ can be written as a differential operator acting on Z. For example, for $\delta_B b_a = -\frac{1}{\xi}(\partial^\mu A_\mu{}^a)\Lambda$ we obtain

$$\frac{i}{\hbar} < \delta_B b_a >= \left(-\frac{1}{\xi}\partial^\mu \frac{\partial}{\partial J_a{}^\mu}\Lambda\right) Z \tag{3.1.15}$$

since $\frac{i}{\hbar} < A_\mu{}^a >= \frac{\partial}{\partial J_a^\mu} Z$. Putting all these results together, the Ward identity for Z simplifies to

$$\int \left(J_a{}^\mu \frac{\partial}{\partial K_a{}^\mu} + \beta_a \frac{\partial}{\partial L_a} + \frac{1}{\xi}\partial^\mu \frac{\partial}{\partial J_a{}^\mu}\gamma^a\right) d^4y\ Z = 0 \tag{3.1.16}$$

(Pulling Λ past γ^a yields an extra minus sign). This Ward identity is a linear first-order partial differential equation with infinitely many variables, a notoriously complicated mathematical object, but we shall be able to extract all information on renormalizability we need from it without actually solving it in general.

We pause at this moment to answer a question the reader may have had from the beginning of this chapter. Namely, why does one not construct the composite operator $\delta_B A_\mu^a(x)$ by differentiating Z simultaneously w.r.t. the sources $J_a^\mu(x)$ and $\beta_a(x)$, instead of using the sources $K_a^\mu(x)$ and $L_a(x)$? The answer is that we can easily apply the Legendre transformation to single derivatives of Z such as $\frac{\partial}{\partial K_a^\mu(x)}Z$, but not to double derivatives such as $\frac{\partial}{\partial J_a^\mu(x)}\frac{\partial}{\partial\beta_a(x)}Z$. We need the Legendre transformation to go from Z to the effective action Γ.

First we go over to connected graphs. They are generated by the generating functional W which is the logarithm of Z

$$Z = \exp\frac{i}{\hbar}W \tag{3.1.17}$$

If no loops were involved, W would simply be equal to the connected tree graphs with sources at the ends, constructed from the action $\mathcal{L}(qu)+\mathcal{L}(extra)+\mathcal{L}(sources)$. Dividing the Ward identity for Z by Z, one finds the Ward identity for W which has the same form since it is only linear in derivatives

$$\int\left(J_a{}^\mu\frac{\partial}{\partial K_a{}^\mu}+\beta_a\frac{\partial}{\partial L_a}+\frac{1}{\xi}\left(\partial^\mu\frac{\partial}{\partial J_a{}^\mu}\right)\gamma^a\right)d^4y\;W=0. \tag{3.1.18}$$

Note that W depends on the same variables as Z

$$W = W(J_a{}^\mu,\beta_a,\gamma^a;K_a{}^\mu,L_a). \tag{3.1.19}$$

Next we go over to the generating functional Γ for proper (one-particle irreducible) graphs. It is related to W by a Legendre transformation

$$\begin{aligned}\Gamma(A_\mu{}^a,c^a,b_a;K_a{}^\mu,L_a) &= W(J_a{}^\mu,\beta_a,\gamma^a,K_a{}^\mu,L_a)\\ &- \int\left(J_a{}^\mu A_\mu{}^a+\beta_a c^a+b_a\gamma^a\right)d^4x\end{aligned} \tag{3.1.20}$$

If there were no loop corrections, Γ would be equal to $S(qu)+S(extra)$. The "$p,\dot q$" of this Legendre transformation are thus the pairs $(A_\mu{}^a,J_a{}^\mu)$, $(-b_a,\gamma^a)$ and (c^a,β_a), but the $(K_a{}^\mu,L_a)$ play the role of the q which are not transformed under the Legendre transformation. If we consider W as the "Lagrangian" and Γ as minus the "Hamiltonian", then the Legendre transform is of the form $-H = -\dot{q}p + L$, and the usual relations of classical mechanics can be written down[6]

$$\begin{aligned}\frac{\partial}{\partial \dot{q}}L = p \Rightarrow \frac{\partial}{\partial J_a{}^\mu}W &= A_\mu{}^a, \frac{\partial}{\partial \beta_a}W = c^a, \frac{\partial}{\partial \gamma^a}W = -b_a \\ \partial H/\partial p = \dot{q} \Rightarrow \partial\Gamma/\partial A_\mu{}^a &= -J_a{}^\mu, \partial\Gamma/\partial c^a = -\beta_a, \partial\Gamma/\partial b_a = \gamma^a\end{aligned} \quad (3.1.21)$$

We indicate right- derivatives by $\partial\Gamma/\partial A_\mu{}^a$ etc., while $\frac{\partial}{\partial A_\mu{}^a}\Gamma$ denotes left- derivatives. (Other notations used in the literature are ${}_A\Gamma$ and Γ_A, or $(\partial_L/\partial A)\Gamma$ and $(\partial_R/\partial A)\Gamma$, or $\frac{\vec{\partial}}{\partial A}\Gamma$ and $\partial\Gamma/\partial \overleftarrow{A}$). For b_a and c^a these derivations differ by a sign: $\frac{\partial}{\partial b_a}\Gamma = -\partial\Gamma/\partial b^a$ and similarly for c. (The easiest way to check such relations is to take $\Gamma = b_a\gamma^a$ as an example; then, since $b_a\gamma^a = -\gamma^a b_a$ one finds $\frac{\partial}{\partial b_a}(b_a\gamma^a) = \gamma^a$ but $\partial(b_a\gamma^a)/\partial b_a = -\gamma^a$.)

The fields A_μ^a, b_a and c^a which appear in (3.1.20) are the path average of the fields A_μ^a which appear in the action. This is clear from the relation $\partial/\partial J_a^\mu W = A_\mu^a$ etc. One calls the former fields A_μ^a sometimes "the classical fields"; not a very clear name because we are at the quantum level. It is customary to use the same notation A_μ^a for both kinds of fields, although the reader may introduce different symbols to avoid confusion. We shall follow the literature and use the same symbols for both kinds of fields.

Another set of identities we shall use correspond to the relation $\partial/\partial q\ L(q,\dot{q}) = -\partial/\partial q\ H(p,q)$ in classical mechanics

$$\frac{\partial}{\partial K_a{}^\mu}\Gamma = \frac{\partial}{\partial K_a{}^\mu}W,\ \frac{\partial}{\partial L_a}\Gamma = \frac{\partial}{\partial L_a}W \quad (3.1.22)$$

[6] For anticommuting variables it matters whether one writes $\dot{q}p$ or $p\dot{q}$, and also the left derivative $\frac{\partial}{\partial \dot{q}}L$ differs from the right derivative $\partial L/\partial\dot{q}$. Defining both for commuting and anticommuting variable $\dot{q}p - L(q,\dot{q}) = H(p,q,\dot{q})$, variation w.r.t. $\dot{q}$ shows that $H(p,q,\dot{q})$ is, in fact, independent of $\dot{q}$ if we define $p = \partial/\partial\dot{q}L$. This shows that if one defines p by left-differentiation of L, then one needs $\dot{q}p$ and not $p\dot{q}$ in the definition of H. Variation w.r.t. q yields $\partial/\partial qL = -\partial/\partial qH$ and variation w.r.t. p yields $\dot{q} = \partial H/\partial p$. (Variation of the left-hand side yields $\delta\dot{q}p + \dot{q}\delta p - \delta q\frac{\partial}{\partial q}L - \delta\dot{q}\frac{\partial}{\partial \dot{q}}L = \dot{q}\delta p - \delta q\frac{\partial}{\partial q}L$. Variation of the right-hand side yields $\delta H = \delta q\frac{\partial}{\partial q}H + \delta p\frac{\partial}{\partial p}H$ and if we replace $\delta p\frac{\partial}{\partial p}H$ by $\partial H/\partial p\delta p$, the Hamiltonian equations of motion follow).

Using these identities, the Ward identity for W goes over into a Ward identity for Γ

$$\int \Big[(-\partial\Gamma/\partial A_\mu{}^a(x)) \ \frac{\partial}{\partial K_a^\mu(x)}\Gamma + (-\partial\Gamma/\partial c^a(x)) \ \frac{\partial}{\partial L_a(x)}\Gamma \\ + \frac{1}{\xi} \ \left(\frac{\partial}{\partial x^\mu} \ A^{\mu a}(x)\right) \partial\Gamma/\partial b_a(x)\Big] \, d^4x = 0 \tag{3.1.23}$$

At tree level (for $\hbar = 0$) this relation reduces to the statement that the quantum action is BRST invariant. Note that an important complication has occurred: the Ward identity for Γ is nonlinear (quadratic) in Γ, whereas the Ward identity for Z (and W) was linear in Z (and W). However, we will only be interested in an analysis of divergences, and for these we shall derive a Ward identity which will again be linear in Γ as we shall see. For comparison, we quote the corresponding Ward identity for linear sigma models with a rigid symmetry $\delta\varphi^i = -\lambda^a(T_a)^i{}_j\varphi^j$ with constant symmetry parameters λ^a (see Chapter IV)

$$(\partial\Gamma/\partial\varphi^i)(T_a)^i{}_j\varphi^j = 0 \tag{3.1.24}$$

Clearly, these Ward identities are linear in Γ. As we already explained, for local nonabelian symmetries, one must use BRST transformations, and these are nonlinear in fields and lead to a Ward identity quadratic in Γ.

At this point, we simplify the Ward identity for Γ by using the knowledge (or, rather, the assumption, to be justified by induction afterwards) that the gauge fixing terms do not renormalize. We subtract them from the effective action, and thus define a functional $\hat{\Gamma}$ by

$$\Gamma = \hat{\Gamma} + \int \mathcal{L}(fix) d^4x \tag{3.1.25}$$

where $\mathcal{L}(fix) = -\frac{1}{2\xi}(\partial^\mu A_\mu{}^a)^2$. Note that at order $\hbar = 0$ (in the absence of loop corrections), $\hat{\Gamma}$ is equal to the quantum action without gauge fixing terms, while at higher order in $\hbar$, there is no difference between Γ and $\hat{\Gamma}$. Since $\mathcal{L}(fix)$ does not depend on $K_a{}^\mu, L_a$ or b_a, we find that only in the term $\partial\Gamma/\partial A_\mu{}^a$ in (3.1.23) it makes a difference whether we replace Γ by $\hat{\Gamma}$ or not, and we claim that this difference precisely cancels the last term in the Ward identity for Γ

$$\int \left[\left(-\partial(\Gamma - \hat{\Gamma})/\partial A_\mu{}^a(x) \right) \frac{\partial}{\partial K_a{}^\mu(x)} \hat{\Gamma} + \frac{1}{\xi} (\partial^\mu A_\mu{}^a(x))\, \partial\hat{\Gamma}/\partial b_a(x) \right] d^4x = 0 \quad (3.1.26)$$

We shall prove this relation in a moment, but accepting this claim, (3.1.23) simplifies to

$$\int \left[\left(\partial\hat{\Gamma}/\partial A_\mu{}^a(x) \right) \left(\frac{\partial}{\partial K_a{}^\mu(x)} \hat{\Gamma} \right) + \left(\partial\hat{\Gamma}/\partial c^a(x) \right) \left(\frac{\partial}{\partial L_a(x)} \hat{\Gamma} \right) \right] d^4x = 0. \quad (3.1.27)$$

To prove the claim in (3.1.26) , we begin with

$$\int dA_\mu{}^a db_a dc^a \frac{\partial}{\partial b_a(y)} e^{\frac{i}{\hbar}[S(qu)+S(extra)+S(sources)]} = 0 \quad (3.1.28)$$

This follows from the property of the Grassman integral that $\int db_a(y) b_a(y) = 1$ but $\int db_a(y) F = 0$ if F is independent of $b_a(y)$. Dividing spacetime into cells, in each cell we have variables $A_\mu{}^a, b_c, c^a$, and since $b_a(y)$ is anticommuting, there are only terms in F which are independent of $b_a(y)$ or linear in $b_a(y)$. Since $\frac{\partial}{\partial b_a(y)} F$ is always independent of $b_a(y)$, the path integral in (3.1.28) vanishes. (In theories with local fermionic invariances such as supergravity and string theory, $b_a(y)$ is commuting, and then one has to argue that the integrand falls off sufficiently fast for large $b_a(y)$. For fermionic $b_a(y)$, the Berezin integral avoids this). Recalling that

$$\frac{\partial}{\partial b_a(y)} S(qu) = \partial^\mu (D_\mu c)^a(y), \frac{\partial}{\partial b_a(y)} S(sources) = \gamma^a(y) \quad (3.1.29)$$

we find the following Ward identity for connected and disconnected graphs

$$< \partial^\mu (D_\mu c)^a(y) + \gamma^a(y) >= 0 \quad (3.1.30)$$

Note that this is a local identity (not an integral over spacetime). We shall give an interpretation of this identity in terms of Feynman diagrams in a moment. Since $\frac{i}{\hbar} < D_\mu c^a(y) >= \frac{\partial}{\partial K_a{}^\mu} Z$, we can rewrite this as

$$\left(\partial^\mu \frac{\partial}{\partial K_a{}^\mu(x)} + \frac{i}{\hbar} \gamma^a(x) \right) Z \quad = \quad 0$$

$$
\begin{aligned}
\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial K_a{}^\mu(x)}W + \gamma^a(x) &= 0 \\
\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial K_a{}^\mu(x)}\Gamma - \frac{\partial}{\partial b_a(x)}\Gamma &= 0 \\
\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial K_a{}^\mu(x)}\hat{\Gamma} - \frac{\partial}{\partial b_a(x)}\hat{\Gamma} &= 0
\end{aligned}
\tag{3.1.31}
$$

Again one may check the signs by taking the $\hbar = 0$ limit, in which case one finds $\partial^\mu D_\mu c^a - \partial^\mu D_\mu c^a = 0$. The result in (3.1.31) is equivalent to (3.1.26) as one may verify by using $\Gamma - \hat{\Gamma} = S(fix)$ and working out the first term in (3.1.26)

$$
-\int \partial S(fix)/\partial A_\mu{}^a(x)\frac{\partial}{\partial K_a{}^\mu(x)}\hat{\Gamma}\, d^4x = \int \frac{1}{\xi}\partial\cdot A^a\partial^\mu\frac{\partial}{\partial K_a{}^\mu}\hat{\Gamma}\, d^4x \tag{3.1.32}
$$

Using (3.1.31) we indeed find that (3.1.26) is correct.

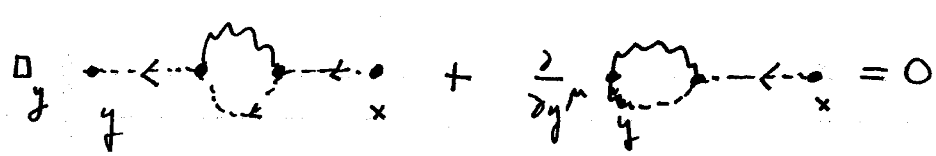
Let us now clarify the meaning of the Ward identity in (3.1.30) by checking that it holds for Feynman diagrams at any loop order. First we can take (3.1.30) and set all external sources to zero; this yields $\langle\delta^\mu D_\mu c^a(x)\rangle = 0$, which is obviously correct because there are no graphs with one external ghost field but no external antighost field (or K source). Next differentiate (3.1.30) with respect to $\gamma^b(x)$, and set afterwards again all external sources to zero. This yields

$$
\langle -\frac{i}{\hbar}b_b(x)(\partial^\mu D_\mu c^a)(y)\rangle + \delta^a_b\delta^4(x-y) = 0\ . \tag{3.1.33}
$$

At tree graph level one finds the relation $\langle\frac{i}{\hbar}b_b(x)\partial^\mu\partial_\mu c^a(y)\rangle = \delta^a_b\delta^y(x-y)$, which is correct since

$$
c^a(y)b_b(x) = \hbar\delta^a_b\int\frac{-i}{k^2}e^{ik(y-x)}\frac{d^4k}{(2\pi)^4} \tag{3.1.34}
$$

and $\int e^{ik(x-y)}d^4k/(2\pi)4 = \delta^4(x-y)$. At the one-loop level, there are two diagrams which contribute; both come from the first term in the Ward identity because the second term is independent of $\hbar$, and has already been used up in the tree graph relation. One must show in diagramatic notation that the following identity holds



(3.1.35)

The vertex on the left-hand side in the second graph comes from expanding $(\partial^\mu D_\mu c^a)(y)$. It is now clear that the sum of both graphs cancels: the $\Box_y$ cancels the propagator on the far left, and the two vertices on the left-hand side of both loops are equal. The reader may check that all signs are such that the identity holds. This concludes the proof of the local Ward identity at the tree and one-loop level.

We have thus obtained two Ward identities for $\hat{\Gamma} = \Gamma - S(fix)$

$$\int \left[\left(\partial\hat{\Gamma}/\partial A_\mu{}^a(x)\right) \left(\frac{\partial}{\partial K_a{}^\mu(x)}\hat{\Gamma}\right) + \left(\partial\hat{\Gamma}/\partial c^a(x)\right) \left(\frac{\partial}{\partial L_a(x)}\hat{\Gamma}\right)\right] d^4x = 0 \qquad (I)$$

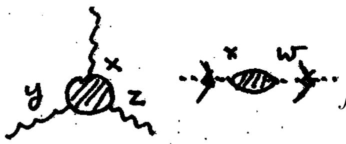
$$\left(\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial K_a{}^\mu(x)} - \frac{\partial}{\partial b_a(x)}\right)\hat{\Gamma} = 0 \qquad (II)$$

The first Ward identity involves an integration over spacetime, whereas the second one is local. The first one is quadratic in $\hat{\Gamma}$, while the second one is linear in $\hat{\Gamma}$. It is not possible to solve the quadratic equation; if one could, one would have the result for the sum of all proper Feynman diagrams. However, we shall soon derive an equation for the divergences in proper Feynman graphs which is linear in Γ and can be solved perturbatively.

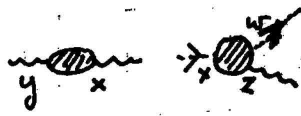
It is clear that (I) by itself gives no information on the b_a dependence of $\hat{\Gamma}$; this information is provided by (II). The need for (II) is not so surprising, since we used so far only the nilpotency of BRST transformations of $A_\mu{}^c$ and c^a to obtain $\mathcal{L}(extra)$, but the information that two BRST transformations of b_a are proportional to the b_a field equation should also be provided, and this is what (II) does. One can use other field equations to derive further Ward identities, but they involve new nonlinear objects for which one must introduce new external sources. These new external sources lead to new Z factors, and because the new nonlinear objects are in general not BRST invariant,[7] one would need even further new external sources for their BRST variations. There is no net gain with this approach.

[7]For example, the ghost field equation reads $< D^\mu\partial_\mu b_a(y) + \beta_a(y) + \cdots >= 0$ but $D^\mu\partial_\mu b_a$ is not BRST invariant.

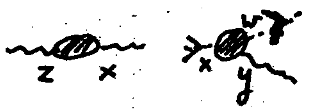
It may be useful to give an example of a Ward identity for proper graphs. By differentiating (I) with respect to ${A_\nu}^b(y), {A_\rho}^c(z)$ and $c^d(w)$, and then setting all fields to zero, one finds the following identity



$$\int \left[\left(\partial^3\hat{\Gamma}/\partial A^a_\mu(x)\partial A^b_\nu(y)\partial A^c_\rho(z)\right)\left(\frac{\partial}{\partial K^\mu_a(x)}\frac{\partial}{\partial c^d(w)}\hat{\Gamma}\right)\right.$$



$$+\left(\partial^2\hat{\Gamma}/\partial A^a_\mu(x)\partial A^b_\nu(y)\right)\left(\frac{\partial}{\partial K^\mu_a(x)}\frac{\partial}{\partial c^d(w)}\frac{\partial}{\partial A^c_\rho(z)}\hat{\Gamma}\right)$$



$$\left.+\left(\partial^2\hat{\Gamma}/\partial A^a_\mu(x)\partial A^c_\rho(z)\right)\left(\frac{\partial}{\partial K^\mu_a(x)}\frac{\partial}{\partial c^d(w)}\frac{\partial}{\partial A^b_\nu(y)}\hat{\Gamma}\right)\right]d^4x = 0 \tag{3.1.36}$$

There are no other terms because ghost number is conserved, and the tadpole graphs $\partial\Gamma/\partial A^a_\mu$ vanish after all remaining fields have been set to zero. Making a Fourier transform of the coordinates y, z, w (using that the Green functions only depend on the differences of the coordinates by translational invariance), we find the corresponding relation in momentum space with momenta p, q and r for the gauge fields. Energy-momentum conservation yields $p+q+r=0$. If we then take all terms at tree level (all terms of order $\hbar = 0$), we find the following identity

$$\begin{aligned} &g f_{abc}(\eta^{\mu\nu}(p-q)^\rho + \eta^{\nu\rho}(q-r)^\mu + \eta^{\rho\mu}(r-p)^\nu)(p_\mu\delta^a_d) \\ &+(\eta^{\mu\nu}q^2 - q^\mu q^\nu)\delta_{ab}(g{f^a}_{cd}\delta^\rho_\mu) \\ &+(\eta^{\mu\rho}r^2 - r^\mu r^\rho)\delta_{ac}(g{f^a}_{bd}\delta^\nu_\mu) = 0 \end{aligned} \tag{3.1.37}$$

It is easy to check that this identity is satisfied, by replacing p by $-q-r$. Thus we have checked the $\hat{\Gamma}\hat{\Gamma}$ equation at the tree level. However, it also holds at any loop level.

As another application of the Ward identity for proper graphs we prove the transversality of the selfenergy of the gauge fields. In this case we differentiate the integrated Ward identity (I) w.r.t. $A_\nu^b(y)$ and $c^d(w)$. Since

$$\frac{\partial^2}{\partial K \partial A}\hat{\Gamma}, \partial\hat{\Gamma}/\partial A, \frac{\partial}{\partial A}\partial\hat{\Gamma}/\partial c, \frac{\partial}{\partial L}\hat{\Gamma}, \partial\hat{\Gamma}/\partial c \tag{3.1.38}$$

all vanish after setting all remaining fields to zero due to ghost number conservation or Lorentz invariance, there is only one term left

$$\int \left(\frac{\partial^2\hat{\Gamma}}{\partial A_\mu^a(x)\partial A_\nu^b(y)}\right)\left(\frac{\partial^2\hat{\Gamma}}{\partial K_a^\mu(x)\partial c^d(w)}\right) d^4x = 0. \tag{3.1.39}$$

The second factor can only be proportional to the momentum p^μ which flows through the graph[8] with external K and c, and the first factor yields the selfenergy of the gauge fields. After Fourier transforming one finds indeed transversality, even off-shell

$$k^\mu \langle A_\mu^a(k) A_\nu^b(-k)\rangle = 0 \tag{3.1.40}$$

We shall encounter many further Ward identites for proper graphs; they are all due to differentiating (I) or (II).

The two Ward identities we have derived hold for the regularized but not yet renormalized theory. (Note that we assumed that we were using such a regularization scheme that the BRST Ward identities were satisfied. One must assume that one is using a regularization scheme because without regularization all path integral manipulations have no meaning). We use dimensional regularization. We must now deduce corresponding Ward identities for the renormalized effective action $\Gamma^{\rm ren}$. We shall prove that the regularized and renormalized effective actions are equal

$$\begin{aligned}&\Gamma^{\rm ren}({A_\mu}^{a,\rm ren}, {b_a}^{\rm ren}, {c_{\rm ren}}^a, {K_a}^{\mu,\rm ren}, {L_a}^{\rm ren}, \mu, u, \epsilon, \hbar, \xi^{\rm ren})\\ &= \Gamma({A_\mu}^a, b_a, c_a, {K_a}^\mu, L_a, g, \epsilon, \hbar, \xi)\end{aligned} \tag{3.1.41}$$

where the relation between the unrenormalized coupling constant g and the dimensionless renormalized coupling constant u is given by $g = Z_1 {Z_3}^{-3/2} \mu^{\frac{1}{2}(4-n)} u$, (see

[8]In noncovariant gauges such as $n \cdot A^a = 0$, the proper two-point function with external K_a^μ and c^b contains also terms proportional to n^μ, and then transversality no longer holds.

(3.1.2)). We shall comment on this relation shortly. The renormalized effective action Γ^{ren} is computed with the renormalized action plus counter terms, while Γ is computed using the bare (unrenormalized) action. In both cases one uses the same regularization scheme with the same regularization parameter ϵ. In practice this is dimensional regularization with $\epsilon \sim n-4$. Then both Γ^{ren} and Γ are first evaluated in n dimensions. We recall the definition of counter terms in a multiplicative renormalizable model

$$\begin{aligned} S({A_\mu}^a, b_a, c^a, {K_a}^\mu, L_a, g) &= S^{\text{ren}} + \Delta S^{\text{ren}} \\ S^{\text{ren}}({A_\mu}^a, b_a, c^a, {K_a}^\mu, L_a, g) &= S({A_\mu}^{a,\text{ren}}, {b_a}^{\text{ren}}, c^a_{\text{ren}}, K_a^{\mu,\text{ren}} {L_a}^{\text{ren}}, u) \\ \Delta S^{\text{ren}} &\equiv S - S^{\text{ren}} \end{aligned} \tag{3.1.42}$$

where $S = S(qu) + S(extra)$. More explicitly, we renormalize, in addition to previous renormalizations, also the external sources ${K_a}^\mu$ and L_a

$${K_a}^\mu = \sqrt{Z_K} {K_a}^{\mu,\text{ren}}, L_a = \sqrt{Z_L} {L_a}^{\text{ren}} \tag{3.1.43}$$

and then we define

$$\begin{aligned} S^{\text{ren}} &= S_{qu}({A_\mu}^{a,\text{ren}}, {b_a}^{\text{ren}}, {c_{\text{ren}}}^a, u) \\ &+ \int \left[{K_a}^{\mu,\text{ren}} (\partial_\mu {c_{\text{ren}}}^a + u f^a{}_{bc} {A_\mu}^{b,\text{ren}} {c_{\text{ren}}}^c) + {L_a}^{\text{ren}} \frac{1}{2} u f^a{}_{bc} {c_{\text{ren}}}^b {c_{\text{ren}}}^c \right] d^4x \end{aligned} \tag{3.1.44}$$

while

$$\begin{aligned} \Delta \mathcal{L}^{\text{ren}} &= -\frac{1}{4}(Z_3 - 1)(\partial_\mu {A_\nu}^{a,\text{ren}} - \partial_\nu {A_\mu}^{a,\text{ren}})^2 + \\ & -\frac{1}{2}(Z_1 - 1) u f^a{}_{bc} (\partial_\mu {A_\nu}^{a,\text{ren}} - \partial_\nu {A_\mu}^{a,\text{ren}}) {A_\mu}^{b,\text{ren}} {A_\nu}^{c,\text{ren}} \\ &+ \ \dots \left(\sqrt{Z_L} Z_1 Z_{\text{gh}} / {Z_3}^{3/2} - 1 \right) {L_a}^{\text{ren}} \frac{1}{2} u f^a{}_{bc} {c_{\text{ren}}}^b {c_{\text{ren}}}^c \end{aligned} \tag{3.1.45}$$

The regularized effective action depends on the regulating parameter ϵ

$$\Gamma({A_\mu}^a, b_a, c^a, {K_a}^\mu, L_a, g, \epsilon, \hbar, \xi) \tag{3.1.46}$$

and is computed using the propagators and vertices of the unrenormalized action

$$S(qu) + S(extra) \tag{3.1.47}$$

while the renormalized effective action

$$\Gamma^{\text{ren}}({A_\mu}^{a,\text{ren}}, {b_a}^{\text{ren}}, {c_{\text{ren}}}^a, {K_a}^{\mu,\text{ren}}, {L_a}^{\text{ren}}, u, \mu, \epsilon, \hbar, \xi^{\text{ren}}) \tag{3.1.48}$$

is computed using

$$S^{\text{ren}}(qu) + S^{\text{ren}}(extra) + \Delta S^{\text{ren}} \tag{3.1.49}$$

In Γ, the limit ϵ to zero keeping unrenormalized quantitites fixed does not exist, but in Γ^{ren} keeping renormalized quantities fixed this limit exists if we have renormalized properly. (Of course, the limit of vanishing ϵ exists also in Γ if one varies A^a_μ etc. in such a way as to keep $A^a_{\mu,\text{ren}}$ fixed).

For nonvanishing ϵ, we have the fundamental identity

$$\Gamma = \Gamma^{\text{ ren}} \tag{3.1.50}$$

Some physicists consider this equality evident since $S = S^{\text{ren}} + \Delta S^{\text{ren}}$, but note that one uses ${J_a}^\mu {A_\mu}^a + \ldots$ in one case and ${J_a}^\mu {A_\mu}^{a,\text{ren}} + \ldots$ (and not ${J_a}^\mu \sqrt{Z_3} {A_\mu}^{a,\text{ren}} + \ldots$) in the other case to define Γ and Γ^{ren}. We give a proof of $\Gamma = \Gamma^{\text{ren}}$ in appendix A to this chapter, starting with $S = S^{\text{ren}} + \Delta S^{\text{ren}}$ as input, and performing the Legendre transformations in both cases to arrive at a relation between Γ and Γ^{ren}.

Since $S(fix) = S^{\text{ren}}(fix)$, as follows from $\frac{1}{2\xi}(\partial^\mu {A_\mu}^a)^2 = \frac{1}{2\xi^{\text{ren}}}(\partial^\mu {A_\mu}^{a,\text{ren}})^2$ due to $Z_3 = Z_\xi$, we also have

$$\hat{\Gamma} = \hat{\Gamma}^{\text{ren}} \tag{3.1.51}$$

where $\hat{\Gamma}^{\text{ren}}$ should be a finite functional (i.e., without divergences) if the theory can be shown to be renormalizable. So, differentiating $\hat{\Gamma}^{\text{ren}}$ w.r.t. its variables $({A_\mu}^{a,\text{ren}}, \ldots {L_a}^{\text{ren}})$ should yield again a finite result. This at once shows that the

Z factors in (I) and (II) due to rescaling must cancel. For example, rewriting (II) in terms of renormalized objects

$$\left(\frac{\partial}{\partial x^\mu} \frac{1}{\sqrt{Z_K}} \frac{\partial}{\partial K_a{}^{\mu,\mathrm{ren}}(x)} - \frac{1}{\sqrt{Z_{gh}}} \frac{\partial}{\partial b_a{}^{\mathrm{ren}}(x)} \right) \hat{\Gamma}^{\mathrm{ren}} = 0 \tag{3.1.52}$$

and using that $\hat{\Gamma}^{\mathrm{ren}}$ is finite, implies[9]

$$Z_K = Z_{gh} \tag{3.1.53}$$

because $\frac{\partial}{\partial K_a^{\mu,\mathrm{ren}}} \hat{\Gamma}^{\mathrm{ren}}$ and $\frac{\partial}{\partial b_a^{\mathrm{ren}}} \hat{\Gamma}^{\mathrm{ren}}$ are both finite. These Z factors are by assumption the Z factors which are needed to make all $(n-1)$ loop proper graphs finite; we shall denote them by $Z^{(n-1)}$ where confusion might arise. (In principle, there is the more general solution $Z_K = \alpha Z_{gh}$ with α a constant, but since for u tending to zero all Z's tend to one, α must be unity as well). Similarly, from (I), we see that

$$Z_3 Z_K = Z_{gh} Z_L \tag{3.1.54}$$

or, combining with $Z_K = Z_{gh}$,

$$Z_L = Z_3 \tag{3.1.55}$$

We could also have renormalized the Ward identity for Γ in (3.1.23), instead of the Ward identity for $\hat{\Gamma}$ and the effective antighost field equation, and then we would have found (3.1.54) and (3.1.55) simultaneously.

Hence, we can only hope to prove renormalizability if we assume from the beginning that $Z_\xi = Z_3, Z_K = Z_{gh}$ and $Z_L = Z_3$. This leaves only three Z factors in pure Yang-Mills theory to absorb infinities, and hence there should not be more than three independent divergences in the proper graphs.

[9] We assume here that we can use multiplicative renormalizability, and the results show that this leads indeed to a finite theory. The Z-factors are due to minimal subtraction (keeping only the pole terms in the divergences due to dimensional regularization). If the theory is first made finite by using these Z factors, a further rescaling by additional finite Z factors will keep the theory finite.

The renormalized Ward identities now read

$$\int \left[\left(\partial \hat{\Gamma}^{\rm ren} / \partial A_\mu{}^{a,\rm ren} \right) \left(\frac{\partial}{\partial K_a{}^{\mu,\rm ren}} \hat{\Gamma}^{\rm ren} \right) + \left(\partial \hat{\Gamma}^{\rm ren} / \partial c_{\rm ren}{}^a \right) \left(\frac{\partial}{\partial L_a{}^{\rm ren}} \hat{\Gamma}^{\rm ren} \right) \right] d^4x = 0$$

$$\left(\frac{\partial}{\partial x^\mu} \frac{\partial}{\partial K_a{}^{\mu,\rm ren}(x)} - \frac{\partial}{\partial b_a{}^{\rm ren}(x)} \right) \hat{\Gamma}^{\rm ren} = 0 \tag{3.1.56}$$

The local Ward identity states that $\hat{\Gamma}^{\rm ren}$ depends on $b_a{}^{\rm ren}$ and $K_a{}^{\mu,\rm ren}$ only in the combination

$$\partial^\mu b_a{}^{\rm ren} - K_a{}^{\mu,\rm ren} \tag{3.1.57}$$

(For example, if $\hat{\Gamma}^{\rm ren} = \int \left(\partial^\mu b_a{}^{\rm ren} - K_a{}^{\mu,\rm ren} \right) F d^4x$ for some b- and K-independent function F, the local Ward identity yields $-\partial_\mu F + \partial_\mu F = 0$). The integrated Ward identity restricts the dependence on the other variables even further, but its complete solution is out of the question. However, the divergences satisfy a much simpler equation which we now derive. Assume by induction that

(i) the theory has been renormalized up to and including $(n-1)$ loops. This means that the terms in $\hat{\Gamma}^{\rm ren}$ of order $\hbar^{n-1}$ and less are finite.

(ii) the equalities $Z_3 = Z_\xi, Z_K = Z_{gh}, Z_L = Z_3$ hold for the terms in the Z factors which are of order $\hbar^{n-1}$ or less.

Then we shall prove that after a further rescaling which removes the n-loop divergences, the same is true to order $\hbar^n$. We start the induction at $n-1=0$, i.e. at the classical level without loops; here (i) and (ii) are obviously satisfied. Since the divergences in n-loop graphs[10] are proportional to $\hbar^n$, whereas the terms of order $\hbar^{n-1}$ and less in $\hat{\Gamma}^{\rm ren}$ are finite by assumption we can decompose $\hat{\Gamma}^{\rm ren}$ as $\hat{\Gamma}^{\rm ren} = \hat{\Gamma}^{{\rm ren},(n)}_{\rm div} + \hat{\Gamma}^{{\rm ren},(n)}_{\rm finite} + \hat{\Gamma}^{{\rm ren},(n-1)}_{\rm finite} + \cdots + \hat{S}^{\rm ren}$. Thus **the divergences in the integrated Ward identity can only be present in the first or in the second factor of the first term, or in the first or in the second factor of the second**

[10] The divergences may, of course, also contain finite parts in addition to divergent parts. One can unambiguously define the divergences as the coefficients of pole terms ϵ^{-n} with $\epsilon = n-4$ in dimensional regularization. However, this is not necessary; finite terms in the Z factors are allowed (sometimes called recalibrations). These finite terms should still be such that (ii) is satisfied and they can be fixed by suitable renormalization conditions.

term in (3.1.56). In each case we have a product of a term with $\hat{\Gamma}^{\mathrm{ren}}(\hbar^n)$ with $\hat{\Gamma}^{\mathrm{ren}}(\hbar = 0)$. Since $\hat{\Gamma}^{\mathrm{ren}}(\hbar = 0) = \hat{S}^{\mathrm{ren}}$, we find then the following equation for the divergences of the n-loop part of the effective action

$$\int \Bigg[\partial\hat{S}^{\mathrm{ren}}/\partial A_\mu{}^{a,\mathrm{ren}} \, \frac{\partial}{\partial K_a{}^{\mu,\mathrm{ren}}} - \partial\hat{S}^{\mathrm{ren}}/\partial K_a{}^{\mu,\mathrm{ren}} \, \frac{\partial}{\partial A_\mu{}^{a,\mathrm{ren}}} + \partial\hat{S}^{\mathrm{ren}}/\partial c^a_{\mathrm{ren}} \frac{\partial}{\partial L_a{}^{\mathrm{ren}}} - \partial\hat{S}^{\mathrm{ren}}/\partial L_a{}^{\mathrm{ren}} \, \frac{\partial}{\partial c^a{}_{\mathrm{ren}}} \Bigg] d^4x \; \hat{\Gamma}^{\mathrm{ren},(n)}{}_{\mathrm{div}} = 0 \quad (3.1.58)$$

Recall that the letters "ren" denote $(n-1)$ loop renormalizability and $\hat{S}$ denotes $S(class) + S(\mathrm{ghost}) + S(extra)$. From now on we will drop the subscripts "ren". The operator between square brackets is called the Slavnov-Taylor operator $\mathcal{S}$. It is given by

$$\mathcal{S} = \int \Bigg[\partial\hat{S}/\partial A_\mu{}^a \, \frac{\partial}{\partial K_a{}^\mu} \; - \partial\hat{S}/\partial K_a{}^\mu \, \frac{\partial}{\partial A_\mu{}^a} + \partial\hat{S}/\partial c^a \, \frac{\partial}{\partial L_a} - \partial\hat{S}/\partial L_a \, \frac{\partial}{\partial c^a} \Bigg] d^4x \quad (3.1.59)$$

When acting on A^a_μ and c^a it generates their BRST transformation. (More precisely, it generates the BRST transformations without the parameter Λ; we denoted these transformations by s, so $sA_\mu = D_\mu c$. The operator $-\Lambda\mathcal{S}$ generates the transformation δ_B with Λ, for example $\delta_B A_\mu = D_\mu c\Lambda$.) Note that this operator is not the BRST Noether charge (although it is closely related); for example it also acts on $K_a{}^\mu$ and L_a, and transforms them into the field equations of A^a_μ and c^a, respectively. Clearly, $\mathcal{S}$ is independent of $\hbar$. It is nilpotent! To prove this we evaluate $\{\mathcal{S}, \mathcal{S}\}$.

We may rewrite the expression for $\mathcal{S}$ by denoting all commuting variables by $x^i = \{A_\mu{}^a, L_a\}$ and all anticommuting variables by $\theta_i = \{K_a{}^\mu, -c^a\}$. Then

$$\mathcal{S} = \left(\partial\hat{S}/\partial x^i \frac{\partial}{\partial \theta_i} - \partial\hat{S}/\partial\theta_i \frac{\partial}{\partial x^i} \right) \quad (3.1.60)$$

For $\frac{1}{2}\{\mathcal{S}, \mathcal{S}\} = \mathcal{S}^2$ we then find four terms

$$\begin{aligned} \mathcal{S}^2 &= \left(\partial\hat{S}/\partial x^i \frac{\partial}{\partial\theta_i} - \partial\hat{S}/\partial\theta_i \frac{\partial}{\partial x^i} \right) \left(\partial\hat{S}/\partial x^j \frac{\partial}{\partial\theta_j} - \partial\hat{S}/\partial\theta_j \frac{\partial}{\partial x^j} \right) \\ &= \partial\hat{S}/\partial x^i \left(\frac{\partial}{\partial\theta_i} \frac{\partial}{\partial x^j} \hat{S} \right) \frac{\partial}{\partial\theta_j} + \partial\hat{S}/\partial\theta_i \left(\frac{\partial}{\partial x^i} \partial\hat{S}/\partial\theta_j \right) \frac{\partial}{\partial x^j} \end{aligned}$$

$$- \quad \left(\partial\hat{S}/\partial x^i \frac{\partial}{\partial\theta_i} \partial\hat{S}/\partial\theta_j \right) \frac{\partial}{\partial x^j} - \partial\hat{S}/\partial\theta_i \left(\frac{\partial}{\partial x^i} \partial\hat{S}/\partial x^j \right) \frac{\partial}{\partial\theta_j} \tag{3.1.61}$$

(summation over i and j is understood and contains integration over x). Terms with two free derivatives all cancel by themselves or pairwise due to symmetry properties. (For example, in $\partial\hat{S}/\partial x^i \partial\hat{S}/\partial x^j \frac{\partial}{\partial\theta_i}\frac{\partial}{\partial\theta_j}$ the first two factors are symmetric in i, j while the last two factors are antisymmetric in i, j). The remaining four terms combine into

$$-\left\{ \frac{\partial}{\partial\theta_j} \left[\partial\hat{S}/\partial x^i \frac{\partial}{\partial\theta_i} \hat{S} \right] \right\} \frac{\partial}{\partial x^j} - \left\{ \frac{\partial}{\partial x^j} \left[\partial\hat{S}/\partial\theta_i \frac{\partial}{\partial x^i} \hat{S} \right] \right\} \frac{\partial}{\partial\theta_j} \tag{3.1.62}$$

These terms vanish since the BRST invariance of the action $\hat{S}$ under variation of the fields A_μ^a and c^a can be written as

$$\partial/\partial x^i \hat{S} \frac{\partial}{\partial\theta_i} \hat{S} = 0 \tag{3.1.63}$$

Hence, the Slavnov-Taylor operator $\mathcal{S}$ is indeed nilpotent, $\mathcal{S}^2 = 0$.

We conclude that the BRST symmetry restricts the n-loop divergences in the $(n-1)$-loop renormalized theory by

$$\mathcal{S}^{\mathrm{ren}} \hat{\Gamma}^{\mathrm{ren}}{}_{\mathrm{div}} = 0, (\mathcal{S}^{\mathrm{ren}})^2 = 0 \tag{3.1.64}$$

where we have reinstated the superscript "ren" to stress once more that the whole renormalization procedure takes place at the $(n-1)$-loop renormalized level. We must now solve this linear Ward identity for the divergences in pure Yang-Mills theory.

2 Multiplicative renormalizability of QCD

We shall now prove that pure Yang-Mills theory, in particular QCD, is multiplicatively renormalizable. We must determine the most general form of the n-loop divergences in proper graphs, and then show that all these divergences can be absorbed by a further multiplicative rescaling of the $(n-1)$-loop renormalized quantities ${A_\mu}^{a,\mathrm{ren}}$, etc. The Slavnov-Taylor operator is to begin with expressed in terms of $(n-1)$-loop renormalized quantities, but after renormalization at the n-loop, it will have the same form in terms of n-loop renormalized quantities because the Z-factors in the $\Gamma\Gamma$ equation were an overall factor which one may omit. Since $(\mathcal{S}^{\text{ren}})^2 = 0$, it is clear that one set of possible divergences is of the form

$$\hat{\Gamma}^{\text{ren}}{}_{\text{div}} = \alpha S(class) + \mathcal{S}^{\text{ren}} X \tag{3.2.1}$$

where X is any Lorentz-invariant and group-invariant polynomial of the correct dimensions and ghost number, and $S(class)$ is any gauge invariant action. Actually, this is the most general solution. To prove this, we must first show that the n-loop divergences are polynomials in the fields and derivatives thereof. This is certainly true for the relativistic gauges with $\partial^\mu A_\mu{}^a$, but it is not always true in noncovariant gauges. To prove the locality of divergences we shall use dispersion relations. Since this requires the same technology as used for unitarity, we shall postpone the proof of locality to the chapter on unitarity, and just assume in this chapter that the divergences are local.

To prove that the general solution of $\mathcal{S}^{\text{ren}}\hat{\Gamma}^{\text{ren}}{}_{\text{div}} = 0$ is the one given above, there are two approaches: a formal one using cohomology of Lie algebras, and a direct one using power counting for proper graphs. The cohomology of polynomials in fields which are integrated over spacetime (corresponding to counter terms) is actually much more complicated than the cohomology of local polynomials. The latter will be studied in detail when we discuss the renormalization of composite operators. For the proof that gauge theories are renormalizable, the direct proof based on combining power counting with BRST symmetry is much simpler. The direct proof proceeds as follows:

(i) determine the set of all proper graphs which could be divergent as far as power counting is concerned

(ii) narrow this set down by requiring that it is annihilated by $\mathcal{S}^{\text{ren}}$

(iii) show that the remaining set of divergences is indeed of the form in (3.2.1).

Power counting shows that a given proper graph with L independent four momenta in the loops, $I_A(I_{bc})$ internal Yang-Mills (ghost-antighost) propagators, n_j vertices with j gauge fields ($j = 3$ or $j = 4$), n_{bAc} ghost vertices of the form ∂bAc, n_K vertices of the form KAc, and n_L vertices of the form Lcc, and E_b external antighosts has the following overall degree of divergence

$$D = 4L - 2I_A - 2I_{bc} + n_3 + n_{bAc} - E_b \tag{3.2.2}$$

The term with E_b is present since in the ghost action b_a appears only as $\partial_\mu b_a$ and hence each external antighost extracts one momentum. We did not take the vertices ${K_a}^\mu \partial_\mu c^a$ into account in this power counting because they cannot appear in proper graphs except at tree level as one vertex by itself. We can eliminate L by observing that to begin with each propagator has its own four momentum, but each vertex gives one energy-momentum conservation law

$$L = I_A + I_{bc} - n_3 - n_4 - n_{bAc} - n_K - n_L + 1 \tag{3.2.3}$$

The reason for the factor $+1$ is that the final integration over coordinates leads to overall energy momentum conservation but does not restrict loop momenta. We eliminate I_A and I_{bc} by observing that each A line or b, c line ends at a vertex. Since propagators end on two vertices but external lines only on one vertex, we obtain

$$\begin{aligned} E_A + 2I_A &= 3n_3 + 4n_4 + n_{bAc} + n_K \\ E_b + E_c + 2I_{bc} &= 2n_{bAc} + n_K + 2n_L \end{aligned} \tag{3.2.4}$$

Substituting these results into the equation for D leads to

$$D = 4 - E_A - 2E_b - E_c - 2n_K - 2n_L \tag{3.2.5}$$

One may check this result in simple one-loop graphs.

The list of all a priori possible divergences with two or more fields is thus

$$A^4, \partial A^3, \partial^2 A^2, \partial^2 bc, \partial bAc, \partial Kc, KAc, Lcc \tag{3.2.6}$$

The derivatives can still be distributed in an arbitrary way over the fields, and the contractions over Lorentz and group indices can be done in any way which yields a scalar. These divergences correspond to the following diagrams where hatched blobs indicate any proper graphs

Terms without any external fields (vacuum selfenergy graphs) have been eliminated in Γ by the normalization factor N in Z. Terms with one gauge field must carry derivatives to obtain a dimension 4 object since we are considering at this point pure gauge theories without masses. These terms are thus total derivatives which we omit. In fact, all tadpole graphs vanish because no field has the quantum numbers of the vacuum.[11]

Note that no divergences with b^2c^2 are possible; this is due to the fact mentioned earlier that each external b_a extracts a momentum. On the other hand, divergences proportional to ${K_a}^\mu \partial_\mu c^a$ can be produced, even though we did not include the vertices ${K_a}^\mu \partial_\mu c^a$ in the power counting.

(3.2.7)

As expected, the vertices in $S = S(qu) + S(\text{extra})$ are contained in this list, but there are many more terms with different contractions of the indices structures which are

[11] For example for a field ϕ with nonzero spin, the vacuum expectation value of the commutator $[\phi, M_{mn}]$ of ϕ with a Lorentz generator M_{mn} must vanish because on the one hand $[\phi, M_{mn}]$ is proportional to ϕ, while on the other hand $< 0|[\phi, M_{mn}]|0 >$ vanishes when Lorentz symmetry is unbroken, $M_{mn}|0 >=< 0|M_{mn} = 0$.

not present in S and which, when actually corresponding to divergences, would make the theory nonrenormalizable. The BRST constraints involving Q must remove all these extra structures.

As far as power counting and (3.1.57) is concerned, the list of possible $n-$ loop divergences is thus given by

$$\begin{aligned}\hat{\Gamma}^{\text{ren}}{}_{\text{div}}(\hbar^n) &= \int \Big[(A^4 + \partial A^3 + \partial^2 A^2) + (K_a{}^\mu - \partial^\mu b_a)(a\partial_\mu c^a \\ &+ \; b g^a{}_{bc} A_\mu{}^b c^c) + \frac{1}{2} c h^a{}_{bc} L_a c^b c^c\Big] d^4x \end{aligned} \tag{3.2.8}$$

where a, b, c contain powers of $(n-4)^{-1}$ and are all proportional to $\hbar^n$, while $g^a{}_{bc}$ and $h^a{}_{bc}$ are invariant tensors of the semisimple gauge group. (Note that b and c are a constant while b^a and c^a denotes ghosts). The tensor $h^a{}_{bc}$ is antisymmetric in its lower two indices. Furthermore, the terms in $A^4 + \partial A^3 + \partial^2 A^2$ contain all possible local divergences with $4, 3$ and 2 gauge fields, contracted in a Lorentz invariant way, and each divergence has its own coefficient with powers of $(n-4)^{-1}$. We recall that the Slavnov-Taylor operator $\mathcal{S}$ is given by

$$\begin{aligned}\mathcal{S}^{\text{ren}} &= \int \Bigg[\left(\frac{\partial}{\partial A_\mu{}^a}\hat{S}\right) \frac{\partial}{\partial K_a{}^\mu} + (D_\mu c)^a \frac{\partial}{\partial A_\mu{}^a} \\ &- \left(\frac{\partial}{\partial c^a}\hat{S}\right) \frac{\partial}{\partial L_a} - \left(\frac{1}{2} u f^a{}_{bc} c^b c^c\right) \frac{\partial}{\partial c^a}\Bigg] d^4x \end{aligned} \tag{3.2.9}$$

where $\hat{S} = S(\text{class}) + S(\text{ghost}) + S(\text{extra})$, and all objects in (3.2.8) and (3.2.9) are $(n-1)$-loop renormalized. We could have written $A^{a,\text{ren}}_{\mu,(n-1)}$ to indicate that we are working with a theory whose proper graphs at $(n-1)$-loops and below are finite, but to simplify the notation we just write A^a_μ, etc. We shall later see how $A^{a,\text{ren}}_{\mu,(n)}$ is obtained from $A^{a,\text{ren}}_{\mu,(n-1)}$. We now work out the consequences of requiring that $\mathcal{S}^{\text{ren}}\hat{\Gamma}^{\text{ren}}{}_{\text{div}}(\hbar^n) = 0$. The reader who is only interested in the result may directly go to (3.2.21).

Since there are many contractions possible in the $A^4 + \partial A^3 + \partial^2 A^2$ terms, we first look at the terms in $\mathcal{S}^{\text{ren}}\hat{\Gamma}^{\text{ren}}_{\text{div}}(\hbar^n)$ which are not of the form $\partial c(A^3 + \partial A^2 + \partial^2 A)$ or

$Ac(A^3 + \partial A^2 + \partial^2 A)$ because the latter are produced by acting with $\mathcal{S}$ on the terms $A^4 + \partial A^3 + \partial^2 A^2$ in (3.2.8)

1) *Lccc* terms. These come only from acting with the last two terms in $\mathcal{S}$ on the last term in (3.2.8) and yield the condition

$$(f^a{}_{rb} h^r{}_{cd} + 2h^a{}_{rb} f^r{}_{cd}) c^b c^c c^d = 0 \tag{3.2.10}$$

We claim that this equation implies that $h^a{}_{cd}$ is an invariant tensor, namely $\delta h^a{}_{cd} = 0$ where $\delta h^a{}_{cd}$ denotes the expression which is obtained if one transforms the indices a, c, and d by an adjoint transformation. The expression within parentheses can be written as an expression which is totally antisymmetric in the indices b, c, d and the 3 terms thus obtained show that $h^a{}_{cd}$ is an invariant tensor. The antisymmetry of $h^a{}_{cd}$ excludes a totally symmetric d-tensor, hence $h^a{}_{cd}$ is proportional to the structure constants.[12]

2) $K\partial cc$ terms. All four terms in $\mathcal{S}$ contribute, but the contributions of the first and last term cancel, and one is left with

$$b g^a{}_{bc} - c h^a{}_{bc} = 0 \tag{3.2.11}$$

Since $h^a{}_{bc}$ is by definition antisymmetric in b, c also $g^a{}_{bc}$ must be antisymmetric in b, c. Thus a d-symbol for $g^a{}_{bc}$ is excluded, and also $g^a{}_{bc}$ is proportional to the structure constants.

3) $KAcc$ terms. Again all four terms in $\mathcal{S}$ contribute and one finds, after using (3.2.11)

$$b\left(f^a{}_{bt} g^b{}_{ps} + \frac{1}{2} f^a{}_{bp} g^b{}_{st}\right) c^s c^t + b\left(g^a{}_{bt} f^b{}_{ps} + \frac{1}{2} g^a{}_{bp} f^b{}_{st}\right) c^s c^t = 0 \tag{3.2.12}$$

The sum of the second and third term vanishes because $g^a{}_{st}$ is an invariant tensor. The remaining terms (the first and fourth term) state that f is an invariant tensor

[12] This statement is equivalent to the statement that in the antisymmetric tensor product of two adjoint representations one finds the adjoint representation only once. One may check this explicitly for each of the simple Lie algebras by looking up the relevant Clebsch-Gordan coefficients.

under rotations with g (instead of g being invariant under rotations with f). Since we already found that b is proportional to the structure constants, also this condition is satisfied.

These results could have been anticipated. Because our gauge fixing term $\mathcal{L}$ (fix) $= -\frac{1}{2\xi}(\partial^\mu A^a_\mu)^2$ is still invariant under rigid group transformations, invariance of the whole action under rigid group transformations allows in general only $h^a{}_{bc} \sim f^a{}_{bc}$ and $g^a{}_{bc} \sim f^a{}_{bc}$. Then (3.2.10) and (3.2.12) are satisfied due to the Jacobi identities, and (3.2.11) yields

$$bg^a{}_{bc} = ch^a{}_{bc} = \gamma u f^a{}_{bc} \tag{3.2.13}$$

with γ a constant to be fixed later.

Finally we consider the terms from $S\int(A^4+\partial A^3+\partial^2 A^2)$. This is the most difficult part of the analysis. These terms contribute due to the second term in S. They must be combined with the result of acting with the first term in S on the divergences in (3.2.8) which depend on $K_a{}^\mu$. The latter read

$$\int \left[\frac{\partial}{\partial A_\mu{}^a} S_{YM}\right](a\partial_\mu c^a + \gamma u f^a{}_{bc} A_\mu{}^b c^c) d^4x \tag{3.2.14}$$

Gauge invariance of S_{YM} allows us to replace the term $\gamma u f^a{}_{bc} A_\mu{}^b c^c$ by $-\gamma\partial_\mu c^a$ (clearly $\int (D_\mu c)^a \frac{\partial}{\partial A^a_\mu} S_{\rm YM} d^4x = 0$ because S_{YM} is gauge invariant). Hence we arrive at the following functional equation for $\int(A^4+\partial A^3+\partial^2 A^2)d^4x$

$$\int (D_\mu c)^a \frac{\partial}{\partial A_\mu{}^a}\left(\int [A^4+\partial A^3+\partial^2 A^2] d^4x\right) d^4y + (a-\gamma)\int \left(\frac{\partial}{\partial A_\mu{}^a} S_{YM}\right)\partial_\mu c^a d^4y = 0 \tag{3.2.15}$$

The general solution for $\int(A^4+\partial A^3+\partial^2 A^2)d^4x$ is a particular solution F of the inhomogeneous equation plus the most general solution of the homogeneous equation

$$\int (A^4+\partial A^3+\partial^2 A^2) d^4x = F + \alpha S_{YM} \tag{3.2.16}$$

We claim that a solution for F is given by

$$F = (\gamma - a)\int A^b_\nu(x) \frac{\partial}{\partial A^b_\nu(x)} S_{YM}\, d^4x \tag{3.2.17}$$

To prove this, consider the operators [8]

$$O_1 = \int D_\mu c^a(y) \frac{\partial}{\partial A^a_\mu(y)} d^4y;\; O_2 = \int A^b_\nu(x) \frac{\partial}{\partial A^b_\nu(x)} d^4x \tag{3.2.18}$$

Since O_2 is a counting operator, the commutator of O_1 and O_2 is given by

$$[O_1, O_2] = \int \partial_\mu c^a \frac{\partial}{\partial A^a_\mu} d^4x \tag{3.2.19}$$

This can, of course, also be checked explicitly. The equation for F can then be written as

$$O_1 F = (\gamma - a)[O_1, O_2] S_{\rm YM} \tag{3.2.20}$$

Since $O_1 S_{\rm YM} = 0$, this equation reduces to $O_1 F = (\gamma - a) O_1 O_2 S_{\rm YM}$. Clearly $F = (\gamma - a) O_2 S_{\rm YM}$ is a solution. Hence, F in (3.2.17) is indeed a particular solution.

We conclude that the general solution of the equation $Q\hat{\Gamma}(\text{div}) = 0$ for pure gauge theories reads (putting $\gamma - a = \beta$ and $c = \gamma$)

$$\begin{aligned}\hat{\Gamma}(div) &= \alpha S_{YM} + \beta \int A_\nu{}^b \frac{\partial}{\partial A_\nu{}^b} S_{YM} d^4x \\ &+ \int (K_a{}^\mu - \partial^\mu b_a) \left[(\gamma - \beta)\partial_\mu c^a + \gamma u f^a{}_{bc} A_\mu{}^b c^c\right] d^4x + \int \gamma L_a \frac{1}{2} u f^a{}_{bc} c^b c^c d^4x\end{aligned} \tag{3.2.21}$$

We draw the reader's attention to the fact that there are no divergences proportional to the gauge fixing term ("transversality"), just as we claimed at the beginning of this chapter. So now this claim is proven

$$\hat{\Gamma}(div) = \Gamma(div) \tag{3.2.22}$$

Let us now compare this result for $\Gamma(div)$ with the expression $\Gamma(div) = \alpha S_{YM} + QX$. The operator Q is Lorentz invariant and has ghost-number $+1$. Hence the most general expression for X is a polynomial of dimension 3 with ghost number -1 which is invariant under Lorentz transformations and rigid group transformations. It is easy to check that the only possibilities are

$$X = A\int(\partial^\mu b_a - K_a{}^\mu)A_\mu{}^a + B\int L_a c^a \tag{3.2.23}$$

where A and B are constants containing powers of $\frac{1}{n-4}$. Evaluating QX, one obtains after several cancellations

$$\begin{aligned}\alpha S_{YM} + QX &= \alpha S_{YM} + (-A-B)\int(\partial^\mu b_c - K_a{}^\mu)\partial_\mu c^a\\ -B\int(\partial^\mu b_a &- K_a{}^\mu)uf^a{}_{bc}A_\mu{}^b c^c + B\int \frac{1}{2}uf^a{}_{bc}L_a c^b c^c\\ -A\int A_\mu{}^a &\frac{\partial}{\partial A_\mu{}^a}S_{YM}\end{aligned} \tag{3.2.24}$$

Comparison with (3.2.21) shows that both expressions agree ($\beta = -A, \gamma = B$). In other words, *the general solution of* $Q\hat{\Gamma}(div) = 0$ *is equal to a sum of gauge-invariant and BRST-exact terms.*

It remains to be shown that the divergences can be absorbed by renormalization. There are three Z factors (Z_3, Z_1, Z_{gh}) and three sets of divergences (α, β, γ). Hence, the numbers of divergences and Z factors match, but we must also satisfy the induction hypotheses.

We rescale the $(n-1)$-loop renormalized objects to n-loop renormalized quantities. Namely in the equation $A_{\mu,\text{ren}} = \sqrt{Z_3}A_\mu$ we want to replace $Z_3^{(n-1)}$ by $Z_3^{(n)}$. At the same time we denote the n-loop renormalized field by $A^{(n)}_{\mu,\text{ren}}$. This is achieved as follows

$$A_{\mu,\text{ren}}{}^{a,(n-1)} = \sqrt{\frac{Z_3^{(n)}}{Z_3^{(n-1)}}}A_{\mu,\text{ren}}{}^{a,(n)} = \left(1 + \frac{1}{2}z_3\hbar^n + \dots\right)A_{\mu,\text{ren}}{}^{a,(n)} \tag{3.2.25}$$

and similarly for L_a, (see (3.1.55)). Here we used that finiteness of $(n-1)$ loops requires that $Z^{(n)}$ agrees with $Z^{(n-1)}$ through order $\hbar^{n-1}$; the ratio of $Z^{(n)}/Z^{(n-1)}$ is then of the form $1 + z\hbar^n + \mathcal{O}(\hbar^{n+1})$. We are only interested in the terms of order $\hbar^n$. For b_a, c^a and $K_a{}^\mu$ we have

$$K_{a,\text{ren}}{}^{\mu,(n-1)} = \left(1 + \frac{1}{2}z_{gh}\hbar^n + \dots\right)K_{a,\text{ren}}{}^{\mu,(n)} \tag{3.2.26}$$

Finally

$$u^{(n-1)} = \left(1 + \left(z_1 - \frac{3}{2}z_3\right)\hbar^n + \dots\right)u^{(n)} \tag{3.2.27}$$

Substituting these relations into $S_{\text{ren}}^{(n-1)}$ and keeping all terms linear in z's, we find a sum of terms linear in z's

$$S_{\text{ren}}^{(n-1)} = S_{\text{ren}}^{(n)} + \text{ terms linear in } z\text{'s} \tag{3.2.28}$$

(Substitution of the rescalings into thecounter terms $\Delta\mathcal{L}^{\text{ren}}$ yields terms of order $\hbar^{n+1}$ and more, which play no role). Adding to this set of terms the divergences $\Gamma(\text{div})$ (more precisely, $\Gamma_{div}^{\text{ren}}(\hbar^n)$), finiteness of $\hat{\Gamma}$ at the n-loop level requires that the sum of z terms and divergences cancels. This yields an overcomplete set of linear relations between the z's and α, β, γ. We must show that there nevertheless exists a solution.

Rescaling $\mathcal{L}(ghost) + \mathcal{L}(extra)$ yields the following new terms in the action which are linear in z_3, z_{gh} and z_1

$$\begin{aligned}
&z_{gh}(K_a{}^\mu - \partial^\mu b_a)\partial_\mu c^a + \\
&(z_{gh} + z_1 - z_3)(K_a{}^\mu - \partial^\mu b_a)uf^a{}_{bc}A_\mu{}^b c^c + \\
&(z_{gh} + z_1 - z_3)L_a \frac{1}{2}uf^a{}_{bc}c^b c^c
\end{aligned} \tag{3.2.29}$$

These terms cancel the ghost-dependent terms in $\Gamma(div)$ in (3.2.21) provided one chooses the z's as follows

$$z_{gh} = \beta - \gamma, z_{gh} + z_1 - z_3 = -\gamma, z_{gh} + z_1 - z_3 = -\gamma \tag{3.2.30}$$

The last relation in (3.2.30) is equal to the second one, and this is a consequence of the BRST Ward identity. In a generic, not gauge invariant, theory we would at this point have found one more independent divergence in the theory.

In a similar manner we can renormalize in the sector without ghost fields. Rescaling of $\mathcal{L}_{YM}$ yields

$$z_3(\partial A)^2 + z_1 u(\partial A)A^2 + (2z_1 - z_3)u^2 A^4 \tag{3.2.31}$$

and adding this to $\alpha S_{YM} + \beta A \frac{\partial}{\partial A} S_{YM}$, the ghost-independent divergences cancel provided

$$\begin{aligned} z_3 &= -(\alpha + 2\beta) \\ z_1 &= -(\alpha + 3\beta) \\ 2z_1 - z_3 &= -(\alpha + 4\beta) \end{aligned} \tag{3.2.32}$$

Again we see the power of BRST symmetry: the last of these relations follows from the previous two, while the difference of the first two yields $z_3 - z_1 = \beta$. This relation agrees with (3.2.30) and hence there is a (unique) solution for the z's.

We conclude that pure Yang-Mills theory is perturbatively multiplicatively renormalizable. Note that we had to satisfy six equations for three variables (z_3, z_{gh} and z_1), hence renormalizability is a nontrivial property.

3 Multiplicative renormalizability of quarks and gluons

Having analyzed pure Yang-Mills theory, the next system to study is the coupling to matter. We shall consider in detail the coupling to nonchiral fermions, for example the coupling of quarks to QCD. We include a mass for the fermions and consider an arbitrary simple gauge group. The proof that this system is also renormalizable follows the same steps as in the case of pure gauge theories, and for a first reading one may skip to (3.3.19). The renormalizability of chiral fermions coupled to gauge fields is more complicated as we shall discuss.

To prove the renormalizability of gauge fields minimally coupled to Dirac fermions with a classical action

$$\begin{aligned} \mathcal{L}(fer) &= -\bar{\psi}_i \gamma^\mu (D_\mu \psi)^i - m \bar{\psi}_i \psi^i \\ (D_\mu \psi)^i &= \partial_\mu \psi^i + g A_\mu{}^a (T_a)^i{}_j \psi^j, \bar{\psi}_i = (\psi^i)^\dagger i \gamma^0 \end{aligned}$$

$$\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}; \eta_{\mu\nu} = (-1, 1, 1, 1); (\gamma_\mu)^\dagger = \gamma^\mu \tag{3.3.1}$$

we go through the same steps as for pure Yang-Mills theory. The gauge transformations of the fermions read

$$\begin{aligned}\delta_{\text{gauge}}\psi^i &= -g(T_a)^i{}_j\psi^j\lambda^a(x), [T_a, T_b] = f_{ab}{}^c T_c \\ \delta_{\text{gauge}}\bar{\psi}_i &= g\bar{\psi}_j(T_a)^j{}_i\lambda^a(x), (T_a)^\dagger = -T_a\end{aligned} \tag{3.3.2}$$

The matrices T_a are a particular representation of the gauge group. The covariant derivative $(D_\mu\psi)^i$ transforms then as ψ^i, which proves the gauge invariance of the minimally coupled Dirac action. This leads to the BRST transformations which are again nilpotent

$$\begin{aligned}\delta_B\psi^i &= -g(T_a)^i{}_j\psi^j c^a\Lambda \\ \delta_B\bar{\psi}_i &= g\bar{\psi}_j(T_a)^j{}_i c^a\Lambda\end{aligned} \tag{3.3.3}$$

Since these BRST transformations are nonlinear in fields, we introduce external sources for them. The action in the path integral Z contains then new source terms for the fermions and their BRST variations

$$\begin{aligned}\mathcal{L}(newfermionicterms) &= \mathcal{L}(fer) - g\bar{N}T_a\psi c^a \\ &- g\bar{\psi}T_a N c^a + \bar{J}_i\psi^i + \bar{\psi}_i J^i\end{aligned} \tag{3.3.4}$$

where N^j and $\bar{N}_j$ are commuting, and J^i and $\bar{J}_i$ anticommuting sources. As for ψ and $\bar{\psi}$, we define $\bar{N}_j = (N^j)^\dagger i\gamma^0$ and $\bar{J}_i = (J^i)^\dagger i\gamma^0$. Note that both N and $\bar{N}$ have ghost number -1. After the Legendre transformation from $J^i, \bar{J}_i$ to $\psi^i, \bar{\psi}_i$, the unrenormalized effective action minus gauge fixing term, $\hat{\Gamma}$, satisfies the Ward identities

$$\begin{aligned}&\left(\frac{\partial}{\partial b_a} - \partial^\mu \frac{\partial}{\partial K_a{}^\mu}\right)\hat{\Gamma} = 0 \\ &\int \Bigg[\left(\partial\hat{\Gamma}/\partial A_\mu{}^a\right)\frac{\partial}{\partial K_a{}^\mu}\hat{\Gamma} + \left(\partial\hat{\Gamma}/\partial c^a\right)\frac{\partial}{\partial L_a}\hat{\Gamma} \\ &+ \left(\partial\hat{\Gamma}/\partial\psi^i\right)\frac{\partial}{\partial\bar{N}_i}\hat{\Gamma} + \left(\partial\hat{\Gamma}/\partial N^i\right)\frac{\partial}{\partial\bar{\psi}_i}\hat{\Gamma}\Bigg]d^4x = 0\end{aligned} \tag{3.3.5}$$

New due to the presence of fermions are the last two terms. The antighost field equation is unchanged because the gauge fixing terms and the ghost action do not depend on the fermions. One may check the signs in these two Ward identities by replacing $\hat{\Gamma}$ by its $\hbar = 0$ part $\hat{S}$.

Since the gauge fixing term is the same as in the case of pure gauge theories, finiteness of $\mathcal{L}(\text{fix})$ requires again $Z_\xi = Z_3$, and since the antighost field equation is unmodified, also the relation $Z_{gh} = Z_K$ is unmodified. The renormalization of the $\hat{\Gamma}\hat{\Gamma}$ Ward identity requires the relations

$$Z_3 Z_K = Z_{gh} Z_L = Z_f Z_N \tag{3.3.6}$$

if we renormalize $\psi = Z_f^{1/2}\psi^{\text{ren}}, \bar{\psi} = Z_f^{1/2}\bar{\psi}^{\text{ren}}, N = Z_N^{1/2}N^{\text{ren}}$ and $\bar{N} = Z_N^{1/2}\bar{N}^{\text{ren}}$. (As only the combinations $\bar{\psi}\psi, \bar{N}\psi$ and $\bar{\psi}N$ appear in S, we can put $Z_\psi = Z_{\bar{\psi}} = Z_f$ and find then that $Z_N = Z_{\bar{N}}$). Furthermore, from (3.3.6) and $Z_{gh} = Z_K$ we obtain

$$Z_N = Z_3 Z_{gh}/Z_f \; ; \; Z_3 = Z_L \tag{3.3.7}$$

We renormalize the fermion mass as

$$m = Z_m m^{\text{ren}} \tag{3.3.8}$$

All Z-factors are now specified and there are 5 independent Z factors: Z_3, Z_1, Z_{gh}, Z_f and Z_m.

The $(n-1)$-loop divergences are local polynomials which satisfy $\mathcal{S}\ \Gamma(\text{div}) = 0$ where $\mathcal{S}$ and $\Gamma(\text{div})$ depend on $(n-1)$-loop renormalized quantities, and $\mathcal{S} = \mathcal{S}\ (\text{pure case}) + \mathcal{S}\ (\text{fer})$ where

$$\begin{aligned}
\mathcal{S}(purecase) = \int \Bigg[& \left(\frac{\partial}{\partial A_\mu{}^a}\hat{S}\right)\frac{\partial}{\partial K_a{}^\mu} + (D_\mu c)^a \frac{\partial}{\partial A_\mu{}^a} \\
& - \left(\frac{\partial}{\partial c^a}\hat{S}\right)\frac{\partial}{\partial L_a} - \left(\frac{1}{2}u f^a{}_{bc}c^b c^c\right)\frac{\partial}{\partial c^a}\Bigg] d^4x \\
\mathcal{S}(fer) = \int \Bigg[& \left(\partial\hat{S}/\partial\psi^i\right)\frac{\partial}{\partial \bar{N}_i} + \left(\partial\hat{S}/\partial N^i\right)\frac{\partial}{\partial\bar{\psi}_i} - \left(\frac{\partial}{\partial\bar{N}_i}\hat{S}\right)\frac{\partial}{\partial\psi^i} + \left(\frac{\partial}{\partial\bar{\psi}_i}\hat{S}\right)\frac{\partial}{\partial N^i}\Bigg] d^4x
\end{aligned} \tag{3.3.9}$$

All objects in (3.3.9) and in what follows below are renormalized through $(n-1)$-loops, but we shall not explicitly write $A^{a,\text{ren}}_{\mu,(n-1)}$ to simplify the notation.

As always $S = \hat{S} + S(\text{fix})$. Again $\mathcal{S}^2 = 0$ because it contains pairs of commuting and anticommuting fields and sources. We claim that the general solution of $\mathcal{S}\,\Gamma(div)$ is again a sum of gauge invariant terms and BRST exact terms

$$\begin{aligned}\Gamma(\text{div}) &= \alpha_1 S(YM) + \alpha_2 S(\text{Dir}) + \alpha_3 S\ (\text{mass}) + \mathcal{S}X \\ X &= \int \left[\beta(K_a{}^\mu - \partial^\mu b_a)A_\mu{}^a + \gamma L_a c^a + \delta \bar{N}_i \psi^i + \epsilon \bar{\psi}_i N^i\right] d^4x \end{aligned} \tag{3.3.10}$$

Although this is a crucial result on which renormalizability rests we shall not record here a proof since the algebra is tedious, straightforward and similar to the case of pure Yang-Mills theory. Briefly, power counting yields for the degree of divergence D of proper graphs

$$D = 4 - E_A - 2E_b - E_c - 2n_K - 2n_L - \frac{3}{2}(n_N + n_{\bar{N}} + E_\psi + E_{\bar{\psi}}) \tag{3.3.11}$$

where n_N and $n_{\bar{N}}$ denotes the number of vertices in (15.6.4). Hence there are new divergences of the form

$$\int \bar{\psi}\partial\psi, \int \bar{\psi}A\psi, \int M\bar{\psi}\psi, \int \bar{N}\psi c, \int \bar{\psi}Nc \tag{3.3.12}$$

(No terms with for example $\bar{N}AN$ are allowed since N and $\bar{N}$ have both ghost number -1.) One should then write down the most general expression for $\Gamma(\text{div})$ compatible with power counting, act on it with $\mathcal{S}$, and require that the result be zero. For the interested reader, a detailed derivation is given in appendix C to this chapter.

Inspection of (3.3.10) seems now to reveal a problem. There seem to be 7 divergent structures (with $\alpha_1, \alpha_2, \alpha_3, \beta, \gamma, \delta, \epsilon$) but only 5 Z factors $(Z_1, Z_3, Z_{gh}, Z_f, Z_m)$. One might expect that $\epsilon \sim \delta$, but then there would still be a mismatch. Clearly, for renormalizability to hold not all 7 divergent parameters can enter in $\Gamma(div)$ in an independent way. Let us see how this comes about.

Evaluating $\mathcal{S}X$, one finds

$$\Gamma(div) = \alpha_1 S(YM) + \alpha_2 S(Dir) + \alpha_3 S(mass) + \beta A_\mu{}^\alpha \left(\frac{\partial}{\partial A_\mu{}^a}\right)\hat{S}$$
$$-\beta(K_a{}^\mu - \partial^\mu b_a)(D_\mu c^a) + \gamma c^a \frac{\partial}{\partial c^a}\hat{S} - \gamma L_a \left(\frac{\partial}{\partial L_a}\hat{S}\right)$$
$$+\delta\left(\psi^i \frac{\partial}{\partial \psi^i} - \bar{N}_i \frac{\partial}{\partial \bar{N}_i}\right)\hat{S} - \epsilon\left(\bar{\psi}_i \frac{\partial}{\partial \bar{\psi}_i} - N^i \frac{\partial}{\partial N^i}\right)\hat{S} \tag{3.3.13}$$

where $S(Dir)$ denotes the Dirac action without mass term. The sum of the counting operators for $\psi, \bar{\psi}, N$ and $\bar{N}$ vanishes on the terms $\bar{N}\psi c$ and $\bar{\psi}Nc$, and the terms due to acting with $\beta A\frac{\partial}{\partial A}$ on the nonclassical part of $\hat{S}$ cancel the Ac part in $(K - \partial b)Dc$. Hence one finds

$$\Gamma(div) = \alpha_1 S(YM) + \alpha_2 S(Dir) + \alpha_3 S(mass) + \beta A_\mu{}^a \frac{\partial}{\partial A_\mu{}^a} S(class)$$
$$+(\gamma - \beta)(K_a{}^\mu - \partial^\mu b_a)\partial_\mu c^a + \gamma(K_a{}^\mu - \partial^\mu b_a) u f^a{}_{bc} A_\mu{}^b c^c + \gamma L_a \frac{1}{2} u f^a{}_{bc} c^b c^c$$
$$+\gamma u(-\bar{N} T_a \psi c^a + \bar{\psi} T_a N c^a) + (-\epsilon \bar{\psi}_i \frac{\partial}{\partial \bar{\psi}_i} + \delta \psi^i \frac{\partial}{\partial \psi^i}) S(Dir) \tag{3.3.14}$$

This is the most general set of divergences; it seems to contain seven independent divergent parameters, namely $\alpha_1, \alpha_2, \alpha_3, \beta, \gamma, \delta$ and ϵ. However,

$$\left(-\epsilon \bar{\psi}_i \frac{\partial}{\partial \bar{\psi}_i} + \delta \psi^i \frac{\partial}{\partial \psi^i}\right) S(\text{fer}) = (-\epsilon + \delta) S(\text{fer}) \tag{3.3.15}$$

where $S_{fer} = S(Dir) + S(mass)$, hence only the combinations $\alpha_2' = \alpha_2 - \epsilon + \delta$ and $\alpha_3' = \alpha_3 - \epsilon + \delta$ appear. So, after all, there are equal numbers of divergences $(\alpha_1, \alpha_2', \alpha_3', \beta, \gamma)$ and Z factors $(Z_1, Z_3, Z_{gh}, Z_f,\ Z_m)$. As noted before, this is a necessary but not sufficient condition for renormalizability. (If the number of divergences would have been less than the number of Z factors, one should be able to reduce the number of independent Z factors).

The issue of renormalizability now boils down to a check that all rescalings can remove all divergences. Substituting

$$A_\mu{}^{a,\text{ren},(n-1)} = \sqrt{\frac{Z_3{}^{(n)}}{Z_3{}^{(n-1)}}} A_\mu{}^{a,\text{ren},(n)} = \left(1 + \frac{1}{2}\hbar^n z_3 + \ldots\right) A_\mu{}^{a,\text{ren},(n)}$$

$$\psi^{i,\mathrm{ren},(n-1)} = \sqrt{\frac{Z_f{}^{(n)}}{Z_f{}^{(n-1)}}}\psi^{i,\mathrm{ren}(n)} = \left(1 + \frac{1}{2}\hbar^n z_f + \dots\right)\psi^{i,\mathrm{ren},(n)} \tag{3.3.16}$$

etc., into S^{ren}, and adding all n-loop counter terms produced in this way to the n-loop divergences, we require that the sum cancels. In the bosonic sector nothing changes, but in the fermionic sector we find new consistency conditions. The fermionic counter terms are spacetime integrals of

$$\begin{aligned} & z_f(-\bar{\psi}\partial\!\!\!/\psi) + (z_1 - z_3 + z_f)(-\bar{\psi}uA\!\!\!/\psi) + (z_m + z_f)(-m\bar{\psi}\psi) \\ & +(z_1 - z_3 + z_{gh})u(-\bar{N}\psi c + \bar{\psi}Nc) \end{aligned} \tag{3.3.17}$$

where we used the expression for Z_N in terms of other Z's given in (3.3.7). The divergences in the fermionic sector are spacetime integrals of

$$(\alpha_2 - \epsilon + \delta)\mathcal{L}(Dir) + (\alpha_3 + \epsilon + \delta)\mathcal{L}(mass) + \beta(-\bar{\psi}uA\!\!\!/\psi) + \gamma u(-\bar{N}\psi c + \bar{\psi}Nc) \tag{3.3.18}$$

Renormalizability requires that the coefficients of each linearly independent monomial in fields vanish. This yields four equations

$$\begin{array}{ll} z_f + \alpha_2 - \epsilon + \delta = 0; & z_m + z_f + \alpha_3 - \epsilon + \delta = 0 \\ z_1 - z_3 + z_f + \alpha_2 + \beta - \epsilon + \delta = 0; & z_1 - z_3 + z_{gh} + \gamma = 0 \end{array} \tag{3.3.19}$$

The two relations on the left lead to $z_1 - z_3 + \beta = 0$ which is indeed a correct relation for the bosonic sector, and the last relation was also earlier found in the bosonic sector, see (3.2.30). This leaves us with two relations which determine z_f and z_m. Hence, there is a solution to the overdetermined set of linear equations between the z's and the α's, $\beta, \gamma, \delta + \epsilon$. We conclude that the theory of Dirac fermions minimally coupled to gauge fields is perturbatively multiplicatively renormalizable.

When there are chiral couplings involving γ_5, the dimensionally regularized theory no longer satisfies the Ward identities. One should then use algebraic regularization (adding by hand suitable local (finite) counter terms to the action such that the regularized Green's functions satisfy the BRST Ward identities. If this cannot be done one has BRST anomalies).

Similar remarks apply to supersymmetric gauge theories. No regularization scheme is known which respects both BRST symmetry and supersymmetry. For example, ordinary dimensional regularization (with $n > 4$) respects BRST symmetry but violates supersymmetry, whereas regularization by dimensional reduction respects supersymmetry (if one works in superspace) but violates BRST symmetry. Again one can then restore both symmetries by adding suitable local counter terms, but this is only possible if there are no anomalies. Anomalies show up in the path-integral as a nontrivial Jacobian. For chiral theories, the regularized BRST Jacobian contains one term

$$Tr(\partial\delta_B\psi^i(x)/\partial\psi^j(y)e^{\not{D}\not{D}/M^2} \sim TrT_a c^a(1+\gamma_5)e^{\not{D}\not{D}} \tag{3.3.20}$$

and if there are triangle anomalies, the result for this Jacobian is not BRST-exact (and vice-versa). In the chapter on anomalies we show that renormalizability requires absence of anomalies, or, phrased differently, in the presence of anomalies, renormalizability breaks down. The anomalies lead to a nonvanishing right-hand side of the Γ -Γ Ward identity

$$\left(\hat{\Gamma},\hat{\Gamma}\right) = \Delta \tag{3.3.21}$$

where $(\hat{\Gamma},\hat{\Gamma})$ denotes the "antifield bracket" (which is just the Ward identity (for pure gauge theory given by $\partial\hat{\Gamma}/\partial A\frac{\partial}{\partial K}\hat{\Gamma} - \partial\hat{\Gamma}/\partial K\frac{\partial}{\partial A}\hat{\Gamma} + \partial\hat{\Gamma}/\partial c\frac{\partial}{\partial L}\hat{\Gamma} - \partial\hat{\Gamma}/\partial L\frac{\partial}{\partial c}\hat{\Gamma}$, and for the coupling to fermions by (3.3.5)) and from this equation it follows that Δ is restricted by BRST symmetry ("consistency conditions for anomalies"), namely $(\hat{\Gamma},\Delta) = 0$. If Δ is BRST exact ($\Delta = \hat{\Gamma}Y$), it can be removed by adding a local counter term to the action, but if Δ is only BRST closed but not BRST exact, one has chiral anomalies, and the theory is nonrenormalizable.

4 On-shell renormalization in QED

Up to this point we determined the renormalized fields and constants by minimal subtraction: by removing poles with $(n-4)^{-1}$ but not adding further finite parts to the Z factors. However, if one defines the masses and coupling constants by experiments at particular kinematical points, one often needs to include finite terms in the Z factors. QED is a case in point. In QED one can impose renormalization conditions which fix the finite parts of Z factors when the electrons and photons are on-shell (on-shell renormalization conditions) because of two reasons: the photon selfenergy corrections are infrared finite at $q^2 = 0$, and $Z_1 = Z_2$ for on-shell renormalization. Neither of these properties holds in QCD.

The photon self energy correction is infrared finite at $q^2 = 0$. For example, at one-loop one finds [13] with ordinary dimensional regularization for arbitrary q^2

$$\begin{aligned} \Pi_{\mu\nu}(q) &= (q^2\eta_{\mu\nu} - q_\mu q_\nu)\Pi(q^2) \\ \Pi^{(1)}(q^2) &= \frac{2\alpha}{\pi}\left(\frac{4\pi\mu^2}{m^2}\right)^{\epsilon}\Gamma(\epsilon)\int_0^1 dx \frac{x(1-x)}{[1+\frac{q^2}{m^2}x(1-x)]^{\epsilon}} \end{aligned} \tag{3.4.1}$$

where $\epsilon = (2 - n/2)$ and m is the mass of the electron while μ is the renormalization mass. The ultraviolet divergent part of $\Pi^{(1)}$ is clearly $\frac{\alpha}{3\pi}\frac{1}{\epsilon}$, and adding the counter term with $Z_3 = 1 - \frac{\alpha}{3\pi}(\frac{1}{\epsilon} + z_3^{\text{finite}})$ where z_3^{finite} is an ultraviolet finite constant, it is clear that the renormalized $\Pi(q^2)$ is finite for generic off-shell momenta. The on-shell renormalization condition for the one-loop correction $\Pi^{(1)}(q^2)$ reads $\Pi^{(1)}(q^2 = 0) = 0$, or, more generally,

$$\Pi(q^2 = 0) = 1 \; . \tag{3.4.2}$$

This is achieved by expanding in terms of ϵ (using $\Gamma(1+\epsilon) = 1 - \epsilon\gamma_E + \ldots$ where $\gamma_E = 0.544$ is Euler's constant) and choosing the finite part of the one-loop contribution to Z_3 appropriately

$$z_3^{\text{finite}} = -\gamma_E + \ln\frac{4\pi\mu^2}{m^2} \tag{3.4.3}$$

Clearly z_3^{finite} is infrared finite when the electron mass m^2 is nonvanishing. Expanding the denominator of (3.4.1) to first order in ϵ when $q^2 \neq 0$, we encounter no infrared divergences when $q^2 \to 0$ either. Thus the renormalized photon selfenergy correction is infrared finite for all q^2, and because Z_3 is infrared finite, also the coupling constant renormalization is infrared finite in QED.

The reason z_3^{finite} contains no infrared divergence is that one finds, upon expanding the integrand, the integral $\int_0^1 dx x(1-x)\ln[1+\frac{q^2}{m^2}x(1-x)]$ which is finite as $q^2 \to 0$. We already see at this point trouble for QCD because in QCD the gluons and ghosts in the loop are massless, and when we set $m^2 = 0$ in the result above we find an infrared divergence in the finite part proportional to $\ln q^2$. This suggests that in QCD one cannot fix Z_3 by on-shell renormalization. We shall discuss this in more detail later in this section.

Next we consider the on-shell renormalization condition for the vertex. In QED, the one-loop renormalized vertex correction for on-shell fermions with momenta p_1 and p_2 satisfying $p_1 = p_2$, is proportional to [13] $\bar{u}(p_2)\Lambda_\mu^{(1)}u(p_1)$ with

$$\begin{aligned}\Lambda_\mu^{(1)} &= \frac{\alpha}{2\pi}\left(\frac{4\pi\mu^2}{m^2}\right)^\epsilon \int_0^1 dz\, z^{1-2\epsilon}\left[(1-\epsilon)^2\Gamma(\epsilon) + \frac{\Gamma(1+\epsilon)}{z^2}(z^2+2z-2-\epsilon z^2)\right] \\ &- \frac{\alpha}{4\pi}\left(\frac{1}{\epsilon} + z_1^{\text{finite}}\right) \qquad (3.4.4)\end{aligned}$$

To obtain this result[13] we used the Dirac equation $\not{p}_1 u(p_1) = imu(p_1)$ and $\bar{u}(p_2)\not{p}_2 = im\bar{u}(p_2)$, and the Gordon identity (which follows by multiplying the Dirac equation with γ^μ)

$$im\gamma^\mu = p_1^\mu + \gamma^{\mu\nu}p_{1,\nu} = p_2^\mu - \gamma^{\mu\nu}p_{2,\nu} \qquad (3.4.5)$$

[13] We could have been more general and consider arbitrary off-shell p_1 and p_2 but since we are interested in studying on-shell renormalization, we only need consider on-shell external momenta. The loop integral reads

$$\int d^n\kappa[\kappa^2 + q^2xy + (x+y)^2m^2 - i\epsilon]^{-3} = \frac{1}{[q^2xy + (x+y)^2m^2]^{1+\epsilon}}$$

For $q^2 = 0$, it yields a factor $z^{-2-2\epsilon}$ where $z = x+y$, and since only $x+y$ appears in the integrand, one can write the result for $q^2 = 0$ as an integral over z. In this way one obtains (3.4.4).

to eliminate factors p_1^μ and p_2^μ. The pole term $-\frac{\alpha}{4\pi}\frac{1}{\epsilon}$ subtracts the ultraviolet divergence and z_1^{finite} is the ultraviolet-finite part of Z_1 which we want to fix by on-shell renormalization

$$Z_1 = 1 - \frac{\alpha}{4\pi}\left(\frac{1}{\epsilon} + z_1^{\text{finite}}\right) + \mathcal{O}(\alpha^2) \tag{3.4.6}$$

Let us now study the infrared singularity structure. The ultraviolet divergence is regulated by taking $n < 4$, and there it is cancelled by the term $-\frac{\alpha}{4\pi}\frac{1}{\epsilon}$. After adding the ultraviolet counterterm one obtains an expression which one can analytically continue to $n > 4$. The z integral over $z^{1-2\epsilon}(\frac{-2}{z^2})$ needs $n > 4$ in order that it be defined. For $n \downarrow 4$ it becomes infrared divergent, and yields an infrared pole $\frac{1}{\epsilon}$. Requiring that the vertex correction vanishes for $q^2 = 0$, i.e., imposing the on-shell renormalization condition

$$\Lambda_\mu(p_1, p_2, m)\mid_{p_1=p_2, p_j^2+m^2=0} = ie\gamma_\mu \tag{3.4.7}$$

fixes z_1^{finite} to

$$z_1^{\text{finite}} = \frac{2}{\epsilon} + 4 - 3\gamma_E + 3\ln\frac{4\pi\mu^2}{m^2} \tag{3.4.8}$$

The pole $2/\epsilon$ is the infrared divergence in Z_1 with on-shell renormalization. Note that both the infrared and the ultraviolet pole contribute to the term with $\ln(4\pi\mu^2/m^2)$, and this yields the coefficient 3. We conclude that Z_1 has both an ultraviolet and an infrared divergence.

$$Z_1 = 1 - \frac{\alpha}{4\pi}\left(\frac{1}{\epsilon_{UV}} + \frac{2}{\epsilon_{IR}} + 4 - 3\gamma_E + 3\ln\frac{4\pi\mu^2}{m^2}\right) \tag{3.4.9}$$

The renormalized vertex correction still has infrared divergences at $q^2 \neq 0$, but they cancel in cross sections, as we discuss later.

Finally, consider the fermion wave function renormalization Z_2 in QED at the one-loop level. We shall derive two results:

(i) the expression for z_2^{finite} contains also an infrared divergence

(ii) both the poles in Z_1 and Z_2 and the finite parts z_1^{finite} and z_2^{finite} are equal.

The one-loop fermion selfenergy correction is given by $\bar{u}(p)\Sigma^{(1)}u(p)$ with [13]

$$\begin{aligned}
\Sigma^{(1)}(\not{p}, m) &= (ie\mu^\epsilon)^2 \int \frac{d^n k}{(2\pi)^n} \frac{\gamma_\alpha(-i\not{k} + m)\gamma^\alpha}{[(k)^2 - i\epsilon][(k-p)^2 + m^2 - i\epsilon]} \\
&= -e^2\mu^{2\epsilon} \int_0^1 dx \int \frac{d^n k}{(2\pi)^n} \frac{[(n-2)i\not{k} + nm]}{[(k-px)^2 + p^2 x(1-x) + m^2(1-x)]^2} \\
&= \frac{-\alpha}{4\pi} \left(\frac{4\pi\mu^2}{m^2}\right)^\epsilon i\Gamma(\epsilon) \int_0^1 dx \frac{[(2-n)mx + nm + (n-2)(i\not{p} + m)x]}{[(1-x)^2 + \frac{p^2+m^2}{m^2} x(1-x)]^\epsilon}
\end{aligned} \tag{3.4.10}$$

We used $\int d^n\kappa/(\kappa^2 + M^2)^2 = i\pi^{n/2}\Gamma(\epsilon)M^{-2\epsilon}$ with $\epsilon = 2 - n/2$ for finite ϵ. There is an ultraviolet divergence due to $\Gamma(\epsilon)$, but the complete selfenergy graph is infrared finite (for $p^2 = 0$ the x-integral is still finite). However, as we shall see, expansion in terms of ϵ will produce infrared divergences in Z_2.

We must fix the finite parts of both Z_2 and Z_m, hence we must impose two conditions on $\sum$. We choose the following on-shell renormalization conditions

$$\sum(\not{p}, m) \mid_{i\not{p}+m=0} = 0, \left(\frac{\partial}{\partial \not{p}} \sum(\not{p}, m)\right) \mid_{i\not{p}+m} = 1 \tag{3.4.11}$$

In other words, we require that near $i\not{p} + m$ (so near $p^2 + m^2 = 0$), the proper fermion selfenergy correction has the form of a free fermion

$$\sum(\not{p}, m) = \not{p} - im + \mathcal{O}(\not{p} + im)^2 \tag{3.4.12}$$

The ultraviolet divergences in $\Sigma^{(1)}(\not{p}, m)$ are cancelled by the counter term

$$\Sigma^{(1)}(\text{counter}) = (-i)(Z_2 i\not{p} + Z_2 Z_m m) \tag{3.4.13}$$

where the pole terms in Z_2 and Z_m follow from the ultraviolet divergent part of the selfenergy

$$\begin{aligned}
\Sigma^{(1)}(\text{div}) &= \frac{-\alpha}{4\pi}\frac{i}{\epsilon}(3m + i\not{p} + m) \\
Z_2 &= 1 - \frac{\alpha}{4\pi}\left(\frac{1}{\epsilon} + z_2^{\text{finite}}\right); Z_m = 1 - \frac{3\alpha}{4\pi}\left(\frac{1}{\epsilon} + z_m^{\text{finite}}\right)
\end{aligned} \tag{3.4.14}$$

The ultraviolet-finite parts z_2^{finite} and z_m^{finite} are fixed by requiring that the terms of order $(i\not{p}+m)^0$ and $(i\not{p}+m)$ in $\sum^{(1)}+\sum^{(1)}$ (counter) cancel. (As before we first renormalize the ultraviolet divergences away at $n<4$, and then continue to $n>4$ to define the x integral). Expanding the denominator of the integrand in (3.4.10) in terms of $i\not{p}+m$ (using $p^2+m^2=(i\not{p}+m)(2m)-(i\not{p}+m)^2$) produces (still for finite ϵ!) the following result

$$\begin{aligned}&\frac{1}{(1-x)^{2\epsilon}}\left[1-\epsilon\frac{p^2+m^2}{m^2}\frac{x(1-x)}{(1-x)^2}+\dots\right]\\&=\frac{1}{(1-x)^{2\epsilon}}-\frac{2\epsilon x}{(1-x)^{1+2\epsilon}}\frac{i\not{p}+m}{m}+\mathcal{O}(i\not{p}+m)^2\end{aligned}\tag{3.4.15}$$

Note that whereas the integral over x of the terms independent of $i\not{p}+m$ is finite, the terms linear in $i\not{p}+m$ give a divergent x-integral as $\epsilon\to 0$. Thus we obtain an infrared divergence in Z_2 but not in Z_m. The integral over x is easy, and the final result is as follows

$$\begin{aligned}Z_2&=1-\frac{\alpha}{4\pi}\left(\frac{1}{\epsilon_{UV}}+\frac{2}{\epsilon_{IR}}+4-3\gamma_E+3\ln\frac{4\pi\mu^2}{m^2}\right)\\Z_m&=1-\frac{3\alpha}{4\pi}\left(\frac{1}{\epsilon_{UV}}+\frac{4}{3}-\gamma_E+\ln\frac{4\pi\mu^2}{m^2}\right)\end{aligned}\tag{3.4.16}$$

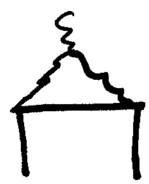
Comparison of the results for Z_1 and Z_2 in QED shows that indeed $Z_1=Z_2$. Hence, even though the factors Z_1 and Z_2 each contain infrared divergences, off-shell the renormalized Green's functions are still infrared finite because the Z_1 and Z_2 factors cancel. A closed fermion loop has as many vertices as propagators, hence for a closed fermion loop all Z_1 and Z_2 cancel. For an open line there is one more Z_1 factor than Z_2 factors in truncated diagrams, but for S-matrix elements one must multiply each external line by $R^{-1/2}$ (where R is the residue of the pole) and this yields for each external line a factor $\sqrt{Z_2}$ (and no further finite terms in on-shell renormalization). Thus also for open lines the Z_1 and Z_2 factors cancel in QED.

Let us now study the same problems in QCD. For the 1-loop vertex correction and fermion selfenergy, the color factors are as follows

$$T_b(F)T_a(F)T_b(F) = \left[-C_2(F) + \tfrac{1}{2}C_2(G)\right] T_a(F) = 1/(2N)T_a(F)$$

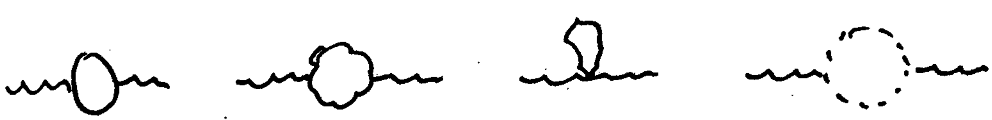
$$T_a(F)T_a(F) = C_2(F) = (N^2-1)/(2N)$$

where $T_a(F)$ denotes the antihermitian group generators in the fundamental representation and G denotes the adjoint representation. We used $T_b(F)T_b(F) = -C_2(F)$ and $T_a(G)T_a(G) = f_{pa}{}^q f_{qa}{}^p = -C_2(G)$. Further $trT_a(F)T_b(F) = -\frac{1}{2}\delta_{ab}$ and tracing over a and b yields $NC_2(F) = -\frac{1}{2}\dim G$ with $\dim G = N^2 - 1$. In addition the QCD selfcoupling gives an additional vertex correction



Hence, in QCD, one has $Z_1 \neq Z_2$ and a given Green function would contain a factor $(\frac{Z_1}{Z_2})^k$ for some integer k. Since both Z_1 and Z_2 contain infrared divergences if one were to use on-shell renormalization (just as in QED), Green functions in QCD would contain infrared divergences for on-shell renormalization.

Although the fact that $Z_1 \neq Z_2$ rules out on-shell renormalization in QCD, it is still interesting to study whether also the Z_3 for the gluon self-energy has infrared divergences. In addition to the quark loop, there are 3 pure- QCD graphs which have no counter part in QED



(3.4.17)

The last graph is due to the ghosts, and the one- but -last graph vanishes in dimensional regularization because it is scaleless. The gluon loop and the ghost loop yield a result proportional to

$$\begin{aligned}\int_0^1 dx \int d^n k \quad &+ \quad \int \frac{1}{[(k-qx)^2 + q^2 x(1-x) - i\epsilon]^2} (N_{\alpha\beta} - M_{\alpha\beta}) \\ N_{\alpha\beta} \quad &= \quad k_\alpha k_\beta (4n-6) + q_\alpha q_\beta (n-6) + (q_\alpha k_\beta + k_\alpha q_\beta)(3-2n) \\ & \qquad + \eta_{\alpha\beta}(2k^2 - 2k\cdot q + 5q^2) \\ M_{\alpha\beta} \quad &= \quad -k_\alpha q_\beta + k_\alpha k_\beta \end{aligned} \tag{3.4.18}$$

The integration over k of the terms which have no k in the numerator yields a factor $[q^2 x(1-x)]^{-\epsilon}$, and thus the ultraviolet finite terms contain a term with $\ln q^2$. This term prevents on-shell renormalization at $q^2 = 0$, even though the x-integral is not divergent.

Since on-shell renormalization is not viable for QCD, one needs other renormalization conditions to fix the finite parts of the Z factors. The most used scheme is minimal subtraction (MS) or modified minimal subtraction ($\overline{\text{MS}}$). Thus one cannot use on-shell renormalization in QCD, and this has actually a physical explanation, namely due to confinement the notion of on-shell incoming and outgoing gluons makes no sense. Note that a perturbative property gives in this case information about a nonperturbative property. To avoid confusion, note that renormalized Green's functions in QED do develop infrared divergences when the momenta go on-shell. The Z_1 and Z_2 still cancel, but the graphs without counter terms themselves develop infrared divergences. These infrared divergences are due to virtual photons which are nearly on-shell, and are cancelled by Bremsstrahlung photons (which are, of course, on-shell). For further details we refer to the chapter on infrared divergences.

5 Nonlinear gauges

Up till now we considered linear gauges and found it convenient to eliminate the auxilary fields d_α by replacing them by $\gamma_{\alpha\beta} F^\beta$. Most useful gauges are linear in fields, for example $\partial^\mu A_\mu{}^a$ for unbroken gauge theories, or $\partial^\mu A_\mu{}^a - \frac{1}{2} g v \chi^a$ for the spontaneously broken SU(2) Higgs model. However, nonlinear gauges can also be used, for example the Dirac gauge $\partial^\mu A_\mu{}^a + \alpha A_\mu A^\mu$ in QED [14]. Variation of this gauge fixing function yields a ghost action with interacting ghosts for QED. A more interesting example is $SU(2)$ gauge theory, with gauge fields $B_\mu = A_\mu^3$ and $W_\mu^\pm = \frac{1}{\sqrt{2}} A_\mu{}^1 \pm i A_\mu^2$. Here one may fix the gauge by

$$\mathcal{L}_{fix} = \frac{1}{2\alpha}(\partial^\mu B_\mu)^2 + \frac{1}{\beta}[(\partial^\mu - igB^\mu)W_\mu^+][(\partial^\nu + igB^\nu)W_\nu] \tag{3.5.1}$$

More generally, in spontaneously broken gauge theories, where the gauge group G is broken down to a subgroup H, one may fix the gauge for the gauge fields associated with the coset G/H by using H-covariant derivatives in $\mathcal{L}_{fix}$, while the gauge of the H-gauge fields B_μ^i may be fixed with the usual gauge $\partial^\mu B_\mu^i$. [15] For applications one may think of $SU(5)$ broken down to $SU(3) \otimes SU(2) \otimes U(1)$, or $SU(3) \otimes SU(2) \otimes U(1)$ broken down to $SU(3) \otimes U(1)_{EM}$.

When F^α is nonlinear in fields, it is better to keep the auxiliary field d_α in the theory. Then one does not need the antighost field equation as a second Ward identity, but we must couple d_α to a source I^α. The quantum action reads now

$$\begin{aligned} S_{qu} \;&=\; S_{cl} + \delta_B \left[b_\alpha F^\alpha - \frac{1}{2} b_\beta d_\alpha \gamma^{\alpha\beta} \right] \Big/ \Lambda \\ &\quad + K_I R_\alpha^I c^\alpha + L_\alpha \frac{1}{2} u f^\alpha{}_{\beta\gamma} c^\gamma c^\beta (-)^{1+\sigma(\beta)} \\ &\quad + J_I \phi^I + \beta_\alpha c^\alpha + b_\alpha \gamma^\alpha + d_\alpha I^\alpha \end{aligned} \tag{3.5.2}$$

where $\sigma(\beta) = 1$ for symmetries with anticommuting parameters (such as local supersymmetry). Since $\delta_B b_\alpha = \Lambda d_\alpha$ is linear in quantum fields, we do not need a new source for the BRST variation of b_α. The last term is BRST invariant by itself. If the BRST Jacobian is really unity (without regularization this is the case, but, as we have stressed, one should regulate or use cohomology) we get the following Ward identity for connected and disconnected graphs

$$\left(J_I \frac{\delta}{\delta K_I} Z + \beta_\alpha \frac{\delta}{\delta L_\alpha} Z - \delta Z / \delta I^\alpha \gamma^\alpha \right) = 0. \tag{3.5.3}$$

Passing to W, and then to $\Gamma = W - \int(J\phi + \beta c + b\gamma + d_\alpha I^\alpha)$ we get, using $\partial W/\partial I^\alpha = d_\alpha$

$$\partial\Gamma/\partial\phi^I \frac{\partial}{\partial K_I}\Gamma + \partial\Gamma/\partial c^\alpha \frac{\partial}{\partial L_\alpha}\Gamma - d_\alpha \frac{\partial}{\partial b_\alpha}\Gamma = 0 \tag{3.5.4}$$

After the Legendre transform, Γ depends on $\phi^I, c^\alpha, b_\alpha, d_\alpha, K_I$ and L_α, but no longer on $J_i, \beta_\alpha, \gamma^\alpha$ and I^α. From (3.5.4) we read off the following relations between the Z factors

$$d_\alpha = \sqrt{Z_d} d_\alpha^{\text{ren}}, K_i = \left(\frac{Z_3^{gh}}{Z_3 Z_d}\right)^{\frac{1}{2}} K_i^{\text{ren}}, L_\alpha = \frac{1}{\sqrt{Z_d}} L_\alpha^{\text{ren}} \tag{3.5.5}$$

and we obtain for the renormalized operator S the following expression (omitting superscripts "ren")

$$\mathcal{S} = \left(\frac{\partial}{\partial\phi^I}S\right)\frac{\partial}{\partial K_I} + \left(\frac{\partial}{\partial K_I}S\right)\frac{\partial}{\partial\phi^I} - \left(\frac{\partial}{\partial c^\alpha}S\right)\frac{\partial}{\partial L_\alpha} - \left(\frac{\partial}{\partial L_\alpha}S\right)\frac{\partial}{\partial c^\alpha} - d_\alpha\frac{\partial}{\partial b_\alpha} \tag{3.5.6}$$

With respect to the Slavnov-Taylor operator used in the approach without auxiliary fields there are two differences: we now have S instead of $\hat{S} = S - S(\text{fix})$, and there is the extra term $-d_\alpha \frac{\partial}{\partial b_\alpha}$.

To prove nilpotency of the Slavnov-Taylor operator, $\mathcal{S}^2 = 0$, we would like to write the term $-d_\alpha \frac{\partial}{\partial b_\alpha}\Gamma$ in (3.5.4) as $\partial\Gamma/\partial H^\alpha \frac{\partial}{\partial b_\alpha}\Gamma$ for some external source H^α. We cannot use I^α, as it is gone after the Legendre transform. Hence we introduce an extra term $-d_\alpha H^\alpha$ into S_{qu}. This term preserves of course BRST invariance. Then the source H^α only appears in Γ at the tree level, as $-d_\alpha H^\alpha$ cannot generate 1PI graphs (except at order $\hbar^0$). The last term of (3.5.4) is now replaced by $\partial\Gamma/\partial H^\alpha \frac{\partial}{\partial b_\alpha}\Gamma$, and in the Slavnov-Taylor operator the term $-d_\alpha \frac{\partial}{\partial b_\alpha}$ is replaced by

$$\left(\frac{\partial}{\partial H^\alpha}S\right)\frac{\partial}{\partial b_\alpha} + \left(\frac{\partial}{\partial b_\alpha}S\right)\frac{\partial}{\partial H^\alpha} \tag{3.5.7}$$

The proof of nilpotency, $\mathcal{S}^2 = 0$, can now be given as before because we have pairs of fields with opposite statistics: $\left(K_I, \phi^I\right), (L_\alpha, c^\alpha)$ and (H^α, b_α). (In the "antifield formalism", K and L are the antifields for ϕ and c, respectively, and H^α is the antifield for the antighost). Having proven nilpotency, we can use $\frac{\partial}{\partial H^\alpha}\Gamma(\text{div}) = 0$ and $\frac{\partial}{\partial H^\alpha}S = -d_\alpha$ to revert to the expression of $\mathcal{S}$ in (3.5.6) with $-d_\alpha \frac{\partial}{\partial b_\alpha}$.

The divergences are now of the form "gauge-invariant terms $+\mathcal{S}X$", and we must write down the most general X which can now depend on d_α (as well as on the other fields, but not on H^α for reasons explained). From here, renormalizability proceeds as before.

6 Noncovariant algebraic gauges

Noncovariant algebraic gauges like the axial gauge $A_3^a = 0$, the temporal gauge $A_0^a = 0$, the light-cone gauge $n \cdot A = 0$ with $n^2 = 0$, and the planar gauge $n \cdot A - \varphi = 0$ play a role in QCD (in particular in the Altarelli-Parisi equation for large $x = Q^2/2P \cdot Q$ and its counterpart, the B-Fadin-Lipatov-Kurayev equation for small x, and the earlier Gribov-Lipatov equation for abelian gauge theories). Also for finite temperature field theory, noncovariant algebraic gauges have been used. A review of the state of the art in 1990 is given in [16–18].

The first three noncovariant algebraic gauges are defined by

$$\mathcal{L}_{\text{fix}} = -\frac{1}{2\xi}(n \cdot A^a)^2 \quad , \xi \to 0 \tag{3.6.1}$$

where $n^2 > 0, n^2 < 0$ and $n^2 = 0$. One calls them "unweighted gauges" because one can also write a delta function $\delta(n \cdot A^a)$ in the path-integral, instead of exponentiating the gauge function $n \cdot A^a$ to obtain $\mathcal{L}_{\text{fix}}$. Another, weighted, noncovariant gauge is obtained from the action $-\frac{1}{2n^2}(\partial_\mu \varphi)^2 - (n \cdot A - \varphi)d$ by integrating out d

$$\mathcal{L}_{\text{fix}} = -\frac{1}{2\xi} n \cdot A^a \frac{\partial^2}{n^2} n \cdot A^a \quad , \xi = 1 \tag{3.6.2}$$

The "generalized planar gauge" corresponds to the case $\xi \neq 1$.

An immediately clear advantage of the gauges $n \cdot A = 0$ is that the Faddeev-Popov ghosts are free: variation of $n \cdot A$ yields $bn \cdot Dc$ for the ghost action, and the interaction term $bn \cdot Ac$ vanishes. In planar gauges, the ghost action is $bn \cdot \partial c + bn \cdot Ac$ so ghosts can still couple to gauge fields, but their propagators are proportional to

$(n \cdot p)^{-1}$, and using dimensional regularization, loops with such ghosts still vanish. In this case, $n \cdot A$ does not couple to ghosts, and, in fact, it is a free field, $\Box n \cdot A = 0$.

The propagator for the class of algebraic gauges in (3.6.1) can be formally obtained by simply inverting the field equation

$$\langle A^a_\mu A^b_\nu \rangle = \frac{-i\delta^{ab}\Pi^{(0)}_{\mu\nu}}{(2\pi)^4(k^2 - i\epsilon)}; \Pi^{(0)}_{\mu\nu} = \left[\eta_{\mu\nu} - \frac{(k_\mu n_\nu + n_\mu k_\nu)}{n \cdot k} + k_\mu k_\nu \frac{n^2 + \xi k^2}{(n \cdot k)^2}\right] \quad (3.6.3)$$

with $\epsilon > 0$ and $k^2 = \vec{k}^2 - k_0^2$. For the general planar gauge one finds

$$\Pi^{(0)}_{\mu\nu} = \eta_{\mu\nu} - \frac{(k_\mu n_\nu + n_\mu k_\nu)}{n \cdot k} + k_\mu k_\nu \frac{(1-\xi)n^2}{(n \cdot k)^2} \quad (3.6.4)$$

while the planar gauge is obtained by setting $\xi = 1$ in this result.

One can apply BRST Ward identities to clarify these results. Making a BRST variation of $\langle b_a(x) A^b_\nu(y)\rangle$, one finds at tree level the following identity for the algebraic gauges, using $\delta b_a = \frac{1}{\xi} n \cdot A^a$ and $\langle b_a c^b \rangle \sim \delta^b_a \frac{1}{n \cdot k}$

$$\frac{1}{\xi} n^\mu \Pi^{(0)}_{\mu\nu} = k_\mu \frac{1}{n \cdot k} \quad (3.6.5)$$

It is straight forward to verify that the propagator in (3.6.3) satisfies this relation. In the unweighted gauges ($\xi \to 0$), one finds instead

$$n^\mu \Pi^{(0)}_{\mu\nu} = 0 \quad (3.6.6)$$

Another property of the propagator is transversality on the mass-shell. As one may directly verify

$$k^\mu \Pi^{(0)}_{\mu\nu} = 0 \text{ at } k^2 = 0 \quad (3.6.7)$$

which holds whether or not $\xi \to 0$. So, for example, in the light-cone gauge the propagator is still transverse. In general, the numerator of the propagator (i.e. the polarization tensor $\Pi_{\mu\nu}$) can be written as a sum over two physical polarizations

$$\Pi^{(0)}_{\mu\nu} = \sum_{m=1}^{2} \epsilon^m_\mu \epsilon^m_\nu \quad (3.6.8)$$

One can obtain propagators by eliminating some field components from the Yang-Mills action (for example, by setting $A_3^a = 0$), and by naively inverting the kinetic operator. One finds then new singularities ("spurious singularities") of the form $1/n \cdot p$ and/or $1/(n \cdot p)^2$. Some early attempts proposed to replace $1/n \cdot p$ by the principal value $P(1/n \cdot p)$ and $1/(n \cdot p)^2$ by $-\frac{\partial}{\partial(n \cdot p)} P(1/n \cdot p)$. [19] However, these propagators lead to a nonunitary theory: they contain ghosts because in the propagator

$$D_{\mu\nu}(p) = \frac{\Pi_{\mu\nu}(p)}{p^2 - i\epsilon} \tag{3.6.9}$$

the polarization tensor $\Pi_{\mu\nu}(p)$ which should be equal to $\sum_{\alpha=1}^{2} \epsilon_\mu^\alpha \epsilon_\nu^\alpha$ at $p^2 = 0$, in fact turns out not to be positive definite. [20]

By applying canonical quantization, one obtains the propagator as a $\int d^3k$ integral, and by extending this result, as usual in field theory, to a $\int d^3k dk_0$ integral, one deduces the correct interpretation of $1/n \cdot p$. Namely now

$$1/n \cdot p \longrightarrow \frac{1}{n \cdot p + i\epsilon \text{ sign } n^\star \cdot p} \tag{3.6.10}$$

where $n^\star$ is defined by $n^\star \cdot n = 1$ and n and $n^\star$ are both orthogonal to ϵ_μ^α. There are still physical and spurious poles, but the latter are derived, not assumed. As far as known, no complete proof of unitarity based on the cutting rules has been given for these propagators, but indirect evidence suggests that unitarity holds. (In the light-cone gauge there are formal perturbative arguments based on the Kugo-Ojima canonical formalism which indicate that unitarity holds [22]). A direct calculation up to two loop in light-cone gauge using the known counter terms (which are nonlocal for proper Green functions) has shown that the theory is unitarity. [27]

Computing loops with these propagators, one finds local but also nonlocal divergences, and in particular local divergences which have a different tensor structure from terms in the action. Thus renormalizability seems violated. However, at least in the light-cone gauge, it is possible to prove, order-by-order in loops, that in the connected Green functions only local divergences remain [22], and these local divergences are of the same form as the terms in the action. (More precisely, the local 3-gluon and 4-gluon divergences are proportional to the 3-gluon and 4-gluon terms in the classical action). This locality of counter terms holds both in pure Yang-Mills theory as well as in the presence of quarks.

In the lightcone gauge, also much is known about the general structure of divergences in proper Green's functions. For example, in pure Yang-Mills theory, having subtracted all divergences at $(n-1)$ loops, one finds at n-loops the usual local divergences plus one particular nonlocal divergent structure. (The reason that only one nonlocal divergent structure remains can be understood from the fact that if one lets $n^\star$ tend to n (which leads, of course, to incorrect propagators) then all divergences should become local. [28] There is only one nonlocal structure with this property.) One can then renormalize Green functions multiplicatively by renormalizing gauge fields as ${A_\mu}^a = {Z_\mu}^\nu {A_\nu}^{a,\text{ren}}$. These matrices ${Z_\mu}^\nu$ are proportional to the one nonlocal structure (so these Z's depend on fields), and further they contain the usual power series in g^2 and $(n-4)^{-1}$. There is also the usual coupling constant renormalization, and if one includes fermions, they also have a wave function renormalization matrix.

In gauges $n \cdot A = 0$ which are not light-cone gauges ($n^2 \neq 0$) the situation is much more complicated. The general structure of divergences of Green's functions is not known. It is known that there are both local and nonlocal divergences, but multiplicative renormalization seems lost.

Why are there nonlocal divergences at all? The existing proofs that counter terms are local are based on standard power counting. Only if a theory is power-counting renormalizable the divergences will be local. These theories are not power-counting renormalizable, essentially because $(n \cdot p)^{-1}$ does not behave like $1/p$ when p does not lie along n. As a result, differentiating a proper graph w.r.t. an external momentum will not always produce a more convergent graph. More detailed analysis confirms this simple argument if one uses the correct propagators with n and $n^\star$.

Explicit one-loop calculations in the light-cone gauge for pure Yang-Mills theory have been completed for the 2, 3 and 4 point 1PI graphs. No problems due to the spurious poles were caused by the Wick rotation (again, using the $n, n^\star$ propagator). At the two-loop level, there exist calculations for the gluon selfenergy; the divergent structure has been checked to agree with the one-loop structure. [27] In other gauges, only one loop calculations have been performed for the gluon and quark selfenergy, and the quark-gluon vertex, and here the complicated structure of divergences was found.

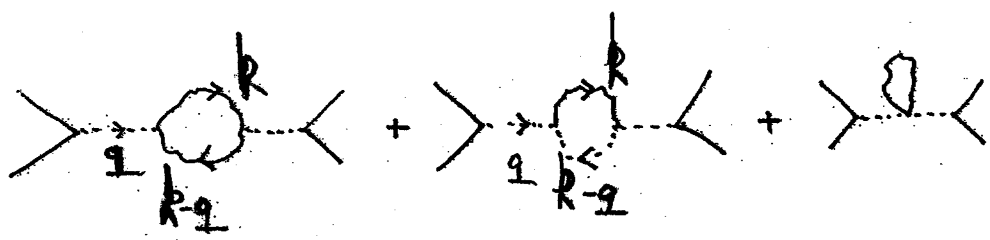
Another approach to gauge field quantization is based on the "light-front approach". This was begun by Dirac in 1949 for QED, and extended to QCD by Kogut and Soper. [29] One chooses as coordinates $x^\pm = (x^3 \pm x^0)1/\sqrt{2}$ and $x_\perp$. Then it is quite natural to also choose the light-cone gauges we have been discussing. Using x^+ (or x^-) as time coordinate, the canonical quantization leads to completely different results (more primary second class constraints, the same first-class constraints). Using for example $A^a_- = 0$ as gauge with x^+ as time, one recovers the old light-front results of the 1960's. The propagators turn out to contain a principal value definition. In this approach one finds only creation and absorption operators for physical (transverse) degrees of freedom, hence $1/\partial_-$ (which is the only nonlocal term in the propagator) must not have an imaginary part. This selects the principal value [29]. (In the non-light front approaches one finds also modes which are nonphysical, and therefore one does not find a principal value prescription for the $(n \cdot p)^{-1}$ terms).

To finish, let us mention that if one uses $A_+ = 0$ (still with x^+ as time) as gauge condition in the abelian case, then the propagator is of the $n, n^\star$ form but with n and $n^\star$ interchanged. In that case one can show that the theory (divergences, unitarity issues, Hilbert spaces) is equivalent to the formulation using t as time and $A_+ = 0$ as gauge [28].

7 Asymptotic freedom in the Coulomb gauge

As an application of the Coulomb gauge we consider asymptotic freedom. Our aim is to illustrate by a realistic example how to evaluate loop diagrams with Lorentz-noncovariant gauges. However, this example has also physical interest by itself because it gives an explanation why asymptotic freedom arises: due to antiscreening.

There are three diagrams which contribute to the 1-loop correction of the β function



(3.7.1)

The dotted lines indicate Coulomb gluons (due to A_0^a) which couple to external on-shell fermions. The wavy lines indicate transversal gluons (more precisely gluons due to A_j^a with $j = 1, 2, 3$). Ghosts do not couple to Coulomb gluons.

In the center-of-mass frame q_0 vanishes, so there is only exchange of the 3-momentum $\vec{q}$. The propagators of a Coulomb gluon and a transversal gluon are, respectively

$$\begin{aligned} \langle A_0^a(\vec{k}) A_0^b(-\vec{k}) \rangle &= \frac{-i\eta_{00}\delta^{ab}}{\vec{k}^2} = \frac{i}{\vec{k}^2}\delta^{ab} \\ \langle A_i^a(\vec{k}) A_j^b(-\vec{k}) \rangle &= \frac{-i}{k^2 - i\epsilon} P_{ij}(\vec{k})\delta^{ab}; P_{ij}(\vec{k}) = \delta_{ij} - \frac{k_i k_j}{\vec{k}^2} \end{aligned} \tag{3.7.2}$$

These propagators can be derived by choosing the gauge fixing term $-\frac{1}{2\xi}(\partial^j A_j^a)^2$, inverting the field operator, and taking the limit $\xi \to 0$. The first graph contains two vertices of the form $\partial_0 A_j A_0 A_j$, while the second graph contains two vertices of the form $\partial_i A_0 A_i A_0$. We use dimensional regularization to evaluate these loops. Then the seagull graph (the third graph) does not contribute since massless tadpoles (scaleless integrals) vanish in dimensional regularization.

Using these expressions, the 1-loop corrections to the Coulomb gluon propagator read [30]

$$D_{00}^{ab}(q) = \frac{-i\delta^{ab}}{\vec{q}^{\,2}} \left[1 - \frac{2ig^2}{\vec{q}^{\,2}} C_2(G) \left\{ A_{tt} + A_{tC} \right\} \right] \tag{3.7.3}$$

where A_{tt} denotes the contribution from the loop with two transversal gluons

$$A_{tt} = \int \frac{d^n k}{(2\pi)^n} (-) \left(k_0 - \frac{1}{2} q_0 \right)^2 \frac{P_{ij}(\vec{k}) P_{ij}(\vec{k} - \vec{q})}{(k^2 - i\epsilon)[(k-q)^2 - i\epsilon]} \tag{3.7.4}$$

while A_{tC} contains the contribution of the loop with one transversal gluon and one Coulomb gluon

$$A_{tC} = 2 \int \frac{d^n k}{(2\pi)^n} q_i q_j \frac{1}{(\vec{k} - \vec{q})^2} \frac{P_{ij}(\vec{k})}{k^2 - i\epsilon} \tag{3.7.5}$$

The symbol $C_2(G)$ denotes the quadratic Casimir operator in the adjoint representation, $f_{pa}{}^q f_{qb}{}^p = -\delta_{ab} C_2(G)$ where the generators in the fundamental representation of $SU(N)$ are normalized to $TrT_a(F)T_b(F) = -\frac{1}{2}\delta_{ab}$.

Setting $q_0 = 0$ and using $\delta_{jj} = n - 1$, we obtain

$$\begin{aligned} A_{tt} &= \int \frac{d^n k}{(2\pi)^n} (-k_0^2) \left\{ n - 3 + \frac{(\vec{k}^2 - \vec{k} \cdot \vec{q})^2}{\vec{k}^2 (\vec{k} - \vec{q})^2} \right\} \frac{1}{(k^2 - i\epsilon)((k-q)^2 - i\epsilon)} \\ A_{tC} &= \int \frac{d^n k}{(2\pi)^n} 2 \left(\vec{q}^{\,2} - \frac{(\vec{q} \cdot \vec{k})^2}{\vec{k}^2} \right) \frac{1}{(k^2 - i\epsilon)(\vec{k} - \vec{q})^2} \end{aligned} \tag{3.7.6}$$

If one adds A_{tt} and A_{tC}, important simplifications occur [31] but we are interested in A_{tt} and A_{tC} separately.[14]

For A_{tC} we need to evaluate the following integral

$$I_2 = \int \frac{\vec{k}^2 \vec{q}^{\,2} - (\vec{k} \cdot \vec{q})^2}{\vec{k}^2 (k^2 - i\epsilon)(\vec{k} - \vec{q})^2} \;, \int \equiv \int \frac{d^n k}{(2\pi)^n} \tag{3.7.7}$$

For A_{tt} we write all terms such that they have the same denominator

$$A_{tt} = \int \frac{(-k_0^2)\{(n-3)\vec{k}^2(\vec{k} - \vec{q})^2 + (\vec{k}^2 - \vec{k} \cdot \vec{q})^2\}}{\vec{k}^2 (\vec{k} - \vec{q})^2 [k^2 - i\epsilon][(k-q)^2 - i\epsilon]} \tag{3.7.8}$$

[14] We thank M. Kreuzer for discussions.

The numerator of A_{tt} can be rewritten as $(-k_0^2)\{(n-2)\vec{k}^2(\vec{k}-\vec{q})^2+(\vec{k}\cdot\vec{q})^2-\vec{k}^2\vec{q}^2\}$. Using this expression, and substituting $-k_0^2 = k^2 - \vec{k}^2$ into the terms without $(n-2)$, A_{tt} decomposes into 3 basic integrals

$$\begin{aligned} A_{tt} &= I_1 - I_2 + I_4 \\ I_1 &= (n-2)\int \frac{(-k_0^2)}{(k^2-i\epsilon)(k-q)^2-i\epsilon)} \\ I_4 &= \int \frac{\vec{k}^2\vec{q}^{\,2}-(\vec{k}\cdot\vec{q})^2}{(k^2-i\epsilon)((k-q)^2-i\epsilon)(\vec{k}-\vec{q})^2} \end{aligned} \tag{3.7.9}$$

We used that the numerator of I_2 does not change if one replaces k by $q-k$. Hence

$$A_{tC} = 2I_2; A_{tt} = I_1 - I_2 + I_4; A_{\text{total}} = A_{tC} + A_{tt} = I_1 + I_2 + I_4 \tag{3.7.10}$$

The calculation of I_1 is standard

$$\begin{aligned} I_1 &= -(n-2)\int \frac{k_0^2 \int_0^1 dx}{[(k-\vec{q}x)^2+\vec{q}^{\,2}x(1-x)-i\epsilon]^2} \\ &= -(n-2)\int \frac{k_0^2 \int_0^1 dx}{(k^2+L)^2} \text{ with } L = \vec{q}^{\,2}x(1-x) \\ &= -\frac{(n-2)}{(2\pi)^n}\cdot i\pi^{n/2}\int_0^1 \frac{(-\frac{1}{2})\Gamma(1-n/2)}{(L)^{1-n/2}}dx \\ &= \frac{i}{2}(n-2)\frac{\pi^{n/2}}{(2\pi)^n}\frac{1}{(1-\frac{n}{2})}\left[\frac{1}{2-\frac{n}{2}}+\dots\right] J \text{ with } J = \int_0^1 \frac{dx\vec{q}^{\,2}x(1-x)}{[\vec{q}^{\,2}x(1-x)]^{2-n/2}} \end{aligned} \tag{3.7.11}$$

We wrote $1/L^{1-n/2}$ as $\vec{q}^{\,2}x(1-x)/L^{2-n/2}$ for later use, and used

$$\int d^n k \frac{k_\mu k_\nu}{(k^2+L-i\epsilon)^\alpha} = i\pi^{\frac{1}{2}n}\frac{1}{2}\eta_{\mu\nu}\frac{\Gamma(\alpha-\frac{1}{2}n-1)}{\Gamma(\alpha)}L^{\frac{1}{2}n-\alpha+1} \tag{3.7.12}$$

By expanding the denominator of J we obtain

$$\begin{aligned} J &= \int_0^1 dx\vec{q}^{\,2}x(1-x) - (2-\frac{n}{2})\int_0^1 dx\vec{q}^{\,2}x(1-x)\ln\{\vec{q}^{\,2}x(1-x)\} + \cdots \\ &= \frac{1}{6}\vec{q}^{\,2} - \left(2-\frac{n}{2}\right)\frac{1}{6}\vec{q}^{\,2}\ln\vec{q}^{\,2} + \text{terms with } \vec{q}^{\,2} \end{aligned} \tag{3.7.13}$$

Hence, the terms with $\ln\vec{q}^{\,2}$ in I_1 are given by

$$I_1 = \frac{i}{2}\cdot 2\cdot\frac{\pi^2}{16\pi^4}(-1)\left(-\frac{1}{6}\right)\vec{q}^{\,2}\ln\vec{q}^{\,2} = \frac{i}{16\pi^2}\left(\frac{1}{6}\vec{q}^{\,2}\ln\vec{q}^{\,2}\right) \tag{3.7.14}$$

The terms with $\vec{q}^2 \ln \vec{q}^2$ will give potentials of the form $\frac{1}{r} \ln r$, and these are the terms we are interested in. Note that these terms contain no divergences.

For the integral I_2 we begin with a contour integral over k_0

$$\begin{aligned} I_2 &= 2\pi i \int \frac{d^{n-1}k}{(2\pi)^n} \left(\frac{1}{2\sqrt{\vec{k}^2}} \right) \frac{\vec{k}^2 \vec{q}^{\,2} - (\vec{k}\cdot\vec{q})^2}{\vec{k}^2 (\vec{k}-\vec{q})^2} \\ &= -i\pi \int \frac{d^{n-1}k}{(2\pi)^n} \left[\frac{(\vec{k}\cdot\vec{q})^2}{(\vec{k}^2)^{3/2}(\vec{k}-\vec{q})^2} - \frac{\vec{q}^{\,2}}{(\vec{k}^2)^{1/2}(\vec{k}-\vec{q})^2} \right] \end{aligned} \tag{3.7.15}$$

Using

$$\frac{1}{a^{\alpha_1} b^{\alpha_2}} = \int_0^1 dx \frac{x^{\alpha_1 - 1}(1-x)^{\alpha_2 - 1}}{[ax + b(1-x)]^{\alpha_1+\alpha_2}} \cdot \frac{\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \tag{3.7.16}$$

and denoting $\vec{k} - \vec{q}x = \vec{\kappa}$, we find

$$\begin{aligned} I_2 &= -i\pi q_i q_j \int \frac{d^{n-1}\kappa}{(2\pi)^n} \frac{(\kappa+qx)_i(\kappa+qx)_j}{[\vec{\kappa}^2 + \vec{q}^{\,2}x(1-x)]^{5/2}} \cdot \frac{\Gamma(5/2)}{\Gamma(3/2)}(1-x)^{\frac{1}{2}} \\ &+ i\pi \int \frac{d^{n-1}\kappa}{(2\pi)^n} \frac{\vec{q}^{\,2}}{[\vec{\kappa}^2 + \vec{q}^{\,2}x(1-x)]^{3/2}} \cdot \frac{\Gamma(\frac{3}{2})}{\Gamma(\frac{1}{2})}(1-x)^{-\frac{1}{2}} \end{aligned} \tag{3.7.17}$$

Using $(\kappa+qx)_i(\kappa+qx)_j = \frac{1}{n-1}\vec{\kappa}^2\delta_{ij} + q_i q_j x^2$ and dropping the term with $q_i q_j x^2$ since it cannot yield a term with $\ln \vec{q}^2$, we replace $\vec{\kappa}^2$ by $\vec{\kappa}^2 + \vec{q}^2 x(1-x)$ in the first integral because this does not change the divergent terms or the terms with $\ln \vec{q}^{\,2}$ terms and facilitates the computation. We arrive at

$$\begin{aligned} I_2 &= i\pi\vec{q}^{\,2} \int_0^1 dx \int \frac{d^{n-1}\kappa}{(2\pi)^n} \left[\frac{\frac{-1}{n-1}\frac{3}{2}(1-x)^{1/2} + \frac{1}{2}(1-x)^{-1/2}}{(\vec{\kappa}^2 + \vec{q}^{\,2}x(1-x))^{3/2}} + \cdots \right] \\ &= i\pi\vec{q}^{\,2} \int_0^1 dx \left[-\frac{3}{2}\frac{(1-x)^{1/2}}{n-1} + \frac{1}{2}\frac{1}{(1-x)^{1/2}} \right] \frac{1}{(\vec{q}^{\,2}x(1-x))^{\frac{3}{2}-\frac{n-1}{2}}} \\ &\quad \frac{\pi^{\frac{n-1}{2}}\Gamma\left(\frac{3}{2}-\frac{n-1}{2}\right)}{(2\pi)^n\Gamma(\frac{3}{2})} + \cdots \end{aligned} \tag{3.7.18}$$

where we used the Euclidean integral

$$\frac{\int d^n\kappa}{(\kappa^2+L)^\alpha} = \pi^{n/2}\frac{\Gamma(\alpha-n/2)}{\Gamma(\alpha)} L^{\frac{n}{2}-\alpha} \tag{3.7.19}$$

There is an ultraviolet divergence due to $\Gamma(2-\frac{n}{2})$, but the terms with $\ln \vec{q}^2$ are obtained by expanding $[\vec{q}^{\,2}x(1-x)]^{-2+\frac{n}{2}}$, and then the ultraviolet divergence is cancelled.

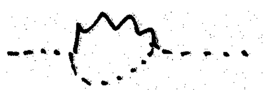
Expanding, setting $n-1=3$ afterwards, and using $\int_0^1 dx(1-x)^{-1/2}=2$ and $\int_0^1 dx(1-x)^{1/2}=\frac{2}{3}$, yields

$$I_2 = \frac{i}{16\pi^2}\left(-\frac{4}{3}\right)\vec{q}^{\,2}\ln\vec{q}^{\,2} + \text{terms without}\ln q^2. \tag{3.7.20}$$

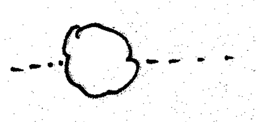
In a similar manner one finds

$$I_4 = \frac{i}{16\pi^2}\left(-\frac{2}{3}\right)\vec{q}^{\,2}\ln\vec{q}^{\,2} + \text{terms without}\ln\vec{q}^{\,2}. \tag{3.7.21}$$

We now see that



$$\sim 2I_2 \sim -\frac{16}{6}\vec{q}^{\,2}\ln\vec{q}^{\,2}$$



$$\sim I_1 - I_2 + I_4 \sim \left(\frac{1}{6}+\frac{8}{6}-\frac{4}{6}\right)\vec{q}^{\,2}\ln\vec{q}^{\,2} = \frac{5}{6}\vec{q}^{\,2}\ln\vec{q}^{\,2} \tag{3.7.22}$$

The sum of both graphs is indeed proportional to $-\frac{11}{6}$, but we also see that the first graph dominates $(-\frac{16}{6})$ over the second graph $(\frac{5}{6})$. This result was first obtained in [32]. Let us now discuss the physical implications.

The correction to the Coulomb potential is proportional to the Fourier transform of $\frac{1}{\vec{q}^{\,2}}(\vec{q}^{\,2}\ln\vec{q}^{\,2})\frac{1}{\vec{q}^{\,2}}$, which yields a potential of the form $\frac{1}{r}\ln r$. This can be shown as follows

$$\begin{aligned} V(r) &= \int\left(\frac{1}{\vec{q}^{\,2}}\ln\vec{q}^{\,2}\right)e^{iqr\cos\theta}\vec{q}^{\,2}dq\,d\cos\theta\,d\varphi \\ &= 2\pi\int_0^\infty\frac{1}{iqr}\left(e^{iqr}-e^{-iqr}\right)\ln q^2 dq \text{ with } q=|\vec{q}| \\ &= \frac{8\pi}{r}\int_0^\infty(\sin qr)\left(\frac{1}{q}\ln q\right)dq \end{aligned} \tag{3.7.23}$$

By writing $\ln q$ as $\ln qr - \ln r$, we find the $\frac{1}{r}\ln r$ term

$$V(r) = \frac{c_1}{r} - \frac{c_2}{r}\ln r,\ c_2 = 8\pi\int_0^\infty\frac{\sin qr}{qr}d(qr) = 4\pi^2\ . \tag{3.7.24}$$

In a similar way one finds that the lowest order term $1/\vec{q}^{\,2}$ yields $V(r)=2\pi^2/r$.

Collecting all constants from (3.7.3), (3.7.14), (3.7.20), (3.7.21) and (3.7.24) we see that the $1/r$ potential between two fermions is modified by 1-loop corrections into

$$V(r) = \frac{g^2}{r}\left(1 + \frac{g^2}{4\pi^2} C_2(G) \frac{11}{6} \ln \frac{r}{r_0}\right) \tag{3.7.25}$$

We have defined g such that at $r = r_0$ the Colomb potential holds. We have shown that the $-11/6$ consists of contributions proportional to $-16/6$ and $5/6$. So, far away there is less suppression of the potential: there is antiscreening.

One sometimes reads in the literature the statement that the relative contributions of the two selfenergy graphs are $-12/6$ and $1/6$, respectively. [33] We have shown that the gluon loop with a Coulomb gluon yields $-16/6$ while the loop with two transversal gluons yields $5/6$. However, a decomposition of $-12/6$ and $1/6$ is found if one uses a Hamiltonian approach instead of the usual Lagrangian approach we have used. One must then solve the Gausz constraint iteratively, and use double dispersion relations. [34] One also finds a $-12/6$ and $1/6$ decomposition if one considers the QCD vacuum as medium with diamagnetic and paramagnetic susceptibility. The vacuum of QED has diamagnetic properties (screening), essentially due to the Pauli exclusion principle of the electrons, but gluons have of course integer spin and do not satisfy the exclusion principle, and this makes it possible for the QCD vacuum to have paramagnetic properties. These ideas were worked out by N.K. Nielsen and R.J. Hughes [35], who found that complex fields with helicities s_3, which couple to QED as $\mathcal{L} = 2eH_3 s_3$ (where H_3 is the z-component of the magnetic field), contribute as follows to the β function

$$\beta \sim (-)^{2s} \sum_{s_3} \left(s_3^2 - \frac{1}{12}\right) \tag{3.7.26}$$

Hence, for pure Yang Mills fields with $s_3 = \pm 1$ one gets a contribution proportional $12/6 - 1/6$. Clearly, the orbital parts of both helicities add up to a contribution proportional to $-1/6$, while the pure spin-one parts yield $12/6$. If one adds spin $1/2$

fields and spin 0 fields one obtains

$$\beta = \frac{-g^3}{16\pi^2}\left[\frac{11}{3}C_2(G) - \frac{2}{3}C_2(R_F) - \frac{1}{6}C_2(R_s)\right] \tag{3.7.27}$$

where $C_2(R_F)$ is the quadratic Casimir operator for the representation R_F of the fermions. Clearly, only nonabelian gauge fields can produce a negative β function. For QCD one has $C_2(G) = N = 3$, while for the fundamental representations $C_2(F) = 1$. Then one finds $\beta \sim 11 - \frac{2}{3}N_{fl}$ for N_{fl} flavours (see next section).

One can also obtain the β function from instanton physics. One requires that the path integral is independent of the Pauli Villars masses which are used to regulate the theory. For pure Yang-Mills theory one gets from the zero modes a contribution 2 to the β function, while the one-loop graphs yield $-1/6$, yielding together again 11/6. [36]

A simple physical way to explain asymptotic freedom is to solve the field equations for QCD coupled to a point particle in the Coulomb gauge by iteration [37]. Gauss's law reads

$$D_i E^{ai} = g\rho^a \quad , \quad E^{ai} = -G^a_{0i} \tag{3.7.28}$$

For a point particle at the origin, with unit charge and orientation $a = 1$ in color space, $\rho^a = \delta^3(x)\delta^{a1}$. If one decomposes the covariant derivative into a free part and an interaction part, and moves the interaction part to the right-hand side, one obtains

$$\partial_i E^{ai} = g\delta^3(x)\delta^{a1} - gf^{abc}A^b_i E^{ci} \tag{3.7.29}$$

The leading term yields a static $1/r$ potential with $a = 1$. A quantum fluctuation A^b_i at a point $\vec{r}$ with $b \neq 1$ induces then a quantum electric field E^{ai} at $\vec{r}$ with $a \neq 1$ and $a \neq b$, as follows by solving the equation $\partial_i E^{ai} = -gf^{ab1}A^b_i E^{1i}$. The time average of this induced electric field vanishes. However, this quantum electric field together with the original quantum fluctuation A^b_i, both at $\vec{r}$, can in turn be inserted into the source term, and solving Gauss's law once again produces then a second-order quantum correction to the static classical field E_i^1 which always has opposite sign. Far away this second-order effect becomes small, but nearby the effective electric field is reduced: there is antiscreening.

8 One-loop Z-factors in QCD

As a hint how to renormalize nonabelian gauge field theories, it helps to look at the results for one-loop calculations, and to spot which Z factors are equal up to this order. In this way one notices that the longitudinal part of the gauge propagators does not renormalize: the quantum corrections to the proper self energy of the gauge fields are proportional to $\eta_{\mu\nu}k^2 - k_\mu k_\nu$. Technically this means that the gauge-fixing term

$$\mathcal{L}(\text{fix}) = -\frac{1}{2\xi}\left(\partial^\mu A_\mu^a\right)^2 \tag{3.8.1}$$

remains finite, and thus that

$$Z_\xi = Z_3 \tag{3.8.2}$$

This was the motivation first to consider $\hat{\Gamma}$ instead of Γ in section 1.

We discuss here the one-loop results for massive fermions coupled to Yang-Mills fields with any simple gauge group G. We could have added the coupling to the K and L sources, and checked that Z_K and Z_L are indeed (at the one-loop level) related to the other Z-factors in the manner we have deduced from the Ward identities. We could also have added scalars.

We consider thus the following action $\mathcal{L}^{\text{ren}} + \Delta\mathcal{L}^{\text{ren}}$ where

$$\begin{aligned}\mathcal{L}^{\text{ren}} \equiv\; & -\frac{1}{4}\left(\partial_\mu A_\nu^{a,\text{ren}} - \partial_\nu A_\mu^{a,\text{ren}} + u^{\text{ren}} f^a{}_{bc} A_\mu^{b,\text{ren}} A_\nu^{c,\text{ren}}\right)^2 \\ & -\frac{1}{2\xi_{\text{ren}}}\left(\partial_\mu A_a^{\mu,\text{ren}}\right)^2 - \bar{\psi}_i^{\text{ren}}\left(\gamma^\mu\left\{\partial_\mu + u_{\text{ren}} A_\mu^a (T_a)^i{}_j\right\}\right)\psi_{\text{ren}}^j \\ & -M_{\text{ren}}\bar{\psi}_i^{\text{ren}}\psi_{\text{ren}}^i - \left(\partial_\mu b_a^{\text{ren}}\right)\left(\partial_\mu c_{\text{ren}}^a + u_{\text{ren}} f^a{}_{bc} A_\mu^{b,\text{ren}} c_{\text{ren}}^c\right)\end{aligned} \tag{3.8.3}$$

We renormalize as follows

$$
\begin{aligned}
&A_\mu^a = \sqrt{Z_3} A_\mu^{a,\mathrm{ren}} \quad , \quad c^a = \sqrt{Z_3^{gh}} c^a_{\mathrm{ren}} \quad , \quad b_a = \sqrt{Z_3^{gh}} b_a^{\mathrm{ren}} \\
&\psi^i = \sqrt{Z_3^f} \psi^i_{\mathrm{ren}} \quad , \quad \bar{\psi}_i = \sqrt{Z_3^f} \bar{\psi}_i^{\mathrm{ren}} \quad , \quad M = Z_M M_{\mathrm{ren}} \\
&\xi = Z_\xi \xi_{\mathrm{ren}} \quad , \quad g = Z_1 Z_3^{-3/2} u_{\mathrm{ren}} \mu^{\frac{1}{2}(4-n)}
\end{aligned}
\tag{3.8.4}
$$

The one-loop divergences do not depend on the renormalization mass μ. Using minimal subtraction, all Z-factors are, in fact, mass independent. A general parametrization of the counter terms which does not assume that all counter terms follow from multiplicative rescalings, is given by

$$
\begin{aligned}
\Delta \mathcal{L}^{\mathrm{ren}} = & -\frac{1}{4}(Z_3 - 1)\left(\partial_\mu A_\nu^{a,\mathrm{ren}} - \partial_\nu A_\mu^a\right)^2 - \left(\frac{Z_3}{Z_\xi} - 1\right)\frac{1}{2\xi_{\mathrm{ren}}}\left(\partial^\mu A_\mu^{a,\mathrm{ren}}\right)^2 \\
& -(Z_1 - 1) u_{\mathrm{ren}} f_{abc} \left(\partial_\mu A_\nu^{a,\mathrm{ren}}\right) A_\mu^{b,\mathrm{ren}} A_\nu^{c,\mathrm{ren}} \\
& -\frac{1}{4} u_{\mathrm{ren}}^2 (Z_4 - 1) f^{abc} f_a{}^{de} A_\mu^{b,\mathrm{ren}} A_\nu^{c,\mathrm{ren}} A_\mu^{d,\mathrm{ren}} A_\nu^{e,\mathrm{ren}} \\
& -\left(Z_3^f - 1\right) \bar{\psi}_i^{\mathrm{ren}} \left(\gamma^\mu \partial_\mu\right) \psi^i_{\mathrm{ren}} \\
& -\left(Z_1^f - 1\right) \bar{\psi}_i^{\mathrm{ren}} \gamma^\mu u_{\mathrm{ren}} A_\mu^{b,\mathrm{ren}} T_b \psi^i_{\mathrm{ren}} \\
& -\left(Z_3^f Z_M - 1\right) M_{\mathrm{ren}} \bar{\psi}_i^{\mathrm{ren}} \psi^i_{\mathrm{ren}} \\
& -\left(Z_3^{gh} - 1\right) \left(\partial_\mu b_a^{\mathrm{ren}}\right) \partial^\mu c^a_{\mathrm{ren}} \\
& -\left(Z_1^{gh} - 1\right) \left(\partial_\mu b^a_{\mathrm{ren}}\right) u_{\mathrm{ren}} f^a{}_{bc} A_\mu^{b,\mathrm{ren}} c^c_{\mathrm{ren}}
\end{aligned}
\tag{3.8.5}
$$

We have only written down all local expressions that appear as divergences; we could have written many more expressions with different index contractions, but then explicit calculation would reveal that they do not show up as counter terms, so one would come back to the set in (3.8.5).

Explicit calculations of the one-loop divergences in proper graphs for arbitrary gauge parameter ξ_{ren} yield (we write ξ instead of ξ_{ren} to simplify the notation)

$$
\begin{aligned}
Z_3 &= 1 + \frac{10}{3} C_2(G) y - \frac{8}{3} T(R) y + (1 - \xi) C_2(G) y \\
Z_3^{gh} &= 1 + C_2(G) y + (1 - \xi) \frac{1}{2} C_2(G) y
\end{aligned}
$$

$$\begin{aligned}
Z_1^{gh} &= 1 - C_2(G)y + (1-\xi)C_2(G)y \\
Z_1^f &= 1 - 2\{C_2(R) + C_2(G)\}y + (1-\xi)\left\{2C_2(R) + \frac{1}{2}C_2(G)\right\}y \\
Z_3^f &= 1 - 2C_2(R)y + (1-\xi)2C_2(R)y \\
Z_M &= 1 - 6C_2(R)y \\
Z_1 &= 1 + \left\{\frac{4}{3}C_2(G) - \frac{8}{3}T(R)\right\}y + (1-\xi)\frac{3}{2}C_2(G)y
\end{aligned} \tag{3.8.6}$$

Here $y = u_{\text{ren}}^2\left[16\pi^2(4-n)\right]^{-1}$ and

$$\begin{aligned}
f_{abc}f_d{}^{bc} &= C_2(G)\delta_{ad} \\
\mathrm{Tr}R_aR_b &= -\delta_{ab}T(R) \\
R_aR^a &= -C_2(R)I
\end{aligned} \tag{3.8.7}$$

Clearly, $\dim R \times C_2(R) = \dim G \times T(R)$ where $\dim G$ is the number of generators of G, and $\dim R$ the size of the matrices R. In particular, $T(G) = C_2(G)$. (For $SU(N)$ one has $C_2(G) = N$ while in the fundamental representation, denoted by $\underline{N}$, one has $C_2(\underline{N}) = (N^2-1)(2N)^{-1}$ and $T(\underline{N}) = \frac{1}{2}$.)

These one-loop results reveal several interesting facts (these facts hold generally, but the one-loop results can only hint at that)

(i)$Z_3^{3/2}/Z_1$ is ξ-independent (gauge independence of the coupling constant renormalization. A better, more precise, statement is "gauge-choice independence").

(ii) Z_M is ξ-independent (the mass is a physical parameter, so it should be independent of the gauge chosen. Indeed, it does not depend on ξ).

(iii) $Z_1^{gh}/Z_3^{gh} = Z_1/Z_3 = Z_1^f/Z_3^f$. This result is a direct consequence of multiplicative renormalizability. The renormalization of the 3-point vertices is proportional to ${Z_1}^{gh}$, Z_1 and ${Z_1}^f$, but if multiplicative renormalization holds, it is also proportional to the renormalization of the coupling constant and the three fields at the 3-point vertex. Therefore, the ratio of a Z factor for a vertex and the Z factors of the three fields at that vertex should be the same for each vertex, (namely, the renormalization of the coupling constant). This leads to ${Z_1}^{gh}/({Z_3}^{gh}{Z_3}^{1/2}) = Z_1/{Z_3}^{3/2} = {Z_1}^f/({Z_3}^f{Z_3}^{1/2})$

and after multiplication by ${Z_3}^{1/2}$ one obtains the Ward identity ${Z_1}^{gh}/{Z_3}^{gh} = Z_1/Z_3 = {Z_1}^f/{Z_3}^f$. In particular, as Z_1^{gh} and Z_3^{gh} are independent of fermion loops at the one-loop level, Z_1/Z_3 and Z_1^f/Z_3^f should not depend on $T(R)$ or $C_2(R)$. This is indeed the case.

(iv) A calculation of all proper graphs with four external A fields gives $Z_4 = Z_1^2/Z_3$. This is in agreement with multiplicative renormalization, because rescaling of u^2A^4 yields a factor $(Z_1/Z_3^{3/2})^2(Z_3^{1/2})^4$. Together with the result in (iii) this proves that all Z-factors in (3.8.5) are due to rescaling of the fields and coupling constant.

(v) There are no divergences proportional to $(\partial_\mu A_a^\mu)^2$. This means that the longitudinal part of the W-boson propagator does not renormalize, and hence (as we already explained) that $Z_\xi = Z_3$.

We conclude that all Z factors in the action are due to rescaling of the fundamental fields and coupling constants.

9 The one-loop beta function and running masses

The calculation of the β function in nonabelian gauge theories started 25 years ago with the one-loop calculation by D.J. Gross and F. Wilczek (*Phys. Rev. Lett.* **30** (1973) 1343) and H.D. Politzer (ibidem, page 1346). Earlier, 't Hooft had noted that the β function for nonabelian gauge theories has an opposite sign from that for abelian gauge theories.

The running of coupling constants follows from the beta function which is defined by the dependence of the renormalized Yang-Mills gauge coupling constant u on the renormalization scale μ

$$\beta(u) = \mu\frac{\partial}{\partial\mu}u \text{ with } g_{un} = \frac{Z_1}{Z_3^{3/2}}\mu^{2-\frac{1}{2}d}u \tag{3.9.1}$$

Here g_{un} is the unrenormalized coupling constant, u the dimensionless renormalized coupling constant and μ is a mass which one must introduce in dimensional regularization because g_{un} is dimensionful. Using that the dimensionful unrenormalized coupling constant is independent of the renormalization scale, $\mu\frac{\partial}{\partial\mu}g_{un}=0$, we obtain

$$\begin{aligned}\beta(u) &= \mu\frac{\partial}{\partial\mu}\left[(Z_3^{3/2}/Z_1)\mu^{\frac{1}{2}d-2}g_{un}\right] \\ &= \left(\frac{1}{2}d-2\right)u+\beta(u)u\frac{d}{du}\ln(Z_3^{3/2}/Z_1)\end{aligned}\tag{3.9.2}$$

where we used that $Z_3^{3/2}/Z_1$ only depends on u (and not on the gauge fixing parameter ξ). It follows that[15]

$$\begin{aligned}\beta(u) &= \frac{\frac{1}{2}(d-4)u}{1+u\frac{d}{du}\ln\frac{Z_1}{Z_3^{3/2}}}\ ;\ \frac{Z_1}{Z_3^{3/2}}=1-\ by\ +\dots \\ y &= \frac{u^2}{16\pi^2}\frac{1}{4-d} \\ b &= \frac{11}{3}C_2(G)-\frac{4}{3}T(R_F)-\frac{1}{3}T(R_S)\end{aligned}\tag{3.9.3}$$

Hence, in the limit d tending to 4, and to lowest order in u

$$\beta(u)=\frac{1}{2}(d-4)u\{1+2yb\}=-\frac{u^3}{16\pi^2}b\tag{3.9.4}$$

Recalling the definition of β, we find how coupling constants run

$$\begin{aligned}&\mu\frac{\partial u}{\partial\mu}=-\frac{u^3}{16\pi^2}b\ ,\ \mu^2\frac{\partial u^2}{\partial\mu^2}=-\frac{u^4}{16\pi^2}b \\ &\mu^2\ \frac{\partial}{\partial\mu^2}\ \frac{1}{\alpha}=\frac{b}{4\pi}\ \text{with}\ \alpha\equiv\frac{u^2}{4\pi} \\ &\frac{1}{\alpha(M^2)}-\frac{1}{\alpha(\mu^2)}=\frac{b}{4\pi}\ln\frac{M^2}{\mu^2}\end{aligned}\tag{3.9.5}$$

[15] More in detail, the one-loop contributions due to loops with gauge fields, complex fermions in a representation R_F and complex scalars in a representation R_S are given by

$$Z_1=1+\frac{4}{3}C_2(G)y-\frac{8}{3}T(R_F)y-\frac{2}{3}T(R_S)y$$

$$Z_3=1+\frac{10}{3}C_2(G)y-\frac{8}{3}T(R_F)y-\frac{2}{3}T(R_S)y$$

where $trT_a^RT_b^R=-T(R)\delta_{ab}$. One normalizes the generators to $T(F)=1/2$ in the fundamental representation F, and $C_2(G)=T$ (adjoint). The coefficients of $T(R_F)$ and $T(R_S)$ in Z_1 and Z_3 are equal due to a Ward identity.

As μ^2 gets large, $\alpha(\mu^2)$ tends to zero if b is positive.

To determine how masses are running, we consider

$$\gamma_m \equiv \frac{\mu}{m_{\rm ren}} \frac{\partial}{\partial \mu} m_{\rm ren}, m_{un} = Z_m m_{\rm ren} \tag{3.9.6}$$

Since also Z_m is only dependent on u, we find in the same way as before

$$\begin{aligned} \gamma_m &= \beta(u) \frac{1}{m_{\rm ren}} \frac{d}{du} \left(Z_m^{-1} m_{un} \right) \\ &= \beta(u) \frac{d}{du} \ln Z_m^{-1} = -\beta(u) \frac{d}{du} \ln Z_m \end{aligned} \tag{3.9.7}$$

Next we use the one-loop result in (3.8.6) for Z_m for complex fermions in a representation R (see below for a derivation)

$$\begin{aligned} Z_m &= 1 - 6C_2(R)y \\ \beta(u) &= \frac{1}{2}(d-4)u + \mathcal{O}(u^3) \end{aligned} \tag{3.9.8}$$

This yields

$$\begin{aligned} \gamma_m &= -\frac{1}{2}(d-4)u \frac{d}{du}(-6C_2(R)y) \\ &= -6C_2(R) \frac{\alpha}{4\pi}. \end{aligned} \tag{3.9.9}$$

The running of the masses is determined by the following equation

$$\mu^2 \frac{\partial}{\partial \mu^2} m_{\rm ren} = \frac{1}{2} \gamma_m m_{\rm ren} \Longrightarrow \frac{\partial \ln m_{\rm ren}}{\partial \ln \mu^2} = \frac{1}{2} \gamma_m \tag{3.9.10}$$

Since γ_m depends on α, which itself depends on μ^2, see (3.9.5), the solution is

$$\frac{m_{\rm ren}(M^2)}{m_{\rm ren}(\mu^2)} = e^{\int_{\ln \mu^2}^{\ln M^2} \ \frac{1}{2} \gamma_m(\alpha(e^t)) dt} \tag{3.9.11}$$

We replaced the integration variable μ^2 by e^t. We go over to $\alpha(e^t)$ as integration variable

$$dt = d\alpha(e^t) \frac{1}{e^t \frac{d\alpha(e^t)}{de^t}} = \frac{d\alpha(e^t)}{(-\frac{b\alpha^2(e^t)}{4\pi})} \tag{3.9.12}$$

The exponent in the solution for $m_{\rm ren}(M^2)/m_{\rm ren}(\mu^2)$ becomes then

$$\begin{aligned} \int_{\alpha(\mu^2)}^{\alpha(M^2)} & \frac{1}{2} 6C_2(R) \frac{\alpha(e^t)}{4\pi} \frac{1}{b\frac{\alpha^2(e^t)}{4\pi}} d\alpha(e^t) \\ = & \frac{3C_2(R)}{b} \ln \frac{\alpha(M^2)}{\alpha(\mu^2)}. \end{aligned} \tag{3.9.13}$$

Hence we find that for complex fermions coupled to a Yang-Mills group in a representation R

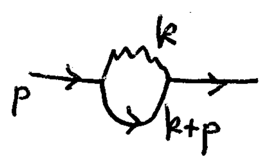
$$\frac{m_{\rm ren}(M^2)}{m_{\rm ren}(\mu^2)} = \left[\frac{\alpha(M^2)}{\alpha(\mu^2)}\right]^{\frac{3C_2(R)}{b}} \tag{3.9.14}$$

where we recall that the one-loop beta function is given by

$$b = \frac{11}{3} C_2(G) - \frac{4}{3} T(R_F) - \frac{1}{3} T(R_S). \tag{3.9.15}$$

This formula yields the running of masses as a function of the renormalization mass μ.

Let us now first check that Z_m is independent of the gauge fixing parameter ξ. First we consider the ξ-dependent divergences in the fermion selfenergy due to a virtual vector boson. They come from the ξ-dependent part of the gauge field propagator $(\eta_{\mu\nu} - k_\mu k_\nu/k^2)\frac{1}{k^2} + \xi k_\mu k_\nu/k^4$ and are proportional to



$$\begin{aligned} & = \int \frac{\not{k}[-i(\not{k} + \not{p}) + m]\not{k} d^n k}{k^4[(k+p)^2 + m^2]} \\ = & \int \frac{(i\not{p} + m) d^n k}{k^2[(k+p)^2 + m^2]} + \int \frac{i(p^2 + m^2)\not{k} d^n k}{k^4[(k+p)^2 + m^2]} \end{aligned} \tag{3.9.16}$$

where we used that $\int d^n k \not{k}/k^4 = 0$. The second term is neither ultraviolet nor infrared divergent, whereas the first term only contributes to the fermion wave function renormalization factor Z_3^f but not to Z_m. Hence, using minimal renormalization, Z_m is indeed independent of ξ.

To compute Z_m note that $Z_3^f = 1 - 2C_2(R)y + (1-\xi)2C_2(R)y$ and in the Feynman gauge $\xi = 1$ the total one-loop result for the fermion selfenergy is proportional to

$$\int [\gamma^\mu(-i\not{k} + m)\gamma_\mu](k-p)^{-2}[k^2 + m^2]^{-1} d^n k$$

$$\sim \int_0^1 dx \int \frac{[2i\not{p}x + 4m]d^n\kappa}{[\kappa^2 + m^2(1-x) + p^2x(1-x)]^2}$$
$$\sim (i\not{p} + 4m) \int \frac{d^n\kappa}{[\kappa^2 + \ldots]^2} \tag{3.9.17}$$

Decomposing $i\not{p} + 4m$ into $(i\not{p} + m) + 3m$, it is clear that the contribution to Z_m is three times that to Z_3^f. This explains the factor 6 in Z_m in (3.9.8).

10 The two-loop β function

The two-loop β function was computed in [38–42]. The three-loop β function for QCD was obtained in [43]. The four-loop result was obtained by van Ritbergen et al. [44].

Below we shall discuss the two-loop β function in more detail, but first we make a few general comments. Higher-loop calculations are tedious, hence any simplification is welcome. For the calculation of β functions one may use any of the following three simplifictions

(i) instead if computing Z_1 and Z_3, one may compute $Z_1(gh), Z_3(gh)$ and Z_3, where $Z_1(gh)$ and $Z_3(gh)$ are the vertex and wave function renormalization constants of the ghosts, which are much simpler to compute that Z_1 and Z_3 for the gauge fields. The Ward identity $Z_1/Z_3 = Z_1(gh)/Z_3(gh)$ may then be used to replace Z_1 in the definition of the β function. Moreover, one may set the momentum of the external gauge field to zero in the calculation of $Z_1(gh)$, which effectively reduces the calculation of $Z_1(gh)$ to the same degree of complexity as Z_3.

(ii) In the background field formalism one only needs to calulate Z_3 because in this formalism $Z_1 = Z_3$. There are, however, extra vertices. Using this approach, L.F. Abbott recalulated the two-loop β function of QCD, as we discuss further in the chapter on the background field method.

(iii) A further simplification which was used in the four-loop calculation follows from the observation that the selfenergy of gauge fields is of the form $Z_3(Q^2\eta_{\mu\nu} - Q_\mu Q_\nu)$.

By differentiating twice w.r.t. Q and then setting Q to zero, the calculation of Z_3 can be reduced to a calculation of vacuum graphs, and by opening one line, one is led to a calculation like Z_3 at one lower loop level. Great care is needed in this approach to split off the infrared divergences which are introduced by setting Q to zero. One gives a small mass to the denominator of the propagators of the massless fields, but this spoils multiplicative renormalizability. For $Z_3(gh)$ similar remarks apply, while for $Z_1(gh)$ one need not even differentiate at all because one already obtains a factor proportional to Q^μ from the vertex where the external antighost couples.

It is well-known that the calculation of the β function is scheme-dependent beyond the first two nonvanishing loop contributions. What then is the use of the three-loop β function?

For gauge fields with a simple gauge group G minimally coupled to complex Dirac fermions in a representation R and complex scalars in a representation S, the β function reads $\mu\frac{\partial}{\partial\mu}g = \beta = \frac{1}{16\pi^2}g^3A + \left(\frac{1}{16\pi^2}\right)^2 g^5B + \cdots$ where the one-loop corrections yield

$$A = -\frac{11}{3}C_2(G) + \frac{4}{3}T(R) + \frac{1}{3}T(S) \tag{3.10.1}$$

while the two-loop corrections yield

$$\begin{aligned} B = & -\frac{34}{3}C_2^2(G) + \left(\frac{20}{3}C_2(G) + 4C_2(R)\right)T(R) \\ & + \left(\frac{2}{3}C_2(G) + 4C_2(S)\right)T(S) \end{aligned} \tag{3.10.2}$$

We recall that $trT_aT_b = -\delta_{ab}T(R)$ and $\delta^{ab}T_aT_b = -C_2(R)I$. Hence

$$C_2(R)\dim R = T(R)\dim G \tag{3.10.3}$$

where $C_2(G) = N$ for $SU(N)$. We normalize the generators T_a to $trT_aT_b = -\frac{1}{2}\delta_{ab}$ in the fundamental representation, hence $T(\text{fund}) = \frac{1}{2}$. Clearly, for the adjoint representation, $T(G) = C_2(G)$.

As applications we consider $N=1$ supersymmetric (susy) models, $N=2$ models and the $N=4$ model. For a general $N=1$ susy model we must include real gluinos in the adjoint representation and chiral superfields (one real fermion and one complex scalar) in representations R. The result for the β function follows straight forwardly from (3.10.1) and (3.10.2). However, we must also consider Yukawa couplings because their coupling constant is proportional to the gauge coupling constant g in susy models. At the two-loop level they contribute to Z_3 but not to Z_1^{gh} and Z_3^{gh}. Only the selfenergy diagram with a matter loop coupled to two external gauge fields and a gluino dividing it in half contributes. It is clear that this two-loop contribution of the Yukawa couplings of a vector multiplet coupled to one chiral multiplet in a representation R is proportional to $tr R_a R_c R_b R_d \delta^{cd}$, and the two-loop contribution to the β function is given by

$$B \text{ (Yukawa) } = -2\left(C_2(G)+C_2(R)\right)T(R) \tag{3.10.4}$$

The β function for an $N=1$ vector multiplet coupled to a chiral multiplet in representation R is then $(S=R)$

$$\begin{aligned}
A(N=1) &= -\frac{11}{3}C_2(G)+\frac{1}{2}\frac{4}{3}C_2(G)+\frac{1}{2}\frac{4}{3}T(R)+\frac{1}{3}T(R) \\
&= -3C_2(G)+T(R) \\
B(N=1) &= \left\{-\frac{34}{3}C_2^2(G)+\frac{1}{2}\left(\frac{20}{3}C_2(G)+4C_2(G)\right)T(G)\right\} \\
&+ \left\{\frac{1}{2}\left(\frac{20}{3}C_2(G)+4C_2(R)\right)T(R)\right\}+\left\{\left(\frac{2}{3}C_2(G)+4C_2(R)\right)T(R)\right\} \\
&- \left\{2\left(C_2(G)+C_2(R)\right)T(R)\right\} \\
&= -6C_2^2(G)+2C_2(G)T(R)+4C_2(R)T(R)
\end{aligned} \tag{3.10.5}$$

We have written the contributions from the gauge bosons plus gauginos, fermions, scalars and Yukawa couplings separately between curly brackets.

In addition to Yukawa couplings with coupling constant g, there are also $\lambda\phi^4$ couplings in susy models with λ proportional to g^2. However these do not contribute at the one-loop and two-loop level, as we now briefly explain. It is obvious that $\lambda\phi^4$ couplings cannot contribute to Z_3 at the one-loop level. At the two-loop level there are three selfenergy graphs for gauge fields with one $\lambda\phi^4$ coupling but the sum of their contributions vanishes also. One graph has the shape of a pair of glasses, but it is not divergent due to the fact that it contains the product of two integrals

$$\int d^4k(2k+p)^\mu(k+m^2)^{-1}[(k+p)^2+m^2]^{-1} \tag{3.10.6}$$

which are convergent. Two other graphs have the form of the number 8 with two gauge fields attached to the lower loop, either at one vertex or at two vertices. Both lower-loop graphs are only logarithmically divergent, while the upper loop graph is proportional to the square of the mass of the scalars. In fact, their sum must cancel (and does cancel) because it should be transversal, but is only proportional to $\delta_{\mu\nu}$. (The graph of the shape of a pair of glasses is separately gauge invariant). There are also one-loop graphs with a one-loop counter term, but their contribution vanishes for the same reasons. For these reasons we did not include the $\lambda\phi^4$ couplings in our one-and two loop considerations above.

In $N=2$ models, the vector multiplet consists of an $N=1$ vector multiplet V coupled to an $N=1$ scalar multiplet Σ, both in the adjoint representation. Matter consists of hypermultiplets which each consist of two $N=1$ chiral multiplets S and $\bar{S}$, one in a representation R and the other in $\bar{R}$. However, in order to obtain $N=2$ susy, there are now, in addition to the Yukawa coulings between V and S and $\bar{S}$, also Yukawa couplings between V and Σ. One finds then

$$\begin{aligned}
A(N=2) &= \left(-\frac{11}{3}+\frac{1}{2}\frac{4}{3}2+\frac{1}{3}\right)C_2(G)+2T(R)\\
&= -2C_2(G)+2T(R)\\
B(N=2) &= \left\{\left(-\frac{34}{3}+\frac{20}{3}+4+\frac{2}{3}+4-2-2\right)C_2^2(G)\right\}\\
&+ \left\{\left(\frac{20}{3}+2\frac{2}{3}-4\cdot 2\right)C_2(G)T(R)\right\}+\{(4+2\cdot 4-4\cdot 2-4)\,C_2(R)T(R)\}\\
&= 0
\end{aligned} \tag{3.10.7}$$

The vanishing of $B(N=2)$ agrees with the nonrenormalization theorem for $N=2$ models, according to which these models contain only divergences at the one-loop level. To obtain $B(N=2)$, we proceeded as follows

(i) in the gauge sector, we find also contributions from the gauge couplings between V and $\sum$, together with one Yukawa coupling between V and Σ; the latter yields $(-2-2)C_2^2(G)$ according to (3.10.4) since $T(G)=C_2(G)$.

(ii) in the sector with $C_2(G)T(R)$ we have 4 Yukawa couplings (between the two gluinos and the two Majorana matter fermions); this yields the term with the factor $4\cdot(-2)$ in $B(N=2)$.

(iii) in the sector with $C_2(R)T(R)$ we find the same term $4\cdot(-2)$ as in the previous sector, see again (3.10.4), but in addition there is one Yukawa coupling between S and $\bar{S}$. The latter yields $-2(C_2(R)+C_2(R))T(R)$ because now all fermions are in the same representation R (or $\bar{R}$). The various Yukawa couplings can be summarized as follows

$$\begin{array}{ccc} -4C_2^2(G)\left\{\begin{array}{l} V \\ \Sigma \end{array}\right. & & \left.\begin{array}{r} S \\ \bar{S} \end{array}\right\} -4C_2(R)T(R) \\ \searrow & & \swarrow \\ & 4\times \text{eq } (3.10.4) & \end{array}$$

Finally we consider the $N=4$ model. It contains an $N=2$ vector multiplet coupled to one hypermultiplet in the adjoint representation. We already know that it is two-loop finite, but also the one-loop β function vanishes as it is given by $-2C_2(G)+2T(R)$ with $T(R)=T(G)=C_2(G)$. As an exercise, one may also derive the one- and two-loop finiteness of the $N=4$ model directly from (3.10.1) and (3.10.2). Then one must include 6 Yukawa couplings between one $N=1$ vector multiplet V and 3 $N=1$ matter multiplets Σ_a in the adjoint representation, as indicated as follows

$$\begin{array}{ccl} & & \Sigma_1 \\ & \nearrow & \\ V & \rightarrow & \Sigma_2 \\ & \searrow & \\ & & \Sigma_3 \end{array}$$

The result for 4 gluinos and 6 real scalars together with the 6 Yukawa couplings reads

$$
\begin{aligned}
A &= \left(-\frac{11}{3} + \frac{1}{2}\frac{4}{3}\cdot 4 + \frac{1}{2}\frac{1}{3}\,6\right) C_2(G) = 0 \\
B &= -\frac{34}{3} + \frac{1}{2}\left(\frac{20}{3}+4\right)4 + \frac{1}{2}\left(\frac{2}{3}+4\right)6 - 6\cdot 4 = 0
\end{aligned}
\tag{3.10.8}
$$

For products groups $G_1 \times G_2$ with simple factors or $U(1)$ factors, the one- and two-loop β function reads

$$
\begin{aligned}
&\beta_{g_1} = \frac{1}{16\pi^2} g_1^3 \left[\frac{4}{3}T(R_1)d(R_2) + \frac{1}{3}T(S_1)d(S_2) - \frac{11}{3}C_2(G)\right] \\
&+\frac{1}{(16\pi^2)^2} g_1^5 \left[\left\{\frac{20}{3}C_2(G_1) + 4C_2(R_1)\right\} T(R_1)d(R_2)\right. \\
&\left.+\left\{\frac{2}{3}C_2(G_1) + 4C_2(S_1)\right\} T(S_1)d(S_2) - \frac{34}{3}C_2^2(G_1)\right] \\
&+\frac{1}{(16\pi^2)^2} g_1^3 g_2^2 \left[4C_2(R_2)d(R_2)T(R_1) + 4C_2(S_2)d(S_2)T(S_1)\right]
\end{aligned}
\tag{3.10.9}
$$

Here $d(R_j)$ is the dimension of the representation R_j of the group G_j. For real or chiral fermions, one needs an extra factor $1/2$. For $U(1)$ one needs $\dim R = 1, C_2(G) = 0$ and $C_2(R) = T(R) = y^2$ where y is the hypercharge.

Applying these formulas to the Standard Model with $SU(3)\times SU(2)\times U(1)$, and normalizing the $U(1)$ such that $y^2 = 3/5$ (as is customary in applications to the grand unified group $SU(5)$), one finds straightforwardly for n_H complex Higgs doublets and n_G fermion generations

$$
\begin{aligned}
& \mu\frac{\partial}{\partial\mu} g_i = \beta_{g_i} = \frac{1}{16\pi^2} g_i^3 b_i + \frac{1}{(16\pi^2)^2}\sum_j b_{ij} g_i^3 g_j^2 \\
b_i &= \begin{pmatrix} 0 \\ -\frac{22}{3} \\ -11 \end{pmatrix} + \begin{pmatrix} \frac{4}{3} \\ \frac{4}{3} \\ \frac{4}{3} \end{pmatrix} n_G + \begin{pmatrix} \frac{1}{10} \\ \frac{1}{6} \\ 0 \end{pmatrix} n_H \\
b_{ij} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\frac{136}{3} & 0 \\ 0 & 0 & -102 \end{pmatrix} + \begin{pmatrix} \frac{9}{50} & \frac{9}{10} & 0 \\ \frac{3}{10} & \frac{13}{16} & 0 \\ 0 & 0 & 0 \end{pmatrix} n_H + \begin{pmatrix} \frac{19}{5} & \frac{3}{5} & \frac{44}{15} \\ \frac{1}{5} & \frac{49}{3} & 4 \\ \frac{11}{30} & \frac{3}{2} & \frac{76}{3} \end{pmatrix} n_G
\end{aligned}
\tag{3.10.10}
$$

For the minimally susy Standard Model (MSSM), one find

$$\begin{aligned} b_i &= \begin{pmatrix} 0 \\ -6 \\ -9 \end{pmatrix} + \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} n_G + \begin{pmatrix} \frac{3}{10} \\ \frac{1}{2} \\ 0 \end{pmatrix} n_H \\ b_{ij} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & -24 & 0 \\ 0 & 0 & -54 \end{pmatrix} + \begin{pmatrix} \frac{38}{15} & \frac{6}{5} & \frac{88}{15} \\ \frac{2}{5} & 14 & 8 \\ \frac{11}{15} & 3 & \frac{68}{3} \end{pmatrix} n_G + \begin{pmatrix} \frac{9}{50} & \frac{9}{10} & 0 \\ \frac{3}{10} & \frac{7}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix} n_H \end{aligned} \tag{3.10.11}$$

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A Proof that $\Gamma = \Gamma^{\text{ren}}$ even with external sources

In this appendix we prove that the unrenormalized and the renormalized action are equal

$$\Gamma = \Gamma^{\text{ren}} \tag{3.A.1}$$

even when the external sources K and L are present. We assure that we are in Minkowski space, as appropriate for particle physics.[16] We define the renormalized action by $S^{\text{ren}} = S(\phi^I_{\text{ren}}, \cdots u)$ where

$$S(\phi^I, \cdots g) \equiv S\left(\phi^I_{\text{ren}}, \cdots, u\right) + \Delta S\left(\phi^I_{\text{ren}}, \cdots, u\right) \tag{3.A.2}$$

[16]For functional methods in Euclidean space, see for example [8].

Here ϕ^I denote all gauge fields and matter fields and u is the renormalized coupling constant. The counter terms ΔS depend on **renormalized** quantities, and are obtained from the unrenormalized action by taking the difference between $[S(\phi^I, \cdots g)$ $= S(\sqrt{Z_3}\phi^I_{\rm ren}, \cdots, \frac{Z_1}{(Z_3)^{3/2}} u^{\rm ren})]$ and $[S\left(\phi^I_{\rm ren}, \cdots, u_{\rm ren}\right)]$. (Using dimensional regularization, the coupling constant g acquires a dimension in n spacetime dimensions, hence one must introduce a new mass parameter μ in order to keep a dimensionless renormalized coupling constant $u_{\rm ren}$. If one computes with $S^{\rm ren}$ and $\Delta S^{\rm ren}$, one must replace u by $u\mu^{\frac{1}{2}(4-n)}$ in n dimensions.) The action contains also the external sources ${K_a}^\mu$ and L_a for the BRST transformations of ${A_\mu}^a$ and c^a, respectively, and these are also renormalized, as explained in the text.

The unrenormalized functional $Z(J_a^\mu, \beta_a, \gamma^a, K_a^\mu, L_a)$ contains the source terms $J_a^\mu A_\mu^a + \beta_a c^a + b_a \gamma^a$. Next we define a renormalized functional $Z^{\rm ren}$ by

$$\begin{aligned} Z^{\rm ren} &= N \int D\phi^I_{\rm ren} D{c_{\rm ren}}^\alpha D{b_\alpha}^{\rm ren} D{d_\alpha}^{\rm ren} \\ &\qquad \exp \frac{i}{\hbar}\left[S_{\rm ren} + \Delta S_{\rm ren} + J_I \phi^I_{\rm ren} + \cdots\right] \end{aligned} \tag{3.A.3}$$

where $S_{\rm ren} + \Delta S_{\rm ren}$ depends on renormalized quantities. (Some people couple ${\phi^I}_{\rm ren}$ to ${J_I}^{\rm ren}$ instead of coupling to J_I and define $\sqrt{Z_3} J_I = {J_I}^{\rm ren}$. Since $J_I \phi^I = J_I(\sqrt{Z_3}{\phi_{\rm ren}}^I) =$ $(\sqrt{Z_3}J_I){\phi_{\rm ren}}^I = {J_I}^{\rm ren}{\phi_{\rm ren}}^I$, they then find $Z(J_I, \ldots) = Z^{\rm ren}({J_I}^{\rm ren}, \ldots)$. We shall only use one symbol J_I). Comparison with the definition of the unrenormalized Z then shows that

$$\begin{aligned} &Z\left(J_a^\mu, \beta_a, \gamma^a, K_a^\mu, L_a, g\right) = \\ &Z^{\rm ren}\left(\sqrt{Z_3}J_a^\mu, \sqrt{Z_{gh}}\beta_a, \sqrt{Z_{gh}}\gamma^a, K_a^{\mu,{\rm ren}}, L_a^{\rm ren}, u\mu^{\frac{1}{2}(4-n)}\right) \end{aligned} \tag{3.A.4}$$

up to products of Z factors from the measure. Taking the logarithm of (3.A.4), we then also get

$$W(J_I, \ldots K \ldots) = W^{\rm ren}\left(\sqrt{Z_3}J_I, \ldots K^{\rm ren} \ldots\right) \tag{3.A.5}$$

(up to an additive constant which is the logarithm of the beforementioned Z-factors in the measure. These Z factors in the measure cancel when one takes the logarithim and differentiates with respect to sources.) The next step is to make a separate Legendre transform for the unrenormalized functionals, and another one for the corresponding renormalized functionals. Hence we define

$$\begin{aligned} &\Gamma(\phi^I, K, \ldots) = W(J_I, K, \ldots) - J_I\phi^I \ldots \\ &\Gamma^{\text{ren}}\left(\phi^I_{\text{ren}}, K^{\text{ren}} \ldots\right) = W^{\text{ren}}\left(J_I, K^{\text{ren}} \ldots\right) - J_I\phi^I_{\text{ren}} \cdots \end{aligned} \tag{3.A.6}$$

(The signs are such that for $\hbar = 0, \Gamma$ reduces to S without source terms with J). However, recalling the relation between $W(J_I, \cdots)$ and $W^{\text{ren}}(\sqrt{Z_3}J_I, \cdots)$, we can also write Γ in terms of renormalized quantities

$$\Gamma\left(\phi^I, K \ldots\right) = W^{\text{ren}}\left(\sqrt{Z_3}J^I, \ldots K^{\text{ren}}_I \ldots\right) - \left(J_I\sqrt{Z_3}\right)\phi^I_{\text{ren}} \cdots \tag{3.A.7}$$

Although the last factor $\sqrt{Z_3}$ came from ϕ^I, we have put it together with J_I because then we see that this relation is identical to the definition of Γ^{ren}, the only difference being that we have used a variable $J'_I \equiv \sqrt{Z_3}J_I$ instead of J_I in the Legendre transformation. Since the Legendre transformation holds for any J_I this makes no difference.

Hence, we find the important relation

$$\Gamma^{\text{ren}}\left(\phi^I{}_{\text{ren}}, c^\alpha_{\text{ren}}, b^{\text{ren}}_\alpha, K^{\text{ren}}_I, L^{\text{ren}}_\alpha, u\right) = \Gamma\left(\phi^I, c^\alpha, b_\alpha, K_I, L_\alpha, g\right) \tag{3.A.8}$$

The one-particle irreducible graphs which represent Γ are computed with unrenormalized quantities, whereas those which yield Γ^{ren} are computed using renormalized quantities and counter terms. Yet, both are formally the same. Since Γ is divergent (and hence does not make sense), we regulate both expressions with the same regularization scheme, with regularization parameter Λ (for example, $\Lambda = \frac{1}{\epsilon} = (n-4)^{-1}$ in dimensional regularization). Then we find at the regularized level the following, now well-defined, relation

$$\Gamma\left(\phi^I \cdots, g, \epsilon\right) = \Gamma^{\text{ren}}\left(\phi^I_{\text{ren}}, \cdots, u, \mu, \epsilon\right) \tag{3.A.9}$$

The limit of ϵ tending to zero exists for $\Gamma^{\text{ren}}[\phi^I_{\text{ren}}, \ldots, \epsilon]$ as a result of the renormalization procedure. This does not mean that also the left-hand side is finite if one lets ϵ tend to zero and keeps $\phi^I \cdots g$ fixed. Rather, because the right-hand side is finite if $\epsilon \to 0$ at fixed ϕ^I_{ren} and u, also the left-hand side is finite if $\epsilon \to 0$ provided one keeps ϕ^I_{ren} and u fixed, but then ϕ^I etc. will change if $\epsilon \to 0$ since $\phi^I = Z_I^{1/2}\phi^I_{\text{ren}}$, and Z_I depends on ϵ.

Although $\Gamma = \Gamma^{\text{ren}}$, this does not mean that the n-point Green functions of the unrenormalized theory are finite. The n-point functions in the unrenormalized theory are obtained by differentiation of Γ w.r.t. ϕ^I etc.

$$\left(\frac{\delta}{\delta\phi^I}\cdots\right)\Gamma = \left(\frac{1}{\sqrt{Z_3}}\frac{\delta}{\delta\phi^I_{\text{ren}}}\cdots\right)\Gamma^{\text{ren}} \tag{3.A.10}$$

The Green functions of the renormalized theory are given by $\frac{\delta}{\delta\phi^I_{\text{ren}}}\cdots\Gamma^{\text{ren}}$. They are finite, and one can thus also obtain them by multiplying the Green's function of the unrenormalized theory by suitable Z factors.

As a graphical check that of (3.A.10), consider a particular proper diagram, for example the triangle diagram for the 3-point function. Recalling $A_\mu{}^a = Z_3{}^{1/2}(A_\mu{}^a)^{\text{ren}}$, we must show that

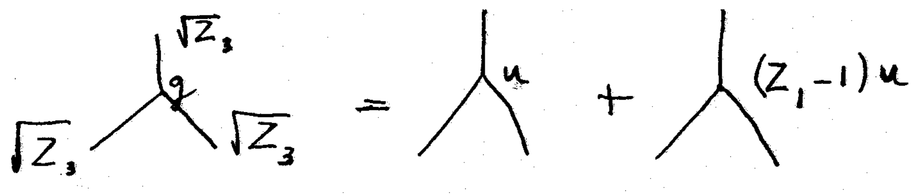
$$Z_3{}^{3/2}\left(\frac{\partial}{\partial A_\mu{}^a}\right)^3 \Gamma\left(A_\mu{}^a, \ldots, g; \epsilon\right) = \left(\frac{\partial}{\partial A_\mu{}^{a,\text{ren}}}\right)^3 \Gamma^{\text{ren}}\left(A_\mu{}^{a,\text{ren}}, \ldots, u\mu^{\frac{1}{2}(4-n)}; \epsilon\right) \tag{3.A.11}$$

We shall compute Γ with $\mathcal{L}(A_\mu{}^a, \ldots, g)$, but Γ^{ren} with $\mathcal{L}(A_\mu{}^{a,\text{ren}}, \ldots, u) + \Delta\mathcal{L}$ $(A_\mu{}^{a,\text{ren}}, \ldots, u)$ where $\Delta\mathcal{L}$ is proportional to $Z_1 - 1, Z_3 - 1$, etc. We shall consider graphs with the same topological structure on the left-hand and right-hand side, but on the right-hand side we allow infinitely many counter term insertions.

First consider the tree graph. On the left-hand side of (3.A.11) one finds only $Z_3{}^{3/2}g$, but on the right-hand side one finds $Z_1 u\mu^{\frac{1}{2}(4-n)}$ because one must add the tree vertex from the action to the tree vertex from the counter terms, and the counter terms for the vertex are proportional to $Z_1 - 1$. The result is the identity

$$Z_3{}^{3/2} g = Z_1 u \mu^{\frac{1}{2}(4-n)} \tag{3.A.12}$$

which is just the definition of u. One can express this relation graphically as follows



Next consider the one-loop triangle graph. On the left-hand side one obtains a factor $Z_3{}^{3/2} g^3$ times a one-loop graph. On the right-hand side, we find the same graph, but now with infinitely many crosses (counter term insertions) both in the propagators and in the vertices. The insertions into the propagator[17] lead to a factor $Z_3{}^{-1}$ because one must sum the geometric series $\sum_{k=0}^{\infty}(1-Z_3)^k = \frac{1}{1+(Z_3-1)} = \frac{1}{Z_3}$. Of course this argument only makes sense if $|Z_3| < 1$.

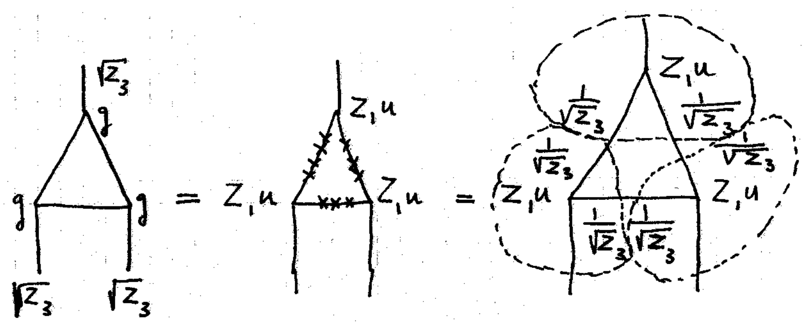


Figure caption: Graphical representation of the coupling constant renormalization. All graphs have the same topological structure (triangles in this example).

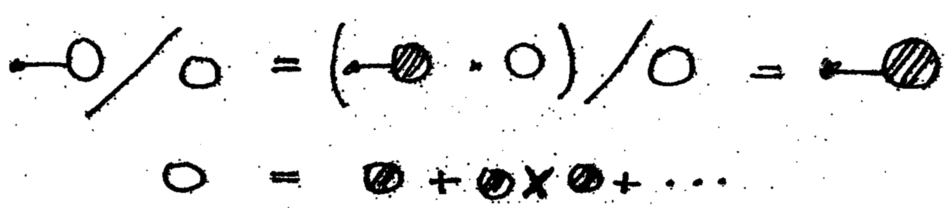
Clearly, each vertex on the right-hand side has one u and one Z_1 and two $Z_3^{-1/2}$, and at each vertex one can use the identity $g = Z_1/Z_3{}^{3/2}u$. In this way the Z factors from propagators, vertices and external lines turn the renormalized coupling constant u into the unrenormalized coupling constant g.

[17] The counter terms are $-\frac{1}{2}(Z_3-1)[(\partial_\mu\varphi)^2+m^2\varphi^2]$, and applying Wick contractions, all corrections to the propagator become proportional to the lowest-order propagator times factors $(1-Z_3)^n$.

B Functional methods for external sources

A well-known theorem states that the functional Γ generates proper graphs (one-particle irreducible diagrams). Several textbooks prove this theorem. [46] We need a generalization when external sources $N(x)$ are present (for example, the BRST sources ${K_a}^\mu(x)$ or $L_a(x)$, or later, when we discuss composite operators, the sources $N^j(x)$ for composite operators $O_j(x)$). We shall now show that also in this case Γ generates proper graphs. Intuitively this seems plausible: the Legendre transformation from J to Q produces graphs Γ which are one-particle irreducible w.r.t. Q (cutting Q, the graphs do not become disconnected). There is no Legendre transformation w.r.t. N, but the external sources do not progagate inside graphs, so the graphs in Γ with external N should remain proper. We want to prove this.

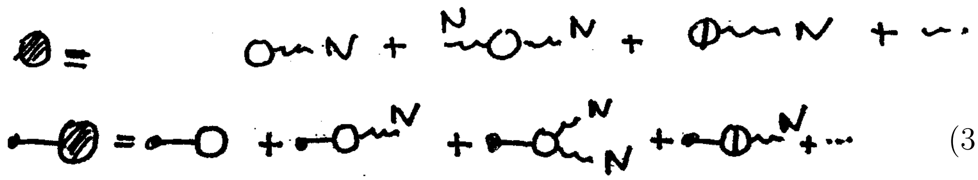
We begin by recalling some results for $W[J]$. We are again in Minkowski space. By definition $Z[J] = \exp \frac{i}{\hbar} W[J]$ where $Z[J]$ generates the sum of connected as well as disconnected graphs. Let the current J couple to fields Q. Consider $\frac{\delta}{\delta J} W = \left(\frac{\hbar}{i}\frac{\delta}{\delta J} Z\right)/Z$ and set $J = 0$ in this expression. In the ratio, the sum of vacuum bubbles cancels, and one finds the set of connected 1-point functions. Denoting connected graphs by hatches, we have graphically, starting with $\left(\frac{\hbar}{i}\frac{\delta}{\delta J} Z\right)_{J=0}/Z_{J=0}$ and ending with $\left(\frac{\delta}{\delta J} W\right)_{J=0}$,



(3.B.1)

Open blobs denote connected and disconnected graphs. The dots denote the point x where a quantum field Q starts to propagate. One can write this result as $< Q >_{\rm dis} - < I >_{\rm dis} = < Q >$, where $< Q >$ denotes the set of connected 1-point function, $< Q >_{\rm dis}$ the set of disconnected as well as connected 1-point functions, and $< I >_{\rm dis}$

the set of all vacuum selfenergy graphs.[18] All of these graphs can have any number of external sources N, but no external sources J because we set $J = 0$. For example

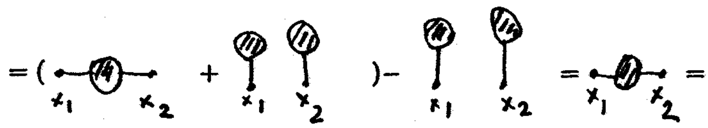


(3.B.2)

where now we have written on the right-hand side examples of Feynman diagrams instead of open blobs.

For the 2-point function we find that also $\frac{\delta^2 W}{\delta J^2}$ yields the connected graphs

$$\frac{\hbar}{i}\frac{\delta}{\delta J(x_2)}\frac{\delta}{\delta J(x_1)}W = \frac{\hbar}{i}\frac{\delta}{\delta J(x_2)}\left(\frac{\frac{\hbar}{i}\frac{\delta}{\delta J(x_1)}Z}{Z}\right) = \frac{< Q_1 Q_2 >_{\rm dis}}{Z} - < Q_1 >< Q_2 >$$



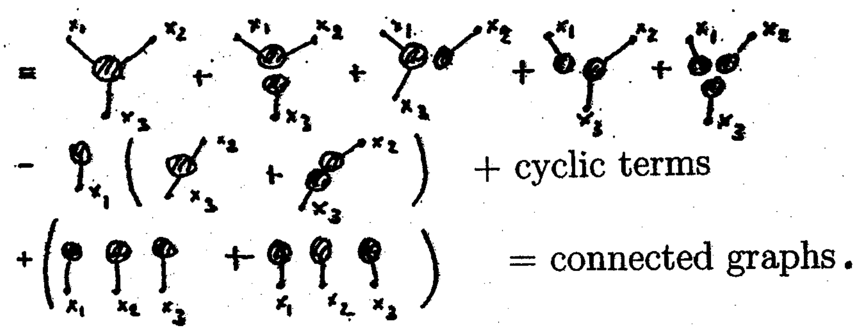
=connected graphs (3.B.3)

We denoted $Q(x_j)$ by Q_j and all terms are taken at $J = 0$. There are no vacuum self-energy graphs because all terms are divided by Z. We set $J = 0$ in this result but all graphs in (3.B.3) can still have any number of external sources N.

Similarly, setting $J = 0$ in all terms below,

$$\left(\frac{\hbar}{i}\right)^2 \frac{\delta^3}{(\delta J)^3}W = \frac{\hbar}{i}\frac{\delta}{\delta J_3}\left(\frac{\frac{\hbar}{i}\frac{\delta}{\delta J_2}\frac{\hbar}{i}\frac{\delta}{\delta J_1}Z}{Z} - \frac{\frac{\hbar}{i}\frac{\delta}{\delta J_2}Z}{Z}\frac{\frac{\hbar}{i}\frac{\delta}{\delta J_1}Z}{Z}\right)$$
$$= \frac{\frac{\hbar}{i}\frac{\delta}{\delta J_3}\frac{\hbar}{i}\frac{\delta}{\delta J_2}\frac{\hbar}{i}\frac{\delta}{\delta J_1}Z}{Z} - \frac{\frac{\hbar}{i}\frac{\delta}{\delta J_3}\frac{\hbar}{i}\frac{\delta}{\delta J_2}Z}{Z}\frac{\frac{\hbar}{i}\frac{\delta}{\delta J_1}Z}{Z} + 2 \text{ cyclic terms}$$
$$+2\frac{\frac{\hbar}{i}\frac{\delta}{\delta J_3}Z}{Z}\frac{\frac{\hbar}{i}\frac{\delta}{\delta J_2}Z}{Z}\frac{\frac{\hbar}{i}\frac{\delta}{\delta J_1}Z}{Z}$$
$$= \frac{< Q_3 Q_2 Q_1 >_{\rm dis}}{Z} - \frac{< Q_3 Q_2 >_{\rm dis}}{Z} < Q_1 > - \frac{< Q_2 Q_1 >_{\rm dis}}{Z} < Q_3 >$$
$$- \frac{< Q_1 Q_3 >_{\rm dis}}{Z} < Q_2 > +2 < Q_3 >< Q_2 >< Q_1 >$$

[18] Usually one normalizes Z by a constant such that $Z[J = 0] = 1$. If external sources are present, one may choose the normalization $Z(J = 0, N = 0) = 1$. Then the open blobs correspond to graphs with only external sources $N(x)$ (at least one external source $N(x)$, but no graphs without any external source).



(3.B.4)

Clearly, only the connected graphs remain. Again, all graphs can still have any number of sources N.

Next we consider $\Gamma(Q) = W(J) - JQ$. (To simplify the notation, we introduced the expression JQ which stands for $\int J_i(x)Q^i(x)d^4x$). From $\frac{\delta W}{\delta J} = Q$ (and not yet setting $J = 0$ in the result) one obtains $Q = Q(J)$ or $J = J(Q)$. When one or more external sources N are present, we have $Q = Q(J, N)$ and (if we can invert this relation, about which more below) $J = J(Q, N)$. Then

$$\begin{aligned} \Gamma(Q, N) &= W(J(Q, N), N) - J(Q, N)Q \\ \Gamma(Q(J, N), N) &= W(J, N) - JQ(J, N) \end{aligned} \tag{3.B.5}$$

It helps to compare with the Legendre transform in classical mechanics where $\Gamma \sim -H, W \sim L, J \sim \dot{q}, Q \sim p$ and $N \sim q$. In the first line Q and N are taken as independent variables, while in the second line J and N are the independent variables. The connected graphs are obtained by differentiating W w.r.t. J. The proper graphs are obtained by differentiating Γ w.r.t. Q, as we shall show.

Let us now first come back to the question whether one can invert $Q(J, N)$. First consider $\frac{\delta W}{\delta J} = Q(J, N)$. The field $Q(J, N)$ is the field which appears in the Legendre transformation $\Gamma = W - QJ$, but it is also the expectation value of the field Q which couples to the source J in the action, $Q(J, N) =< Q >_{J,N}$. From the Legendre transformation we find the important relation

$$J = -\frac{\delta\Gamma}{\delta Q} \tag{3.B.6}$$

Without external sources N one usually assumes that $J = 0$ implies $Q(J = 0) = 0$, and $Q = 0$ implies $J(Q = 0) = 0$, and that $(\partial^2 W/\delta J^2)_{J=0} \neq 0$. Then one can invert $Q(J)$. In theories with spontaneous symmetry breaking one achieves this by shifting $Q(J) = q(J) + < Q >_{J=0}$ where $< Q >_{J=0}$ is a **constant**, and working with q. With external sources N, $J = 0$ does not imply $Q(N, J = 0) = 0$, and $Q = 0$ does not imply $J(N, Q = 0) = 0$. We again shift $Q = q + < Q >_{J=0}$, even though now $< Q >_{J=0}$ is not a constant but x-dependent and a functional of N. We view Γ as a function of q and N (not Q and N). At $J = 0$, we then still have

$$< q >_{J=0} = 0 \qquad \frac{\delta \Gamma}{\delta q}_{J=0} = 0. \tag{3.B.7}$$

This is a complicated relation, because Q and N in this relation are not independent, but constrained by $J(Q, N) = 0$.

We always redefine the quantum fields by shifts such that tadpoles vanish, but from now on we revert to the usual notation, and use Q instead of q. Hence

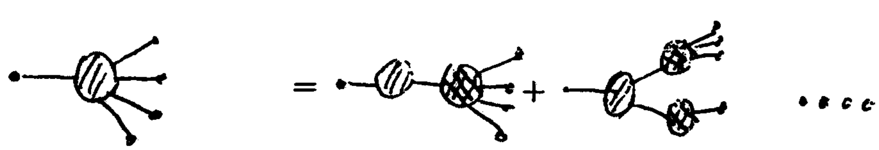
$$\left(\frac{\delta W[J, N]}{\delta J(x)}\right)_{J=0} = (< Q(J, N, x) >)_{J=0} =$$

$$= 0 \tag{3.B.8}$$

Then, by a Taylor expansion in J at fixed N

$$q(J) = \left(\frac{\delta^2 W}{\delta J^2}\right)_{J=0} J + \frac{1}{2!}\left(\frac{\delta^3 W}{\delta J^3}\right)_{J=0} J^2 + \ldots \tag{3.B.9}$$

(Again the notation $\left(\frac{\delta^2 W}{\delta J^2}\right)_{J=0} J$ includes an integration over spacetime). The factors $\delta^n W/\delta^n J$ at $J = 0$ yield connected n-point functions, so $\delta W/\delta J$ is a sum of connected graphs



The doubly-hatched graphs denote proper graphs. The dot on the left in each graph indicates the point from where a Q-propagator starts, and the dots on the right in each graph indicate sources $\int J d^4x$. This proves that $-J \equiv \{\partial\Gamma[Q(J,N),N]/\partial Q\}$ is the generator of proper graphs.

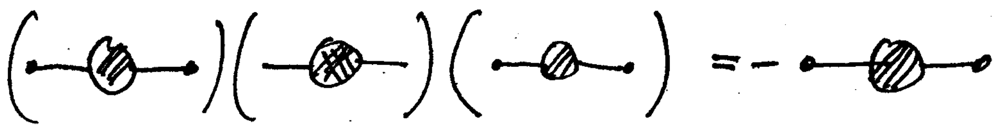
It is instructive to consider the 2, 3, and 4 point functions separately. First consider $(\delta^2\Gamma/\delta Q_I \delta Q_K)_{Q=0} = -\left(\frac{\delta}{\delta Q_I} J_K\right)_{Q=0} = -\left(\frac{\delta}{\delta J} Q\right)^{-1}_{IK}$ at $J=0$. Since $Q = \frac{\delta}{\delta J} W$ we find

$$\left(\frac{\delta^2\Gamma}{\delta Q^2}\right)_{Q=0} = -\left(\frac{\delta^2 W}{\delta J^2}\right)^{-1}_{J=0} \quad \text{(matrix notation)} \tag{3.B.10}$$

or

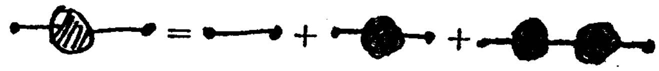
$$\left(\frac{\delta^2 W}{\delta J^2}\right)_{J=0}\left(\frac{\delta^2\Gamma}{\delta Q^2}\right)_{Q=0}\left(\frac{\delta^2 W}{\delta J^2}\right)_{J=0} = -\left(\frac{\delta^2 W}{\delta J^2}\right)_{J=0} \tag{3.B.11}$$

In diagrammatical notation

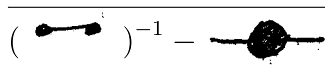
 (3.B.12)

One can gain further insight in this result by summing the set of connected 2-point functions

$$\left(\frac{\delta^2 W}{\delta J^2}\right)_{J=0} =$$



$$= \frac{1}{(\quad)^{-1} - \quad}$$

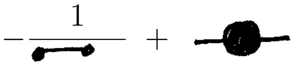
 (3.B.13)

The solid blobs denote genuine loop corrections to the proper selfenergy, but not the $\hbar = 0$ part. We used

$$x + x(\pi x) + x(\pi x \pi x)\cdots = \left(\frac{1}{x}\right)^{-1}\left(\frac{1}{1-\pi x}\right) = \frac{1}{(1-\pi x)x^{-1}} = \frac{1}{\frac{1}{x} - \pi} \tag{3.B.14}$$

By taking the inverse we find

$$\left(\frac{\delta^2\Gamma}{\delta Q^2}\right)_{Q=0} = -\left(\frac{\delta^2 W}{\delta J^2}\right)^{-1}_{J=0} = -\frac{1}{\quad} + \quad \tag{3.B.15}$$



This proves that $\frac{\delta^2\Gamma}{\delta Q^2}$ is equal to the amputated set of proper graphs. The first term is the kinetic term in the action, and the next term is the sum of all proper selfenergy corrections with $\hbar \neq 0$.

As an example, consider a real scalar field φ with kinetic action $\frac{1}{2}\varphi A\varphi$ and source term $J\varphi$. After completing squares one obtains the path integral $Z = e^{-\frac{1}{2}\frac{i}{\hbar}\int JA^{-1}J}$ $\left(\int d\varphi e^{\frac{i}{\hbar}\int \frac{1}{2}\varphi' A\varphi'}\right)$ where φ' is defined by $\varphi' = \varphi + A^{-1}J$. For $A = \Box - m^2$ we get

$$W = -\frac{1}{2}\int JA^{-1}J = -\frac{1}{2}\int J(x)\left[\int \frac{e^{ik(x-y)}}{k^2+m^2-i\epsilon}\frac{-1}{(2\pi)^4}d^4k\right]J(y)d^4xd^4y \quad (3.B.16)$$

Then to lowest order in J

$$\frac{\delta^2 W}{\delta J^2}\mid_{J=0} = \frac{\delta}{\delta J(x)}\frac{\delta}{\delta J(y)}W_{J=0} = \frac{1}{(2\pi)^4}\int \frac{e^{ik(x-y)}}{k^2+m^2-i\epsilon}d^4k \quad (3.B.17)$$

So to lowest order in $\hbar$, Γ is indeed equal to the action, and to lowest order in φ,

$$\frac{\delta^2\Gamma}{\delta\varphi^2} = A(x)\delta(x-y) = (\Box - m^2)\delta(x-y) = \int(-k^2-m^2)\frac{1}{(2\pi)^4}e^{ik(x-y)}d^4k. \quad (3.B.18)$$

The relation

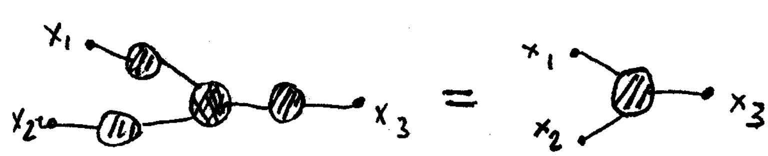
$$\left(\frac{\delta^2 W}{\delta J^2}\right)^{-1}_{J=0} = -\left(\frac{\delta^2\Gamma}{\delta Q^2}\right)_{Q=0} \quad (3.B.19)$$

clearly holds.

We want to go to 4-point functions for reasons to become clear, so we do some more graphical analysis, using all the time $\frac{\delta}{\delta Q} = \frac{\delta J}{\delta Q}\frac{\delta}{\delta J}$ and $\frac{\delta J}{\delta Q} = \left(\frac{\delta^2 W}{\delta J^2}\right)^{-1}$. Consider

$$\begin{aligned}
\frac{\delta^3}{\delta Q^3}\Gamma &= \frac{\delta J_{1'}}{\delta Q_1}\frac{\delta}{\delta J_{1'}}\left(\frac{\delta^2\Gamma}{\delta Q^2}\right)_{23} = \left(\frac{\delta^2 W}{\delta J^2}\right)^{-1}_{11'}\frac{\delta}{\delta J_{1'}}\left(-\frac{\delta^2 W}{\delta J^2}\right)^{-1}_{23} \\
&= \left(\frac{\delta^2 W}{\delta J^2}\right)^{-1}_{11'}\left(\frac{\delta^2 W}{\delta J^2}\right)^{-1}_{22'}\left(\frac{\delta}{\delta J_{1'}}\frac{\delta}{\delta J_{2'}}\frac{\delta}{\delta J_{3'}}W\right)\left(\frac{\delta^2 W}{\delta J^2}\right)^{-1}_{3'3}
\end{aligned} \quad (3.B.20)$$

After multiplication by three factors $\frac{\delta^2 W}{\delta J^2}$ one finds in graphical notation



Hence, peeling off as many connected two-point functions as possible, one is left with the proper three-point functions.

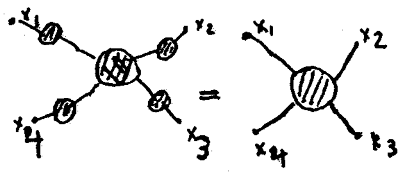
Finally, for the 4-point functions

$$\frac{\delta^4}{\delta Q^4}\Gamma^4 = \left(\frac{\delta^2 W}{\delta J^2}\right)^{-1}_{44'} \frac{\delta}{\delta J_{4'}}\left(\frac{\delta^3\Gamma}{\delta Q_1\delta Q_2\delta Q_3}\right) \tag{3.B.21}$$

and after inserting the result for $\frac{\delta^3\Gamma}{\delta Q^3}$ we find for the right-hand side

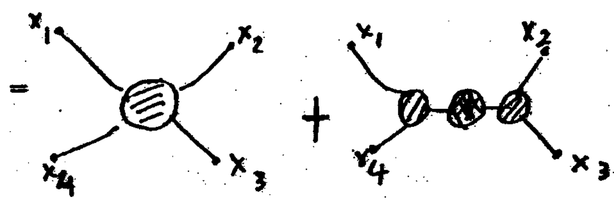
$$\left(\frac{\delta^2 W}{\delta J^2}\right)^{-1}_{44'}\left(\frac{\delta^2 W}{\delta J^2}\right)^{-1}_{11'}\cdots\left(\frac{\delta^4 W}{\delta J^4}\right)_{1'2'3'4'} - \left(\frac{\delta^2 W}{\delta J^2}\right)^{-1}_{44'}\left(\frac{\delta^2 W}{\delta J^2}\right)^{-1}_{1p}\left(\frac{\delta^3 W}{\delta J^3}\right)_{p4'q}\left(\frac{\delta^2 W}{\delta J^2}\right)^{-1}_{q1'}$$
$$\left(\frac{\delta^2 W}{\delta J^2}\right)^{-1}_{22'}\frac{\delta^3 W}{\delta J_{1'}\delta J_{2'}\delta J_{3'}}\left(\frac{\delta^2 W}{\delta J^2}\right)^{-1}_{3'3} + 2 \text{ cyclic terms} \tag{3.B.22}$$

After multiplication by 4 factors $\frac{\delta^2 W}{\delta J^2}$ we obtain in graphical notation



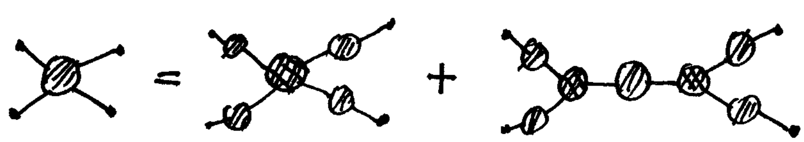
$$- \left(\frac{\delta^3 W}{\delta J^3}\right)_{14q}\left(\frac{\delta^2 W}{\delta J^2}\right)^{-1}_{q1'}\frac{\delta^3 W}{\delta J_{1'}\delta J_2\delta J_3}$$

$$+2 \text{ cyclic terms} =$$



$$+ 2 \text{ cyclic terms} \tag{3.B.23}$$

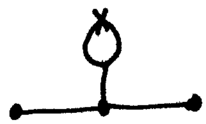
We used (3.B.10) in the last line. Inserting the result for the connected 3-point functions, we find, using also (3.B.12),



$$+ 2 \text{ cyclic terms} \tag{3.B.24}$$

Note that from the 4-point function on, amputation (removal of connected 2-point functions from the external lines) does not in general lead to proper graphs.

We now generalize these results to the case that external fields $N(x)$ are present. Consider the following one-loop connected graph with one insertion of the composite operator and two external fields.



$$= \frac{\delta^3 W}{\delta N(x)\delta J(y)\delta J(z)} \tag{3.B.25}$$

Then we claim that $\delta\Gamma/\delta N(x)$ corresponds to the following proper graph

$$= \delta\Gamma/\delta N(x) \tag{3.B.26}$$

To derive expressions for Γ_N, consider W_N, for example the expression $\frac{\partial}{\partial K_a{}^\mu(x)} W$ which one encounters in the proof of renormalization. Differentiation of

$$\Gamma[Q,N] = W[J(Q,N),N] - J(Q,N)Q \tag{3.B.27}$$

w.r.t. N at fixed Q yields

$$\Gamma_{N(x)} = \int \left[\frac{\partial W}{\partial J(y)} \frac{\partial J(y)}{\partial N(x)} + \frac{\partial W}{\partial N(x)} - \frac{\partial J(y)}{\partial N(x)} Q(y) \right] d^4 y = W_{N(x)} \tag{3.B.28}$$

since $\frac{\partial W}{\partial J}[J,N] = Q$. Differentiation of W_N w.r.t. J at fixed N yields all connected n-point functions with one N vertex at the point x. We claim that differentiation of $\Gamma_N[Q,N]$ w.r.t. Q at fixed N yields proper graphs, as suggested by $\Gamma_N = W_N$. (Note that Γ and hence Γ_N depends on Q and N.) To prove this, we use the chain rule repeatedly. For example,

$$\frac{\partial}{\partial Q(y)}\Gamma_N = \int \left(\frac{\partial}{\partial Q(y)} J(z) \right) \left(\frac{\partial}{\partial J(z)} W_N \right) d^4 z = \left(\frac{\delta^2 W}{\delta J^2} \right)^{-1} \left(\frac{\partial}{\partial J} W_N \right) \tag{3.B.29}$$

(Note that this relation between Γ_N and W_N is not valid for Γ and W in which case one finds $\frac{\partial}{\partial Q}\Gamma = -J$). After multiplication by the connected propagator we obtain

$$\left(\frac{\delta^2 W}{\delta J^2} \frac{\partial}{\partial Q} \Gamma_N \right)_{J=Q=0} =$$

$$=$$

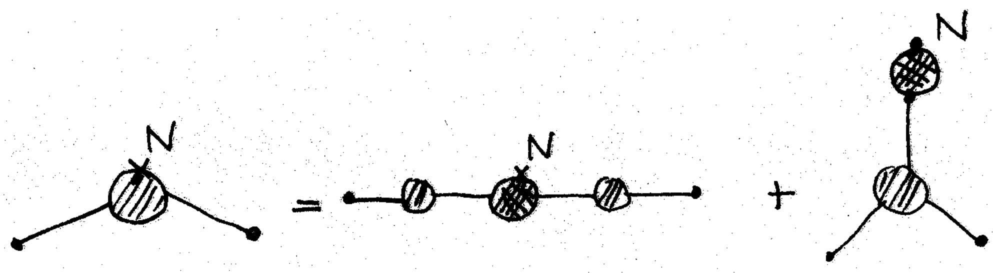
$$= \left(\frac{\delta W_N}{\delta J} \right)_{J=0} \tag{3.B.30}$$

This proves graphically that $\partial/\partial Q\ \Gamma_{N|Q=0}$ consists of proper graphs.

Similarly, by differentiating $W_N = \Gamma_N$ twice w.r.t. J and using $\partial Q/\partial J = \partial^2 W/\partial J^2$ one obtains

$$\begin{aligned} \frac{\delta^2 W_N}{\delta J^2} &= \frac{\delta^3 W}{\delta J \delta J \delta N} = \frac{\delta}{\delta J}\left(\frac{\delta^2 W}{\delta J^2}\frac{\delta \Gamma_N}{\delta Q}\right) \\ &= \frac{\delta^2 W}{\delta J^2}\frac{\delta^2 \Gamma_N}{\delta Q^2}\frac{\delta^2 W}{\delta J^2} + \frac{\delta^3 W}{\delta J^3}\frac{\delta^2 \Gamma}{\delta Q \delta N} \end{aligned} \tag{3.B.31}$$

In graphical notation this expression becomes



(3.B.32)

For example, the BRST source K_a^μ which couples to the BRST variation of A_μ^a has vertices $K_a^\mu \partial_\mu c^a$ and $K_a^\mu g f^{abc} A_\mu^b c^c$. The vertex $K_a^\mu \partial_\mu c^a$ is then equal to $\delta^2\Gamma/\delta Q \partial N$ at tree level, and everything else belongs to $\delta^3 W/\delta J^3$.

It is clear from these examples that $\Gamma_N[Q, N]$ as a function of Q yields proper graphs. This result is crucial for the proofs of renormalizability and also for the study of divergences in graphs with one insertion of a composite operator. The proper graphs are then $\frac{\partial}{\partial Q_1} \cdots \frac{\partial}{\partial Q_k}\Gamma_N$, as we have shown.

C Details of the renormalization of the Dirac-Yang-Mills system

The most general divergences as far as power counting, ghost number and dimensions are concerned are given by

$$\begin{aligned} \Gamma^{\text{ren}}_{\text{div}} = \int \Big[& D(A, \psi, M) + ({K_a}^\mu - \partial^\mu b_a)(\alpha_4 \partial_\mu c^a + \alpha_5 {g^a}_{bc} {A_\mu}^b c^c + \alpha_7 M c^a) \\ & + \alpha_8 M^2 b_a c^a + \alpha_6 {h^a}_{bc} \frac{1}{2} g L_a c^b c^c + \alpha_9 \bar{N} q_a \psi c^a + \alpha_{10} \bar{\psi} q'_a {N_c}^a \Big] \end{aligned} \tag{3.C.1}$$

where D is the most general polynomial constructed from ${A_\mu}^a, \psi$ and $\bar{\psi}$ and M with the usual properties (Lorentz invariance, group invariance, dimension four). The tensors ${g^a}_{bc}$ and ${h^a}_{bc}$ are general invariant tensors, so in principle they are combinations of f- and d- symbols, and q_a, q'_a are matrices which will soon be shown to be equal to the generators T_a of the Lie algebra.

By brute force one can evaluate $Q\Gamma^{\text{ren}}_{\text{div}} = 0$, and finds then the following results in the various sectors

$\bar{N}\psi c^2$: using that $[T_a, q_b] = f_{ab}q^c$ because q_a is an invariant tensor, and similarly for q'_a, one finds $q'_a = \sigma T_a$ and $g\alpha_6 = -\sigma\alpha_9$.

$\bar{\psi}Nc^2$: Similarly $q'_a = \sigma' T_a$ and $g\alpha_6 = \sigma'\alpha_{10}$. Eliminating α_9 and α_{10} in terms of α_6, also σ and σ' are eliminated.

$$
\begin{aligned}
Lc^3 &: ({h^a}_{bc}{f^{a'}}_{ab'} + {f^a}_{bc}{h^{a'}}_{ab'})\, c^a c^b c^c = 0 \\
KAc^2 &: {h^a}_{bc} \sim {f^a}_{bc} \\
K\partial c^2 &: \alpha_5 {g^a}_{bc} = \alpha_6 g {h^a}_{bc} \\
KMc^2 &: \alpha_7 = 0 \\
Mbcc &: \alpha_8 = 0
\end{aligned}
\tag{3.C.2}
$$

The determination of $D(A, \psi, M)$ is more complicated. Using the same methods as in the pure gauge case, one finds

$$
\partial S(\text{class})/\partial {A_\mu}^a(\alpha_4 - \alpha_6)\partial_\mu c^a + D_\mu c^a \left(\frac{\partial}{\partial {A_\mu}^a} D\right) - \partial D/\partial\psi(gT_a\psi)c^a
$$
$$
+ g\bar{\psi}T_a \left(\frac{\partial}{\partial\bar{\psi}} D\right) c^a = 0. \tag{3.C.3}
$$

A special solution of the inhomogeneous equation for D is given by

$$
D(\text{inh}) = (\alpha_4 - \alpha_6) \int \left(A^b_\nu \frac{\partial}{\partial A^b_\nu} + \psi\frac{\partial}{\partial\psi} + \bar{\psi}\frac{\partial}{\partial\bar{\psi}} \right) S(\text{class}) \tag{3.C.4}
$$

The most general solution of the homogeneous equation for D is given by

$$
D(\text{hom}) = \alpha_m M \int \bar{\psi}\psi + \alpha_D \int \bar{\psi}\not{D}\psi + \alpha_{YM} S_{YM} + \alpha_\theta \int F\tilde{F} \tag{3.C.5}
$$

The θ-term is excluded if parity is preserved, and is not produced by perturbation theory. Therefore, the total number of possible divergences (excluding the θ-term) is 5, parametrized by $\alpha_4, \alpha_6, \alpha_m, \alpha_D$ and α_{YM}. This result was assumed below (3.3.15).

Chapter 4

Renormalization of Higgs models

In this chapter we extend the study of the renormalization of gauge theories from the case of unbroken gauge theories to the case of spontaneously broken gauge theories, i.e., gauge theories with a Higgs effect. [1]

In the Standard Model, all particles (quarks, leptons and gauge bosons) get their mass from the Higgs effect. The masses are of the form gv, where g is the coupling to the Higgs boson (a Yukawa coupling for the fermions or the electroweak $SU(2)$ and $U(1)$ gauge couplings for the gauge bosons) and $v = 246$ GeV is the vacuum expectation value of the Higgs scalar field.[1] Thus heavier particles couple more strongly to the Higgs boson. For a Higgs boson heavier than 200 GeV, experimental searches concentrate on the decay of a Higgs boson into two Z particles, each of which in turn decays into two quark jets or two lepton jets. On the other hand, for a Higgs boson of much lower mass, the decay into two photons via a (top) quark triangle loop will be looked for. Current estimates are that the Higgs boson has a mass of around 115 GeV. If this is the case, detection of the Higgs boson at the LHC (the large hadron collider) at CERN should be possible.

[1] From $m_W = \frac{1}{2}g_2 v = \frac{1}{2}ev/\sin\theta_w = 80.4$ GeV and $m_Z = m_W/\cos\theta_w = 91.18$ GeV one finds $\cos^2\theta_w = 0.23$ and $v = 246$ GeV. To get good agreement, one must use $\alpha(QED) = 1/128.5$ at the Z mass, instead of $\alpha(QED) = 1/137$ at $Q^2 = 0$.

One needs one (or more) Higgs bosons to give mass to the quarks and leptons in the Standard Model. One cannot construct directly mass terms by multiplying some left-handed spinors with right-handed spinors because in the Standard Model left-handed spinors and right-handed spinors transform in different representations of $SU_2 \times U(1)$. Thus such mass terms would violate gauge invariance. Also for the gauge fields it is not possible to add a mass term by hand because a theory with such massive vector bosons is not renormalizable. Thus **the Higgs mechanism to give mass to the particles of the Standard Model is unavoidable.**

The Higgs boson appears in the Standard Model as a real scalar field, and is part of a complex $SU(2)$ doublet. The other three real scalar fields in this doublet give mass to the vector bosons W^+, W^- and Z. Because the $SU(2)$ isospin of this Higgs multiplet is 1/2, and the action should be $SU(3) \times SU(2) \times U(1)$ invariant, the fermions with left-handed chirality necessarily belong to different $SU(2)$ multiplets than the fermions with right-handed chirality. This implies parity violation. Thus **the Higgs mechanism (with a Higgs doublet) requires that parity is broken in Nature**, as it indeed is in the weak interactions. There is no alternative to a doublet for the Higgs field because the ρ parameter, defined by $\rho = m_W/(m_Z \cos\theta_w)$, is experimentally very near unity, and only Higgs doublets can achieve this (except for some highly exotic Higgs multiplets).

As a warming up exercise we begin in section 1 with a discussion of the Goldstone theorem [2] in spontaneously broken field theories with scalar fields but without gauge fields. We consider the linear $O(2)$ sigma model with a massless Goldstone boson π ("the pion") and a massive field σ ("the Higgs boson"). [3] The action is invariant under the rotational symmetry $SO(2)$ and the reflection symmetry $\pi \to -\pi$, yielding together the $O(2)$ symmetry group after which it is named. The Goldstone theorem states that if a rigid continuous symmetry is spontaneously broken, massless scalars called Goldstone bosons emerge. An even simpler model with spontaneous symmetry breaking than the $O(2)$ linear sigma model is $\lambda\varphi^4$ theory with one real scalar field φ.

However, in this model only the discrete Z_2 symmetry $\varphi \to -\varphi$, not a **continuous** symmetry, is spontaneously broken, and this explains why the field φ can remain massive after spontaneous symmetry breaking.

In section 2 we check by explicit calculation that the pion remains massless at the one-loop and two-loop level, and explain why the masslessness of the pion does not lead to infrared divergences in Green functions. As a result the S matrix exists for this model. We also investigate whether the pion remains massless at the quantum level if one adds finite nonminimal renormalization terms (which we call recalibrations) to the Z factors. This leads us into a discussion of renormalization conditions. We discuss whether the finite renormalization Z_v of v (the vacuum expectation value of the Higgs scalar σ) and the wave function renormalizations Z_σ and Z_π should still be related by the $O(2)$ symmetry. Our main result is that the Goldstone theorem holds under very general conditions, but one should impose the renormalization condition that tadpoles cancel.

In section 3 we discuss the $SU(2)$ Higgs model at the classical level. In the Standard Model the Higgs sector has an $SU(2) \times U(1)$ symmetry group, but for reasons given below we only consider $SU(2)$. For the scalars we choose a complex $SU(2)$ doublet, as in the Standard Model, containing one Higgs boson σ and three would-be Goldstone bosons χ^a with $a = 1, 3$. The χ^a are called would-be Goldstone bosons because they cease to be physical after coupling to gauge fields. They can be gauged away (the unitary gauge), or if one keeps them in the theory, they can be "eaten" by the $SU(2)$ gauge fields $\chi^a = 0$, which then "become massive as a result of this banquet" (Coleman). This explains their name: they would be Goldstone bosons if there were no gauge fields to eat them. The best way to deal with Goldstone bosons is to use an R_ξ gauge (a renormalizable gauge with a gauge-fixing parameter ξ, see below); the χ^a become then propagating fields. We shall show that there exists an $SO(4)$ symmetry for complex $SU(2)$ Higgs doublets which consists of a local left-handed $SU(2)$ symmetry and a rigid right-handed $SU(2)$ symmetry. The local symmetry is fixed by a suitable gauge fixing term, but the rigid symmetry is also present at the quantum level. We discuss $R(\xi)$ gauges, in particular the renormalizable $\xi = 0$ Landau gauge with $\eta_{\mu\nu} - k_\mu k_\nu/k^2$ in the propagator, the Feynman- 't Hooft gauge $\xi = 1$ with $\eta_{\mu\nu}$ in the propagator, and the unitary $\xi \to \infty$ gauge with $\eta_{\mu\nu} + k_\mu k_\nu/m^2$ in the propagator. (On-shell, $k^2 + m^2 = 0$ in our conventions).

Finally we bring in section 4 the results of chapter III on renormalization of pure gauge theories together with the results of section 1 of this chapter on renormalization of Goldstone models. We study the renormalization of the $SU(2)$ Higgs model. The more realistic case of $SU(2) \times U(1)$ with chiral fermions is much more complicated. Dimensional regularization cannot be used in a direct and well-defined way for theories with chiral fermions. In that case one must use algebraic renormalization program which uses additive renormalization [5]. There occur then two new problems: new divergences might occur which lead to different couplings between the fermions and $U(1)$ gauge fields, and new anomalies might appear due to CP violation. One can still use ordinary field theory, but the analysis becomes tremendously complicated [6]. The new divergences can be eliminated by considering the (Γ, Γ) equation at one level higher than usual [7]. A great simplification occurs if one uses instead the background field formalism [8]. We refer to the introduction of [8] for further discussion and references.

We study instead an $SU(2)$ model without fermions. We use a particular $R(\xi)$ gauge which removes the off-diagonal kinetic terms between the gauge fields and the would-be Goldstone bosons. We use the same set-up as for pure gauge theories. Namely, we derive BRST Ward identities, which lead to various relations between Z factors and to the equation $\mathcal{S}^{\text{ren}}\hat{\Gamma}^{\text{ren}}_{\text{div}} = 0$ (where $\mathcal{S}^{\text{ren}}$ is the Slavnor-Taylor operator [9]) for the n-loop divergences in proper graphs which result if the theory has been renormalized through $(n-1)$ loops. An important technical result is that Z_v is in general not equal to Z_σ in linear R_ξ gauges, even though in the classical action only the combination $v + \sigma$ appears (σ is the Higgs field and v its vacuum expection value). In the analysis of divergences a new technical complication is encountered:[2] there seem to be more divergences allowed by BRST symmetry than there are Z factors to absorb divergences, even after using the $SO(4)$ symmetry to relate certain divergences. (The renormalization of v into $Z_v v^{\text{ren}}$ yields one new Z factor, but there is also one new divergent structure, $\mathcal{S}Kv$, so the extra renormalization of v does not help.) The resolution is that two of the coefficients of divergences only appear in one particular combination and not separately. We prove this by deriving yet another Ward identity which follows from the fact that in the matter sector v and σ only appear in the combination $v+\sigma$. This rescues the renormalizability of spontaneously broken gauge theories. One might have expected that the theory is renormalizable because renormalization is a high-energy process and at high energy spontaneous symmetry breaking effects should become unimportant, but we prove this with a detailed exposition.

1 Renormalization of Goldstone models

The Goldstone theorem has been known for more than forty years. [2] It states that when a rigid continuous symmetry (a symmetry with a constant parameter) is spontaneously broken, massless spin zero bosons will be present in the spectrum. The number of these massless scalars, called Goldstone bosons, is equal to the number of spontaneously broken continuous symmetries. This theorem plays a central role in the renormalizability of spontaneously broken gauge theories. It is proved either by analyzing the Hilbert space of states with a vacuum not invariant under rigid symmetry transformations, or from a functional point of view by using a Ward identity for the effective action (see below). In particular the path integral approach allows a

[2]By linear gauges we mean gauges which are linear in quantum fields such as $\partial^\mu A^a_\mu = 0$ or $\partial^\mu A^a_\mu - \frac{1}{2}\xi g v \chi^a = 0$. One can choose the nonlinear gauge $\partial^\mu A^a_\mu - \frac{1}{2}\xi g(v+\sigma) = 0$. then the divergences can only depend on the combination $v+\sigma$, so $Z_v = Z_\sigma$ in this case, and the new technical complication is absent.

very clear presentation of the Goldstone theorem.

For the analysis of certain experiments testing the Standard Model, one must evaluate Feynman diagrams at higher- (two-) loop level. In this section, we discuss some aspects of such computations in the Higgs sector without gauge fields; this is the sector with a rigid symmetry where the original Goldstone theorem applies. At the one-loop level no complications arise and many one-loop calculations in spontaneously broken theories have been performed. On the other hand, for two-loop calculations in spontaneously broken field theories some subtle issues arise. Do infrared divergences associated with the massless Goldstone bosons arise? Do finite renormalizations which are needed to satisfy the renormalization conditions but break the gauge symmetry, lead to nonrenormalizable divergences at higher loops? Clearly, one should in general require that proper tadpole graphs with a massless external scalar sum up to zero, since otherwise one would encounter connected tadpole graphs with massless propagator $1/k^2$ at $k = 0$. However, due to the symmetry of the action under $\pi \to -\pi$ there are no tadpole graphs with external Goldstone bosons possible. Tadpole graphs with an external Higgs scalar do exist, but the Higgs scalar is massive, so in principle one could allow such tadpole graphs.

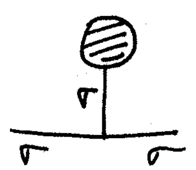


Figure caption: Tadpoles with massive scalars do not lead to infrared divergences, but should cancel for the Goldstone theorem to hold.

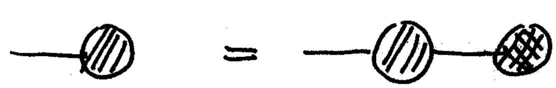
Nevertheless, as we shall see, one must require that the sum of tadpole graphs vanishes in order that Goldstone bosons remain massless. This fixes some of the finite renormalizations, and raises the question which further finite terms are allowed in the renormalization conditions without violating the Goldstone theorem.

Given the wide applicability of the Goldstone theorem to various areas in physics, one might expect that the Goldstone theorem is so general that one should be able to prove it in an algebraic manner, without having to specify which regularization scheme one employs or which renormalization conditions one imposes. This will indeed be the case. There are two steps in the renormalization program: first one removes the tadpoles order by order in loops. Then one renormalizes the rest of the theory as usual. The Goldstone theorem holds already after the first step.

The path integral approach shows in a very clear and simple way that cancellation of tadpoles is the crucial requirement for the Goldstone theorem. [10] Consider an action with scalar fields ϕ^i which is invariant under the continuous rigid linear symmetry $\delta\phi^i = \lambda^a (T_a)^i{}_j \phi^j$ with λ^a some rigid (constant) parameters. Let the scalar fields have vacuum expectation values v^k and denote by φ^k the quantum fields. So, $\phi^k = \varphi^k + v^k$ and $< \varphi^k >= 0$. The effective action then satisfies the Ward identity $\int (\delta\Gamma/\delta\varphi^j(x))(T_a)^j{}_k(\varphi^k(x)+v^k)d^4x = 0$,[3] and by taking another derivative w.r.t. $\phi^i(y)$ (or $\varphi^i(y)$) and then putting $\varphi = 0$ one finds

$$\int \frac{\delta^2\Gamma}{\delta\varphi^i(x)\delta\varphi^j(y)}(T_a)^j{}_k v^k d^4x = 0 \text{ at } \varphi = 0 \tag{4.1.1}$$

To obtain this result it is crucial that v^k be chosen such that tadpoles vanish, $\frac{\delta\Gamma}{\delta\varphi^j(y)} = 0$ at $\varphi = 0$. [10] One sometimes writes this condition as $< \varphi >_{1PI} = 0$ by which is meant that the sum of all proper graphs with one external quantum field φ should vanish. The condition $\delta\Gamma/\delta\varphi[\varphi = 0]$ is already needed to prove that Γ generates proper graphs, see the appendix B of chapter III. It is therefore very natural to impose it, and we shall not study the consequences of not imposing it. Vanishing of the 1-point proper graphs implies vanishing of the 1-point connected graphs because of the identity



where the hatched blob denotes connected graphs and the doubly-hatched blob denotes proper graphs. The integration over d^4x in (4.1.1) projects out the zero mo-

[3]This Ward identity follows straightforwardly by making a change of integration variables in the path integral Z from ϕ^i to $\phi^i + \lambda^a(T_a)^i{}_j\phi^j$, then taking the logarithm to obtain the generating functional for connected graphs, and finally making the Legendre transformation.

mentum part of the effective action (since it yields the Fourier transform $\int d^4x \exp ipx$ at $p = 0$) yielding the effective potential. For example terms such as $(\partial_\mu \varphi)^2$ in the action do not contribute since their Fourier transform is proportional to $(p_\mu)^2$. Since the terms in the effective potential quadratic in fields yield the effective mass matrix M_{ij}, (4.1.1) shows that for each symmetry with $T_a v \neq 0$ there are massless bosons

$$M_{ij}(T_a)^j{}_k v^k = 0 \tag{4.1.2}$$

where $M_{ij} = \int \frac{\delta^2 \Gamma}{\delta\varphi^i(x)\delta\varphi^j(y)} d^4x$ at $\varphi = 0$. Using

$$(T_a)^j{}_k v^k = (T_a)^j{}_k < 0|\phi^k|0 >=< 0 \mid [\phi^k, \hat{T}_a]|0 > \tag{4.1.3}$$

it is clear that these symmetry operators $\hat{T}_a$ do not leave the vacuum invariant. In other words, there is spontaneous symmetry breaking.

In the next section we give a simple combinatorial proof of the Goldstone theorem at the two-loop level for the $O(2)$ linear σ-model [11] which only uses the trivial algebraic identity

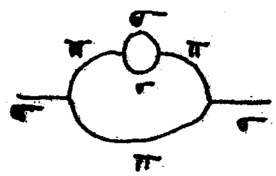
$$m^2 \frac{1}{k^2} \frac{1}{k^2 + m^2} = \frac{1}{k^2} - \frac{1}{k^2 + m^2}. \tag{4.1.4}$$

Extension to the $O(N)$ linear σ-model is straightforward and does not lead to new insight. To keep the discussion as simple as possible, we therefore restrict our attention to the $O(2)$ linear σ-model. Using this identity, we shall show that the **integrands** of the Feynman graphs for the proper two-point function of the Goldstone boson (the “pion”) at vanishing external momentum cancel each other algebraically. This means that the mass matrix in (4.1.1) has a zero eigenvalue and this proves that the Goldstone boson remains massless at the quantum level.

Hence this proof holds for any regularization scheme as long as it respects (4.1.4) and is valid without having to require that the integrals $\int \frac{d^4k}{k^2}$ or $\int \frac{d^4k}{k^4}$ vanish as in dimensional regularization. In particular, graphs with counterterms cancel separately which demonstrates that the pion remains massless before or after renormalization.

The proof can also be applied to other models (such as the complex Higgs doublet of the Standard Model) as it only depends on (4.1.4). Crucial in these proofs is again the requirement that tadpoles are cancelled.

The Goldstone boson does not lead to infrared divergences in Green functions in $d = 4$ dimensions. For example, a self-energy insertion into the propagator of a massless pion leads to a factor p^{-4} from the two massless propagators



but the Goldstone theorem guarantees that the proper self-energy itself (due to the σ-loop in the figure) provides another factor p^2.[4] We note that at the unrenormalized level our proof even holds in any dimension.

The action of the $O(2)$ linear σ-model reads

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu \hat{\sigma})^2 - \frac{1}{2}(\partial_\mu \pi)^2 + \frac{1}{2}\mu^2(\hat{\sigma}^2 + \pi^2) - \frac{1}{4}\lambda(\hat{\sigma}^2 + \pi^2)^2 \tag{4.1.5}$$

It is clearly invariant under rotations and reflections. We choose an unphysical sign for the mass term in order that spontaneous symmetry breaking occurs. Decomposing $\hat{\sigma} = \sigma + v$ one obtains

$$\begin{aligned} \mathcal{L} &= -\frac{1}{2}(\partial_\mu \sigma)^2 - \frac{1}{2}(\partial_\mu \pi)^2 - \lambda v^2 \sigma^2 - \beta v \sigma \\ &\quad -\frac{1}{2}\beta(\sigma^2 + \pi^2) - \lambda v \sigma(\sigma^2 + \pi^2) - \frac{1}{4}\lambda(\sigma^4 + 2\sigma^2\pi^2 + \pi^4) \end{aligned} \tag{4.1.6}$$

where $\beta = -\mu^2 + \lambda v^2$. We have replaced the unphysical mass μ^2 as an independent variable by the variable β whose renormalized value we require to vanish in order to exclude tadpoles at the tree graph level

$$\beta^{(0)}_{\text{ren}} = -\mu^2_{\text{ren}} + \lambda_{\text{ren}} v^2_{\text{ren}} = 0. \tag{4.1.7}$$

[4]In QED massless virtual photons can also lead to similar infrared divergences in loops, but they are canceled in the cross section by soft real photons (Bremsstrahlung). Such a cancellation is not possible for would-be Goldstone bosons because they are not physical, and thus cannot be radiated.

Then there is no term linear in σ in the action, and the coefficient of the term with $\frac{1}{2}\sigma^2$ determines the Higgs mass which we denote by m

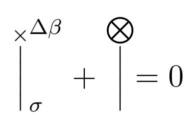
$$m^2_{\text{ren}} = 2\lambda_{\text{ren}} v^2_{\text{ren}} \tag{4.1.8}$$

To avoid confusion note that we do not consider explicit symmetry breaking by adding an extra term proportional to σ to the action. An extra term linear in σ leads to soft explicit symmetry breaking and the Goldstone bosons acquire a mass. One can still prove renormalizability in this case. [4, 10] However we do not study renormalization in the presence of explicit symmetry breaking and base our discussion on the action in (4.1.5).

We renormalize the model by setting $\sigma = Z_\sigma^{1/2}\sigma^{\text{ren}}$; $v = Z_v^{1/2} v^{\text{ren}}$; $\pi = Z_\pi^{1/2}\pi^{\text{ren}}$, $\lambda = Z_\lambda \lambda^{\text{ren}}/Z_\sigma^2$ but we renormalize β additively, by making a loop expansion of β as

$$\beta = \beta^{(0)}_{\text{ren}} + \Delta\beta^{(1)}_{\text{ren}} + \dots \tag{4.1.9}$$

We fix $\Delta\beta^{(j)}_{\text{ren}}$ ($j = 1, 2, \dots$ denotes the order of loops) by requiring that tadpole tree graphs with vertex $\Delta\beta^{(j)}_{\text{ren}} v_{\text{ren}}\sigma_{\text{ren}}$ cancel the sum of tadpole Feynman diagrams with j loops.



(4.1.10)

Hence $\Delta\beta^{(j)}_{\text{ren}}$ is unambiguously determined by the vanishing of tadpoles at the j-loop level, and will in general contain infinite as well as finite parts. We cannot renormalize β multiplicatively because we set $\beta^{\text{ren}} = 0$, so that the relation $\beta = Z_\beta \beta^{\text{ren}}$ would not make sense. We could have kept μ^2 as an independent variable and used multiplicative renormalization for μ^2, but the choice of β as independent parameter is more convenient. For our purposes it is sufficient to keep writing these $\Delta\beta^{(j)}_{\text{ren}}$ as sums of Feynman diagrams without explicitly evaluating them. Fixing $\Delta\beta_{\text{ren}}$ by requiring tadpoles to vanish, these same $\Delta\beta$ contribute to proper graphs due to the mass term $-\frac{1}{2}\beta(\sigma^2 + \pi^2)$.

Before we check the Goldstone theorem explicitly at the one-loop and two-loop level, we shall give a proof that the $O(2)$ linear σ-model with spontaneous symmetry breaking is renormalizable. We shall prove that one can set the Z factors for the fields σ and π and the vacuum expectation value v all equal

$$Z_\sigma = Z_\pi = Z_v \tag{4.1.11}$$

This is suggested by the $O(2)$ symmetry of the action in terms of $\hat{\sigma} = \sigma + v$ and π. Expanding the action in terms of $\hat{\sigma}$ and π, one never breaks the $O(2)$ symmetry. We expand the action in terms of σ and π, but because renormalization is a high-energy effect, one would expect that the presence of v does not alter the results of renormalization. This is correct for the present model, but it ceases to be correct in the case of the Higgs model because we shall use a so-called renormalizable R_ξ gauge which breaks the $O(2)$ symmetry. In that case $Z_\sigma = Z_\pi$, but $Z_v \neq Z_\sigma = Z_\pi$ as we shall see.

The proof of the renormalizability of the $O(2)$ linear sigma model follows the same steps as the proof of the renormalization of unbroken gauge theories: one begins with a change of integration variables in the path integral Z, leading to $< \delta\varphi^i J_i >= 0$ where φ^i denotes σ and π. Then we make the usual Legendre transformation, and obtain the Ward identity $\int (T_a)^i{}_j \phi^j (\partial/\partial\phi^i \Gamma) = 0$. This is the Ward identity which we use to analyze the divergences. First we renormalize. Using

$$\Gamma(\varphi^i, \beta, \lambda, v, \hbar, \epsilon) = \Gamma^{\text{ren}}(\varphi^i_{\text{ren}}, \Delta\beta_{\text{ren}}, \lambda_{\text{ren}}, v_{\text{ren}}, \hbar, \epsilon) \tag{4.1.12}$$

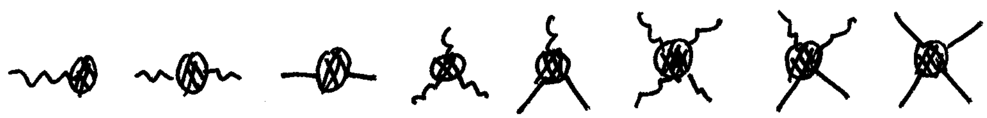
(the proof of this equality is the same as in appendix A of chapter III) and assuming (to be proven by induction) that $Z_v = Z_\sigma = Z_\pi$, we find the renormalized Ward identity

$$\int (T_a)_i{}^j (\varphi^j_{\text{ren}} + v^j_{\text{ren}}) \frac{\delta}{\delta\varphi^i_{\text{ren}}} \Gamma^{\text{ren}} = 0 \tag{4.1.13}$$

(There is of course no $\hat{\Gamma}$ since there is no gauge fixing term in this model). The possible divergences follow from power counting. The degree of divergence D of L-loop graphs is determined by

$$D = 4L - 2I; \quad L = I - \sum_i n_i + 1; \quad 2I + E = \sum_i i n_i$$
$$D = 4 - E - \sum (4-i) n_i \tag{4.1.14}$$

where E is the number of external σ and π lines, and n_i is the number of vertices in a given graph with i lines. There are only divergences in the following proper graphs



(4.1.15)

where solid lines denote the field π and wiggly lines denote σ. The Z_2 reflection symmetry $\pi \to -\pi$ excludes a pion tadpole, and restricts the n-loop divergences in the $(n-1)$-loop renormalized effective action to the form

$$\Gamma^{\text{ren}}_{\text{div},(n-1)} = \int \Big[a_1 (\partial_\mu \sigma)^2 + a_2 (\partial_\mu \pi)^2 + a_3 \sigma^2 + a_4 \pi^2 + a_5 \sigma$$
$$+ b_1 \sigma^3 + b_2 \sigma \pi^2 + c_1 \sigma^4 + c_2 \sigma^2 \pi^2 + c_3 \pi^4 \Big] d^4 x \tag{4.1.16}$$

(To simplify the notation we have dropped the subscripts "ren" on the right-hand side). Imposing the Ward identity for the rotational $O(2)$ symmetry between $\sigma + v$ and π restricts this further to [12]

$$\Gamma^{\text{ren}}_{\text{div},(n-1)} = \int \Big[A\{(\partial_\mu \sigma)^2 + (\partial_\mu \pi)^2\} + B m^2 \{(\sigma + v)^2 + \pi^2\}$$
$$+ C\{(\sigma+v)^2 + \pi^2\}^2 \Big] d^4 x, \quad m^2 = 2\lambda v^2 \tag{4.1.17}$$

This shows that at the n-loop level the divergences indeed are $O(2)$ invariant, implying in particular that $Z \equiv Z_\sigma = Z_\pi = Z_v$. Choosing $\beta^{(n)}_{\text{ren}}$ such that tadpoles cancel, and Z and Z_λ such that they cancel these n-loop divergences, we have achieved finiteness of the effective action at the n-loop level.

What happens if we make at some loop level, say the n-loop level, further non-minimal finite rescalings which violate the $O(2)$ symmetry? We shall not restrict ourselves to the $O(2)$ model but consider a general $O(N)$ linear σ model. Then

$$\delta \varphi^i = (T_a)^i{}_j (\varphi^j + v^j) \tag{4.1.18}$$

where T_a are the generators of $SO(N)$ and $<\varphi^j>=0$. Since $\delta\varphi^i$ is at most linear in quantum fields we obtain

$$(T_a)^i{}_j(<\varphi^j + v^j>_J)J_i = 0 \tag{4.1.19}$$

where $<\varphi^j>_J$ denotes the path integral expectation value in the presence of a source term $J_j\varphi^j$ in the action. (For all $J_j = 0$ we have $<\varphi_j>= 0$). As usual $<\varphi^j>_J = \frac{\hbar}{i}\frac{\delta}{\delta J_j}Z$ and hence

$$(T_a)^i{}_j\left(\frac{\hbar}{i}\frac{\delta}{\delta J_i}Z + v^j Z\right)J_i = 0 \tag{4.1.20}$$

Taking the logarithm leads to

$$(T_a)^i{}_j\left(\frac{\delta}{\delta J_j}W + v^j\right)J_i = 0 \tag{4.1.21}$$

Making the Legendre transform

$$\Gamma = W - \int J_j\varphi^j d^4x \tag{4.1.22}$$

we obtain

$$(T_a)^i{}_j(\varphi^j + v^j)(-\frac{\delta}{\delta\varphi^i}\Gamma) = 0 \tag{4.1.23}$$

Will finite rescalings at lower loop levels which break the $O(2)$ symmetry lead to violations of the $O(2)$ symmetry in the divergent part of the Z factors at still-higher loops? Recall that we did assume that $Z_\sigma = Z_\pi = Z_v$ in the proof by induction when we derived the renormalized Ward identity.

Suppose we make minimal $O(2)$ symmetric renormalizations at $(n-1)$-loops. But now in addition to the required minimal $O(2)$ symmetric renormalizations at n-loops (namely $\sigma = (Z_\sigma^{(n)})^{\frac{1}{2}}\sigma_{\text{ren}}^{(n)}$, $\pi = (Z_\pi^{(n)})^{\frac{1}{2}}\pi_{\text{ren}}^{(n)}$, $v = (Z_v^{(n)})^{\frac{1}{2}}v_{\text{ren}}^{(n)}$ with $Z_\sigma = Z_\pi = Z_v$ and $\lambda = (Z_\lambda^{(n)})^{\frac{1}{2}}(Z_v^{(n)})^{-2}\lambda_{\text{ren}}^{(n)}$) we renormalize all parameters by additional finite amounts which without loss of generality we write as finite rescalings $\sigma = (z_\sigma^{(n)}Z_\sigma^{(n)})^{\frac{1}{2}}\sigma_{\text{ren}}^{(n)}$,

$\pi = (z_\pi^{(n)} Z_\pi^{(n)})^{\frac{1}{2}} \pi_{\rm ren}^{(n)}$, $v = (z_v^{(n)} Z_v^{(n)})^{\frac{1}{2}} v_{\rm ren}^{(n)}$ and $\lambda = (z_\lambda^{(n)} Z_\lambda^{(n)})^{\frac{1}{2}} \lambda_{\rm ren}^{(n)}$. The renormalization of $\beta, \Delta\beta_{\rm ren}^{(n)}$, is determined as before, by requiring tadpoles to vanish. Finiteness at $(n+1)$-loop level can still be achieved by renormalizing the redefined fields $\sigma_{\rm ren}'^{(n)} \equiv (z_\sigma^{(n)})^{\frac{1}{2}} \sigma_{\rm ren}^{(n)}$, $\pi_{\rm ren}'^{(n)} \equiv (z_\pi^{(n)})^{\frac{1}{2}} \pi_{\rm ren}^{(n)}$, $v_{\rm ren}'^{(n)} \equiv (z_v^{(n)})^{\frac{1}{2}} v_{\rm ren}^{(n)}$ and $\lambda_{\rm ren}'^{(n)} \equiv (z_\lambda^{(n)})^{\frac{1}{2}} \lambda_{\rm ren}^{(n)}$ in the usual $O(2)$ symmetric way. After the theory has been made finite at $(n+1)$ loops one can transform back to the original fields $\sigma_{\rm ren}^{(n)}$, $\pi_{\rm ren}^{(n)}$ and $v_{\rm ren}^{(n)}$ without losing finiteness. All one has to do is expand the factors $(z_\sigma^{(n)})^{1/2}$ etc. Successions of such field redefinitions clearly prove the renormalizability of the theory for all n. Thus we need not renormalize such that $Z_\sigma = Z_\pi$ at each stage (although that is certainly the simplest way to proceed). If $Z_\sigma \neq Z_\pi$ the $O(2)$ symmetry is not broken, it is only hidden by finite non-symmetric rescalings.

However we must still verify that this procedure respects the Goldstone theorem. The Ward identity after recalibrations (finite renormalizations with Z factors which we denote by lower case z) reads

$$\frac{\delta\Gamma^{\rm ren}}{\delta\pi_{\rm ren}} z_\pi^{-\frac{1}{2}} (z_\sigma^{\frac{1}{2}} \sigma_{\rm ren} + z_v^{\frac{1}{2}} v_{\rm ren}) = \frac{\delta\Gamma^{\rm ren}}{\delta\sigma_{\rm ren}} z_\sigma^{-\frac{1}{2}} (z_\pi^{\frac{1}{2}} \pi_{\rm ren}) \qquad (4.1.24)$$

Taking another derivative w.r.t. $\pi_{\rm ren}$ and afterwards setting $\sigma_{\rm ren} = \pi_{\rm ren} = 0$ gives

$$\frac{\delta^2\Gamma^{\rm ren}}{\delta\pi_{\rm ren}^2}\Big|_{(\sigma_{\rm ren}=0=\pi_{\rm ren})} z_\pi^{-\frac{1}{2}} z_v^{\frac{1}{2}} v_{\rm ren} = \frac{\delta\Gamma^{\rm ren}}{\delta\sigma_{\rm ren}}\Big|_{(\sigma_{\rm ren}=0=\pi_{\rm ren})} z_\sigma^{-\frac{1}{2}} z_\pi^{\frac{1}{2}}. \qquad (4.1.25)$$

But the requirement that tadpoles vanish, $(\delta\Gamma^{\rm ren}/\delta\sigma_{\rm ren})|_{(\sigma_{\rm ren}=0=\pi_{\rm ren})} = 0$, shows the vanishing of the Goldstone boson mass even when the Z-factors are unequal due to finite renormalizations.

One can add fermions to the $O(2)$ linear sigma model [3,4]. One obtains then a toy model for the quark-Higgs sector of the Standard Model. For example, the action $\mathcal{L} = \mathcal{L}(O(2)\text{ model}) - \bar\psi \partial\!\!\!/ \psi - g\bar\psi(\hat\sigma - i\gamma_5\pi)\psi$ has the chiral $O(2)$ symmetry $\hat\sigma' = \cos\alpha\ \hat\sigma - \sin\alpha\ \pi, \pi' = \sin\alpha\ \hat\sigma + \cos\alpha\ \pi, \psi' = \exp(\frac{i}{2}\alpha\gamma_5)\psi$. One can prove the renormalizability of this model (the one-loop corrections to the fermion mass $m_f = gv$ are finite) [12].

2 The Goldstone theorem at one- and higher-loop level

Having discussed the renormalization of linear σ-models, we now turn our attention to the Goldstone theorem. By combining the integrands of the pion self-energy diagrams we shall use the identity (4.1.4) to prove the Goldstone theorem diagrammatically. [11]

To illustrate the method, we first go through the one-loop case. We shall work in Euclidean space to avoid factors of i and we will use dimensional regularization for the one-loop counterterms (although this is not needed since the contributions of the counterterms to the Goldstone theorem cancel separately). The Feynman rules are follow from (4.1.6) and are given by

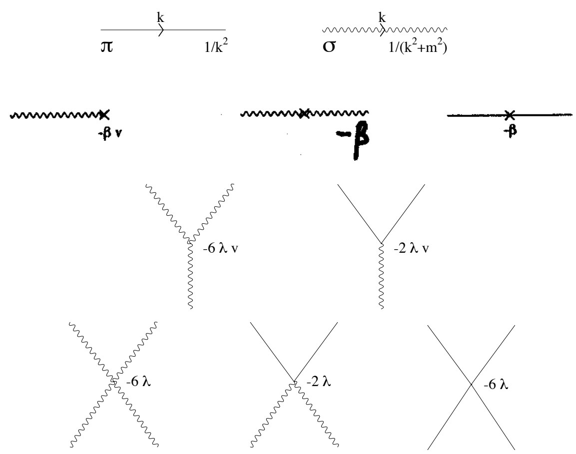
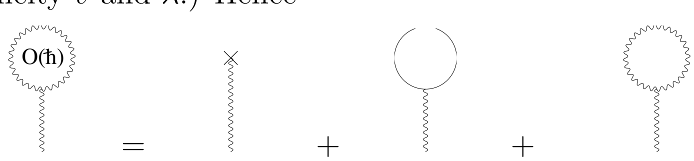


Figure caption: Feynman rules for the $O(2)$ model. Wiggly lines denote σ and solid lines denote π.

We do not expand $\lambda\mu^{\frac{1}{2}(4-n)}$ but rather keep this exponent multiplied by the renormalized coupling constant as one object. All cancellations will take place without having to expand this composite object. First we determine $\Delta\beta^{(1)}_{\text{ren}}$ by requiring that $\Delta\beta^{(1)}_{\text{ren}}v$ cancels the one loop tadpoles. (To this order, $v = v_{\text{ren}}$ and $\lambda = \lambda_{\text{ren}}$ and we write for simplicity v and λ.) Hence

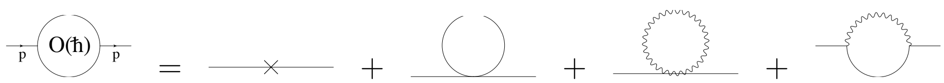


$$0 = -\Delta\beta^{(1)}_{\text{ren}}v \quad -\lambda v\int\frac{1}{k^2} \quad -3\lambda v\int\frac{1}{k^2+m^2}. \tag{4.2.1}$$

The symbol $\int$ stands for $\int d^n k (2\pi)^{-n}$. Thus

$$\Delta\beta^{(1)}_{\mathrm{ren}} = -\lambda \int \frac{1}{k^2} - 3\lambda \int \frac{1}{k^2+m^2}. \tag{4.2.2}$$

Replacing β in the action (4.1.6) by $\Delta\beta^{(1)}_{\mathrm{ren}}$, and dropping the term linear in σ, we now evaluate the proper graphs for the pion self-energy to order $\hbar$. One finds

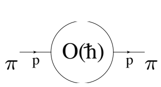


$$= (-\Delta\beta^{(1)}_{\mathrm{ren}} - Z^{(1)}_{\pi} p^2) \quad -3\lambda \int \tfrac{1}{k^2} \quad -\lambda \int \tfrac{1}{k^2+m^2} \quad +2\lambda \int \tfrac{m^2}{k^2[(k+p)^2+m^2]}. \tag{4.2.3}$$

where $Z^{(1)}_{\pi} p^2$ is the one-loop contribution from the wave function renormalization of the pion. To prove the Goldstone theorem at one loop level we must show that this expression vanishes at $p^2 = 0$ because then the Goldstone boson will remain massless. Using (4.1.4) for the last term, and inserting the value for $\Delta\beta^{(1)}_{\mathrm{ren}}$ in (4.2.2), it is clear that the selfenergy of the Goldstone boson indeed vanishes at $p = 0$.

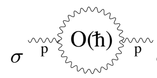
Note that this result holds irrespectively of whether $Z^{(1)}_{\pi}$ vanishes or not. There are clearly no divergences in $Z^{(1)}_{\pi}$ as only the last graph in (4.2.3) contributes to $Z^{(1)}_{\pi} p^2$, and the difference between this graph at nonvanishing p and this graph at $p = 0$ is finite. However, a nonzero finite $Z^{(1)}_{\pi}$ is allowed at this point. This value must be determined by a suitable renormalization condition, for example the condition in (4.2.10). Crucial for the vanishing of (4.2.3) is that $\beta^{(0)}_{\mathrm{ren}} = 0$. If $\beta^{(0)}_{\mathrm{ren}}$ would not have been made to vanish (leading to tadpoles at tree graph level), a contribution proportional to $Z^{(1)}_{\pi} \beta^{(0)}_{\mathrm{ren}}$ would have been present, and if $Z^{(1)}_{\pi}$ would not vanish, the pion would have acquired a mass.

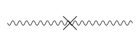
For our two-loop calculation it will be useful to have a simple expression for the one-loop pion self energy at arbitrary p. It is clearly given by



$$= 2\lambda m^2 \left(\int \frac{1}{k^2[(k+p)^2+m^2]} - \int \frac{1}{k^2[k^2+m^2]} \right) - Z^{(1)}_{\pi} p^2 \tag{4.2.4}$$

Similarly, it will be useful to have an expression for the complete one-loop self-energy of the field σ. Denoting the one-loop renormalization of $m^2 = 2\lambda v^2$ by $(\Delta m^2)^{(1)}$ we find



$=$

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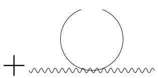
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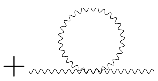
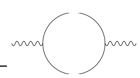


$$\left[-(\Delta m^2)^{(1)} - \Delta\beta^{(1)}_{\text{ren}} - Z^{(1)}_\sigma(p^2+m^2)\right] \qquad -\lambda \int \tfrac{1}{k^2} \qquad -3\lambda \int \tfrac{1}{k^2+m^2}$$

$+$

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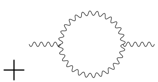
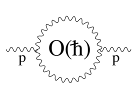


$$+\lambda \int \tfrac{m^2}{k^2(k+p)^2} \qquad +9\lambda \int \tfrac{m^2}{[k^2+m^2][(k+p)^2+m^2]}. \tag{4.2.5}$$

Using $\Delta\beta^{(1)}_{\text{ren}}$ in (4.2.2), the two graphs with $\int \frac{1}{k^2}$ and $\int \frac{1}{k^2+m^2}$ cancel. The p^2 dependent parts of the last two graphs are finite, hence $Z^{(1)}_\sigma$ is finite, but the p^2-independent parts are divergent and should be canceled by the mass counter term $(\Delta m^2)^{(1)}$. Since $m^2 = 2\lambda v^2$, the one-loop correction of m^2 follows from the one-loop term in the product of the coupling constant renormalization ($\lambda\sigma^4 = Z_\lambda \lambda_{\text{ren}}\sigma^4_{\text{ren}}$ so $\lambda = Z_\lambda Z_\sigma^{-2}\lambda_{\text{ren}}$) and the renormalization of v^2 (namely $v^2 = Z_v v^2_{\text{ren}}$). This leads to $(\Delta m^2)^{(1)} = (Z^{(1)}_\lambda - 2Z^{(1)}_\sigma + Z^{(1)}_v)m^2_{\text{ren}}$. For notational simplicity we shall write m^2 for m^2_{ren}. The result for the σ self-energy is then



$$= \lambda m^2 \int \frac{1}{k^2(k+p)^2} + 9\lambda m^2 \int \frac{1}{[k^2+m^2][(k+p)^2+m^2]}$$
$$-(Z^{(1)}_\lambda - 2Z^{(1)}_\sigma + Z^{(1)}_v)m^2 - Z^{(1)}_\sigma(p^2+m^2) \tag{4.2.6}$$

We must now fix the finite parts of the one-loop corrections $Z^{(1)}_\pi, Z^{(1)}_\sigma, Z^{(1)}_v$ and $Z^{(1)}_\lambda$ by suitable renormalization conditions. We shall formulate these renormalization conditions in terms of mass-shell conditions, so in Minkowski space. This means that we continue our Euclidean momenta with $p^2 > 0$ to Minkowski values with $p^2 < 0$. We could have stayed all the time in Minkowski space at the expense of extra factors of i at various places. The remaining one-loop correction (that of β, or, equivalently, that of μ^2) has already been fixed by requiring absence of tadpoles at the one-loop level.

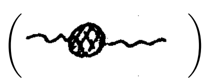
We already saw from the path integral proof that if this requirement is not met, then the pion is no longer massless, so absence of tadpoles is one of our renormalization conditions.

We fix Z_λ by requiring that the proper graphs for elastic σ scattering at threshold be given by λ_{ren}

$$= -6\lambda_{\text{ren}} \text{ at } s = 4m^2, t = 0 \tag{4.2.7}$$

Furthermore, we fix Z_v and Z_σ by requiring that the physical mass of the σ field be $m_{\text{ren}}^2 = 2\lambda_{\text{ren}} v_{\text{ren}}^2$ and that the residue of the pole in the 2-point function for the σ -field be equal to unity. Denoting proper graphs by doubly-hatched blobs, these requirements read

$$= 0 \qquad \text{at } p^2 + m_{\text{ren}}^2 = 0$$

$$\frac{d}{dp^2}$$ 

$$= 0 \qquad \text{at } p^2 + m_{\text{ren}}^2 = 0 \tag{4.2.8}$$

The connected 2-point function is a geometric series, which reads in Minkowski space

$$\begin{aligned} &= \frac{-i}{p^2+m_{\text{ren}}^2} + \frac{-i}{(p^2+m_{\text{ren}}^2)}[i(\Pi(p^2, m_{\text{ren}}^2)]\frac{-i}{p^2+m_{\text{ren}}^2} + \cdots \\ &= \frac{-i}{p^2+m_{\text{ren}}^2-\Pi(p^2,m_{\text{ren}}^2)} \end{aligned} \tag{4.2.9}$$

The renormalization condition that the physical mass of the σ field be m_{ren}^2 means that the pole of the connected two-point function be at $p^2+m_{\text{ren}}^2 = 0$, and leads to the requirement $\Pi(p^2 = -m_{\text{ren}}^2, m_{\text{ren}}^2) = 0$. It is clear from (4.2.6) that this condition fixes $Z_v^{(1)} - 2Z_\sigma^{(1)}$. Since $\Pi(p^2 = -m_{\text{ren}}^2, m_{\text{ren}}^2) = 0$, it follows that $\Pi(p^2 = -m_{\text{ren}}^2, m_{\text{ren}}^2) = a(p^2 + m_{\text{ren}}^2) + b(p^2 + m_{\text{ren}}^2)^2 + \cdots$. The second renormalization condition then states that $a = 0$ and hence that the propagator is given by $\frac{-i}{p^2+m_{\text{ren}}^2-i\epsilon}[1 + \mathcal{O}(p^2 + m_{\text{ren}}^2)]$. This fixes $Z_\sigma^{(1)}$.

We have now fixed the finite parts of most Z factors by renormalization conditions. The $\sigma - \sigma$ scattering amplitude at threshold is $-6\lambda_{\text{ren}}$ which fixes Z_λ, the physical mass of the σ boson is m_{ren} which fixes Z_v and the renormalized propagator is $-i(p^2 + m^2)^{-1}$ which fixes Z_σ. Absence of tadpoles fixes $\Delta\beta$ (or Z_{μ^2}). That only leaves Z_π. It is clear from (4.2.3) that we can fix Z_π by requiring that the residue of the pion pole be unity at $p^2 = 0$

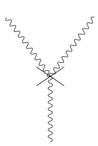
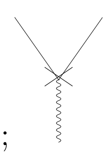
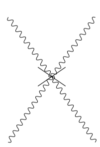
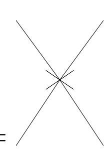
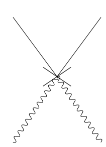
$\frac{\partial}{\partial p^2}\Bigg($  $\Bigg) = 0 \qquad \text{at } p^2 = 0$ (4.2.10)

It is clear from (4.2.3) that $Z_\pi^{(1)}$ is finite. This does not fix the mass of the pion, in fact, the whole point of the Goldstone theorem is that $m_\pi^2 = 0$ can be proven, instead of being one more renormalization condition.

We shall first determine the divergent parts of $Z_\lambda^{(1)}, Z_\sigma^{(1)}, Z_v^{(1)}$ and $Z_\pi^{(1)}$. Then we shall prove the Goldstone theorem at the 2-loop level by only using these divergent parts of the Z factors. To parametrize divergences, we denote the divergent part of $\lambda_{\text{ren}} \int [k^2(k+p)^2]^{-1}$ by δ

$$\delta \equiv \lambda_{\text{ren}} \int [k^2(k+p)^2]^{-1}_{\text{div}} \tag{4.2.11}$$

We shall need the one-loop counterterms for the two three-point vertices in order to determine $\Delta\beta^{(2)}$. Since we also need the one-loop counterterms for two of the three four-point vertices, we just give all one-loop vertex counter terms[5]

 $= -60\lambda v\delta;$  $= -20\lambda v\delta;$  $= -60\lambda\delta =$  ;  $= -20\lambda\delta.$

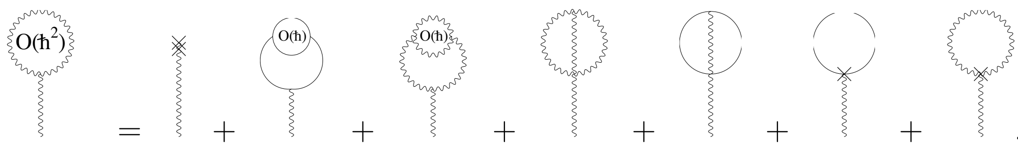
(4.2.12)

This implies that $\lambda = (1 + 10\delta)\lambda_{\text{ren}}$ as far as divergent terms are concerned. This value of $Z_\lambda^{(1)}$ cancels the divergences in (4.2.6), hence we conclude that not only $Z_\pi^{(1)}$

[5]Only one-loop graphs with two vertices are divergent, and it is easy to determine their overall divergences. For example the factor 60 for the counter term with σ^4 is due to a virtual σ loop (giving a factor 18) and a virtual π loop (giving a factor 2) in three pairs of diagrams related by crossing (giving a factor 3). Details are given in [4].

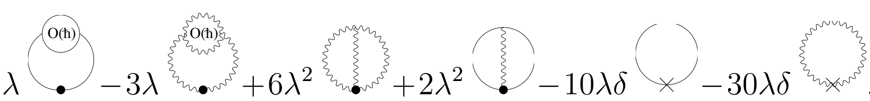
but also $Z_\sigma^{(1)}$ and $Z_v^{(1)}$ are finite. We set $Z_\pi^{(1)}, Z_\sigma^{(1)}$ and $Z_v^{(1)}$ equal to zero, and will discuss at the end of this section what happens if they are not vanishing.

We now start the analysis of the two-loop Goldstone theorem. First we shall determine $\Delta\beta^{(2)}$, and then use the result in the pion self-energy. From now on we shall use a diagrammatic notation to be explained, and drop the subscripts "ren". The set of diagrams contributing to $\Delta\beta^{(2)}$ is given by



(4.2.13)

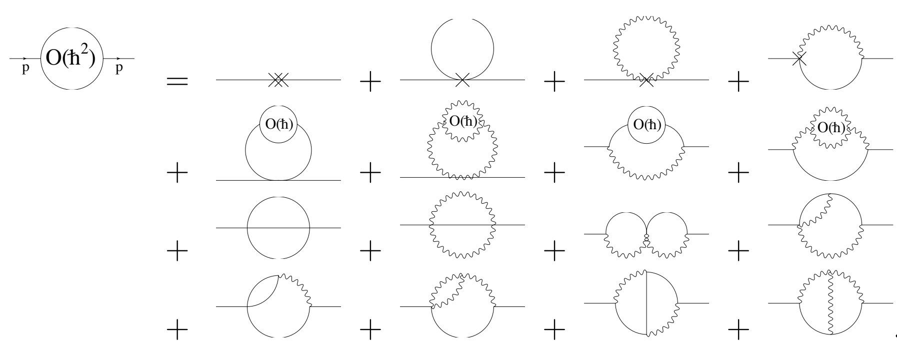
The diagram with the double cross is equal to $-\Delta\beta^{(2)}v$ (recall $Z_v^{(1)} = Z_\sigma^{(1)} = \beta^{(0)} = 0$), and $\Delta\beta^{(2)}$ is determined by requiring the sum of the above diagrams to vanish. We now write each diagram as a product of a numerical factor (due to symmetry factors, Feynman rules etc.) and a picture representing an integral over a product of propagators. The dots or crosses indicate where external lines were attached.

$\Delta\beta^{(2)} =$ 

(4.2.14)

For example, the first picture stands for $\int l^{-4}\Pi(l)$ where $\Pi(p)$ is given in (4.2.4). As mentioned above we shall not need to evaluate these diagrams.

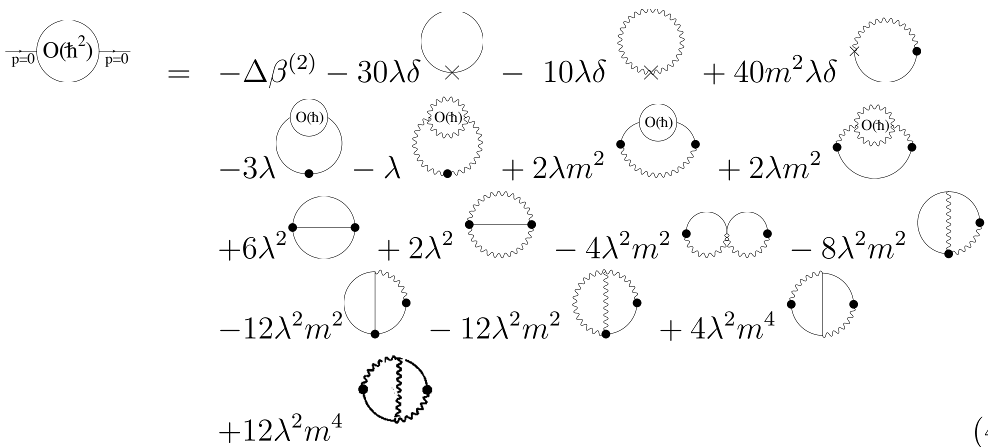
Next we consider the two-loop pion self-energy at $p = 0$. The graphs contributing are given by



(4.2.15)

The diagram with the double cross contains again the contribution from $\Delta\beta^{(2)}$. (There is no graph proportional to $Z_\pi^{(2)}$ since we work at p=0).

We write down all graphs at $p = 0$ (except that with the double cross) as products of numerical factors times pictures (graphs without external lines denoting again integrals over products of propagators).

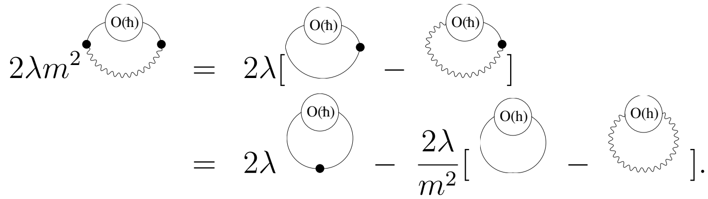


(4.2.16)

The two loop Goldstone theorem requires that $\Delta\beta^{(2)}$ in (4.2.14) equals the sum of all above graphs in (4.2.16) (except the term $\Delta\beta^{(2)}$) on the right-hand side, of course. Our strategy will be to repeatedly use (4.1.4), which in diagrammatic notation can be written as

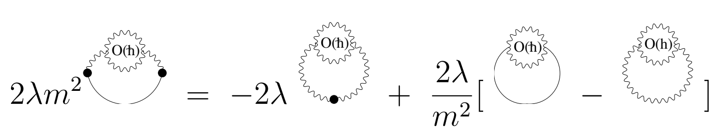
(4.2.17)

As an example consider the following pion self-energy graph with a pion self-energy insertion



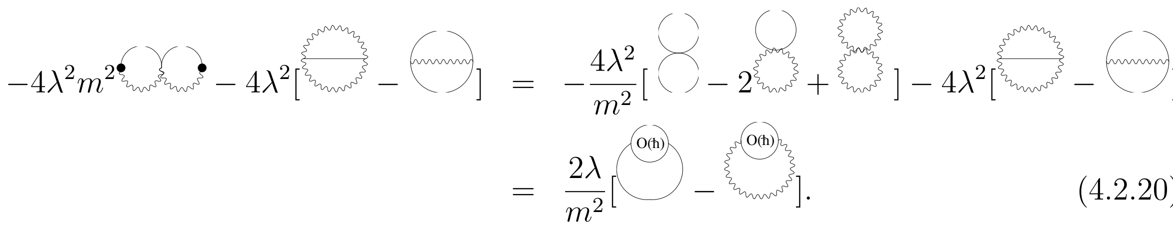
(4.2.18)

In a similar fashion we have, again using only (4.2.17),

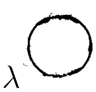
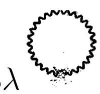


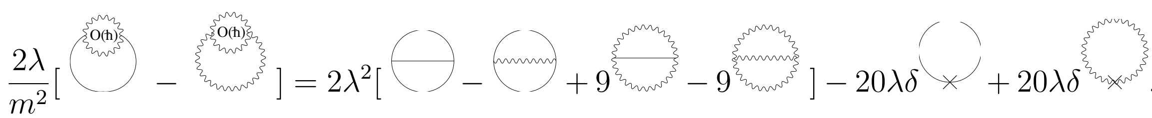
(4.2.19)

The first terms on the right-hand sides of (4.2.18) and (4.2.19) combine with the first two graphs in the second line of (4.2.16) to exactly agree with the first two graphs of $\Delta\beta^{(2)}$ in (4.2.14). The remaining four terms on the right-hand sides of (4.2.18) and (4.2.19) do not appear in $\Delta\beta^{(2)}$. However notice that using (4.2.3), and of course (4.2.17), we have



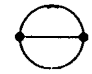
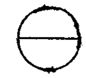
(4.2.20)

where we added the two graphs on the left-hand side to produce the right-hand side. The first equality in (4.2.20) follows directly from (4.2.17), but the second equality requires to use (4.2.3) and to write $\Delta\beta^{(1)}$ in (4.2.3) as λ  $- 3\lambda$ . The result in (4.2.20) clearly cancels the remaining two graphs in (4.2.18). To cancel the remaining two graphs in (4.2.19), we substitute (4.2.6) into these graphs and expand.

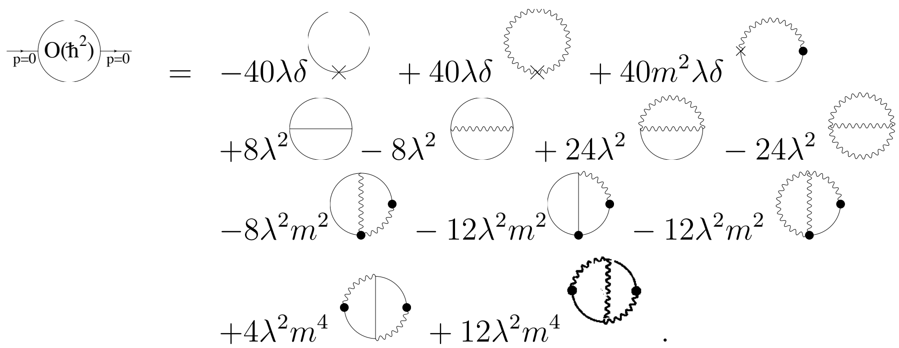


(4.2.21)

(We replaced $-(Z_\lambda^{(1)} - Z_v^{(1)})m_{\text{ren}}^2$ by its divergent part $-10m_{\text{ren}}^2\delta$, and set $Z_\sigma^{(1)}$ to zero as it has no divergent part.)

At this moment we are left with the following set of diagrams: the last four in $\Delta\beta^{(2)}$, the two setting-sun diagrams at the left-hand side of (4.2.20) (which we added to produce the right-hand side of (4.2.20)), the diagrams on the right-hand side of (4.2.21) and further in (4.2.16) the three diagrams with a cross and the seven setting-sun diagrams (of which five have a vertical rather than horizontal line). That is (after only trivial additions of identical diagrams, and identifying  $=$ 

etc.) to obtain the second line in the equation below



(4.2.22)

Notice that, as promised, using (4.2.17) the three diagrams with a cross in the first line (i.e., those with a factor $\frac{1}{\epsilon}$) cancel separately. Finally, after repeated applications of (4.2.17), all remaining diagrams cancel. This concludes our two-loop proof of the Goldstone theorem where we used minimal subtractions at the one-loop level ($Z_\lambda^{(1)} = 10\delta$ and $Z_\sigma^{(1)} = Z_\pi^{(1)} = Z_v^{(1)} = 0$).

3 The spontaneously broken $SU(2)$ Higgs model

Since we are going to discuss renormalization and unitarity in spontaneously broken gauge theories, we give here a short discussion of an example of such theories. Consider the coupling of an $SU(2)$ Higgs doublet $\varphi^a (a = 1, 2)$ to Yang-Mills theory

$$\mathcal{L} = -\frac{1}{4}(F_{\mu\nu}{}^a)^2 - (D_\mu\phi)^\dagger D^\mu\phi - V(\phi^\dagger\phi) \tag{4.3.1}$$

where $\phi = \binom{\varphi^1}{\varphi^2}$ and $V(\phi^\dagger\phi) = -\mu^2\phi^\dagger\phi + \lambda(\phi^\dagger\phi)^2$. The minus sign in front of the term $\mu^2\phi^\dagger\phi$ leads to spontaneous symmetry breaking. For reasons to be explained below, we parametrize ϕ as $\phi = \frac{1}{\sqrt{2}}(\psi + i\chi^a\tau_a)\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}\psi + i\chi^3\\ i\chi^1 - \chi^2\end{pmatrix}$ where τ_a are the Pauli matrices (three hermitian 2×2 matrices). If $<\phi> \neq 0$, we can use a rigid $SU(2)$ rotation to achieve that only $< Re\varphi^1 >= \frac{1}{\sqrt{2}}v \neq 0$. (Under an $SU(2)$ transformation ϕ transforms into $(\exp i\frac{\vec{\omega}}{2}\cdot\vec{\tau})\phi = (\cos\frac{\omega}{2} + i\sin\frac{\omega}{2}\frac{\vec{\omega}\cdot\vec{\tau}}{\omega})\phi$ and given any vacuum expectation value $<\phi>$ of ϕ, one can always find angles $\vec{\omega}$ such that only

$<\psi>$ is non zero. For example, we can first make the phases of φ^1 and φ^2 equal by a rotation with τ_3, next rotate $<\varphi_2>$ to zero by a rotation with τ_1, and then rotate the phase of $<\varphi_1>$ away by another rotation with τ_3. Putting $\psi = v + \sigma$, we get then

$$\phi = \frac{1}{\sqrt{2}}(v + \sigma + i\vec{\tau}\cdot\vec{\chi})\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} v + \sigma + i\chi^3 \\ i\chi^1 - \chi^2 \end{pmatrix} \tag{4.3.2}$$

where $<\sigma>=<\chi^a>=0$. The covariant derivative of ϕ is defined by

$$D_\mu\phi = \partial_\mu\phi - \frac{i}{2}g\vec{A}_\mu\cdot\vec{\tau}\phi \tag{4.3.3}$$

which is of the general form $D_\mu\phi = \partial_\mu\phi + gA^a_\mu T_a\phi$ and $T_a = \frac{-i\tau_a}{2}$. Due to the normalization factor $-i/2$, the T_a satisfy $[T_a, T_b] = \epsilon_{abc}T_c$.

The parametrization of ϕ as a Lie algebra valued field acting on the spinor $\binom{1}{0}$ needs a short explanation. The form of the covariant derivative in (4.3.3) is standard; in general one has $D_\mu\phi = \partial_\mu\phi + gA_\mu{}^aT_a\phi$ where ϕ forms a representation of the antihermitian T_a, and the only thing one needs to know is that the T_a satisfy $[T_a, T_b] = f_{ab}{}^cT_c$. However, the expression in (4.3.2) is very special: it is written as a linear combination of generators of the group $U(2)$ acting on a constant spinor. This is possible because $U(2)$ has 4 generators and a complex scalar doublet has also 4 real fields. For general groups and a general representation there is no natural way of writing ϕ itself as $T_a\zeta^a(x)\eta$ with η some constant vector. (For general groups with ϕ in the adjoint representation, one may consider Lie algebra valued fields $\phi = \sum\phi^aT_a$ and then (4.3.3) can be rewritten as $D_\mu\phi = \partial_\mu\phi + g[A_\mu, \phi]$.) This explains why only for the doublet representation of $U(2)$ the parametrization in (4.3.2) makes sense. Of course, it is precisely this representation which Nature seems to choose in the Standard Model for the Higgs fields. For simplicity we shall only consider $SU(2)$ gauge fields although we could have added a $U(1)$ gauge field.

The fact that ϕ can be expanded into $U(2)$ generators does not explain why it is useful to do so. The reason is that it allows to explicitly exhibit further symmetries, as we shall discuss below (4.3.8).

Since $D_\mu\phi$ can be written as $\frac{1}{\sqrt{2}}(D_\mu\sigma + i\tau_a D_\mu\chi^a)\binom{1}{0}$ with $D_\mu\sigma$ and $D_\mu\chi^a$ to be given in (4.3.12), the matrix which appears in $(D_\mu\phi)^\dagger D^\mu\phi$ is proportional to the unit matrix. This is again particular for $U(2)$; for $U(n)$ it is not true that the square of each generator is proportional to the unit matrix and that cross terms with two generators cancel. We could have taken a trace over the matrix in $(D_\mu\phi)^\dagger D^\mu\phi$, but we prefer to take the matrix element corresponding to the spinor $(1,0)$ because then ϕ becomes an $SU(2)$ spinor in terms of which Higgs scalars are often formulated. (Of course, a spinor (α,β) instead of $(1,0)$ gives the same result for the action up to an overall factor $|\alpha|^2+|\beta|^2$).

Substituting (4.3.3) and (4.3.2) into (4.3.1) we obtain for the action

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}(F_{\mu\nu}{}^a)^2 - V(\phi^\dagger\phi)\\ &- \frac{1}{2}(1,0)\left[\partial^\mu\sigma - i\partial^\mu\vec{\chi}\cdot\vec{\tau} + (v+\sigma-i\vec{\chi}\cdot\vec{\tau})\left(\frac{ig}{2}\vec{A}^\mu\cdot\vec{\tau}\right)\right]\\ &\quad\left[\partial_\mu\sigma + i\partial_\mu\vec{\chi}\cdot\vec{\tau} - \left(\frac{ig}{2}\vec{A}_\mu\cdot\vec{\tau}\right)(v+\sigma+i\vec{\chi}\cdot\vec{\tau})\right]\begin{pmatrix}1\\0\end{pmatrix}\end{aligned} \tag{4.3.4}$$

From (4.3.4) we can read off the terms in the action. The kinetic terms for the scalars are standard

$$\mathcal{L}(\mathrm{kin}) = -\frac{1}{2}(\partial_\mu\sigma\partial^\mu\sigma + \partial_\mu\chi^a\partial^\mu\chi^a) \tag{4.3.5}$$

The terms linear in $A_\mu{}^a$ are given by

$$\begin{aligned}&-\frac{1}{2}(1,0)\Big[(v+\sigma-i\vec{\chi}\cdot\vec{\tau})\frac{ig}{2}(\vec{A}^\mu\cdot\vec{\tau})(\partial_\mu\sigma + i\partial_\mu\vec{\chi}\cdot\vec{\tau})\\ &-(\partial^\mu\sigma - i\partial^\mu\vec{\chi}\cdot\vec{\tau})\left(\frac{ig}{2}\vec{A}_\mu\cdot\vec{\tau}\right)(v+\sigma+i\vec{\chi}\cdot\vec{\tau})\Big]\begin{pmatrix}1\\0\end{pmatrix}\end{aligned} \tag{4.3.6}$$

Working this further out, one finds the coupling of $A_\mu{}^a$ to the Noether current for the rigid SU(2) transformations in (4.3.11) denoted by δ_L

$$\begin{aligned}\mathcal{L}(A) &= \frac{1}{2}gv(A_\mu{}^a\partial^\mu\chi^a) + \frac{1}{2}gA_\mu{}^a(\sigma\overleftrightarrow{\partial}^\mu\chi^a)\\ &+ \frac{1}{2}g\epsilon_{abc}(\chi^a A_\mu{}^b\partial^\mu\chi^c)\end{aligned} \tag{4.3.7}$$

Note that in this expression only the combination $\sigma + v$ occurs. (We used the identity $(\tau_a\tau_b\tau_c - \tau_c\tau_b\tau_a) = 2i\epsilon_{abc}$). We draw the reader's attention to the off-diagonal kinetic term $\frac{1}{2}gvA^a_\mu\partial^\mu\chi^a$. Finally, the A^2 terms are given by

$$\begin{aligned}
\mathcal{L}(A^2) &= -\frac{1}{2}(1,0)\,(v+\sigma-i\vec{\chi}\cdot\vec{\tau})\left(\frac{ig}{2}\vec{A}_\mu\cdot\vec{\tau}\right)\left(\frac{-ig}{2}\vec{A}_\mu\cdot\vec{\tau}\right)(v+\sigma+i\vec{\chi}\cdot\vec{\tau})\begin{pmatrix}1\\0\end{pmatrix} \\
&= -\frac{g^2}{8}({A_\mu}^a A^{\mu a})[(v+\sigma)^2+\chi^b\chi^b] \qquad (4.3.8)
\end{aligned}$$

They give the gauge fields a mass: the Higgs effect.

The reason we parametrized ϕ as $\frac{1}{\sqrt{2}}M\begin{pmatrix}1\\0\end{pmatrix}$ with $M = \psi + i\chi^a\tau_a$ is that the fields ψ and χ^a appear in the kinetic terms in (4.3.5) with a rigid $SO(4)$ symmetry, and in the potential and (4.3.8) with a local $SO(4)$ symmetry. The terms in (4.3.7) have a rigid $SU(2)$ symmetry which only acts in the indices a of χ^a and ${A_\mu}^a$. The origin of these symmetries is as follows. The two commuting $SU(2)$ groups into which $SO(4)$ decomposes act on M by left- or right-multiplication: $M' = UMV$ with U and V in $SU(2)$. (Since $\det M = \psi^2 + (\chi^a)^2$, and $\det UMV = \det M$, the matrices U and V generate the two $SU(2)$ subgroups of $SO(4)$, and the action of U and V on M clearly commutes). The $SO(4)$ symmetry can be made manifest by writing the action in terms of the matrix M as

$$\begin{aligned}
&tr\left\{\left[\left(\partial^\mu - \frac{ig}{2}A^\mu\right)M\right]^\dagger\left(\partial_\mu - \frac{ig}{2}A_\mu\right)M\right\} \\
&= tr\left\{M^\dagger\left(\overleftarrow{\partial}^\mu + \frac{ig}{2}A^\mu\right)\left(\partial_\mu - \frac{ig}{2}A_\mu\right)M\right\} \qquad (4.3.9)
\end{aligned}$$

where $A_\mu = {A_\mu}^a\tau_a$. It is clear that the SU(2) symmetry due to $M \to UM$ can be made local provided A_μ transforms as $\partial_\mu - \frac{ig}{2}A_\mu \to U(\partial_\mu - \frac{ig}{2}A_\mu)U^{-1}$ (which is the usual law for gauge fields), while the other rigid $SU(2)$ symmetry due to $M \to MV$ remains a rigid symmetry in the gauged model. (This follows from cyclicity of the trace. The symmetries $M \to MV$ can only be rigid because the derivative ∂_μ in (4.3.9) acts also on V). Thus the symmetries of this Higgs model are: $SU_L(2$, local) $\otimes SU_R(2$, rigid), where $L(R)$ indicates left(right). Under the diagonal rigid $SU(2)$

subgroup $M' = V^{-1}MV$ the fields χ^a and ${A_\mu}^a$ rotate as vectors ($\delta\chi^a = \epsilon^{abc}\chi^b\lambda^c$) while ψ is inert.

The local $SU(2)_L$ transformation laws are given by $\delta\phi = U\phi$ and $\delta(-ig/2A_\mu) = U(\partial_\mu - ig/2A_\mu)U^{-1}$ with $U = \exp ig\vec{\lambda}\cdot\frac{\vec{\tau}}{2}$. Infinitesimally one finds

$$\delta\phi = \frac{ig}{2}\lambda\phi \quad , \quad \delta A_\mu = \partial_\mu\lambda - \frac{ig}{2}[A_\mu, \lambda] \quad , \quad \lambda = \lambda^a\tau_a, A_\mu = A_\mu^a\tau_a \tag{4.3.10}$$

and $D_\mu\phi = \partial_\mu\phi - \frac{i}{2}gA_\mu\phi$ transforms as $\delta(D_\mu\phi) = \frac{ig}{2}\lambda D_\mu\phi$. Since this implies that $\delta(D_\mu\phi)^\dagger = (D_\mu\phi)^\dagger\ (\frac{-ig}{2}\lambda)$, the gauge invariance of the classical action is manifest. In terms of components the $SU(2)_L$ gauge transformations and the rigid $SU(2)_R$ symmetry read

$$\begin{aligned}
&\delta_L\sigma = -\tfrac{1}{2}g\chi_a\lambda^a && \delta_R\sigma = -\tfrac{1}{2}g\chi_a\lambda^a \\
&\delta_L\chi^a = \tfrac{1}{2}g(v+\sigma)\lambda^a + \tfrac{1}{2}g{\epsilon^a}_{bc}\chi^b\lambda^c && \delta_R\chi^a = \tfrac{1}{2}g(v+\sigma)\lambda^a - \tfrac{1}{2}g{\epsilon^a}_{bc}\chi^b\lambda^c \\
&\delta_L{A_\mu}^a = \partial_\mu\lambda^a + g{\epsilon^a}_{bc}{A_\mu}^b\lambda^c && \delta_R{A_\mu}^a = 0
\end{aligned} \tag{4.3.11}$$

The reader may check that the action is invariant under these transformation rules. Note that for constant λ^a the diagonal rigid subgroup symmetry $\delta_L - \delta_R$ rotates χ^a and A_μ^a as vectors and leaves σ invariant; this explains the factor 1/2 in front of ϵ^{abc} in $\delta_L\chi^a$ and $\delta_R\chi^a$.

The $SU(2)_L$ covariant derivative $D_\mu\phi$ decomposes into the components

$$\begin{aligned}
D_\mu\sigma &= \partial_\mu\sigma + \frac{1}{2}g{A_\mu}^a\chi_a \\
D_\mu\chi^a &= \partial_\mu\chi^a + \frac{1}{2}g\epsilon^{abc}{A_\mu}^b\chi^c - \frac{1}{2}g{A_\mu}^a(v+\sigma)
\end{aligned} \tag{4.3.12}$$

and $D_\mu\phi = \frac{1}{\sqrt{2}}(D_\mu\sigma + i\tau_a D_\mu\chi^a)$. The total action for the scalars can then be written in the simple form

$$\mathcal{L} = -\frac{1}{2}(D_\mu\sigma)^2 - \frac{1}{2}(D_\mu\chi^a)^2 \tag{4.3.13}$$

but this form hides a subtlety: the off-diagonal kinetic term $\frac{1}{2}gv{A_\mu}^a\partial^\mu\chi^a$ in (4.3.7). To cancel the off-diagonal kinetic term, we fix the local $SU(2)$ symmetry by a suitably-chosen "'t Hooft gauge-fixing term" [16]

$$\mathcal{L}_{\text{fix}} = -\frac{1}{2}\left(\partial^\mu{A_\mu}^a - \frac{1}{2}gv\chi^a\right)^2 \tag{4.3.14}$$

The cross term in this expression indeed cancels the off-diagonal kinetic term. As a result, the would-be Goldstone bosons χ^a acquire a mass $m = \frac{1}{2}gv$, namely[6]

$$\mathcal{L}(\text{kin}, \chi) = -\frac{1}{2}(\partial_\mu \chi^a)^2 - \frac{1}{2}\left(\frac{1}{2}gv\right)^2 (\chi^a)^2 \tag{4.3.15}$$

It is clear from (4.3.8) that also the Yang-Mills bosons have a mass $m = \frac{1}{2}gv$

$$\mathcal{L}(\text{kin}, YM) = -\frac{1}{4}(F_{\mu\nu}{}^a)^2 - \frac{1}{2}\left(\frac{1}{2}gv\right)^2 (A_\mu{}^a)^2 \tag{4.3.16}$$

The Faddeev-Popov ghost action, obtained as usual by varying the gauge-fixing term, also gets a mass term with $m = \frac{1}{2}gv$

$$\begin{aligned} \mathcal{L}\,(\text{ghost}) \quad &= \quad b_a\left[\partial^\mu \delta A_\mu{}^a - \frac{1}{2}gv\delta\chi^a\right] \text{ with } \lambda^a \text{ in (4.3.11) replaced by } c^a \\ &= b_a \partial^\mu D_\mu c^a - \left(\frac{1}{2}gv\right)^2 b_a c^a - \frac{1}{4}g^2 v b_a(\sigma c^a + \epsilon^a{}_{bc}\chi^b c^c) \end{aligned} \tag{4.3.17}$$

The Faddeev-Popov antighost b_a, the Faddeev-Popov ghost c^a, the would-be Goldstone bosons χ^a and the Yang-Mills bosons with an unphysical polarization $\partial^\mu A^a_\mu$ all have the same mass. They form a quartet of unphysical states with common mass $m = \frac{1}{2}gv$. A more general class of gauges in which the off-diagonal kinetic term $A_\mu{}^a \partial^\mu \chi^a$ still is canceled, is given by $\mathcal{L}\,(\text{fix}) = -\frac{1}{2\xi}\left(\partial^\mu A_\mu{}^a - \frac{1}{2}\xi gv\chi^a\right)^2$. Also in these so-called $R(\xi)$ gauges (renormalizable gauges with a parameter ξ) the quartet has the same mass. For example, the kinetic terms are $-\frac{1}{2\xi}(\partial^\mu A_\mu{}^a)^2$ for the longitudinal vector bosons and $-\frac{1}{2}(\partial_\mu A_\nu{}^a - \partial_\nu A_\mu{}^a)^2$ for the transversal bosons. The mass term is still $-\frac{1}{8}(gv)^2(A_\mu{}^a)^2$ in (4.3.8). If we write $(A^a_\mu)^2$ and $\left(A^a_\mu - \partial_\mu \frac{\partial\cdot A}{\Box} + \partial_\mu \frac{\partial\cdot A}{\Box}\right)^2$ and decompose this into $\left(A^a_\mu - \partial_\mu \frac{\partial\cdot A}{\Box}\right)^2 + \partial\cdot A\left(\frac{1}{-\Box}\right)\partial\cdot A$, then we can identity $(-\Box)^{-1/2}\partial\cdot A$ with the longitudinal part of A_μ, and $A_\mu - \partial_\mu \frac{1}{\beta}\partial\cdot A$ with the transversal part. (It is simpler to see this decomposition arising in the propagator, see (4.3.19)). Then the

[6]The reader may wonder whether no further mass term for χ^a is produced by $V(\phi^\dagger\phi)$. Expanding one finds $-\frac{\mu^2}{2}[(v+\sigma)^2 + (\chi^a)^2] + \frac{\lambda}{4}[(v+\sigma)^2 + (\chi^a)^2]^2$ and the condition that terms linear in σ be absent (which is necessary in order to be at the minimum of V) is $\mu^2 - \lambda v^2 = 0$. This is also the condition that the $(\chi^a)^2$ contributions from V cancel. Hence, the mass of the Goldstone bosons is entirely due to the gauge fixing term. This is a general result.

unphysical longitudinal vector bosons have a mass $\frac{1}{2}\sqrt{\xi}gv$, as do the (anti)ghosts and the would-be Goldstone bosons, while the three physical transversal bosons have a mass $\frac{1}{2}gv$. Even if one uses a gauge fixing term $-\frac{1}{2\alpha}(\partial \cdot A^a - \frac{1}{2}\xi gv\chi^a)^2$ with $\alpha \neq \xi$, the four unphysical fields $(\partial \cdot A^a, b_a, c^a, \chi^a)$ still form a quartet; they still transform on-shell into each other under linearized (asymptotic) BRST transformations even though their masses are no longer equal

$$\begin{aligned} \delta_B \chi^a &\sim \frac{1}{2} gvc^a; \delta_b c^a \sim 0 \\ \delta_B b^a &\sim \frac{1}{\alpha}(\partial \cdot A^a - \frac{1}{2}\xi gv\chi^a); \delta_B(\partial \cdot A^a - \frac{1}{2}\xi gv\chi^a) \sim 0 \end{aligned} \tag{4.3.18}$$

The complete gauge field propagator for arbitrary ξ can be written in three equivalent ways

$$\begin{aligned} D_{\mu\nu} &= \frac{\eta_{\mu\nu} - k_\mu k_\nu/k^2}{k^2 + m^2} + \frac{\xi k_\mu k_\nu/k^2}{k^2 + \xi m^2} \\ &= \frac{\eta_{\mu\nu} + k_\mu k_\nu/m^2}{k^2 + m^2} - \frac{k_\mu k_\nu/m^2}{k^2 + \xi m^2} \\ &= \left[\eta_{\mu\nu} + \frac{(\xi - 1)k_\mu k_\nu}{k^2 + \xi m^2}\right]/(k^2 + m^2) \end{aligned} \tag{4.3.19}$$

We see again that the transversal boson has mass m, while the longitudinal boson has mass $\sqrt{\xi}m$. For $\xi = 1$ one obtains the Feynman - 't Hooft propagator with $\eta_{\mu\nu}$, while for $\xi = 0$ one obtains the Landau propagator with $\eta_{\mu\nu} - k_\mu k_\nu/k^2$. This latter propagator is renormalizable (by which is meant that it leads to a power-counting renormalizable theory), but the two poles, each with an $-i\epsilon$ prescription as follows from requiring unitarity, make calculations in this gauge cumbersome. In the limit $\xi \to \infty$ one finds the propagator in the unitary gauge, $D_{\mu\nu} = [\eta_{\mu\nu} + k_\mu k_\nu/m^2]/(k^2+m^2)$ with three physical degrees of freedom. This propagator is nonrenormalizable as far as power counting is concerned, but calculations of S-matrix elements show (after much work) that the nonrenormalizable terms cancel in physical processes. [13] For $\xi \to \infty$ the quartet becomes infinitely heavy but note that some interactions of the ghosts in (4.3.17) become infinitely strong. Thus one cannot naively omit all ξ-dependent terms in the action. [14]

The χ^a are sometimes called Higgs ghosts, because they are unphysical since they can be gauged away, using $\delta_L\chi^a = \frac{1}{2}gv\lambda^a(x)+$ more. We prefer the name, "would-be Goldstone bosons", because they are not ghosts in the sense of unusual statistics and would be Goldstone bosons if gauge fields were absent. When the gauge fields ${A_\mu}^a$ are present, one may introduce redefined gauge fields ${W_\mu}^a = {A_\mu}^a - \frac{2}{gv}D_\mu\chi^a$. For abelian theories the gauge action is clearly invariant under this redefinition, while the mass term $-\frac{1}{8}g^2v^2(W_\mu^a)^2$ accounts for both the kinetic terms $-\frac{1}{2}(D_\mu\chi^a)^2$ and the $A_\mu^a\partial_\mu\chi^a$ cross term. So for the abelian Higgs model (with only a field χ and no ψ) one can in this way completely remove the would-be Goldstone bosons by redefinitions, and the field W_μ is then itself both massive and gauge invariant. One might say that the vector fields W_μ have "eaten" the χ fields and become massive as a result of this banquet. For the nonabelian Higgs model one cannot find gauge invariant fields by local redefinitions but one can choose the gauge $\chi^a = 0$. The remaining physical particles are then the three polarizations of the massive Yang-Mills field, and the real Higgs scalar σ. [2]

It is better to keep the would-be Goldstone bosons χ^a in the theory at the quantum level, instead of choosing the unitary gauge $\chi^a = 0$. Let us analyze in more detail why one should avoid the unitary gauge in loop calculations. The gauge choice $\chi^a = 0$ is achieved in a path integral approach by only adding a term $d_a\chi^a$ to the action but no term $\frac{1}{2}(d_a)^2$. The field d_a is the BRST auxiliary field. Integrating out d_a, one obtains a delta function $\delta(\chi^a)$. In addition, the variation of χ^a gives the ghost action

$$\mathcal{L}\,(\text{ghost}) = \frac{1}{2}b_a g(\sigma+v)c^a \tag{4.3.20}$$

Integrating the ghosts b and c out, one obtains the determinant $\det[g(\sigma+v)(x)\delta(x-y){\delta^a}_b]$. The terms with gauge fields reduce for $\chi^a = 0$ to

$$\mathcal{L} = -\frac{1}{4}{F_{\mu\nu}}^2 - \frac{1}{8}g^2(v+\sigma)^2{A_\mu}^2 \tag{4.3.21}$$

which leads again to the unitary propagator for the gauge fields with $\eta_{\mu\nu} + k_\mu k_\nu/m^2$

in the numerator.

In the gauge $\chi^a = 0$ we have found the extra factor with $\det g(v+\sigma)$. It is possible to find a formulation where only physical particles are present and no complicated remnants of effects due to the limit $\xi \to 0$ remain. This is the canonical formulation. [15] Starting with the path integral over $\vec{E}^a, A_k{}^a, A_0^a, \chi^a, \pi_\chi^a, \sigma$ and π with action $\mathcal{L} = -\dot{A}_k^a E_k^a - A_0^a \div E^a + \dot{\sigma}\pi + \dot{\chi}^a \pi_\chi^a - \mathcal{H}$, and choosing the gauge $\chi^a = 0$ as discussed before, one gets the factor $\det[g(v+\sigma)]^3$. However, integrating also over $A_0{}^a$ gives another determinant which precisely cancels the original determinant, as is clear from the A_0 dependent terms in the action

$$\mathcal{L}(A_0) = \frac{1}{8}[g(v+\sigma)]^2 (A_0{}^a)^2 - A_0(\text{div } \vec{E}) \tag{4.3.22}$$

The final result is an action which only depends on the canonical variables $\vec{E}^a, \vec{A}^a$ and π, ψ describing a massive spin 1 and a massive spin 0 particle

$$\begin{aligned}
\mathcal{L} &= -\dot{A}_k{}^a E_k + \dot{\sigma}\pi + \dot{\chi}^a \pi_\chi^a \\
\mathcal{H} &= \frac{1}{2}(E_k{}^a)^2 + \frac{1}{4}(F_{ij}{}^a)^2 + \frac{1}{8}g^2(v+\sigma)^2(A_k{}^a)^2 + \frac{1}{2}(D_k\sigma)^2 \\
&+ \frac{1}{2}\pi^2 + V(\sigma) + (\text{div } \vec{E}^a)^2 / \left[\frac{1}{8}g(v+\sigma)\right]^2
\end{aligned} \tag{4.3.23}$$

Since there are no ghosts left, this model describes a unitary theory, but renormalizability is hard to prove in this formulation. As we already noted, the propagator of A_μ is of the non-renormalizable form $\eta_{\mu\nu} - k_\mu k_\nu / m^2$ with $m = \frac{1}{2}gv$. Furthermore, the term with $(\text{div}\vec{E})^2$ contains the factor $[\frac{1}{8}g(v+\sigma)]^2$ in the denominator, which leads to a nonpolynomial action. Integrating out the momenta $\vec{E}$, one finds the troublesome determinant back. The preferred way to deal with Higgs models is to keep the would-be Goldstone bosons in the theory; then unitarity and renormalizability can both be proven as we show in the next section and in chapter V.

One can add fermions with chiral couplings to this model. Taking complex Dirac fermions λ in an $SU(2)$ doublet we obtain for the kinetic term and Yukawa coupling

$$\mathcal{L}_Y = -\bar{\lambda}\not{D}\lambda - h\bar{\lambda}[\psi + i\vec{\tau}\cdot\vec{\chi}\gamma_5]\lambda \tag{4.3.24}$$

with $D_\mu\lambda = \partial_\mu\lambda - \frac{i}{2}g\vec{A}_\mu\cdot\vec{\tau}\lambda$ and h the real Yukawa coupling constant. Setting again $\psi = \sigma + v$, the fermions get a mass by the Higgs mechanism. Introducing chiral fermions $\lambda_L = \frac{1}{2}(1+\gamma_5)\lambda$ and $\lambda_R = \frac{1}{2}(1-\gamma_5)\lambda$, we can rewrite the Yukawa interaction as

$$-h\bar{\lambda}_R M \lambda_L + h.c. \tag{4.3.25}$$

The matrix $M = \psi + i\vec{\chi}\cdot\vec{\tau}$ transforms as $M' = UMV$, see (4.3.9). It is now clear that the fermionic interactions are invariant under $SU_L(2, \text{ local}) \otimes SU_R(2, \text{ rigid})$ if we choose the transformation laws for λ_L and λ_R suitably[7]

$$\lambda'_L = V^{-1}\lambda_L, \lambda'_R = U\lambda_R \tag{4.3.26}$$

The kinetic terms

$$\mathcal{L}_\lambda \text{ (kin) } = -\bar{\lambda}_L \not{\partial} \lambda_L - \bar{\lambda}_R \not{D} \lambda_R \tag{4.3.27}$$

are also invariant under the same symmetry group.[8] This model was constructed by Gell-Mann and Levy as a simple model exhibiting spontaneous symmetry breaking. [3] In their formulation the Higgs field σ and the would-be Goldstone bosons χ^a appeared in an $O(4)$ invariant fashion, as

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu\sigma + \partial_\mu\vec{\pi})^2 + \frac{\mu^2}{2}(\psi^2 + \vec{\pi}^2) - \frac{\lambda}{4}(\psi^2 + \vec{\pi}^2)^2 \tag{4.3.28}$$

where $\psi = \sigma + v$. We have formulated it in terms of $M = \psi + \vec{\chi}\cdot\vec{\tau}$ in order to use complex Higgs doublets, but the formulations are, of course, equivalent. We shall not discuss this fermionic extension of the $SU(2)$ Higgs model further.

[7]In a representation in which γ_5 is diagonal, λ_L and λ_R are the upper and lower two-component spinors whose transformation laws are clearly unrelated.

[8]In QCD with massless (u, d) quark doublet, the kinetic terms are $\bar{u}_L \not{D} u_L + \bar{d}_L \not{D} d_L + \bar{u}_R \not{D} u_R + \bar{d}_R \not{D} d_R$. Then there is an even bigger symmetry group: $U(2)_L \otimes U(2)_R$. It decomposes into $U(1)_V \otimes SU(2)_V \otimes U(1)_A \otimes SU(2)_A$. The same can be done for (u, d, s) quarks and $U(3)$. We discuss this in the chapter on instantons.

4 Renormalization of the SU(2) Higgs model

Since the electroweak sector of the Standard Model is a spontaneously broken gauge theory, the renormalization of spontaneously broken gauge theories (Higgs models) is an important subject in quantum gauge field theory. In the Standard Model, the symmetry group that is spontaneously broken is $SU(2) \times U(1)$ and the Higgs scalars form a complex doublet. Here we consider instead the $SU(2)$ Higgs model, still with a complex Higgs doublet.

Consider then the spontaneously broken $SU(2)$ gauge theory coupled to the Higgs sector of the Standard Model [16], with σ the Higgs scalar and χ^a the would-be Goldstone bosons. One might expect that the approach to prove renormalizability which we discussed in the previous chapter for unbroken gauge theories goes through without any new complications. Namely, all divergences should be of the form "gauge invariant terms plus SX", and the number of arbitrary parameters in this expression should be equal to the number of Z factors, the latter being restricted by relations which follow from renormalizing the two Ward identities. This program was worked out in [19, 20]. Surprisingly, according to this way of counting there is one more divergent structure allowed by the BRST Ward identities than there are Z factors. This problem is resolved by a new Ward identity for the effective action of spontaneously broken gauge theories, which holds in addition to the BRST Ward identities. It originates from the observation that in the matter sector only the combination $\sigma + v$ appears. [20] As we already mentioned in footnote (), choosing a gauge fixing term which only depends on $v+\sigma$ one finds directly equal numbers of divergences and Z factors, but such nonlinear gauges have the drawback that they introduce extra vertices in the theory.

Since the theory combines pure gauge theory with a Goldstone model, we shall encounter a subtle interplay between the renormalization of gauge theories as expounded in chapter III, and the renormalization of a spontaneously broken model of scalars as discussed in detail in section 1 of this chapter. The Lagrangian is given by

$$\mathcal{L} = \mathcal{L}(\text{gauge}) + \mathcal{L}(\text{matter}) + \mathcal{L}(\text{fix}) + \mathcal{L}(\text{ghost}) + \mathcal{L}(\text{extra}) \tag{4.4.1}$$

where

$$\begin{aligned}
\mathcal{L}(\text{gauge}) &= -\frac{1}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^a{}_{bc} A^b_\mu A^c_\nu)^2 \\
\mathcal{L}(\text{matter}) &= -\frac{1}{2}(D_\mu \sigma)^2 - \frac{1}{2}(D_\mu \chi^a)^2 + \frac{1}{2}\mu^2\{(\sigma+v)^2 + (\chi^a)^2\} \\
&\quad -\frac{1}{4}\lambda\{(\sigma+v)^2 + (\chi^a)^2\}^2 \\
&= -\frac{1}{2}(D_\mu \sigma)^2 - \frac{1}{2}(D_\mu \chi^a)^2 - \frac{1}{4}\lambda\{\sigma^2 + (\chi^a)^2\}^2 - \lambda v\sigma[\sigma^2 + (\chi^a)^2] \\
&\quad -\lambda v^2\sigma^2 - \beta v\sigma - \frac{1}{2}\beta(\sigma^2 + (\chi^a)^2) \\
\mathcal{L}(\text{fix}) &= -\frac{1}{2\alpha}\left(\partial^\mu A^a_\mu - \xi\left(\frac{1}{2}gv\right)\chi^a\right)^2 \\
\mathcal{L}(\text{ghost}) &= b_a\left\{\partial^\mu D_\mu c^a - \xi\frac{1}{2}gv\left(\frac{1}{2}g(\sigma+v)c^a + \frac{1}{2}g f^a{}_{bc}\chi^b c^c\right)\right\} \\
\mathcal{L}(\text{extra}) &= K^\mu_a D_\mu c^a - K\left(\frac{1}{2}g\chi_a c^a\right) + K_a\left(\frac{1}{2}g(\sigma+v)c^a + \frac{1}{2}g f^a{}_{bc}\chi^b c^c\right) \\
&\quad + L_a\left(\frac{1}{2}g f^a{}_{bc} c^b c^c\right)
\end{aligned} \tag{4.4.2}$$

Note that we have two constants in $\mathcal{L}$ (fix), namely α and ξ. This is necessary to achieve that $\mathcal{L}_{\text{fix}}$ does not renormalize. We can, of course, take the renormalized values equal, $\alpha_{\text{ren}} = \Xi_{\text{ren}}$, in order to eliminate cross terms in the kinetic terms. The generators of $SU(2)$ are $T_a = -\frac{i}{2}\tau_a$ with τ_a the Pauli matrices and $f^a{}_{bc} = \epsilon^a{}_{bc}$. We recall from section 1 that the parameter β is given by

$$\beta = -\mu^2 + \lambda v^2$$

and the unphysical sign in front of mass term $+\frac{1}{2}\mu^2((\sigma+v)^2 + (\chi^a)^2)$ leads to spontaneous symmetry breaking. We recall the following observations made in section 1. Since μ^2 and λv^2 will in general renormalize differently, one cannot expect that β renormalizes multiplicatively. It is very convenient to require that the value of the renormalized β be zero at the tree graph level

$$\beta^{(0)}_{\text{ren}} = -\mu^2_{\text{ren}} + \lambda_{\text{ren}} v^2_{\text{ren}} = 0 \tag{4.4.3}$$

since this eliminates terms linear in σ from the quantum action. However, $\beta^{(0)}_{\rm ren} = 0$ excludes multiplicative renormalizability of β. Taking μ^2 as independent variable saves multiplicative renormalizability, but we prefer to renormalize β additively: $\beta = \beta^{(0)}_{\rm ren} + \Delta\beta^{(1)}_{\rm ren} + \cdots = 0 + \Delta\beta^{(1)}_{\rm ren} + \cdots$

The external sources K, K_a and K^μ_a multiply the BRST variations of σ,χ^a and A^a_μ, see (4.3.11), and the theory with (and hence without) them will be shown to be renormalizable. The covariant derivatives were given in (4.3.12)

$$D_\mu\sigma = \partial_\mu\sigma + \frac{1}{2}gA^a_\mu\chi_a \tag{4.4.4}$$

$$D_\mu\chi^a = \partial_\mu\chi^a - \frac{1}{2}gA^a_\mu(\sigma+v) + \frac{1}{2}gf^a{}_{bc}A^b_\mu\chi^c. \tag{4.4.5}$$

Clearly, $\mathcal{L}$(matter) depends only on $\sigma + v$, but $\mathcal{L}$(fix) and $\mathcal{L}$(ghost) violate this property for $\xi \neq 0$. Hence we expect that σ and v will renormalize differently if $\xi \neq 0$. We shall choose the renormalized ξ and α to be given by $\xi_{\rm ren} = \alpha_{\rm ren} = 1$ in order that the propagators be diagonal and simple. In fact, the propagators are diagonal for any $\xi_{\rm ren} = \alpha_{\rm ren}$.

We could have used a gauge-fixing term which also only depends on $\sigma+v$, namely $\mathcal{L}_{\rm fix} = -\frac{1}{2\alpha}(\partial\cdot A^a - \frac{1}{2}\xi g(\sigma+v)\chi^a)^2$. Then also the ghost action would only depend on $\sigma + v$, and the same reasoning as used in section 1 for the $O(2)$ model shows that in this case $Z_v = Z_\sigma$. Because the gauge fixing term with $\sigma + v$ contains terms cubic and quartic in quantum fields, we should use the auxiliary field d_a in the derivation of the Ward identities as explained in chapter III. The advantage of the gauge-fixing we use is that it is quadratic in fields and does not lead to additional vertices.

We couple the fields $\sigma, \chi^a, A^a_\mu, b_a$ and c^a to external sources, and perform the Legendre transformation as in the unbroken case discussed in chapter III. The two Ward identities for the effective action Γ read before renormalization

$$\int \partial\hat\Gamma/\partial\Phi^I \frac{\partial}{\partial K^I}\hat\Gamma = 0 \tag{4.4.6}$$

$$\left(\partial^\mu\frac{\partial}{\partial K^\mu_a} - \xi\frac{1}{2}gv\frac{\partial}{\partial K_a} - \frac{\partial}{\partial b_a}\right)\hat\Gamma = 0 \tag{4.4.7}$$

where $\Phi^I = \{\sigma, \chi^a, A^a_\mu, c^a\}, K_I = \{K, K_a, K^\mu_a, L_a\}$ and $\hat{\Gamma} = \Gamma - \int \mathcal{L}(\text{fix}) d^4x$. In addition we shall use below two further identities related to ghost number conservation and to the symmetry of $\mathcal{L}$(matter) under $\sigma \to \sigma + \Delta v, v \to v - \Delta v$.

The Ward identities in (4.4.6) and (4.4.7) remain valid after renormalization if all rescalings are such that they amount to an overall factor. Choosing $A^a_\mu = (Z_3)^{\frac{1}{2}} A^a_{\mu,\text{ren}}$ and $c^a = (Z_{\text{gh}})^{\frac{1}{2}} c^a_{\text{ren}}$, and furthermore $\sigma = (Z_\sigma)^{\frac{1}{2}} \sigma_{\text{ren}}$ and $\chi^a = (Z_\chi)^{\frac{1}{2}} \chi^a_{\text{ren}}$ (Z_χ is independent of the $SU(2)$ index a since $\mathcal{L}$(fix) and $\mathcal{L}$(ghost) are invariant under rigid diagonal $SU(2)$ transformations), we assume, to be proven by induction with respect to the number of loops, the following properties:

Step 1. $\hat{\Gamma}$ is made finite by multiplicative rescalings of all objects except β which is renormalized additively such that tadpoles are made to vanish. In particular, K^μ_a and b_a scale like c^a, while L_a scales like A^a_μ. Furthermore the rescalings of K and K_a are such that σK and $\chi^a K_a$ have the same overall Z factor as $A^a_\mu K^\mu_a$.

Step 2. α and ξ scale such that $\mathcal{L}$(fix) is finite by itself. Then Γ^{ren} will be finite if $\hat{\Gamma}^{\text{ren}}$ has been made finite because $\hat{\Gamma}$ equals Γ minus $\int \mathcal{L}(\text{fix}) d^4x$. This leads to $Z_\alpha = Z_3$ where $\alpha = Z_\alpha \alpha_{\text{ren}}$, and furthermore fixes Z_ξ.

We also renormalize $g = Z_g u\, \mu^{\frac{1}{2}(4-n)}$ with u the renormalized coupling constant, $v = (Z_v)^{\frac{1}{2}} v_{\text{ren}}$ and $\lambda = Z_\lambda Z_\sigma^{-2} \lambda_{\text{ren}} \mu^{(4-n)}$. Hence, in addition to the two coupling constant renormalizations, we find the following wave function and gauge fixing parameter renormalizations[9]

$$\begin{array}{rcl}
A^a_\mu = Z_3^{1/2} A^a_{\mu,\text{ren}} & K = (Z_3 Z_{\text{gh}}/Z_\sigma)^{\frac{1}{2}} K_{\text{ren}} & K^a_\mu = Z_{gh}^{1/2} K^a_{\mu,\text{ren}} \\
c^a = Z_{gh}^{1/2} c^a_{\text{ren}} & K^a = (Z_3 Z_{\text{gh}}/Z_\chi)^{\frac{1}{2}} K^a_{\text{ren}} & L^a = Z_3^{1/2} L_{a,\text{ren}} \\
b_a = Z_{gh}^{1/2} b_{a,\text{ren}} & \xi = Z_3^{\frac{1}{2}} Z_g^{-1} Z_v^{-\frac{1}{2}} Z_\chi^{-\frac{1}{2}} \xi_{\text{ren}} & \sigma = Z_\sigma^{1/2} \sigma_{\text{ren}} \\
\alpha = Z_3^{1/2} \alpha_{\text{ren}} & v = Z_v^{1/2} v_{\text{ren}} & \chi^a = Z_\chi^{1/2} \chi^a_{\text{ren}}
\end{array} \tag{4.4.8}$$

[9] In early studies of Z factors in Yang-Mills theory, one introduced separate Z factors for vertex renormalization (Z_1 for the 3 gluon vertex, and $Z_{1,gh}$ for the gluon-ghost vertex). One then tried to prove that $\frac{Z_1}{Z_3} = \frac{Z_{1,gh}}{Z_{gh}}$, and $\frac{Z_1}{Z_3} = \frac{Z_{1,f}}{Z_{3,f}}$ if there were fermions present. [24] These relations are contained in the renormalization approach we follow.

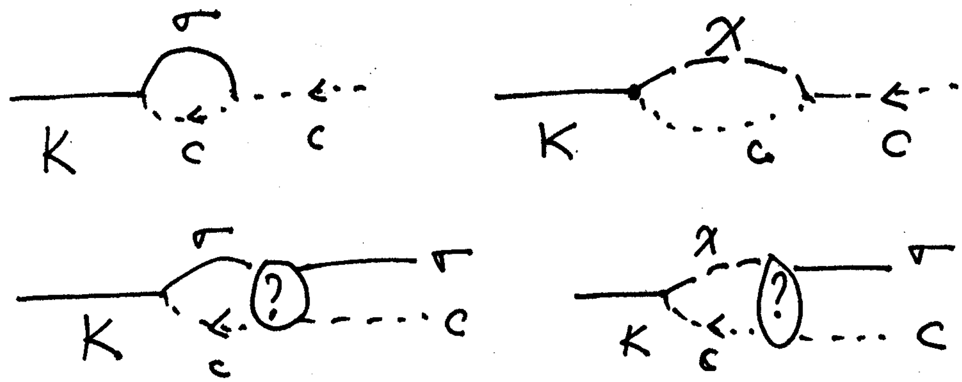
Since the action conserves ghost number, we can equate the Z-factors of b and c without loss of generality. For practical calculations it is useful to set α_{ren} and ξ_{ren} both equal to unity, but note that their Z factors are different. This shows that we need separate parameters α and ξ in $\mathcal{L}$ (fix) in order that $\mathcal{L}$ (fix) remains finite after renormalization.

It is instructive to do a quick one-loop calculation to prove that Z_v is indeed different from Z_σ if one uses the gauge fixing term in (4.4.1). The most direct calculation would be to determine the divergences proportional to $\frac{1}{4}\lambda\sigma^4$ and $\lambda v\sigma^3$. If $Z_\sigma = Z_v$, the coefficients of these divergences should be equal. One-loop graphs with only matter loops yield $Z_\sigma = Z_v$ as we have seen in the Goldstone model. (The fact that χ^a becomes massive due to the gauge fixing term does not alter this fact because the divergences are only logarithmic hence a mass in propagators does not change them). However, there are many graphs with virtual gauge fields and ghosts, and a simpler process is called for. We choose to study the $K_a c^a$ 2-point correlator and the $K_a \sigma c^a$ 3-point correlator.

Consider the divergences proportional to

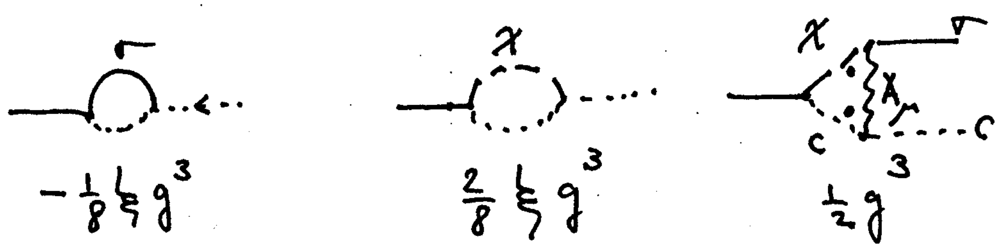
$$K_a \frac{1}{2} g v c^a \quad \text{and} \quad K_a \frac{1}{2} g \sigma c^a \tag{4.4.9}$$

If $Z_v = Z_\sigma$, these divergences should be equal. Only the vertices $K_a \frac{1}{2} g\sigma c^a$ and $K_a \frac{1}{2} g\epsilon^{abc}\chi^b c^c$, but not the vertices $K_a \frac{1}{2} g v c^a$, can contribute to proper graphs with an external source K_a. Thus we need to construct all proper one-loop graphs of the following form



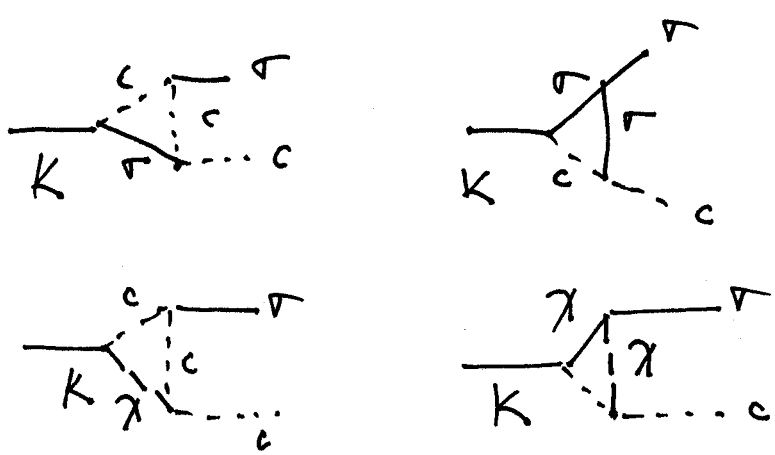
(4.4.10)

This yields the following set of divergent graphs, with the coefficients of the divergences $\frac{1}{\epsilon}$ indicated below



(4.4.11)

The two dots in the last graph denote derivatives. So, $Z_v \neq Z_\sigma$. Other graphs are finite, for example



(4.4.12)

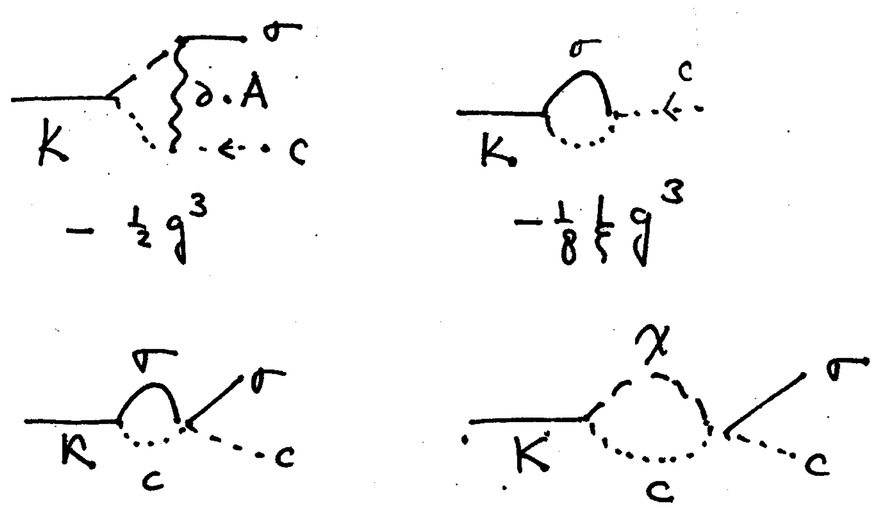
As a check we repeat the calculation in the corresponding theory in which only the combination $\sigma + v$ occurs. In this case we should find that $Z_v = Z_\sigma$. Thus we take in this case

$$\mathcal{L}_{\text{fix}} = -\frac{1}{2\alpha}\left(\partial\cdot A^a - \frac{1}{2}g\xi \overbracket{v\chi^a - \frac{\mathbf{1}}{\mathbf{2}}\boldsymbol{g\xi}}\boldsymbol{\sigma\chi^a}\right)^2 \tag{4.4.13}$$

which leads to the following ghost action

$$\begin{aligned}
\mathcal{L}^{\text{old}}_{\text{ghost}} &= b_a\left[\partial^\mu D_\mu c^a - \frac{1}{2}g\xi v\left\{\frac{1}{2}g(\sigma+v)c^a + \frac{1}{2}g\epsilon^{abc}\chi^b c^c\right\}\right]\\
& \qquad\qquad\qquad\qquad\qquad \updownarrow \qquad\qquad\qquad \updownarrow\\
\mathcal{L}^{\text{new}}_{\text{ghost}} &= \frac{1}{4}g^2\xi\left[\boldsymbol{b_a\chi^a\chi^b c^b - b_a\sigma(\sigma+v)c^a - \epsilon^{abc}b_a\sigma\chi^b c^c}\right]
\end{aligned} \tag{4.4.14}$$

The bold terms are new, and the arrows combine terms such that only the combination $\sigma + v$ occurs. There are 3 new vertices from (4.4.13), and also 3 new vertices from (4.4.14). The latter yield 3 new graphs, but only the new vertex indicated by the hook in (4.4.13) gives a new graph. So there are four new graphs, and we write again the coefficients of the divergence $\frac{1}{\epsilon}$ below them.



(4.4.15)

The last two new graphs cancel each other. The second new graph cancels the old two graphs with the same structure. The first new graph cancels the corresponding old graph. Thus now we find that indeed $Z_\sigma = Z_v$.

We now start the renormalization program. Assuming $(n-1)$-loop finiteness of $\hat{\Gamma}$ (and hence of Γ), the n-loop proper (1PI, one-particle irreducible) divergences satisfy the equations

$$\mathcal{S}_{\rm ren}\hat{\Gamma}^{(n),\rm div}_{\rm ren} = 0 \tag{4.4.16}$$

where $\mathcal{S}_{\rm ren} = \left(\partial\hat{\Gamma}^{(0)}_{\rm ren}/\partial\Phi^I_{\rm ren}\right)\frac{\partial}{\partial K^{\rm ren}_I} - \left(\partial\hat{\Gamma}^{(0)}_{\rm ren}/\partial K^{\rm ren}_I\right)\frac{\partial}{\partial\Phi^I_{\rm ren}}$ and

$$\left(\partial^\mu \frac{\partial}{\partial K^\mu_{a,\rm ren}} - \xi_{\rm ren}\frac{1}{2}uv_{\rm ren}\frac{\partial}{\partial K^{\rm ren}_a} - \frac{\partial}{\partial b^{\rm ren}_a}\right)\hat{\Gamma}^{(n),\rm div}_{\rm ren} = 0 \tag{4.4.17}$$

We recall that u is the renormalized gauge coupling constant and $\hat{\Gamma}^{(0)}_{\rm ren}$ equals the quantum action minus $\int\mathcal{L}(\text{fix})d^4x$. All objects have been multiplicatively renormalized such that all proper graphs with $(n-1)$ loops are finite. We shall drop the subscripts "ren", understanding that from now on all objects are $(n-1)$-loop renormalized.

The n-loop divergences are local (see chapter 5 for a proof), and (4.4.17) states that b_a can only appear in the divergences in the combination $K^\mu_a - \partial^\mu b_a$ or $K_a - \xi\frac{1}{2}uvb_a$. One might expect that the general form of the n-loop divergences is given by

$$\hat{\Gamma}^{(n),\rm div}_{\rm ren} = \sum_{i=1}^{4} a_i(\epsilon)G^i + \mathcal{S}_{\rm ren}\sum_{j=1}^{5} b_j(\epsilon)X^j \tag{4.4.18}$$

where the first term contains all possible gauge-invariant local expressions, see (4.4.2)

$$\sum_i G^i = a_1 S(\text{gauge}) + a_2\{S(\text{kin},\sigma) + S(\text{kin},\chi)\} + a_3 S(\text{mass matter}) + a_4 S(\text{pot}) \tag{4.4.19}$$

Note that only the sum of the kinetic terms for σ and χ is gauge invariant. (A complex doublet forms an irreducible representation of $SU(2)_L$). The $\mathcal{S}$-exact term is given by

$$\sum b_j X^j = b_1(K^\mu_a - \partial^\mu b_a)A^a_\mu + b_2\left(K_a - \frac{1}{2}uv\xi b_a\right)\chi^a + b_3 K\sigma + b_4 L_a c^a + b_5 Kv. \tag{4.4.20}$$

Since the terms with X^j must have dimensions 3 and ghost number -1, they must depend on b_a or the K and L sources, and then the terms in (4.4.20) are the only possibilities.

This would seem to indicate that there are 9 divergences (corresponding to the a_i and b_j) but only 8 Z factors ($Z_3, Z_{gh}, Z_\sigma, Z_\chi, Z_g, Z_v, Z_{\mu^2}$ or rather $\Delta\beta^{(n)}_{\rm ren}, Z_\lambda$). Note that Z_σ and Z_χ are different because $\mathcal{L}_{\text{fix}}$ and $\mathcal{L}_{\text{ghost}}$ break the classical $SO(4)$ symmetry between σ and χ. Actually, the fact that the parameters g, μ^2 and λ are physical and appear in S (mass matter) and S (pot), while the wave function renormalizations of σ and χ are unphysical and correspond to $S(\text{kin}, \sigma)$ and $S(\text{kin}, \chi)$, may suggest that the kinetic terms of the scalars in (4.4.19) are unphysical, namely BRST exact. Once one entertains this idea, it is not hard to find such a relation

$$\begin{aligned}&\mathcal{S}[K(\sigma+v)+K_a\chi^a] = v\partial/\partial v\hat{S} + 2\mathcal{L}(\text{mass, matter})\\&+2\mathcal{L}(\text{kin},\sigma)+2\mathcal{L}(\text{kin},\chi)+4\mathcal{L}(\text{pot})\end{aligned} \tag{4.4.21}$$

Thus we could replace the kinetic terms proportional to a_2 in (4.4.19) by $a_2(v\partial/\partial v)\hat{S}$. Since the number of divergences is unchanged we shall not make this replacement.

It is easy to see that (4.4.18) is a solution of (4.4.16) and (4.4.17) since $\mathcal{S}$, the Slavnov-Taylor charge, acting on a gauge invariant term is zero and $\mathcal{S}^2 = 0$. One can give a rather general but rather complicated proof that the general solution of (4.4.16) is a sum of gauge invariant terms and $\mathcal{S}$-exact terms as in (4.4.18). We recall from chapter III that it is possible to prove this for a given model in a simple and direct way as follows:

1. write down all local expressions with dimension four and ghost-number zero which can be a priori divergent according to power counting
2. use the fact that their sum must be annihilated by $\mathcal{S}_{\text{ren}}$.

For the model in (4.4.1), the result is (4.4.18).

We have seen that there seem to be nine divergent structures but only eight Z-factors. In pure unbroken Yang-Mills theory there is no such mismatch, but in the matter coupled case with unbroken symmetry the same mismatch occurs. As in that case, also here renormalizability is still possible because the nine divergences only occur in eight combinations.

To prove multiplicative renormalizability, it is useful to write each of the local divergences as a counting operator $x\frac{\partial}{\partial x}$ acting on $\hat{\Gamma}^{(0),\text{ren}} = \hat{S}^{\text{ren}}$ where x denotes all fields, sources and parameters in the theory. For most terms, the analysis has already been given in [17]. In particular for the gauge invariant terms we find

$$\begin{aligned} S(\text{gauge}) &= \frac{1}{g^2}S(gA^a_\mu) = \left(-\frac{1}{2}g\frac{\partial}{\partial g} + \frac{1}{2}A^a_\mu\frac{\partial}{\partial A^a_\mu} + \frac{1}{2}L_a\frac{\partial}{\partial L_a}\right. \\ &\left. + \frac{1}{2}K\frac{\partial}{\partial K} + \frac{1}{2}K_a\frac{\partial}{\partial K_a} + \xi\frac{\partial}{\partial \xi}\right)\hat{\Gamma}^{(0)}_{\text{ren}} \qquad (4.4.22) \\ S(\text{kin. matter}) &= \left(\frac{1}{2}\sigma\frac{\partial}{\partial\sigma} + \frac{1}{2}v\frac{\partial}{\partial v} + \frac{1}{2}\chi_a\frac{\partial}{\partial\chi_a} - \mu^2\frac{\partial}{\partial\mu^2}\right. \\ &\left. -2\lambda\frac{\partial}{\partial\lambda} - \frac{1}{2}K\frac{\partial}{\partial K} - \frac{1}{2}K_a\frac{\partial}{\partial K_a} - \xi\frac{\partial}{\partial\xi}\right)\hat{\Gamma}^{(0)}_{\text{ren}} \\ S(\text{pot.}) &= \lambda\frac{\partial}{\partial\lambda}\hat{\Gamma}^{(0)}_{\text{ren}}\ ;\ S(\text{mass matter}) = \mu^2\frac{\partial}{\partial\mu^2}\hat{\Gamma}^{(0)}_{\text{ren}} \qquad (4.4.23) \end{aligned}$$

One may check that acting with these counting operators on the action in (4.4.2) projects out the gauge invariant terms.

Most terms in $\mathcal{S}X$ can also be written as $x\frac{\partial}{\partial x}\hat{S}$. For example, $\mathcal{S}b_1(K^\mu_a - \partial^\mu b_a)A^a_\mu$ is equal to

$$\begin{aligned} &\left(-\partial\hat{S}/\partial K^\mu_a\frac{\partial}{\partial A^a_\mu} + \partial\hat{S}/\partial A^a_\mu\frac{\partial}{\partial K^\mu_a}\right)b_1(K^\mu_a - \partial^\mu b_a)A^a_\mu \\ &= b_1[-\partial\hat{S}/\partial K^\mu_a(K^a_\mu - \partial^\mu b^a) + (\partial\hat{S}/\partial A^a_\mu)A^a_\mu] \\ &= b_1\left[-(K^\mu_a - \partial^\mu b_a)\frac{\partial}{\partial K^\mu_a} + A^a_\mu\frac{\partial}{\partial A^a_\mu}\right]\hat{S} \qquad (4.4.24) \end{aligned}$$

In this way the terms from $\mathcal{S}X$ can be written as follows

$$\begin{aligned} &\left[b_1\left(A^a_\mu\frac{\partial}{\partial A^a_\mu} - (K^\mu_a - \partial^\mu b_a)\frac{\partial}{\partial K^\mu_a}\right) + b_2\left(\chi_a\frac{\partial}{\partial\chi_a} - \left(K_a - \frac{1}{2}uv\xi b_a\right)\frac{\partial}{\partial K_a}\right)\right. \\ &\left. +b_3\left(\sigma\frac{\partial}{\partial\sigma} - K\frac{\partial}{\partial K}\right) + b_4\left(c_a\frac{\partial}{\partial c_a} - L_a\frac{\partial}{\partial L_a}\right) + b_5 v\frac{\partial}{\partial\sigma}\right]\hat{\Gamma}^{(0)}_{\text{ren}} \qquad (4.4.25) \end{aligned}$$

Most terms in $\mathcal{S}X$ are now of the form $x\frac{\partial}{\partial x}\tilde{\Gamma}^{(0)}_{\text{ren}}$. We next analyze the terms which are not yet cast into this form. These are

$$\left(b_1\partial^\mu b_a\frac{\partial}{\partial K^\mu_a} + b_2\xi\frac{1}{2}uvb_a\frac{\partial}{\partial K_a} + b_5 v\frac{\partial}{\partial\sigma}\right)\hat{\Gamma}^{(0)}_{\text{ren}}. \qquad (4.4.26)$$

The first term equals $-S(\text{ghost})$ at $\xi = 0$ (times b_1), and can be written as (b_1 times) $-\frac{1}{2}b_a\frac{\partial}{\partial b_a} - \frac{1}{2}c_a\frac{\partial}{\partial c_a} + \frac{1}{2}K\frac{\partial}{\partial K} + \frac{1}{2}K_a\frac{\partial}{\partial K_a} + \frac{1}{2}K^\mu_a\frac{\partial}{\partial K^\mu_a} + L_a\frac{\partial}{\partial L_a} + \xi\frac{\partial}{\partial\xi}$ acting on $\hat{\Gamma}^{(0)}_{\text{ren}}$. The second term in (4.4.26) is (b_2 times) minus the ξ term in S(ghost), hence it equals (b_2 times) $-\xi\frac{\partial}{\partial\xi}$ acting on $\tilde{\Gamma}^{(0)}_{\rm ren}$. The last term we deal with later.

Looking at these results, we see that the combination $c^a\frac{\partial}{\partial c^a}+b_a\frac{\partial}{\partial b_a}$ appears everywhere except in the term with b_4. However, ghost number conservation leads to the identity

$$\left(b_a\frac{\partial}{\partial b_a}-c^a\frac{\partial}{\partial c^a}+K\frac{\partial}{\partial K}+K_a\frac{\partial}{\partial K_a}+K^\mu_a\frac{\partial}{\partial K^\mu_a}+2L_a\frac{\partial}{\partial L_a}\right)\hat{\Gamma}^{(0)}_{\rm ren}=0 \tag{4.4.27}$$

and using this identity to convert half of the $c^a\frac{\partial}{\partial c^a}$ terms in the b_4 term into $b_a\frac{\partial}{\partial b_a}$ terms, we find also in the b_4 term the desired combination $c^a\frac{\partial}{\partial c^a}+b_a\frac{\partial}{\partial b_a}$. At this point the n-loop divergences can be written as follows

$$\begin{aligned}
\hat{\Gamma}^{(n),{\rm div}}_{\rm ren} \;=\;& \Big[\left(\frac{1}{2}a_1+b_1\right)\left(A^a_\mu\frac{\partial}{\partial A^a_\mu}+L_a\frac{\partial}{\partial L_a}\right)\\
&+\left(-\frac{1}{2}b_1+\frac{1}{2}b_4\right)\left(c_a\frac{\partial}{\partial c_a}+b_a\frac{\partial}{\partial b_a}+K^\mu_a\frac{\partial}{\partial K^\mu_a}\right)\\
&+\left(\frac{1}{2}a_2+b_3\right)\sigma\frac{\partial}{\partial\sigma}+\left(\frac{1}{2}a_2+b_2\right)\chi_a\frac{\partial}{\partial\chi_a}\\
&+\left(\frac{1}{2}a_1-\frac{1}{2}a_2+\frac{1}{2}b_1-b_3+\frac{1}{2}b_4\right)K\frac{\partial}{\partial K}\\
&+\left(\frac{1}{2}a_1-\frac{1}{2}a_2+\frac{1}{2}b_1-b_2+\frac{1}{2}b_4\right)K_a\frac{\partial}{\partial K_a}\\
&-\frac{1}{2}a_1 g\frac{\partial}{\partial g}+(a_1-a_2+b_1-b_2)\xi\frac{\partial}{\partial\xi}\\
&+(-a_2+a_3)\mu^2\frac{\partial}{\partial\mu^2}+(-2a_2+a_4)\lambda\frac{\partial}{\partial\lambda}\\
&+\frac{1}{2}a_2 v\frac{\partial}{\partial v}+b_5 v\frac{\partial}{\partial\sigma}\Big]\tilde{\Gamma}^{(0)}_{\rm ren}.
\end{aligned} \tag{4.4.28}$$

We see that A^a_μ and L_a scale the same way, as do c^a, b_a and K^μ_a. Furthermore, the factors in front of $K\frac{\partial}{\partial K}$ (or $K_a\frac{\partial}{\partial K_a}$) depend linearly on those corresponding to A^a_μ, c^a and σ (or χ^a), namely in agreement with (4.4.8). This confirms our first induction hypothesis below (4.4.7).

As usual, $\alpha=Z_3\alpha_{\rm ren}$ fulfills step 2 of the induction as far as the ξ-independent terms are concerned. We are left with the only nontrivial part of the proof of renormalizability, the proof that the rescaling of ξ in (4.4.8) is consistent with the rescaling of v which has been left unspecified so far. The key to this compatibility lies in the last term in (4.4.28), the term with $b_5 v \frac{\partial}{\partial \sigma} \hat{\Gamma}^{(0)}_{\text{ren}}$. To write it, too, as a counting operator, we recall that the matter action is annihilated by $v\frac{\partial}{\partial \sigma} - v\frac{\partial}{\partial v}$ (since it only depends on $\sigma + v$). There is only one place in $\hat{\Gamma}^{(0)}_{\text{ren}}$ where v appears separately, namely in the term with ξ in $\mathcal{L}$(ghost) where the combination ξv occurs. (Recall that in $\hat{\Gamma}$ there is no $\mathcal{L}$(fix)). Clearly then, the following identity holds

$$\left(\xi_{\text{ren}} \frac{\partial}{\partial \xi_{\text{ren}}} - v_{\text{ren}} \frac{\partial}{\partial v_{\text{ren}}} + v_{\text{ren}} \frac{\partial}{\partial \sigma_{\text{ren}}} \right) \hat{\Gamma}^{(0)}_{\text{ren}} = 0. \tag{4.4.29}$$

Using this identity to eliminate $v\frac{\partial}{\partial \sigma}$ from (4.4.28), we find that v rescales with $\frac{1}{2}a_2 + b_5$ and ξ with $(a_1 - a_2 + b_1 - b_2 - b_5)$. These renormalizations of ξ and v are in agreement with the rescaling of ξ in (4.4.8). Hence the multiplicative renormalizability of the spontaneously broken $SU(2)$ Higgs model is proven.

We end with some comments.

1. In [19] one can find explicit expressions for the one-loop contributions to all Z factors for arbitrary gauge parameters α and ξ. The Ward identity in (4.4.29) was obtained and used in [20].

2. In [10] the renormalizability of spontaneously models has been proven by using a gauge fixing term $-\frac{1}{2}(\partial^\mu A^a_\mu)^2$. Then the $SO(4)$ symmetry is not broken, but in practical calculations one uses an $R(\xi)$ gauge to diagonalize the kinetic terms. This breaks the $SO(4)$ symmetry, and then our proof is applicable.

3. One often uses the background field method in field theories which are obtained from dimensional reduction of a field theory in higher dimensions. Examples are the supersymmetric $N = 2$ and $N = 4$ Yang-Mills theories which are obtained from corresponding field theories in 6 and 10 dimensions, respectively. Then a useful background-invariant R_ξ gauge is $-\frac{1}{2\xi}(D^M(A)a_M)^2$ where M is the higher-dimensional vector index, and A_M denotes the background field and a_M the quantum field in higher dimensions. This preserves the rigid Lorentz symmetry of the higher-dimensional theory, but one has to pay a price: for $\xi \neq 1$ the kinetic terms are no

longer diagonal [21].

4. Adding a term linear in σ to the action, $\mathcal{L}_{\text{break}} = c\sigma$, the $SO(4)$ symmetry of the classical action is broken, but only softly. One can then still prove renormalizability [10]. As we already mentioned in section 1, the pion gets then a mass. However, in the real world the mass of physical pions is not thought to be due to this explicit soft symmetry breaking, but rather due to dynamical symmetry breaking [26] in which case "condensates" appear due to nonperturbative quantum effects. For example, one may begin with a theory in which the electron mass vanishes classically, and then generate an electron mass due to quantum effects [22].

5. As we already mentioned in chapter III one can also use the formulation with the BRST auxiliary field for linear gauges, for example linear R_ξ gauges. One can then begin with the $\Gamma\Gamma$ equation and the d field equation instead of the antighost field equation

$$\delta\Gamma/\delta d_a = \alpha d_a + \partial \cdot A^a - \frac{1}{2} g\xi v \chi^a \tag{4.4.30}$$

The meaning of the d field equation is clearly that it specifies the gauge choice. One can perform the renormalization using the $\Gamma\Gamma$ equation and the d field equation. The antighost field equation is still indirectly present, since the commutator $\mathcal{S}\frac{\delta\Gamma}{\delta d} - \frac{\delta}{\delta d}\mathcal{S}\Gamma = 0$ is the antighost field equation.

6. In sections 1 and 2 we discussed Goldstone bosons. There exist also pseudo-Goldstone bosons. These are bosons which are massless at tree level, usually because the potential for the scalar fields has a larger symmetry group than the rest of the action. In this case radiative corrections can generate a mass for these pseudo-Goldstone bosons, and then the graph below (4.1.4) leads to fatal infrared divergences in ordinary perturbation theory. One solution is to use these masses of order $\hbar$ (and higher) in the propagators; there are then no longer infrared divergences but the perturbation series is resummed in a nontrivial way [23].

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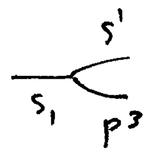

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Chapter 5

Perturbative unitarity from the cutting rules

Unitarity is, together with renormalizability, one of the central properties of a quantum field theory. The concept of unitarity was introduced by Heisenberg [1] as we have discussed in the first chapter. As in the case of renormalizability, also unitarity is more complicated for gauge theories. We shall only discuss perturbative unitarity, i.e., unitarity of the matrix elements of the S matrix order-by-order in the number of coupling constants. To prove unitarity for gauge theories we shall again use BRST Ward identities, but this time for connected graphs instead of proper graphs. These are of course the graphs which are needed for the S matrix.

One can already discover the need for ghosts by studying unitarity at the one-loop level: if one adds a gauge-fixing term but not a ghost action, one can write down one-loop graphs in Yang-Mills theory, for example for the self energy, but these graphs violate unitarity. By adding ghosts, Feynman [2] outlined how unitarity might be restored. Continuing this program, Veltman developed a Stückelberg formalism for nonabelian gauge theories, and it was found that there is a mass discontinuity at the quantum level between massive and massless gauge theories [3]. Since the massive theories were not renormalizable, from then on only massless or spontaneously broken gauge theories were considered viable. We shall discuss separately the unitarity of massless gauge theories and theories with a Higgs mechanism.

There exist various approaches to proving unitarity of gauge theories. We shall follow the approach originally begun by Cutkosky [4], and modified by Veltman in ways we shall explain [5,6]. In particular we shall use his proof based on the so-called largest-time equation. The approach of Cutkosky is based on analytic continuation of momenta, and uses Feynman parameters to combine the denominators in graphs. It leads to an expression for the discontinuity of any Feynman amplitude across a cut, and the unitarity equation is claimed to be a special case of the general discontinuity equation. The approach of Veltman does not use analytically continued momenta or Feynman parameters, but instead expresses the matrix elements of the T matrix as well as those of the $T^\dagger$ matrix in terms of the propagators Δ^+ and Δ^- into which any Feynman propagator can be decomposed. The propagators used for T are the usual Feynman propagators, but the propagators used for $T^\dagger$ are antipropagators (to be explained). Since continuation in momenta requires complex function theory with several variables, which is a very complicated branch of mathematics, the direct approach based only on combinatorics of Δ^+ and Δ^- propagators with real momenta is much simpler.

Unitarity of quantum electrodynamics was proven by Feynman [7] who used gauge invariance of S-matrix elements to prove that if one replaces a polarization vector $\epsilon_\mu^m(m=1,2)$ of a photon in an S-matrix element $S=M^\mu\epsilon_\mu^m$ by the four-momentum k_μ of the photon, one obtains zero: $k_\mu M^\mu=0$. Hence in the polarization sums $\sum_{m=1,2}\epsilon_\mu^m(\vec{k})\epsilon_\nu^m(\vec{k})$ which appear in the square of matrix elements, one may add the terms $(k_\mu\bar{k}_\nu+k_\nu\bar{k}_\mu)/k\cdot\bar{k}$ where $k_\mu=(\vec{k},k_0)$ and $\bar{k}_\mu=(\vec{k},-k_0)$. Since $\sum_{m=1,2}\epsilon_\mu^m\epsilon_\nu^m+(k_\mu\bar{k}_\nu+k_\nu\bar{k}_\mu)/k\cdot\bar{k}=\eta_{\mu\nu}$ for $k^2=0$, one may extend the sum over $m=1,2$ in the polarization sums to a sum over $m=1,2,3,0$ where $\epsilon_\mu^3=(\bar{k},0)/k|\bar{k}|$ and $\epsilon_\mu^0=(0,1)$. For QED this rule even holds when there are several photons: one may extend the summation over m in any number of polarization sums. For nonabelian gauge theories this is no longer true, and one needs ghosts.

The meaning of this result for unitarity is that in the expression $< f \mid S^\dagger \mid n >$ $< n \mid S \mid i >$ one may replace the sum over physical polarizations ($m = 1, 2$) by a sum over all polarizations ($m = 1, 2, 3, 0$), and unitarity (which requires for QED as we shall discuss below, that the sum over physical polarizations be equal to the sum over all polarizations) holds. We shall obtain Feynman's substitution equation for QED ($k_\mu M^\mu = 0$) as a special (and simple) case from the more general Ward identities for nonabelian gauge theories which we shall derive using BRST methods.

Another approach to unitarity, based on operator methods and developed in particular by Japanese physicists, starts from the BRST operator Q which is hermitian and satisfies $Q^2 = 0$ [8]. This charge is time-independent, and it acts linearly on the asymptotic in- and out-states. Let the linear action of Q on these states be denoted by Q_0. Also $Q_0^2 = 0$. Physical states $\mid \text{phys} >$ are by definition states which are BRST invariant (gauge invariant at the quantum level) but not BRST exact: $Q_0 \mid \text{phys} >= 0$ but not $\mid \text{phys} >= Q_0 \mid \chi >$. States of the form $\mid \psi >= Q_0 \mid \chi >$ have the property that they have vanishing norm: $< \psi \mid \psi >=< \chi \mid Q_0^2 \mid \chi >= 0$, and they are orthogonal to physical states: $< \psi \mid \text{phys} >=< \chi \mid Q_0 \mid \text{phys} >= 0$. Suppose one starts with a physical state $\mid \text{phys} >$ and considers the transition to a state $\mid \psi >$. Assuming that BRST symmetry holds at the quantum level, the BRST charge (which may depend on ghosts) commutes with the Hamiltonian,[1] $[Q, H] = 0$ and hence with the S matrix, $[Q, S] = 0$. From $SQ \mid \text{phys} >= 0$ we learn then that $QS \mid \text{phys} >= 0$. This means that either $S \mid \text{phys} >$ is again physical, or BRST-exact. In the unitarity sum $\sum_n < \text{phys}_I \mid S^\dagger \mid n >< n \mid S \mid \text{phys}_{II} >$ we may then restrict the sum over all states $\mid n >$ to a sum over physical states or BRST exact states. However, the latter do not contribute because BRST exact states have vanishing inner products with physical states and with other BRST exact states. Thus one need only sum over physical states, and this proves unitarity.

[1]In general the precise expressions for Q and H are **constructed** by requiring that they commute.

There exists also an indirect proof of unitarity at the level of Feynman graphs for spontaneously broken gauge theories with a so-called $R(\xi)$ gauge [9]. (A renormalizable gauge with a free gauge parameter ξ). As we have shown in chapter IV, provided the $R(\xi)$ gauge is chosen such that all kinetic terms are diagonal, the quartet of unphysical particles (ghosts, antighosts, would-be Goldstone bosons and time-like polarizations of the gauge bosons) all have the same ξ-dependent mass $\frac{1}{2}\xi(gv)$. Since the S-matrix should be independent of the gauge chosen, the poles in the propagators should not depend on ξ, but because the poles of the unphysical particles are on the other hand explicitly ξ dependent, there are simply no such poles. One can also go to the $\xi \to \infty$ limit (the unitarity limit) where only physical polarizations of the vector bosons remain with propagators $(\eta_{\mu\nu} + k_\mu k_\nu/m^2)(k^2+m^2)^{-1}$ (the term $k_\mu k_\nu/m^2$ is equal to $\epsilon^0_\mu \epsilon^0_\nu$ for massive vectors). This proof can be found in [10] and has often been used to explain why a renormalizable theory can at the same time be unitary: in one gauge ($\xi = 1$) renormalizability is manifest and in another gauge ($\xi \to \infty$) unitarity is manifest. However, these proofs use gauge-choice independence of the S matrix, which is at least as hard to prove as unitarity, and can only be used for $R(\xi)$ gauges, so they cannot be used for unbroken gauge theories.

Also path integral proofs of the unitarity of renormalizable gauge theories have been given [11]. One starts with only physical degrees of freedom, and first and second class constraints (see the chapter on the Dirac formalism). In this form the theory is manifestly unitary. One identifies then Lagrange multipliers for the constraints with unphysical components of the gauge fields, and after a series of functional manipulations one ends up with a manifestly relativistically invariant theory. Since the starting point was manifestly unitary (it contained only physical p's and q's), also the end result should should be unitary. This is also not a direct proof of unitarity.

We shall give a direct proof of unitarity using the cutting rules which result from the so-called largest-time equation. For a consistent quantum field theory one requires, in addition, that causality holds and that physical states have positive energy and positive norms, and that tachyons (particles with $M^2 < 0$ which move faster than light) are absent. Vector bosons with unphysical polarizations have either positive energy or positive norms but not both. The same applies to the Faddeev-Popov ghosts. We shall choose a vacuum which is Lorentz invariant; then all states in Fock space have positive energy (including ghost states and states with timelike polarizations) but not all states have positive norms. In the Fock space of all states (including unphysical polarizations and ghosts) the S matrix is still unitary as we shall show. Thus the S matrix is unitary **in the space of physical states** at the perturbative level. In fact, we shall turn matters around, and find a **definition of physical states** such that the S matrix is unitary. In the subspace of physical states the inner product is positive definite (hence this subspace is a Hilbert space) and, as we shall prove, the S matrix is unitary in this Hilbert space. In this form the theory is manifestly unitary. We shall only consider field theories whose propagators contain single poles; for these we shall in general prove perturbative unitarity. We do not consider theories with double poles because these "dipole" theories [12] violate the requirements of positive energy or positive norms [8].

The largest-time equation is sufficient to prove unitarity for scalar fields. It is an identity for the integrand F of a given Feynman diagram, and its content is (roughly) that $F - F^*$ equals a sum of "cut diagrams". For a given integrand F of a Feynman diagram, there are in general many more cut diagrams in the largest-time equation than there are intermediate physical states in the unitarity relation. However, by integrating F over the spacetime positions of the vertices, energy conservation at all vertices shows that the extra cut diagrams vanish. The integrated largest-time equation becomes then precisely the unitarity equation.

We present in section 1 the proof of unitarity for scalars in the regularized but not yet renormalized theory. The fundamental property of field theory on which this proof of unitarity rests is the decomposition of the Feynman propagator into a part Δ^+ with positive time and a part Δ^- with negative time. In section 5 we repeat the analysis for the renormalized theory. By requiring that the renormalized S matrix be unitary, one immediately reads off the proper definition of the S matrix, in particular the factors $\sqrt{R}$ where R is the finite residue of the full connected renormalized propagator. Usually these factors are found from the LSZ reduction formalism [13] but perturbative unitarity gives a much clearer (and, incidentally, more rigorous) derivation. On the other hand, the LSZ approach can also be used at the nonperturbative level.

For spin 1/2 fields, which are discussed in section 2, the proof is similar to that for scalar fields because there are no unphysical states in Fock space. Various minus signs occur because fermionic fields anticommute and one must sum over the helicities of those fermion propagators which are cut, but the crucial property on which unitarity rests, namely the decomposition of the Feynman propagator into Δ^+ and Δ^-, still holds.

In section 3 we consider pure (unbroken) Yang-Mills theory. The proof of unitarity proceeds in two steps: first one uses again the largest-time equation for Feynman diagrams where one sums over all intermediate states in Fock space (those with physical and unphysical polarizations, and ghost and antighost states, all with $p^2 = 0$). The Faddeev-Popov ghosts violate the spin-statistics relation, but their propagator decomposes just as the propagator for scalars, and thus also for graphs with ghosts, the largest-time equation holds. At this point one has proven that the S matrix is perturbatively unitary in Fock space. Then one shows by means of Ward identities that the contributions of unphysical polarizations (the longitudinal and timelike parts) cancel those of the Faddeev-Popov ghosts and antighosts. This proves that the S matrix remains unitary in the space of physical states.

The case of spontaneously broken Yang-Mills theory, discussed in section 4, resembles that of pure Yang-Mills theory, but now the longitudinal modes become physical and their role as unphysical modes is taken over by the would-be Goldstone bosons.

The Ward identities are then used to show that the contributions of the remaining unphysical Yang-Mills modes, would-be Goldstone bosons and Faddeev-Popov ghosts and antighosts (all with $p^2 + \xi M^2 = 0$), cancel in the unitarity equation. Note that in both cases the sum of four unphysical degrees of freedom cancels. This is a general feature, and called the quartet mechanism by Kugo and Ojima [14].

In section 5 we finally study the modifications of the proof of unitarity, introduced by the renormalization procedure. We derive BRST transformation laws for renormalized fields, and show that certain selfenergy-like graphs which are sometimes omitted in simplified accounts of cutting rules, are actually needed to prove unitarity. This section is the most technical part of this chapter, but any complete proof of unitarity for the renormalized theory has to confront the issues discussed in this section.

We end this chapter with a proof that the counter terms needed to renormalize field theories, are actually local, and we also prove causality of field theories with Feynman propagators. Key to these proofs is an extension of the largest-time equation, called the two largest times equation.

In the rest of this introduction we make some general remarks concerning unitarity, and we discuss antipropagators.

In all cases we begin by writing the S-matrix in the interaction picture as

$$S = \mathcal{T} e^{i \int \mathcal{L}_{int}(x) d^4 x} \tag{5.0.1}$$

where $\mathcal{T}$ is the time-ordering symbol. (Strictly speaking we use the $\mathcal{T}^\star$ symbol because we have replaced $\mathcal{H}_{int}$ by $-\mathcal{L}_{int}$ and use Lorentz-covariant Feynman rules to construct amplitudes). That the S-matrix should be unitary without subtleties for scalar fields is already clear from (5.0.1), since $S^\dagger S = I$ holds as long as $\mathcal{L}_{int}$ is hermitian, even with $\mathcal{T}$ present.[2] However, a direct proof at the level of Feynman diagrams gives

[2]It is instructive to check this to second order in $\mathcal{L}_{int}$. One gets terms with $\theta(x^0 - y^0)$ and

considerable further insight in the way unitarity is achieved; moreover, our direct analysis will reveal that each given diagram separately satisfies the unitarity relation, and for each cut separately.

By expanding the exponent in (5.0.1) and taking matrix elements between *physical states* we obtain the S matrix expressed in terms of Feynman diagrams. It is customary to decompose S into a part that does nothing to the eigenstates of the free Hamiltonian, and a part that is due to interactions between the asymptotic states

$$S = I + iT. \tag{5.0.2}$$

Usually one splits off a delta function for energy-momentum conservation, but since evaluation of the Feynman diagrams automatically produces this delta function we do not write it explicitly but leave it in the definition of T.

Unitarity of the S matrix means that

$$\sum_n{}' < f|S|n >< n|S^\dagger|i >= \sum_n{}' < f|S^\dagger|n >< n|S|i >=< f|i > \tag{5.0.3}$$

where the prime next to the summation symbol indicates that one should only sum over physical states. In the interaction picture, these physical states are eigenstates of the free Hamiltonian. Taking physical intermediate states $\mid n >$, one identifies $< n|S|i >$ again as an S-matrix element in the interaction picture. Later we shall

$\theta(y^0 - x^0)$ from S and $S^\dagger$, while the cross term contains no θ's. The θ's combine to unity, and the sum of all terms cancels after integration over spacetime.

$$I - i\mathcal{L}(x^0) + i\mathcal{L}(y^0) + \left(-\frac{1}{2}\theta(x^0 - y^0)\mathcal{L}(y^0)\mathcal{L}(x^0) - \frac{1}{2}\theta(y^0 - x^0)\mathcal{L}(x^0)\mathcal{L}(y^0)\right)$$
$$+(-i\mathcal{L}(x^0))(i\mathcal{L}(y^0)) + \left(-\frac{1}{2}\theta(x^0 - y^0)\mathcal{L}(x^0)\mathcal{L}(y^0) - \frac{1}{2}\theta(y^0 - x^0)\mathcal{L}(y^0)\mathcal{L}(x^0)\right) = I$$

It is amusing but tedious to repeat this analysis to third order. There are now 6 terms in the third-order terms of S, each with two θ-terms. In the product $S^\dagger S$ one looks at all terms with 3 $\mathcal{L}$'s. The exercise is then to convert single θ's due to cross terms into two-theta terms as occur in the third-order terms of S (and $S^\dagger$). Using the identity $\theta_{12} - \theta_{13} = \theta_{12}\theta_{31} - \theta_{13}\theta_{21}$ one indeed finds that $S^\dagger S = I$ to third order.

discuss the definition of physical states in general; in practice, one uses Feynman-like gauges and then the physical states for gauge fields correspond to transversal polarizations. Always, the momenta of physical states satisfy $p^2 = 0$, or $p^2 + M^2 = 0$. In terms of the T matrix, the unitarity condition becomes

$$< f \mid iT \mid i > + < f \mid (iT)^\dagger \mid i >= -\sum_n{}' < f \mid (iT)^\dagger \mid n >< n \mid (iT) \mid i > \quad (5.0.4)$$

where $|i>$ and $<f|$ are the initial and final states, both physical.

To evaluate the matrix elements of $T^\dagger$ one might begin by using that $< n|T^\dagger|i >= < i|T|n >^*$, and then proceed by bringing $< i|$ to $|i >$ by using time-reversal invariance. However, one expects also to need parity and charge-conjugation invariance, since for scalar particles unitarity follows from the hermiticity of $\mathcal{L}_{int}$ as we shall show, which suggests that CPT invariance should be sufficient. In fact, one would expect that theories which violate time-reversal invariance but which are still CPT invariant, can be unitary. Rather than using C, P and T transformations to bring $|i >$ to $< i|$ and $< f|$ to $|f >$, we want to proceed in a simpler way, by keeping the states $< n|$ and $|i >$ fixed. This we can indeed achieve as follows.

To evaluate $< n|T^\dagger|i >$ (and $< f|T^\dagger|i >$) we use that

$$\begin{aligned} < n|S^\dagger|i > &= < n|[\mathcal{T} e^{i\int \mathcal{L}_{int}(x) d^4x}]^\dagger|i > \\ &= < n| \sum_{m=0}^{\infty} \frac{(-i)^m}{m!} \int d^4x_1 \dots d^4x_m [\mathcal{T}\mathcal{L}_{int}(x_1)\dots\mathcal{L}_{int}(x_m)]^\dagger|i > \end{aligned} \quad (5.0.5)$$

Hence, for each vertex one gets an extra minus sign, and further the operators $\mathcal{L}_{int}(x_j)$ are anti-chronologically ordered. To spell this out in detail: $\mathcal{T}$ leads to chronological (time) ordering, but hermitian conjugation reverses the ordering and leads thus to anti-chronological ordering. It is useful to introduce an anti-chronological ordering symbol $\mathcal{T}^\dagger$. Then $[\mathcal{T}\mathcal{L}_{int}(x_1)\dots\mathcal{L}_{int}(x_m)]^\dagger$ is equal to $\mathcal{T}^\dagger\mathcal{L}_{int}(x_m)\dots\mathcal{L}_{int}(x_1)$ and this is also equal to $\mathcal{T}^\dagger\mathcal{L}_{int}(x_1)\dots\mathcal{L}_{int}(x_m)$.

We should stress at this point that we have assumed that $\mathcal{L}_{int}$ is hermitian. If it were not hermitian, we would get the anti-chronological ordering of ${\mathcal{L}_{int}}^\dagger$, and the vertices in $< n|T^\dagger|i >$ would be different from those in $< f|T|n >$. In such a case the relation (5.0.4) would not hold in general since on the left-hand side one would find vertices from $\mathcal{L}$ **or** $\mathcal{L}^\dagger$ while on the right-hand side one would find vertices of $\mathcal{L}$ **times** vertices of $\mathcal{L}^\dagger$. We shall therefore require $\mathcal{L}$ to be hermitian.[3] For this reason we take the ghost fields as hermitian and the antighosts as anti-hermitian, because then the ghost action is hermitian. Note that hermiticity of the ghost action is not a sufficient condition; for example, multiplying the bcA_μ interaction term by a factor α, the ghost action remains hermitian but unitarity is violated (because BRST symmetry is violated, so the BRST Ward identities by which one proves unitarity, are violated).

We define normal ordering as usual, and can then prove the Wick expansion theorems for $T^\dagger$. We recall the relation $AB = N(AB) + < 0|AB|0 >$ where N denotes normal ordering and A and B are two hermitian fields whose (anti)commutator is a c-number. We define the $\mathcal{T}^\dagger$ operation on two fields by

$$\mathcal{T}^\dagger A(x_1)B(x_2) \equiv \pm\theta(x_1^0 - x_2^0)B(x_2)A(x_1) + \theta(x_2^0 - x_1^0)A(x_1)B(x_2) \qquad (5.0.6)$$

where the minus sign is needed if both A and B are anticommuting. One may then prove the following identity

$$\mathcal{T}^\dagger A(x_1)B(x_2) = N(A(x_1)B(x_2)) + < 0|\mathcal{T}^\dagger A(x_1)B(x_2)|0 > \qquad (5.0.7)$$

[3]There is an exception: one can use complex gauge fixing terms, and corresponding complex ghost actions, and still satisfy unitarity. [15] The reason unitarity is not violated in this case is that the S-matrix is gauge-choice independent. In the unitary equation one finds one gauge-fixing term to the left of the cut, and the complex conjugate of this gauge-fixing term to the right of the cut. Unitarity can then still be proven because in cut graphs one may replace the complex conjugated gauge fixing term by the original gauge fixing term (because a cut diagram is the product of two S matrix elements). Choosing suitable complex gauge fixing terms, one can cancel some terms of the classical gauge action, and this can be used to simplify loop calculations.

The last term defines the antipropagator.[4] In this way the matrix elements of $T^\dagger$ are calculated as usual (with the Wick contraction rules), but with antipropagators instead of propagators, and with an extra minus sign for each vertex.

For fermions one could have defined anti-time ordering with an extra overall minus sign. With our present definition, this minus sign is absent and $\mathcal{T}^\dagger(AB) = [\mathcal{T}(BA)]^\dagger$ for hermitian operators $A(x)$ and $B(y)$. All results for bosons generalize straightforwardly to fermions without extra signs.

For real scalars one has

$$\mathcal{T}^\dagger\varphi(x)\varphi(y) = \theta(y^0 - x^0)\varphi(x)\varphi(y) + \theta(x^0 - y^0)\varphi(y)\varphi(x) \tag{5.0.8}$$

The anti-propagator is then given by

$$< 0|\mathcal{T}^\dagger\varphi(x)\varphi(y)|0 >= \int \frac{d^3k}{(2\pi)^3}\left[\frac{e^{ik(x-y)}}{2\omega}\theta(y^0 - x^0) + \frac{e^{ik(y-x)}}{2\omega}\theta(x^0 - y^0)\right] \tag{5.0.9}$$

where $kx = \vec{k}\cdot\vec{x} - \omega t$ and $\omega^2 = (\vec{k})^2$. For ordinary propagators, one chooses the k_0 contour for the first term in the upper half plane, and in the lower half plane for the second term. For the anti-propagators one must close the contour in the opposite way (in the lower half plane for the first term, and in the upper half plane for the second term). This yields then the following relativistically invariant expression

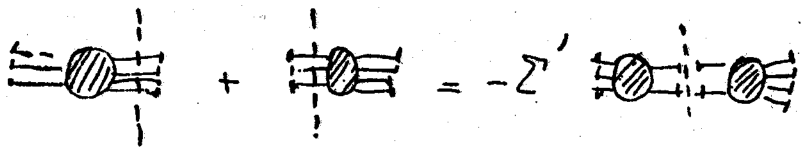
$$< 0|\mathcal{T}^\dagger\varphi(x)\varphi(y)|0 >= \frac{1}{(2\pi)^4}\int d^4k \frac{+i}{k^2 + M^2 + i\epsilon} e^{i(\vec{k}\cdot(\vec{x}-\vec{y})+k_0(x^0-y^0))} \tag{5.0.10}$$

where the symbol d^4k denotes d^3kdk_0. (We use the convention that $k^2 = \vec{k}^2 - k_0^2$). Hence, antipropagators have $+i\epsilon$ and $+i$ where propagators have of $-i\epsilon$ and $-i$. Since the propagator and the antipropagator with $x - y$ are equal to those with $y - x$ as the change $k \to -k$ of the integration variable k shows, the antipropagators for

[4]The proof is the same as for the usual case with time ordering. On the left-hand side one finds terms with $aa, aa^\dagger, a^\dagger a$ and $a^\dagger a^\dagger$, where a and $a^\dagger$ denote annihilation and creation operators. Only the terms with $aa^\dagger$ are not normal-ordered. Writing these terms in normal-ordered forms as $a^\dagger a$ on the right-hand side, one obtains compensating terms proportional to the c-numbers $[a, a^\dagger]$. These yield the propagator.

scalar fields are the complex conjugate of the propagators. This is also clear from (5.0.9).

The condition in (5.0.4) for the S-matrix to be unitary can graphically be written as follows:



(5.0.11)

where to the left of the cut one should use ordinary Feynman rules, while to the right of the cut one uses anti-Feynman rules. The intermediate states $|n><n|$ have small transversal bars on the lines to indicate that these states are physical (on-shell with physical polarization vectors, and $p^2+M^2=0$ or $p^2=0$). As is well-known, summing the product of the wave-functions for emission and absorption of a scalar over the 3-momenta $\vec{k}$ produces a factor

$$\int \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega}e^{\pm ik(x-y)} = \Delta^{\pm}(x-y) \tag{5.0.12}$$

Since the Feynman propagator is given by

$$\Delta_F(x-y) = \theta(x^0-y^0)\Delta^+(x-y) + \theta(y^0-x^0)\Delta^-(x-y) \tag{5.0.13}$$

while from (5.0.9) the anti-Feynman propagator is given by

$$\Delta_F^*(x-y) = \theta(y^0-x^0)\Delta^+(x-y) + \theta(x^0-y^0)\Delta^-(x-y) \tag{5.0.14}$$

(which is the complex conjugate of Δ_F because $(\Delta^+)^* = \Delta^-$) we are led to consider "cut-diagrams", i.e., diagrams where in one part one uses Δ_F, in another part Δ_F^*, while for the cut lines one uses Δ^+ or Δ^-. For such diagrams, there exists an identity, the so-called largest-time equation, from which unitarity follows. We shall derive this identity shortly. [5]

[5]Incidentally, we should really consider disconnected diagrams as well, but our analysis and hence unitarity holds for each connected part. Alternatively one could define the T matrix by $S = \exp iT$. This T automatically picks out the connected graphs.

More precisely the unitarity relation $\sum_n < f \mid S^\dagger \mid n >< n \mid S \mid i >=< f \mid i >$ leads to diagrams where the incoming particles are incoming from the left and one uses ordinary propagators in the left-hand part of the cut diagram but "antipropagators" in the right-hand side of the cut diagram. On the other hand, the relation $\sum_n < f \mid S \mid n >< n \mid S^\dagger \mid i >=< f \mid i >$ corresponds to a cut diagram where the incoming particles are incoming from the right-hand side if one still uses ordinary propagators in the left-hand part of the cut diagram. We shall mainly discuss the former case; the latter case follows the same steps. At the perturbative level the relation $S^\dagger S = I$ implies $SS^\dagger = I$.

Unitarity boils down to identities between diagrams which follow from the basic decomposition of Δ_F into a term with Δ^+ and a term with Δ^-. No decomposition of Δ_F into a Dirac delta part and a principal value part will be needed, and the whole proof of unitarity is purely algebraic. We have not worked out the details of cutting rules for nonplanar diagrams, the consistency of cutting rules when infrared divergences are present, or the application of dimensional regularization to cutting rules ($\int d^4k$ becomes $\int d^n k$ but $\int \frac{d^3k}{2\omega}$ becomes a $n-1$ dimensional integral). These are research topics the reader may want to tackle.

1 The largest-time equation:unitarity for scalars

Consider N points $x_1, \ldots x_N$ in Minkowski spacetime, connected by lines and multiplied by some real coupling constants which we omit writing. The corresponding Feynman integrand $F(x_i)$ is then by definition the product of a factor i for each of the N vertices, and a factor $\Delta_F(x_i - x_j)$ for each line joining x_i and x_j. (The factor i comes from the expansion of the time ordered product $\exp iS_{int}$, which yields the S-matrix in perturbation theory). Since $\Delta_F(-x) = \Delta_F(x)$, one does not have to specify whether a line connecting x_i and x_j runs from x_i to x_j, or from x_j to x_i. For that reason we said that the line "joins" x_i and x_j. We shall later integrate over

x_i, and add external factors for the emission or absorption of particles, but now we consider $F(x_i)$ with fixed x_i. The Feynman propagator for a scalar field is given by

$$\Delta_F(x-y) = \theta(x^0 - y^0)\Delta^+(x-y) + \theta(y^0 - x^0)\Delta^-(x-y) \tag{5.1.1}$$

where

$$\begin{aligned} \Delta^\pm(x) &= \sum_{\vec{k}} \frac{1}{V}\,\frac{1}{2\omega} e^{\pm ikx} \text{ (with } \omega = (\vec{k}^{\,2} + m^2)^{1/2}) \\ &= \int \frac{d^4k}{(2\pi)^4} 2\pi\theta(\mp k_0)\delta(k^2+m^2)e^{\pm ikx} \end{aligned} \tag{5.1.2}$$

Usually one writes the Feynman propagator as a contour integral

$$\Delta_F(x) = \int \frac{d^4k}{(2\pi)^4}\frac{-i}{k^2+m^2-i\epsilon}e^{ikx} \tag{5.1.3}$$

but in this section we shall rather use the expression (5.1.1) in terms of $\Delta^+(x)$ and $\Delta^-(x)$. We need a few properties of these invariant Δ functions.

$$\begin{aligned} \Delta^+(x)^* &= \Delta^-(x)\ ;\ \Delta^+(-x) = \Delta^-(x) \\ \Delta_F(x)^* &= \theta(x^0)\Delta^-(x) + \theta(-x^0)\Delta^+(x) \end{aligned} \tag{5.1.4}$$

These properties follow from the definitions of $\Delta_F, \Delta^+, \Delta^-$. (For massless fields one often uses the symbol D instead of Δ, so D_F, D^+ and D^-).

Given an N-vertex amplitude $F(x_i)$, one can construct $2^N - 1$ other amplitudes ("cut graphs") by the following recipe

(i) draw circles around some (or all) vertices,

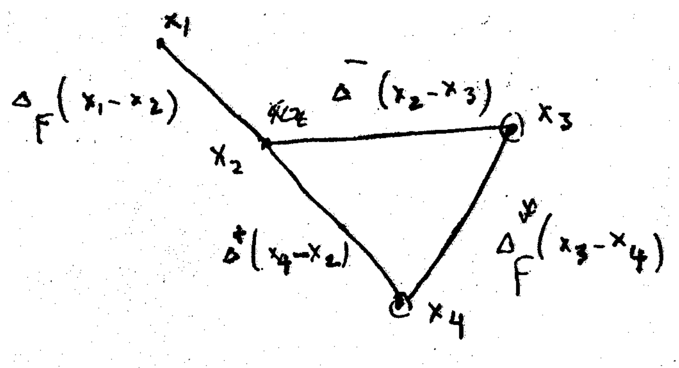
(ii) replace in the circled vertices the factor i by $-i$ (circling will be related to complex conjugation),

(iii) write $\Delta_F(x_i - x_j)$ for lines connecting an uncircled x_i to uncircled x_j.

$\Delta^+(x_i - x_j)$	a circled x_i to uncircled x_j.
$\Delta^-(x_i - x_j)$	an uncircled x_i to circled x_j.
${\Delta_F}^*(x_i - x_j)$	a circled x_i to circled x_j.

Note that we do not write $\Delta^+(x_i - x_j)\theta(x_i^0 - x_j^0)$ or $\Delta^-(x_i - x_j)\theta(x_j^0 - x_i^0)$; therefore our recipe does not correspond to simply decomposing the Feynman propagators in the original diagram in terms of $\Delta^+\theta$ and $\Delta^-\theta$ terms.

The following example illustrates these rules



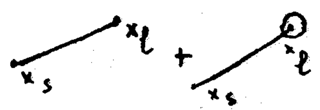
(5.1.5)

These rules are consistent with the relation $\Delta^+(x) = \Delta^-(-x)$. For example, instead of $\Delta^+(x_4 - x_2)$ we could have written $\Delta^-(x_2 - x_4)$. The reason we write $\Delta^+(x_i - x_j)$ when x_i is circled but x_j is not, instead of writing $\Delta^-(x_i - x_j)$, is that with this rule the fundamental largest-time relation in (5.1.6) holds. If we had chosen $\Delta^-(x_i - x_j)$, we would have obtained the smallest-time equation (which we could equally well have used to prove unitarity).

The operation of putting circles around vertices is related to complex conjugation as follows. Given a graph with some vertices encircled, consider a new graph by removing all existing circles and putting new circles around all vertices which were previously uncircled. Then the corresponding amplitudes are each other's complex conjugates. This follows from $\Delta^+(x)^* = \Delta^-(x)$. In particular, the graph with all vertices circled is the complex conjugate of $F(x_i)$. We shall now derive an equation for $F(x_i) + F(x_i)^*$. (Later we shall relate this to ImT where $S = I + iT$). These graphs are not yet the full S-matrix amplitude. To get the full S-matrix amplitude one must multiply certain vertices with factors $(2\pi)^{-3/2}(2\omega)^{-1/2}e^{\pm ikx}$ for absorption or emission of the scalars in $|i>$ and $<f|$ and integrate over all points x_i. So far we have not yet incorporated in our rules the information of which particles come in or

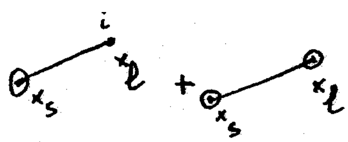
go out.

Consider a graph, and select two points: the point x_ℓ (ℓ for largest) which has the largest time component of all points, and any other point x_s (s for smaller). Then one has always ${x_\ell}^0 > {x_s}^0$. Of course, if one boosts to another Lorentz frame, another point may have in that frame the largest time equation, and then one can start with that point and prove unitarity. So in any frame one can prove unitarity. Consider now a given graph with x_ℓ not circled and the same graph but with x_ℓ circled. We claim that the sum of these graphs cancels, irrespective of whether x_s is circled or not. To prove this, we consider separately the cases that x_s is circled or not circled. If x_s is not circled one has



$$= 0 \text{ since } \Delta_F(x_\ell - x_s) = \Delta^+(x_\ell - x_s) \text{ if } {x_\ell}^0 > {x_s}^0. \quad (5.1.6)$$

If x_s is circled, one finds



$$= 0 \text{ since } \Delta^-(x_\ell - x_s) = {\Delta_F}^*(x_\ell - x_s) \text{ if } {x_\ell}^0 > {x_s}^0 \quad (5.1.7)$$

The reason the graph with x_ℓ circled and the graph with x_ℓ not circled sum up to zero is that all propagators are equal but the vertices of x_ℓ differ by a minus sign. The equality holds for all x_s with ${x_\ell}^0 > {x_s}^0$, hence for all points $x_i \neq x_\ell$ since x_ℓ has the largest time. This does not cover the cases that the times of x_ℓ and another point coincide, but this set of configurations has measure zero, and we shall assume that we may safely disregard these cases.[6] (One can check this in simple examples).

Hence: for any graph with x_ℓ not circled there is another graph with x_ℓ circled, and their sum cancels. (If x_ℓ is connected to more than one other point, the cancellation still holds, since all propagators and all other vertices are the same while the circle at the largest-time vertex gives one minus sign.)

[6] Feynman graphs involve, of course, products of distributions, but a consistent scheme of perturbation theory exists after making a Wick rotation. The short distance singularities for $x_i \to x_j$ can be handled in this approach.

It follows that the sum of all 2^N graphs, with circles in all possible ways, combines into 2^{N-1} pairs such that the two graphs of a given pair sum up to zero. Writing the graphs with none and with N circles on the left-hand side we find the largest-time equation

$$F(x_i) + F(x_i)^* = -\underline{F}(x_i) \tag{5.1.8}$$

Here $\underline{F}(x_i)$ denotes the set of $2^N - 2$ graphs with one up to and including $N-1$ circles. (The graph without any circles gives $F(x_i)$ and the graph with N circles gives $F(x_i)^*$). We recall that $F(x)$ is the original Feynman diagram, $F(x)^*$ the same diagram but with antipropagators (and minus signs at the vertices), and $\underline{F}(x)$ will be related to the graphs with intermediate states in the unitarity relation.

The S-matrix is obtained by multiplying $F(x_i)$ by plane waves e^{ipx} for absorption and emission of particles, and integrating over x^i. However, **these factors and integrations are the same for all diagrams** because the states $|i>$ and $<f|$ in (5.0.4) are always the same, only the operator T is replaced by $T^\dagger$. The $\dagger$ and cutting only refer to the parts $[k^2+m^2-i\epsilon]^{-1}$ and $[2\omega]^{-1}$ of the propagators. We shall see that (5.1.8) is not yet precisely the unitarity equation because there are more terms on the right-hand side of (5.1.8) than there can be intermediate states $|n>$. We shall show that these extra terms on the right-hand side of (5.1.8) cancel after integration over x_i as a consequence of energy conservation. First we discuss how dispersion relations follow from the largest time equation. Readers who prefer to go on with the discussion of unitarity may skip to below (5.1.18).

Often the unitarity relation is written as $ImT = TT^\dagger$. To derive this relation, we note that Feynman diagrams for the S-matrix are obtained by multiplying $F(x_i)$ with the wave functions (plane waves) for the incoming and outgoing particles, and integrating over all x_i. Recalling $S = I + iT$, we thus have

$$T(p_j, q_k) = \frac{1}{i}\int (Dx) F(x_i) \tag{5.1.9}$$

where

$$(Dx) = \left(\Pi_{i=1}^{N} d^4 x_i\right)\left(\Pi_{j=1}^{P} \frac{e^{ip_j x_{m_j}}}{\sqrt{2\omega_j V}}\right)\left(\Pi_{k=1}^{Q} \frac{e^{-iq_k x_{m_k}}}{\sqrt{2\omega_k V}}\right) \tag{5.1.10}$$

Here the momenta p_j are coming in at the vertices x_{m_j}, and the momenta q_k are going out from the vertices x_{m_k}. Now, $F(x_i)$ has an obvious but important symmetry

$$F(-x_i) = F(x_i) \tag{5.1.11}$$

This follows from $\Delta_F(-x) = \Delta_F(x)$. (Note that we are still only dealing with scalars). By changing integration variables from x_i to $-x_i$ we then find

$$T(p_j, q_k) = \frac{1}{i}\int (Dx)^* F(x_i) \tag{5.1.12}$$

By subtracting the imaginary parts of (5.1.9) and (5.1.12) we find

$$0 = \int (ImDx) ImF \tag{5.1.13}$$

while adding the imaginary parts yields

$$ImT = -\int (ReDx) ReF \tag{5.1.14}$$

On the other hand, subtracting the real parts of (5.1.9) and (5.1.12) gives

$$0 = i\int (ImDx) ReF \tag{5.1.15}$$

Subtracting the last two results we obtain

$$ImT = -\int (Dx) ReF \tag{5.1.16}$$

For ReF we have found in (5.1.8) an expression given by the largest-time equation: $2ReF = -\underline{F}(x_i)$. Hence

$$\begin{aligned} 2ImT &= \int (Dx)\underline{F}(x_i) \\ &= \sum_n i < f|T|n >< n|T^\dagger|i > \end{aligned} \tag{5.1.17}$$

Using the unitarity equation in (5.0.4) for the right-hand side we see that $2ImT$ is indeed equal to $T - T^\dagger$

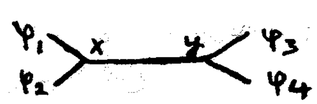
$$\begin{aligned} 2Im < f|T|i > &= < f|T|i > - < f|T^\dagger|i > \\ &= \sum_n i < f|T^\dagger|n >< n|T|i > \end{aligned} \tag{5.1.18}$$

This relation forms the starting point for dispersion relations. A very clear exposition can be found in [6]. We shall not pursue this subject any further, and return to unitarity.

As a simple example of the largest time equation we consider the Born graph for the scattering of two scalar fields φ_1 and φ_2 into fields φ_3 and φ_4 with an intermediary field φ. We use this example for pedagogical reasons although there is the complication that if φ can occur as a physical intermediate state, it is unstable and hence it cannot be one of the incoming or outgoing particles at $t = \pm\infty$. There exist discussions extending the concept of unitarity for scalar particles to unstable particles, but we shall not venture into these more complicated areas. [16] In perturbation theory based on the interaction picture, there are no problems: one need not consider complex masses.

Consider the action with $\mathcal{L}(int) = g_1\varphi\varphi_1\varphi_2 + g_2\varphi\varphi_3\varphi_4$. The tree graph S-matrix element reads

$$S = \int ig_1\varphi_1(x)\varphi_2(x) \int \frac{d^4k}{(2\pi)^4} \frac{-i}{k^2 + m^2 - i\epsilon} e^{ik(x-y)} \int ig_2\varphi_3(y)\varphi_4(y) d^4x d^4y \tag{5.1.19}$$



The corresponding amplitude $F(x_i)$ is thus

$$F(x_i) = ig_1 \int \frac{d^4k}{(2\pi)^4} \frac{-i}{k^2 + m^2 - i\epsilon} e^{ik(x-y)}\, ig_2 = ig_1\Delta_F(x-y)ig_2 \tag{5.1.20}$$

The largest time equation for this graph reads

$$i\Delta_F(x-y)i + (-i)\Delta_F{}^*(x-y)(-i) = -i\Delta^-(x-y)(-i) - (-i)\Delta^+(x-y)i \tag{5.1.21}$$

Expressing Δ_F and $\Delta_F{}^*$ into Δ^+ and Δ^-, this equation is readily verified. (Alternatively, we can extract an integration over $d^3k/(2\pi)^3$ and find then

$$\int \frac{ie^{ik_0(x^0-y^0)}}{\vec{k}^2 - k_0{}^2 - i\epsilon}\,\frac{dk_0}{2\pi} + \int \frac{-ie^{ik_0(x^0-y^0)}}{\vec{k}^2 - k_0{}^2 + i\epsilon}\,\frac{dk_0}{2\pi} = -\frac{e^{+i\omega(x^0-y^0)}}{2\omega} - \frac{e^{-i\omega(x^0-y^0)}}{2\omega} \tag{5.1.22}$$

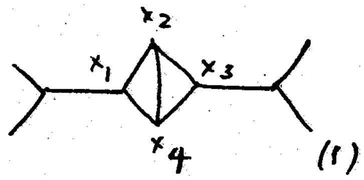
where $\omega = -k_0$. The proof of this equation rests on the identity

$$\int dk_0 f(k_0)\frac{1}{\vec{k}^2 - k_0{}^2 - i\epsilon} = P\int dk_0 f(k_0)\frac{1}{k^2 - k_0{}^2} + i\pi\delta(\vec{k}^2 - k_0{}^2)f(k_0) \tag{5.1.23}$$

where P denotes the principal value. The two principal value contributions cancel, and the two contributions from the delta function add up, yielding a simple delta function which has two zeros and indeed produces the right-hand side of the largest-time equation.)

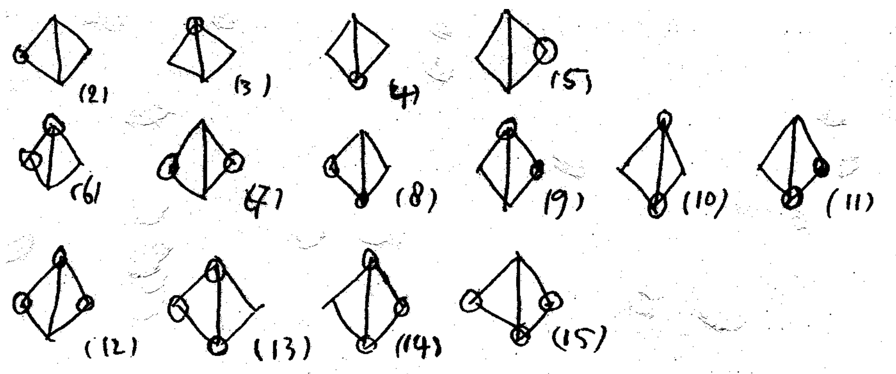
If we disregard for a moment the instability of the φ particles, we can go on and see how unitarity is achieved. Integration of $F(x_i)$ over the points x_i then produces energy-momentum conservation at each vertex. Physically we expect that a sum over intermediate scalars yields **one** factor $\int d^3k(2\omega)^{-1}$, not two such factors. Namely, either the virtual particle is emitted from the left or from the right, but not both. At first sight this seems to be in disagreement with the largest-time equation where we found two such factors (corresponding to Δ^+ and Δ^- in (5.1.21)). However, due to the delta functions for energy-momentum conservation, one of the terms corresponding to Δ^+ or Δ^- vanishes. For example, if energy flows in from the left then the term with $\Delta^+(x-y)$ vanishes after integration over the x_i since $\Delta^+(x-y)$ has a factor $\exp -i\omega(x^0 - y^0)$ which describes energy flow from y^0 to x^0. Conversely, if energy flows in from the right, then $\Delta^-(x-y)$ does not contribute. Thus energy conservation ensures that (5.1.8) agrees with the unitarity relation. We now give a really consistent example of unitarity, namely an example without unstable particles.

The example is the scattering of two scalar particles, with a two-loop selfenergy correction.



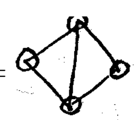
(5.1.24)

To simplify the notation, we shall only write down the proper selfenergy itself and omit the external lines. We label this graph by (1), to distinguish it from the following 14 graphs in $\underline{F}(x_i)$



(5.1.25)

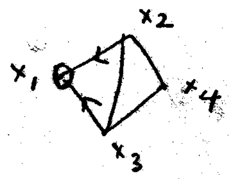
To this set we can add the 16th graph

$F(x_i)^* =$  (16) (5.1.26)

Since $\Delta^+(x-y)$ contains a factor $\exp ip_0(x^0-y^0) = \exp -ip^0(x^0-y^0)$ and p^0 is positive, $p^0 = \omega$, the energy p^0 flows from y to x.[7] So we give the line corresponding to $\Delta^+(x_i - x_j)$ an arrow, pointing from x_j to x_i. Similarly we give a line with $\Delta^-(x_i - x_j)$ an arrow pointing from x_i to x_j. Lines with $\Delta_F(x_i - x_j)$ or $\Delta_F(x_i - x_j)^*$ do not get arrows, because energy flows on these lines in both directions (there are terms in Δ_F

[7]For a particle energy flows in one direction, and for an antiparticle in the opposite direction, but for physical particles (corresponding to cut propagators) energy cannot flow in both directions.

and $\Delta_F{}^*$ with $\theta(p_0)$ and $\theta(-p_0)$). For graph (2) we then get

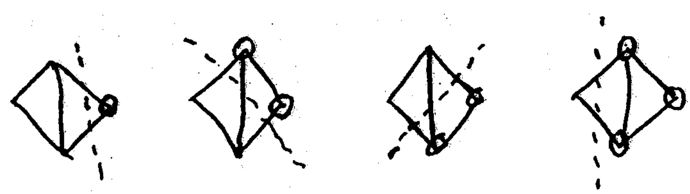


(5.1.27)

In general, the arrows go from an uncircled to a circled vertex. If energy flows in from the left into graph (2), this graph vanishes, since at vertex x_1 all three energies are flowing in. However, if energy were to flow out from x_1, graph (2) would be nonvanishing, but graph (5) would then vanish. Graph (3) is always vanishing.

(5.1.28)

Draw now a continuous line (a "cut") through all lines with an arrow, but not through lines without an arrow. Then only those graphs are nonvanishing where energy flows from one side of the cut to the other side, and all vertices on one side of the cut carry circles. This part of the diagram is thus complex-conjugated. When energy flows in from the left, the nonvanishing cut graphs in $\underline{F}(x_i)$ are



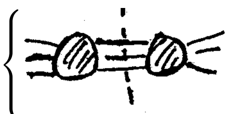
(5.1.29)

and energy flows from left to right across the cuts. If energy were to flow from right to left (thus incoming at vertex x_2) then the same cut graphs would be found, but now they would correspond to graphs (2), (6), (8) and (13).

It is now clear that if we consider the Feynman graph in (5.1.24) with two incoming particles and two outgoing particles, we do not get cuts through the lines which connect the selfenergy to the external particles if all particles have the same mass (then all particles are stable, and one particle cannot decay into two other particles).

The unitarity equation (V.4) requires that the sum over all physical states in the unitarity equation be equal to the sum over all cuts

$$\sum_n{}' < f|T^\dagger|n >< n|T|i >= \sum_{\text{all cuts}} i \left\{ \quad \right\}_f \tag{5.1.30}$$



(The minus signs in (V.4) and (5.1.8) cancel. Furthermore, since there are no unphysical states in scalar field theory, one can drop the prime in this equation.) States $|n>$ with two intermediate particles on the left correspond to two-particle cuts and states $|n>$ with three intermediate particles correspond to three-particle cuts. It is clear that the right-hand sides of the cut diagrams are all complex conjugated, as required by the factor $< f|T^\dagger|n >$ in the unitarity relation, because all vertices are circled and hence complex-conjugated.

If $|n>$ stands for a scalar particle which is absorbed by $T^\dagger$ in $< f|T^\dagger|n >$ at a point x and emitted by T in $< n|T|i >$ at a point y we find wave functions $e^{ipx}[(2\pi)^3 2\omega]^{-1/2}$ and $e^{-ipy}[(2\pi)^3 2\omega]^{-1/2}$ (note that $T^\dagger$ produces antichronological time ordering only for the vertices in $< n|T^\dagger|i >$ but does not act on the external wave functions). The summation over all its 3-momenta yields then

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega} e^{ip(x-y)} \tag{5.1.31}$$

In the cut graph one finds the same factor for the line joining x and y because this factor is $\Delta^+(x-y)$. In a similar way, one may show that all other intermediate particles in $|n>$ correspond to cut lines. Thus, each term on the left of the unitarity equation is equal to precisely one term on the right-hand side, and thus the unitarity equation is satisfied.

When one considers the relation $SS^\dagger = I$ instead of $S^\dagger S = I$, one obtains the expression $< f|T|n >< n|T^\dagger|i >$. Now the circled vertices lie in the left-hand side of the diagram, namely near the incoming state. One obtains the same factor (5.1.31)[8]

[8]The reason one obtains the same factors e^{ikx} and e^{-iky} is that $\mathcal{T}^\dagger$ only changes the ordering of

but because now y is circled while x is not, one gets $\Delta^-(x-y)$ instead of $\Delta^+(x-y)$. It is clear that one should reverse the roles of Δ^+ and Δ^- in the recipe below (5.1.4) when one considers the case $SS^\dagger = I$. We shall continue with the case $S^\dagger S = I$ and the rules as we have formulated them. Readers who want to work with the case $SS^\dagger = I$ should interchange everywhere Δ^+ with Δ^-.

We now generalize from these two examples to the general case. Given a Feynman diagram, one first writes down the largest-time equation for it. Then one draws arrows indicating the flow of energy, and deletes the diagrams which violate energy conservation. The remaining set of cut diagrams correspond in 1-1 fashion to the possible intermediate states in the unitarity relation, and unitarity is proven for this diagram. We have thus shown that for scalar fields, unitarity of the S matrix in terms of Feynman diagrams follows directly from the largest-time equation. Note that the largest-time equation holds actually *diagram by diagram*. This is as expected, since different diagrams will in general depend on the external and cut momenta in a different way, and hence one can distinguish diagrams by varying the external and cut momenta. Each term in $\underline{F}(x_i)$ corresponds to a particular cut of the diagram, and the total set of states $|n>$ follows by cutting a *given* diagram in all possible ways. This is of course more than is needed because unitarity need only hold for the sum of all graphs of a given order in the coupling constant.

We now extend the discussion to the case when regularization is taken into account. Suppose one has a set of Pauli-Villars regulators with coupling constants e_i and masses M_i. When the propagator contains the wrong sign in the numerator, one finds $-|n><n|$ in the cut graphs instead of $|n><n|$. There would then seem to be a ghost in the theory. However, since the energy which flows across the cut in a given diagram must be larger than the regulator mass if a regulator line has been cut

the interaction terms $\mathcal{L}(x_j)$, but the rules for emission and absorption are unchanged. One can also see this from $<n \mid T^\dagger \mid i>=<i \mid T \mid n>^*$: in one case n is emitted and in the other case absorbed, but complex conjugation undoes this difference.

(the total incoming energy must be equal to the total intermediate energy which is larger than the regulator mass), no cuts through regulators need be considered since we always take the limit $M_i \to \infty$, keeping the total incoming (= outgoing) energy fixed. A similar effect occurs if one uses the higher-derivative regularization scheme with propagators like $(k^2 + m^2 + k^4/M_i{}^2)^{-1}$ which can be written as the difference

$$\left(k^2 + m^2 + \mathcal{O}\frac{1}{M_i^2}\right)^{-1} - (k^2 + M_i^2 + \mathcal{O}(1))^{-1}. \qquad (5.1.32)$$

In the interaction picture both propagators have the same $\int d^n k \exp ikx$, and cuts through the second propagator will not contribute when E (incoming) $< M_i$.

If one uses dimensional regularization, the Feynman propagator in n dimensions is still a sum of Δ^+ and Δ^-. Thus in dimensional regularization, unitarity is of scalar field theory preserved at the regularized level.

This concludes the proof of unitarity for scalar particles with propagators $\Delta_F(x)$. We shall now extend this proof to spin 1/2 and spin 1 fields.

2 Unitarity for spin 1/2 fields

The proof of unitarity for scalars which we gave before, can straightforwardly be extended to spin 1/2 fields. As always, unitarity means

$$< f \mid iT \mid i > + < f \mid (iT)^\dagger \mid i > = -\sum < f \mid (iT)^\dagger \mid n >< n \mid (iT) \mid i > \qquad (5.2.1)$$

but now the intermediate states are spin 1/2 states with momenta satisfying $p^2 + m^2 = 0$, and with spinor wave functions $u^{\pm r}(\vec{p})$ with $r = 1, 2$ satisfying the Dirac equation. As we shall show, summation over the two helicities of $|n>$ produces the propagator $S_F{}^+$ or $S_F{}^-$ while in $< f|T^\dagger|i >$ one uses antipropagators $S_F{}^*$. There is one thing to be stressed: we do not need to show that $S_F{}^*$ is related to S_F by complex conjugation (in fact, it is not; for that one also needs the charge conjugation matrix) but one only needs to use that in $S_F{}^*$ the $\theta(x^0 - y^0)$ and $\theta(y^0 - x^0)$ are interchanged, just as in the scalar case. In particular, we do not need to show that $< f|T|i > - < f|T^\dagger|i >$ is equal to twice the imaginary part of $< f|T|i >$, but instead we only need to consider the largest time equation, obtained by decomposing S_F into S^+ and S^- and considering vertices with circles around them. The reason we did more in the scalar case, is that an expression for ImT is a useful starting point for dispersion relations. In this section we shall be content with only a proof of unitarity. The analysis of ImT will not be given, as it is more complicated for spinors.

For those readers who are interested in details we discuss a few aspects further. The Dirac equation reads $(\gamma^\mu\partial_\mu + \frac{Mc}{\hbar})\psi = 0$. We use the following expression for a complex spin 1/2 field in second quantization

$$\psi(x) = \frac{\sqrt{\hbar c}}{\sqrt{V}} \sum_{\vec{q},r=1,2} [c^r(\vec{q})u^{+r}(\vec{q})e^{\frac{i}{\hbar}qx} + d^r(\vec{q})^\dagger u^{-r}(-\vec{q})e^{-\frac{i}{\hbar}qx}] \tag{5.2.2}$$

where $qx = q_j x^j + q_0 x^0$ and $q_0 = -\sqrt{\vec{q}^{\,2} + m^2}$. We set $\hbar = 1$. The spinors u^{+r} and u^{-r} are orthonormal

$$\begin{aligned} \sum_{\alpha=1}^{4} \quad & u_\alpha{}^{+r}(\vec{q})^* u_\alpha{}^{+s}(\vec{q}) = \sum_{\alpha=1}^{4} u_\alpha^{-r}(\vec{q})^* u_\alpha{}^{-s}(\vec{q}) = \delta^{rs} \\ \sum_{\alpha=1}^{4} \quad & u_\alpha^{+r}(\vec{q})^* u_\alpha{}^{-s}(\vec{q}) = 0 \end{aligned} \tag{5.2.3}$$

Hence, since column-orthonormality implies row-orthonormality

$$\sum_{r=1}^{2} \left[u_\alpha{}^{+r}(\vec{q}) u_\beta{}^{+r}(\vec{q})^* + u_\alpha{}^{-r}(\vec{q}) u_\beta{}^{-r}(\vec{q})^* \right] = \delta_\alpha{}^\beta \tag{5.2.4}$$

An explicit expression of the spinors in the representation

$$\gamma^k = \begin{pmatrix} 0 & -i\sigma_k \\ i\sigma_k & 0 \end{pmatrix} \text{ and } \gamma^0 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \tag{5.2.5}$$

is given by

$$u^{+r}(\vec{q}) = \sqrt{\frac{E + Mc^2}{2E}} \begin{pmatrix} \xi^{(r)} \\ \frac{c\vec{\sigma}\cdot\vec{q}}{E+Mc^2}\, \xi^{(r)} \end{pmatrix} \tag{5.2.6}$$

where $\xi^{(1)} = \binom{1}{0}$ and $\xi^{(2)} = \binom{0}{1}$. Further,

$$u^{-r}(-\vec{q}) = (-)^s \sqrt{\frac{E+Mc^2}{2E}} \begin{pmatrix} \frac{c\vec{\sigma}\cdot\vec{q}}{E+Mc^2}\xi^{(s)} \\ \xi^{(s)} \end{pmatrix}, s \neq r. \tag{5.2.7}$$

In all these equations, E is always positive, and for massless spinors one simply puts $M = 0$ (for this reason we do not use spinors which contain normalization factors proportional to $M^{-1/2}$).

When evaluating $|n><n|$ for intermediate spinors, we need sums over polarizations of u^+ or u^- spinors only. They are given by (putting $c = 1$)

$$\sum_{r=1}^{2} u_\alpha{}^{+r}(\vec{q})\bar{u}_\beta{}^{+r}(\vec{q}) = \frac{E+M}{2E} \begin{pmatrix} I & -q_k\sigma^k/(E+M) \\ q_k\sigma^k/(E+M) & -\vec{q}^{\,2}\, I/(E+M)^2 \end{pmatrix}_{\alpha\beta}$$

$$= \left(\frac{-i\not{q}+M}{2E}\right)_{\alpha\beta} \quad ; \quad \sum_{s=1}^{2} u_\alpha^{-s}(-\vec{q})\bar{u}_\beta^{-s}(-\vec{q}) = \left(\frac{-i\not{q}-M}{2E}\right)_{\alpha\beta} \tag{5.2.8}$$

where $\bar{u} = u^\dagger\gamma^4$ with $\gamma^4 = i\gamma^0$ (like $x^4 = ix^0$). Adding these two relations (but with $-\vec{q}$ in the second relation) one recovers the row-orthogonality relation. Again, the explicit E is positive, and

$$\not{q} = q_k\gamma^k + q_0\gamma^0 \text{ with } q_0 = -q^0 = -E < 0. \tag{5.2.9}$$

The relations in (5.2.8) can also be proven in general without using a particular representation by multiplying (5.2.4) with $-i\not{q} - M$. This annihilates the first term, while in the second term one may define $\vec{q} = -\vec{q}\,'$. Writing $-iq_0\gamma^0$ as $+iq_0\gamma^0 - 2iq_0\gamma^0$ one uses the Dirac equation for the spinors $u^{-r}(-\vec{q}\,')$, and (5.2.8) follows.

Using the mode decomposition of ψ_α, the propagator becomes

$$S_F(x-y) \equiv <0|\mathcal{T}\psi_\alpha(x)\bar{\psi}_\beta(y)|0> = \frac{1}{V}\sum_{\vec{q}} \left[\theta(x^0-y^0)\left(\frac{-i\not{q}+M}{2E}\right)_{\alpha\beta} e^{iq(x-y)}\right.$$
$$\left. -\theta(y^0-x^0)\left(\frac{-i\not{q}-M}{2E}\right)_{\alpha\beta} e^{-iq(x-y)}\right] = \theta(x^0-y^0)S^+(x-y) + \theta(y^0-x^0)S^-(x-y) \tag{5.2.10}$$

It is easy to check that

$$\begin{aligned} S_F(x-y) &= (-\gamma^\mu\partial_\mu + M)\,[\Delta_F(x-y)] \\ &= (-\gamma^\mu\partial_\mu + M)[\theta(x^0-y^0)\Delta^+(x-y) + \theta(y^0-x^0)\Delta^-(x-y)] \end{aligned} \quad (5.2.11)$$

Hence, $(-\gamma^\mu\partial_\mu + M)\Delta^+(x-y) = S^+(x-y)$ and $(-\gamma^\mu\partial_\mu + M)\Delta^-(x-y) = S^-(x-y)$. Closing the q_0 contour as in the scalar case, we obtain

$$< 0|\mathcal{T}\psi_\alpha(x)\bar\psi_\beta(y)|0 >= \frac{-1}{(2\pi)^4}\int d^4q \frac{(\not q + iM)_{\alpha\beta}}{q^2+M^2-i\epsilon}\, e^{iq(x-y)} \quad (5.2.12)$$

where $d^4q = d^3q dq_0$ and $qx = \vec q \vec x + q_0 x^0$.

These results are of course well-known, but we reviewed them in order to use the same steps to obtain the antipropagator for Dirac fermions. Recalling our definition of anti-time ordering of section 1, it is given by

$$\begin{aligned} S_F^*(x-y) \equiv < 0|\mathcal{T}^\dagger\psi_\alpha(x)\bar\psi_\beta(y)|0 >= \frac{1}{V}\sum_{\vec q}\Bigg[\theta(y^0-x^0)\left(\frac{-i\not q + M}{2E}\right)_{\alpha\beta} e^{iq(x-y)} \\ -\theta(x^0-y^0)\left(\frac{-i\not q - M}{2E}\right)_{\alpha\beta} e^{-iq(x-y)}\Bigg] = \theta(y^0-x^0)S^+(x-y) + \theta(x^0-y^0)S^-(x-y) \end{aligned} \quad (5.2.13)$$

Again it is easy to check that

$$\begin{aligned} S_F{}^*(x-y) &= (-\gamma^\mu\partial_\mu + M)[\theta(y^0-x^0)\Delta^+(x-y) + \theta(x^0-y^0)\Delta^-(x-y)] \\ &= (-\gamma^\mu\partial_\mu + M)\Delta_F^*(x-y). \end{aligned} \quad (5.2.14)$$

The contour is now closed in the opposite way and we obtain

$$< 0|\mathcal{T}^\dagger\psi_\alpha(x)\bar\psi_\beta(y)|0 >= \frac{+1}{(2\pi)^4}\int d^4q\, \frac{(\not q + iM)_{\alpha\beta}}{q^2+M^2+i\epsilon}\, e^{iq(x-y)} \quad (5.2.15)$$

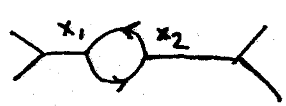
Hence, we get again an extra minus sign and $-i\epsilon$ becomes $+i\epsilon$. A simple way to remember the correct signs is to note that the trace over spinor indices yields $4M$ times the scalar case. (To be totally correct in the notation, we should have written ψ^α and $\bar\psi_\alpha$. Then the trace can be taken without any additional metric and yields $\psi^\alpha\bar\psi_\alpha$, which is clearly a Lorentz scalar). The matrix elements $< f|S^\dagger|n >$ are thus again obtained from those of $< f|S|n >$ by replacing each propagator by the antipropagator, and giving each vertex a minus sign.

A graph consists again of N points $x_1, \dots x_N$, connected by lines which now carry an arrow if we are dealing with complex (Dirac) fermions. These arrows will be present in all diagrams which appear in the largest-time equation, and have, of course, no relation to the arrows we use to indicate energy flow. The Lagrangian must be hermitian as we discussed, but may break C, P or T invariance. Decomposing S_F into a term with S^+ and a term with S^-, the largest time equation follows as before: the diagram with only S_F propagators plus the diagram with only $S_F{}^*$ propagators (and an extra factor -1 at each vertex) is again equal to minus the set of all diagrams with one up to $N-1$ circles. This is just a consequence of the θ functions. However, if we compare the cut diagrams involving S^+ and S^- with the contributions from intermediate physical states in the unitarity relation, minus signs occur at various places (in closed loops, for example).

We illustrate this with an instructive example, the scalar selfenergy due to a fermion loop. As action we take

$$\mathcal{L} = -\frac{1}{2}\varphi(\Box + m^2)\varphi - \frac{1}{2}\bar{\psi}(\partial\!\!\!/ + M)\psi + g\varphi\bar{\psi}\psi \tag{5.2.16}$$

It is hermitian. (To make this example physically correct, one could hook up the external φ lines to a pair of incoming and a pair of outgoing physical particles, just as we did in the example. The S matrix integrand is then given by



$$= -(ig)S_F(x_1 - x_2)$$
$$S_F(x_2 - x_1)(ig) \tag{5.2.17}$$

where the minus sign is due to the closed fermion loop. The diagram with two circles is obtained by replacing both S_F by $S_F{}^*$. The diagram with a circle on the left is obtained by replacing the upper S_F by S^+ and the lower S_F by S^- and adding one minus sign. Similarly for the diagram with a circle on the right. This yields the largest time equation which reads (omitting a factor g^2)

$$\begin{aligned}&S_F(x_1-x_2)S_F(x_2-x_1)+S_F{}^*(x_1-x_2)S_F{}^*(x_2-x_1)=\\&-[-S^+(x_1-x_2)S^-(x_2-x_1)-S^-(x_1-x_2)S^+(x_2-x_1)]\end{aligned} \tag{5.2.18}$$

Taking into account that $\theta(x^0-y^0)\theta(y^0-x^0)=0$, this identity is easily verified. Note that as a consequence of $\Delta^+(y-x)=\Delta^-(x-y)$ and the expressions for $S^\pm$ in terms of $\Delta^\pm$, also $S^+(y-x)=S^-(x-y)$. It makes therefore no difference whether one writes $S^+(x_1-x_2)$ or $S^-(x_2-x_1)$ in (5.2.17); all that matters is that this propagator connects an uncircled vertex to a circled vertex.

We shall now check the unitarity relation, by showing that the right-hand side of the largest-time equation agrees with the emission and absorption of an electron-positron pair. In the unitarity relation, there is only one intermediate state, with an electron of momentum $\vec{p}$ and positron with momentum $\vec{q}$, and it contributes

$$\begin{aligned}< f|iT|r,\vec{p}\,;\; s,\vec{q}> &= ig\bar{u}^{-s}(-\vec{q})u^{+r}(\vec{p})\\< s,\vec{q}\,;\; r,\vec{p}|(iT)^\dagger|i> &= -ig\;\bar{u}^{+r}(\vec{p})u^{-s}(-\vec{q})\end{aligned} \tag{5.2.19}$$

The ordering of the fermionic creation operators must be chosen such that $< s,\vec{q}\;;\; r,\vec{p}\mid r,\vec{p}\;;\; s,\vec{q}>$ equals unity. For $-\Sigma < f|iT|n>< n|(iT)^\dagger|i>$ we find (omitting again a factor g^2 and denoting the momenta of the incoming and outgoing scalar by k and k')

$$-\int Tr\left[\left(\frac{-i\not{p}+M}{2E}\right)\left(\frac{-i\not{q}-M}{2E}\right)\right]\frac{d^3p}{(2\pi)^3}\,\frac{d^3q}{(2\pi)^2}(2\pi)^8\delta^4(k'-p-q)\delta^4(p+q-k) \tag{5.2.20}$$

Consider next the largest time equation (5.2.15). If energy flows in at the vertex x_1 on the left, only the term $S^-(x_1-x_2)S^+(x_2-x_1)$ contributes in the integrated largest-time equation. The contribution of this term is given by

$$\int d^4x_1d^4x_2\varphi(x_2)S^-(x_1-x_2)S^+(x_2-x_1)\varphi(x_1)=$$

$$\int d^4x_1 d^4x_2 e^{-ik'x_2} \int \frac{d^3q}{(2\pi)^3}(-)\left(\frac{-i\not{q}-M}{2E}\right)e^{-iq(x_1-x_2)}$$
$$\int \frac{d^3p}{(2\pi)^3}\left(\frac{-i\not{p}+M}{2E}\right)e^{ip(x_2-x_1)}e^{ikx_1} \quad (5.2.21)$$

The expressions in (317) and (318) agree, and hence unitarity holds.

Is there any connection between statistics and unitarity? This is an interesting question in its own right, but it has also a practical aspect. Namely, the Faddeev-Popov ghosts are anticommuting scalars, and since we want to use the largest time equation for quantum Yang-Mills theory, we must make sure that the cutting equations also hold for anticommuting scalars. The propagators Δ_F of Faddeev-Popov ghosts can be decomposed into a term with Δ^+ and a term with Δ^-, and this is already enough for the largest time equation. For unitarity we then have to show that the integral over $\vec{k}$ of products of $(\exp \pm ikx)/\sqrt{2\omega}$ for emission and annihilation in $|n><n|$ are equal to Δ^+ or Δ^-. These are the same signs as we encountered for the fermions in closed loops, hence the cutting equations, and hence perturbative unitarity, hold also for anticommuting scalars. Something goes wrong, of course, for anticommuting scalars, and that is causality. (Also, the requirement that physical states have positive energy or positive norms is violated.) Similarly, commuting fermions do not violate perturbative unitarity, but they violate causality and positive energy, or causality and positive norms.

3 Unitarity for massless spin 1 fields

For massless vector bosons we add a gauge fixing term $\mathcal{L}(\text{fix}) = -\frac{1}{2\xi}(\partial^\mu A_\mu{}^a)^2$ with $\xi_{\text{ren}} = 1$. Later we shall consider the case of spontaneously broken theories and theories with $\xi_{\text{ren}} \neq 1$, but we begin with pure (unbroken) QED and Yang-Mills theory. Actually, we begin by neglecting renormalization, and prove unitarity for the unrenormalized theory. Only in a later section do we consider the modifications brought about by taking renormalization into account. Of course one expects that "somehow" the longitudinal and timelike modes of the vector fields cancel against the ghost and antighost modes but the precise meaning of this "somehow" needs detailed Ward identities. These we shall derive from the BRST symmetry of the quantum action, using functional methods.

We begin again by deriving the largest-time equation. Thus we consider vertices with a circle around them and antipropagators. We decompose the Feynman propagators $(\Delta_F)_{\mu\nu}$ for the gauge fields into $\theta(x^0-y^0)\Delta^+_{\mu\nu}+\theta(y^0-x^0)\Delta^-_{\mu\nu}$. The antipropagators are obtained by interchanging the two θ functions, and then one finds again that the antipropagators are the complex conjugates of the propagators. In QED there is a factor $\eta_{\mu\nu}$ in the numerator and for Yang-Mills theory there is an extra δ^{ab} for internal space, but these tensors $\eta_{\mu\nu}$ and δ^{ab} are real and can be factored out. Hence for real vector fields the decomposition of the Feynman propagator is exactly the same as for scalars (except for the extra factors $\eta_{\mu\nu}$ and δ^{ab}).

For the ghosts the propagator is

$$\begin{aligned}&< 0 \mid \mathcal{T} c(x)b(y) \mid 0 >= \int \frac{d^4k}{(2\pi)^4}\frac{-ie^{ik(x-y)}}{k^2-i\epsilon}\\&=\theta(x^0-y^0)\Delta^+(x-y)+\theta(y^0-x^0)\Delta^-(x-y)\\&=\theta(x^0-y^0)< 0 \mid c(x)b(y)\mid 0 > -\theta(y^0-x^0)< 0\mid b(y)c(x)\mid 0 >\end{aligned} \tag{5.3.1}$$

The decomposition into Δ^+ and Δ^- is the same as for scalars because the integral with d^4k is the same. To obtain the result for the integral with d^4k it is easiest to use path integral methods, but one may also use canonical quantization. The latter yields directly the result with Δ^+ and Δ^-, and gives a simple direct justification of the $-i\epsilon$ in the propagator with $\int d^4k$.

The field $c^a(x)$ is real, but the antighost field $b_a(x)$ is antihermitian. We recall that we made these assignments in order that $\mathcal{L}(\text{ghost}) = -(\partial^\mu b_a)(D_\mu c^a)$ be hermitian. Recall that hermiticity is a necessary requirement for unitarity. (With one exception as we already discussed: one may use complex gauge fixing terms and corresponding complex ghost actions. But taking the usual real gauge fixing term and a complex ghost action violates unitarity). We now evaluate the antipropagator. We recall that $(\mathcal{T}c(x)b(y))^\dagger \equiv \mathcal{T}^\dagger c(x)b(y)$. Hence

$$\mathcal{T}^\dagger(c(x)b(y)) = \theta(x^0 - y^0)(-b(y))c(x) - \theta(y^0 - x^0)c(x)(-b(y)) \tag{5.3.2}$$

The antipropagator then becomes

$$< 0 \mid \mathcal{T}^\dagger c(x)b(y) \mid 0 >= \theta(x^0 - y^0)\Delta^-(x-y) + \theta(y^0 - x^0)\Delta^+(x-y) \tag{5.3.3}$$

This is the same result as for scalars. Thus, the minus signs due to the anticommutativity and due to the antihermiticity of b have conspired to give the same result as for real commuting fields.

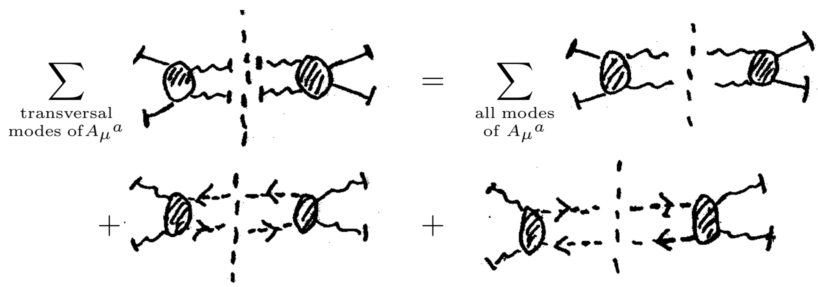
The combinatorics for the gauge fields and the ghost fields which leads to the largest time equation is thus the same as for scalar fields.

At first sight the situation seems identical to the case of scalar fields which we treated before, but there is one difference: in the largest-time equation we "cut" covariant Feynman diagrams whose propagators contain $\eta_{\mu\nu}$ (for $\xi = 1$) and ghost propagators, whereas in the unitarity condition one sums only over physical intermediate states. The physical states in the gauge $\xi = 1$ of the vector fields have $k^2 = 0$ and transverse polarization vectors $\epsilon_\mu{}^m (m = 1, 2)$ which are spacelike and orthogonal to the three-momentum $\vec{k}$. We must thus show that the contributions from the longitudinal and timelike parts in the propagators of the vector bosons and those from the (anti)ghosts cancel

$$\left(\sum_{\text{non}-\text{transverse pol.}} + \sum_{(\text{anti})-\text{ghosts}} \right) < f \mid T^\dagger \mid n >< n \mid T \mid i >= 0 \tag{5.3.4}$$

Again we shall prove this cancellation diagram-by-diagram (i.e., for a given Feynman diagram contributing to the process $< f|T|i >$) and cut by cut (i.e., for a given particular cut through n Yang-Mills propagators. One must then separately consider all other n-particle cuts with $n-2p$ gauge fields and $2p$ (anti)ghosts for $p = 0, 1, 2, \ldots$).

We specialize to two-particle cuts for simplicity. Then what we have to prove is the following equality:



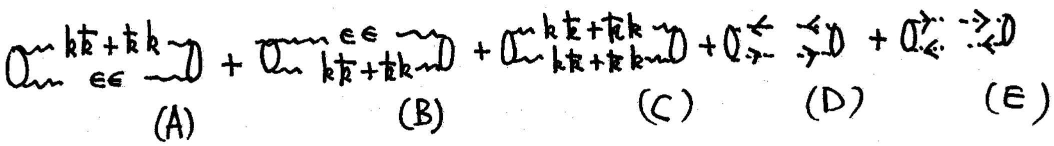
(5.3.5)

The left-hand side comes from the unitarity equation, the right-hand side from the cutting equation. The external lines are denoted by ⊣, and play no role as we shall prove; it does not matter how many there are, as long as they correspond to physical particles (gauge fields with transverse polarizations and $k^2 = 0$). We shall therefore omit them in the following graphs. Furthermore, there are two graphs with a cut through a ghost and antighost line, because ghosts and antighosts are distinguishable particles (their propagators carry an arrow). There is, of course, no graph with a cut through one gauge field and one ghost because ghost number is conserved.

As we already discussed at some length, what does change is the fact that on the left-hand side one sums over two physical polarizations but on the right-hand side over four polarizations and (anti) ghosts. To identify the terms on the right-hand side which are equal to the left-hand side, we shall use the following identity, valid for 4-momenta satisfying $k^2 = 0$.

$$\eta_{\mu\nu} = \sum_{m=1}^{2} \epsilon_\mu{}^m (\epsilon_\nu{}^m) + (k_\mu \bar{k}_\nu + \bar{k}_\mu k_\nu)(k \cdot \bar{k})^{-1} \tag{5.3.6}$$

Here $\epsilon_\mu{}^m$ are two real transversal 3-dimensional polarization tensors, $k_\mu = (\vec{k}, k_0)$ while $\bar{k}_\mu = (\vec{k}, -k_0)$. Thus $\bar{k}_\mu$ is the time-reversal of k_μ. (One can also rewrite the $\epsilon\epsilon$ term in terms of complex polarization vectors with definite helicities). To prove this identity one may contract it in turn with $\epsilon_\nu{}^m, k_\nu$ and $\bar{k}_\nu$, using $\epsilon \cdot k = \epsilon \cdot \bar{k} = k^2 = \bar{k}^2 = 0$. Note that there is no factor $\frac{1}{2}$ in the normalization of the last term. We decompose each $\eta_{\mu\nu}$ in the vector propagators on the right-hand side of the unitarity equation into a transversal part and the rest. The purely transversal part on the right-hand side agrees with the left-hand side, so we must show that the sum of all other terms vanishes. This means diagrammatically that one has to prove the following equation



(5.3.7)

For simplicity of notation we have omitted the denominators $(k \cdot \bar{k})^{-1}$ and vector indices μ and ν and the superscripts m of ϵ_μ^m in the figure, but they will be reinstated at the appropriate moment.

To prove this equation we need a series of Ward identities for **connected** graphs. These Ward identities follow from the fact that the BRST variation of a particular Green's function with any number of unphysical fields and any number of physical fields, vanishes if one varies the unphysical fields under BRST transformations and takes the vacuum expectation value. For two fields $B(x)$ and $C(x)$ one obtains

$$\delta_{BRST} < B(x)C(y) >= 0 \tag{5.3.8}$$

This expression vanishes because $\delta_{BRST}B(x)C(y)$ can also be written as the (anti) commutator of the BRST charge Q with $B(x)C(y)$, namely as $\{Q, B(x)C(y)\}$ and the vacuum is assumed to be BRST invariant, $Q|0 >= 0$.[9]

We neglect at this point renormalization; later we shall study the renormalized BRST transformation, and how to handle cut lines with self-energy insertions. The proof of (5.3.8) follows from writing this Green function as

$$< B(x)C(y) >\equiv \int [d\phi] B(x)C(y)\varphi(x_1)\ldots\varphi(x_n) \exp\frac{i}{\hbar}S \tag{5.3.9}$$

Here $[d\phi]$ indicates integration over all fields and $\varphi(x_j)$ are the physical fields. Making

[9]Physical states are states which are annihilated by Q (and which are not of the form $Q|\chi >$), and the vacuum is the physical state with lowest energy.

a change of integration variables $\phi \to \phi + \delta\phi$, where $\delta\phi$ is a BRST variation, the result in (5.3.8) follows provided

(i) the quantum action is BRST invariant

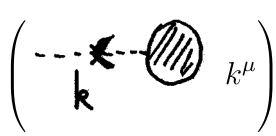
(ii) the Jacobian is unity

(iii) the BRST variations of the physical fields do not contribute

The quantum action is always BRST invariant (by construction), but the Jacobian is unity only if there are no chiral anomalies.

The physical fields are BRST invariant, really because this is the definition of physical states (their BRST invariance is a generalization to the quantum level of the requirement that physical quantities should be gauge-invariant). However, assuming that we already know that physical states have transversal polarization tensors and on-shell momenta ($k^2 = 0$), we can check that their BRST variation does not contribute to the Ward identity. To obtain an S-matrix element, we must amputate the external lines: multiply the full connected (not proper!) two-point function by $p^2 + m_{\text{phys}}^2$ and take the limit $p^2 \to m_{\text{phys}}^2$. (By m_{phys}^2 we mean the physical (pole) mass, not the renormalized mass). We must also multiply by a residue factor $\sqrt{R}$, but this factor we shall derive by requiring that unitarity holds.

Consider now a physical state corresponding to $\varphi(x_j)$. The leading term in the BRST variation of a gauge field is $\partial_\mu c^a(x)$, and contracting this with a physical polarization tensor, one finds zero ($\epsilon^m \cdot k = 0$).



$$\times\, \epsilon_\mu^m = 0 \qquad (5.3.10)$$

The dotted line denotes the ghost field c^a, and the shaded blob denotes the rest of an S-matrix element. The nonlinear terms do not give a pole since "amputating" the

Green function by multiplying with inverse propagators $p^2 + m^2_{\rm phys}$ and putting the momenta on-shell, these contributions vanish since they do not contain poles

$$\left(\quad \times k^\mu \right) \times \epsilon^m_\mu = 0 \tag{5.3.11}$$

In addition there are terms with a selfenergy structure, due to the nonlinear term in $D_\mu c^a$.[10] However, also these graphs are proportional to k_μ, and vanish when contracted with ϵ_μ.

$$\epsilon_\mu \text{ times } k^2 \text{ times } \quad = 0 \tag{5.3.12}$$

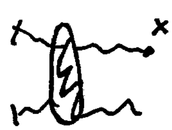
This proves that one may disregard physical states in the BRST variation of correlation functions. We shall now derive the series of Ward identities which we will need, by taking suitable choices for the fields $B(x)$ and $C(x)$. We recall the BRST rules

$$\begin{aligned} \mathcal{L}(qu) &= \mathcal{L}(\text{class}) - \frac{1}{2}(\partial^\mu A_\mu{}^a)^2 - (\partial_\mu b^a) D_\mu c^a \\ \delta A_\mu{}^a &= D_\mu c^a \Lambda \quad , \quad \delta b_a = -\Lambda(\frac{1}{\xi}\partial^\mu A_\mu{}^a) \\ \delta c^a &= \frac{1}{2} g f^a{}_{bc} c^b c^c \Lambda \end{aligned} \tag{5.3.13}$$

We shall work in the gauge with $\xi = 1$. The first Ward identity we will need is obtained from varying $< b^a >$. It yields

$$\delta_{\rm BRST} < b^a(x) > = < -\partial^\mu A_\mu(x) > = 0 \tag{5.3.14}$$

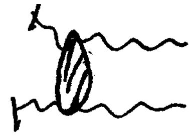
It states that in a graph where *all* lines are physical except one vector boson, the divergence of this line vanishes ("transversality"). In particular



$$\left(\quad \begin{matrix} k, \mu, a \\ \\ k', \nu, b \end{matrix} \right) k_\mu \epsilon_\nu(k') = 0 \tag{5.3.15}$$

[10]The proper subdiagram containing the $A_\mu c$ vertex due to the BRST variation is connected to the rest of the diagram by a connected 2-point function. Multiplication by p^2 yields a finite nonvanishing result proportional to the residue R.

In this graph, there is a propagator $1/k^2$ which propagates $A^a_\mu(x)$ from the point x to the blob. We can multiply the graph by k^2 (truncation) to remove this propagator, and we obtain then the Ward identity in the form we need it



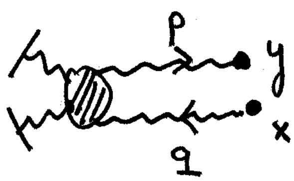
$$\left(\quad \begin{matrix} k, \mu, a \\ \\ k', \nu, b \end{matrix} \right) k_\mu \epsilon_\nu(k') = 0 \tag{5.3.16}$$

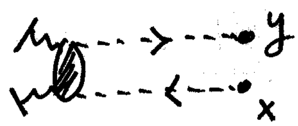
Using this Ward identity, the graphs (A) and (B) in (5.3.7) vanish.

The second Ward identity is obtained by taking $C = b_a$ and $B = A_\nu{}^b$ in (5.3.8). Omitting the constant anticommuting BRST parameter, one obtains

$$-\delta_{\rm BRST} < A_\nu{}^b(y) b_a(x) > = < A_\nu{}^b(y) \frac{1}{\xi} \partial^\mu A_\mu{}^a(x) > + < D_\nu c^b(y) b_a(x) > = 0 \tag{5.3.17}$$

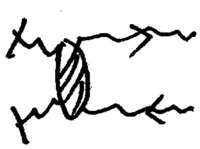
At the point x a gauge boson or a ghost with momentum q is emitted, while at the point y a gauge boson or a ghost with momentum p is absorbed. (For emission we get a factor $-iq_\mu$ while absorption gives a factor $+ip_\nu$).



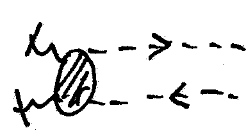


(5.3.18)

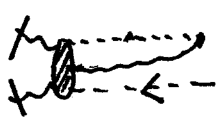
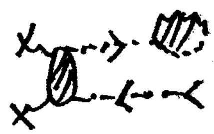
Now $D_\nu c^b(y)$ contains a term $\partial_\nu c^b(y)$ and a term $g f_{pq}{}^b A_\nu{}^p c^q$. To obtain the S-matrix elements we must amputate the Green's functions (meaning that one must multiply by inverse propagators). We set $\xi = 1$ (in section 5 we discuss the renormalized theory where we use $\xi_{\rm ren} = 1$). Then we find the following amputated graphs



p, ν, b
$q, \mu, a \times (-iq_\mu)$ +



$\times (ip_\nu)$

+  $\times (p^2)$ + 

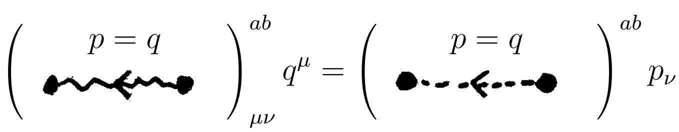
(5.3.19)

where after the amputation at the bottom a gauge boson or ghost is incoming, while at the top a gauge boson or ghost is emitted. (For emission we get a factor $-iq_\mu$ while absorption gives a factor $+ip_\nu$). The third graph contains no pole in p^2 while the fourth graph contains a pole in p^2. Multiplication by p^2 and then putting $p^2 = 0$ gives only a contribution in the latter case but not in the former case. We shall show in section 5 that the last graph (whose contribution is proportional to p_ν, but nonvanishing) is canceled by the factor $\sqrt{R}$ which is needed in the definition of the S matrix. The Ward identity then reduces to

$$q^\mu < A_\nu{}^b(p) A_\mu{}^a(q) >= p_\nu < b_b(p) c^a(q) > \tag{5.3.20}$$

where we recall that we have suppressed the physical external fields.

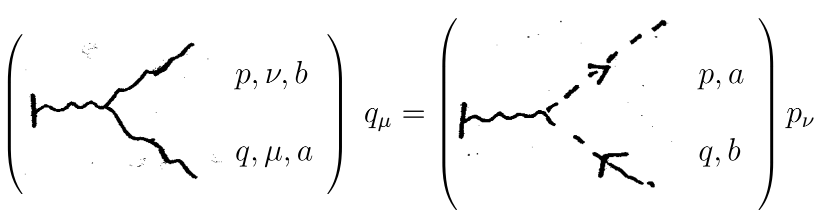
To lowest order this Ward identity yields a relation between propagators. This is most easily seen from the relation (5.3.17) which relates diagrams before truncation.



(5.3.21)

The first term yields a factor $-iq_\mu \eta^{\mu\nu}$, while the second term yields $+ip^\nu = +iq^\nu$.

A less trivial check involves the tree graphs with one physical Yang-Mills field. Then we must show that the following relation between vertices holds



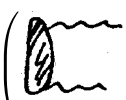
(5.3.22)

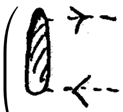
where q is incoming and p is outgoing. The vertices which contribute are given by

$$\mathcal{L}(int) = -\partial^\rho A_d^\sigma g f^d{}_{ef} A_\rho{}^e A_\sigma{}^f - (\partial^\rho b_d) g f^d{}_{ef} A_\rho{}^e c^f \tag{5.3.23}$$

and one may check that the Ward identity holds. (This is not a very strong check, since the on-shell conditions imply that $(p-q)^2 = 0$ and $\epsilon^\rho (p-q)_\rho = 0$ where ϵ^ρ is the polarization tensor of the physical vector boson. In the end the relation boils down to $f_{abd}\epsilon^\rho(-p_\rho p_\nu) = f_{abd}\epsilon^\rho(-p_\rho p_\nu)$. Still, the Ward identity holds).

For general connected graphs with on-shell momenta the Ward identity in (5.3.20) is diagrammatically given by





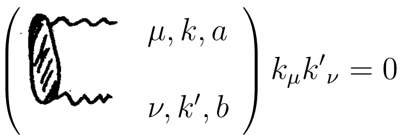
$$\left(\begin{matrix} k,\nu,b \\ k',\mu,a \end{matrix} \right) k'_\mu = \left(\begin{matrix} k,a \\ k',b \end{matrix} \right) k_\nu \qquad (5.3.24)$$

where k' are incoming and k are outgoing momenta, and the ghost at the bottom is incoming. This Ward identity replaces a cut graph with gauge bosons by a cut graph with ghosts. It is a kind of slot machine: hit one side with one momentum (the divergence with k'_μ), and out comes on the other side the other momentum (k_ν). That other momentum (k_ν) can contract with the factor $\frac{\bar{k}_\nu k_\rho}{k\cdot\bar{k}}$ from the numerator of the cut gauge propagator to yield k_ρ, and k_ρ acts on the other half of the cut diagram and one may repeat the procedure. The net effect is a cycling around which replaces some diagrams with unphysical polarizations by minus the same diagrams with ghosts. (The minus sign is due to the diagrams with a closed ghost loop, as we shall discuss). This is a first example of how the contributions from quartets of unphysical intermediate particles (longitudinal and timelike vector bosons, ghosts and antighosts) cancel in the unitarity relation. We shall soon apply this identity to the graphs in C.

The last Ward identity we need follows from taking $C = b_a(x)$ and $B = \partial^\nu A^b_\nu(y)$ in (5.3.8). This is a special case of (5.3.17). One obtains

$$\delta < \partial^\nu A_\nu{}^b(y) b_a(x) >= - < \partial^\nu A_\nu{}^b(y)\partial^\mu A_\mu{}^a(x) > - < \partial^\nu D_\nu c^b(y) b^a(x) >= 0 \qquad (5.3.25)$$

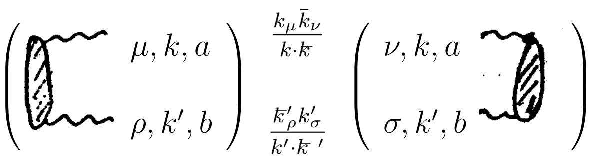
We truncate again by multiplying by inverse propagators. Again the nonlinear terms in $D_\nu c^b$ can be dropped, but also the term in $\partial^\nu D_\nu c^b$ which is linear in fields can now be dropped.[11] Indeed, this term is $\Box c^b(y)$, and on-shell it yields an extra zero ($k^2 = 0$). (The cut lines are on-shell with their momenta since Δ^+ and Δ^- can be written in a form with a factor $\delta(k^2+m^2)$). All that remains is transversality in both lines.



(5.3.26)

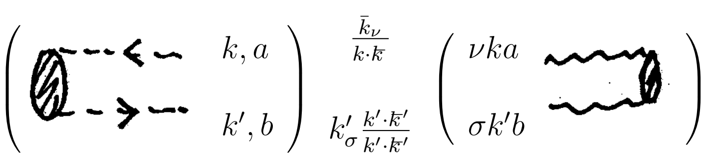
(Although we do not need it let us mention that transversality in n lines holds: keeping all other lines physical, and contracting *all* n lines with their own momentum yields zero. This was actually a crucial identity in 't Hoofts first paper on renormalizability [12]). Using this Ward identity, two of the four graphs in (C) cancel (one graph with $k_\mu k'_\rho$ and $k_\nu k'_\sigma$, and another graph with $k_{\mu'} k_\rho$ and $k'_\nu k_\sigma$).

As promised before, we shall now work out one of the two cross terms in (C). We start from the expression



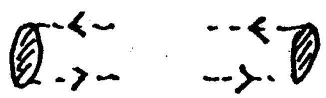
(5.3.27)

Using the Ward identity in (5.3.24) for k_μ on the left-hand side, we obtain



(5.3.28)

The factor $(\bar{k}_\nu / k \cdot \bar{k})$ is left over at the top, while at the bottom k'_ρ has contracted with $\bar{k}'_\rho k'_\sigma / k' \cdot \bar{k}'$ to yield k'_σ. We can then once more apply the second Ward identity, this time to k'_σ and the right-hand side of the diagram. The result is


(5.3.29)

[11] In a path integral approach, the composite operator $\partial^\nu (D_\nu c)^b e^{iS}$ can be written as $\frac{\partial}{\partial b^b} e^{iS}$, and partially integrating $\int b_a(x) \frac{\partial}{\partial b_b}(y) e^{iS}$ one finds a constant $\delta_b{}^a \delta(x-y)$ which is evidently $\hbar$ independent. Because the nonlinear terms lead to loops which are $\hbar$ dependent, this again shows that one may drop the term with $< b_a \partial^\nu (D_\nu c)^b >$.

This is equal to minus diagram D. The minus sign is due to the fact that a closed ghost loop has an extra minus sign whereas application of the Ward identities does not provide such a minus sign. Readers who are not satisfied with this short argument may check it in a simple example.

Similarly, one can evaluate the other cross term in (C) and show that it cancels against the other diagram with ghosts, namely diagram (E). Hence, in QCD, the graphs with one longitudinal boson (graphs (A) and (B)) or those with two longitudinal bosons (the two diagonal terms in (C)) all cancel by themselves (transversality). The graphs with one longitudinal and one timelike boson in (C) cancel against the two graphs with ghosts. Of course, the concept of unitarity is not meaningful in QCD, because due to confinement no in- and out- states can be defined,[12] but we will use the same methods for the case of spontaneously broken gauge theories, where unitarity does make sense.

As an exercise the reader may check unitarity (and the minus signs in the ghost loops) for the vector boson selfenergy by using the tree graphs given in (5.3.22)). Instead of putting the external gauge fields on-shell, one may couple each to two on-shell fermions. Then one obtains a check of unitarity for fermion-fermion scattering already at the level of one particular graph.

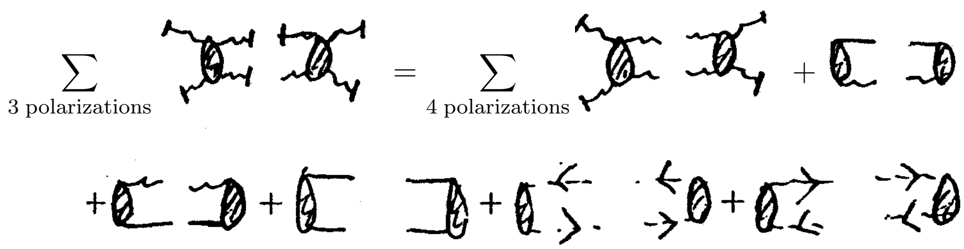
4 Unitarity for spontaneously broken gauge theories

For spontaneously broken gauge theories, the vector bosons are massive and thus have *three* physical polarizations. Only the polarization with $\epsilon_\mu = k_\mu$ is still unphysical. There is now a longitudinal polarization vector $\epsilon^3_\mu = (E\ \vec{k}/|\vec{k}|, -|\vec{k}|)/M$ with is orthogonal to $k_\mu = (\vec{k}, k_0)$ when $k^2 + M^2 = 0$. Its role in the spontaneously broken theory

[12]Actually, in PQCD (perturbative QCD) the cutting rules themselves are widely used for other purposes.

is taken over by the would-be Goldstone boson, which we denote by χ^a. In addition, there are the Faddeev-Popov ghosts c^a and antighosts b_a. We expect that again the contributions from this quartet will cancel. This is plausible if one remembers from chapter III that all four fields $A_\mu{}^a, \chi^a, b_a$ and c^a carry the same mass $M = \frac{1}{2}\xi g v$ if the two parameters in the gauge-fixing term (α and ξ) are equal. We shall show that the cuts through intermediate $A_\mu{}^a$ and χ^a are canceled by cuts through the ghost and antighost lines, but the Ward identities we need are this time those of the spontaneously broken theory.

Let us recall that we have to show that



(5.4.1)

Wiggly lines denote massive gauge bosons, straight lines indicate the would-be Goldstone bosons and dotted lines indicate the ghosts. There are no graphs with one cut ghost line because ghost number is conserved. We consider only cut graphs in which both internal lines carry a Yang-Mills index. This excludes graphs with a cut physical Higgs propagator. We leave as an exercise to analyze unitarity in the sector with cut Higgs bosons.[13] Again we only analyze cuts through two internal lines and leave the generalization to n-particle cuts as an exercise.

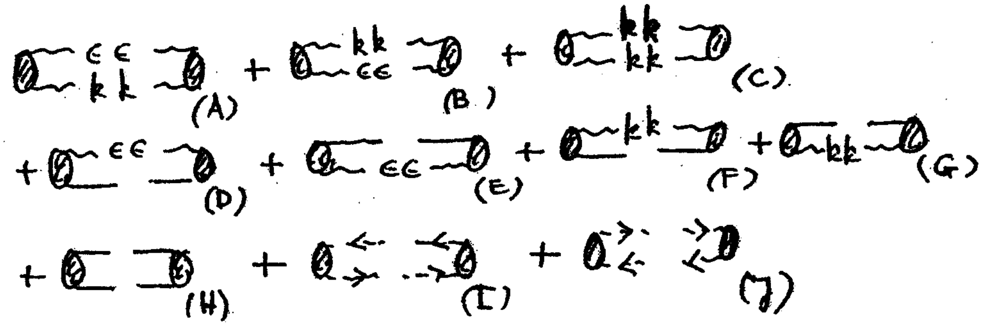
We begin again with a suitable decomposition of the metric $\eta_{\mu\nu}$, but since now $k^2 + M^2 = 0$ instead of $k^2 = 0$, we have three physical polarization tensors $\epsilon_\mu{}^m(m =$

[13]Cutting a graph with an intermediate massive vector boson and a Higgs field, the physical part comes from the 3 transversal polarizations of the vector field, but there is also an unphysical part due to the polarization vectors k_μ. The exercise is then to show that the unphysical graphs with one longitudinal vector boson cancel with the graphs, with one would-be Goldstone boson. This is easy if one notes that a cut Higgs field is a physical field which one may omit from consideration for the same reasons as one may omit the physical incoming and outgoing states from consideration.

1, 2, 3) and one unphysical $\epsilon_\mu{}^0 \sim k_\mu$. Hence

$$\eta_{\mu\nu} = \sum_{m=1}^{3} \epsilon_\mu{}^m (\epsilon_\nu{}^m) - \frac{k_\mu k_\nu}{M^2} \tag{5.4.2}$$

Again one may verify this decomposition by contraction with $\epsilon_\nu{}^n$ or k_ν, using $k^2 = -M^2$ and $k \cdot \epsilon^m = 0$ for $m = 1, 2, 3$. The 3 physical polarization tensors give the same contributions on the left-hand side as on the right-hand side of the unitarity equation, and hence the remainder has to cancel by itself. Using the decomposition of $\eta_{\mu\nu}$ given above, we must prove the following diagrammatic equation



(5.4.3)

Again we have not explicitly shown the external lines; all we need to know is that they are physical, i.e., invariant under BRST transformations. (Note that at this point we specialize to the S-matrix, instead of Green's functions). As before we have omitted signs and superscripts (for example, $k_\mu k_\nu$ stands for $-k_\mu k_\nu / M^2$), but we shall reinstate them later.

The Ward identities which we need for these cancellations follow from the BRST transformations of the spontaneously broken theory. We shall consider the $SU(2)$ Higgs model of chapter III. The local $SU(2)$ symmetry (the left-handed $SU(2)$ group) leads to the BRST transformations. For the classical fields $A_\mu{}^a, \chi^a, \psi$ one has

$$\begin{aligned} \delta A_\mu{}^a &= \left(\partial_\mu c^a + g f^a{}_{bc} A_\mu{}^b c^c\right) \Lambda \\ \delta \chi^a &= \left(\frac{1}{2} g (v + \psi) c^a + \frac{1}{2} g f^a{}_{bc} \chi^b c^c\right) \Lambda \\ \delta \psi &= -\frac{1}{2} g \chi_a c^a \Lambda \end{aligned} \tag{5.4.4}$$

In addition, the ghosts transform as usual, while the antighosts transform into the gauge-fixing term which has now an extra term

$$
\begin{aligned}
\delta c^a &= \frac{1}{2} g f^a{}_{bc} c^b c^c \Lambda \\
\delta b_a &= -\Lambda(\partial^\mu A_\mu{}^a - M\chi^a).
\end{aligned}
\tag{5.4.5}
$$

(We set $\xi = 1$ and denote $\frac{1}{2}gv$ by M. Furthermore we do not consider graphs with cut Higgs boson propagators and so we omit the Higgs field as explained before).

For the graphs in (A) and (B) we need a Ward identity containing $< \partial^\mu A_\mu(x) >$. (Recall that contracting an on-shell Yang-Mills field with a physical polarization vector $\epsilon_\mu{}^m$ makes it inert under BRST transformations. Thus the cut lines contracted with $\epsilon_\mu{}^m$ are on the same footing as the external lines and can be omitted from consideration). Clearly, this Ward identity is obtained by varying b_a

$$
\delta_{\text{BRST}} < b_a(x) > = < \partial^\mu A_\mu{}^a(x) - M\chi^a > = 0 \tag{5.4.6}
$$

Diagramatically we find then for incoming k after truncation

ik_μ  k, μ, a $= M$  k, a (5.4.7)

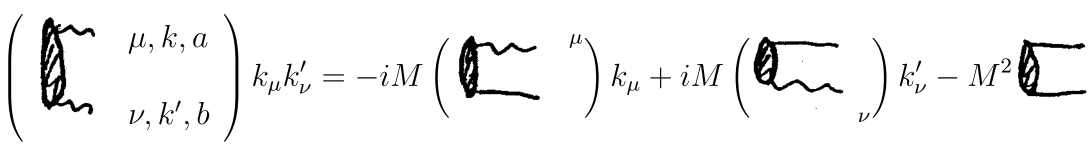
Thus in the spontaneously broken theory, transversality is broken by terms involving the would-be Goldstone boson. This shows that the sum of (A) and (D) cancels, as does the sum of (B) and (E).

Next we turn to the graphs (C) in (5.4.3). We now need a Ward identity involving $< \partial^\mu A_\mu(x) \partial^\nu A_\nu(y))$. Clearly the appropriate Ward identity is obtained by varying the product of the antighost and the gauge fixing term[14]

$$
\begin{aligned}
&\delta_{\text{BRST}} < (\partial^\nu A_\nu{}^b - M\chi^b)(y) b_a(x) > = \\
&- < (\partial^\nu A_\nu{}^b - M\chi^b)(y)(\partial^\mu A_\mu{}^a - M\chi^a)(x) > \\
&+ < \delta_{\text{BRST}}(\partial^\nu A_\nu{}^b - M\chi^b)(y) b_a(x) > = 0
\end{aligned}
\tag{5.4.8}
$$

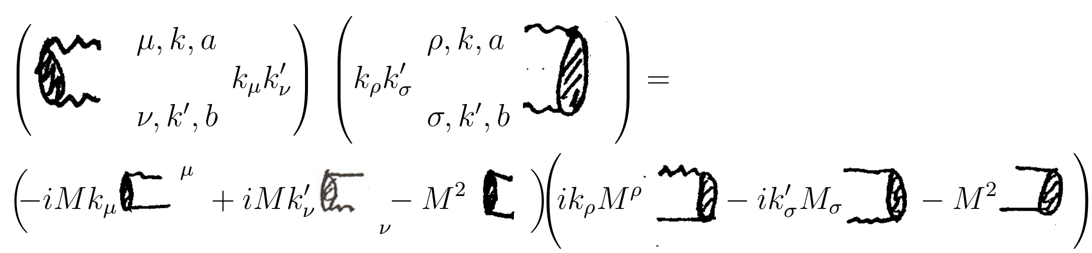
[14]In the case of QCD we needed also the Ward identity $\delta < A^b_\nu(y) b_a(x) > = 0$ because there were two-boson cuts, one contracted with k_μ and the other with $\bar{k}'_{\mu'}$. In the present case there are no $\bar{k}_\mu$ so we do not need this Ward identity.

The variation of the gauge fixing term in the last expression does not contribute. It reads $\delta_{\text{BRST}}(\partial^\nu A_\nu{}^b - M\chi^b)(y) = \partial^\nu D_\nu c^b - M^2 c^b +$ nonlinear terms, and the nonlinear terms do not contribute when one truncates, while the linear terms are proportional to $(\Box - M^2)c^b$ which vanishes as c^a satisfies the massive Klein-Gordon equation. (Again the same result follows from a path integral approach where the variation of the gauge-fixing term yields the antighost equation $(\partial^\nu D_\nu - M^2)c^c e^{iS} = \frac{\partial}{\partial b_c} e^{iS}$, and partial integration of $\int b_a(x)\partial/\partial b_c(y) \exp iS$ yields $\delta_a{}^c \delta(x-y)$.) Hence only the first term in the Ward identity is nonvanishing. Diagrammatically we have after truncation the following identity for outgoing k_μ and incoming $k'_\nu{}^a$



(5.4.9)

This Ward identity states that again transversality is broken by terms involving the would-be Goldstone bosons. Using this Ward identity, (C) becomes equal to the square of (5.4.9).



(5.4.10)

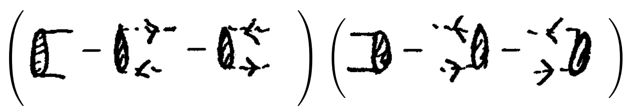
We have taken the flow of all momenta in a clockwise direction. The four graphs in this expression with one divergence of the Yang-Mills field and one would-be Goldstone boson can be further simplified by yet another Ward identity. It follows from varying $< \chi^b(y) b_a(x) >$ since $\delta_{\text{BRST}} b_a(x)$ contains a term $\partial^\mu A_\mu(x)$. One finds

$$< \chi^b(y)(\partial^\mu A_\mu{}^a - M\chi^a)(x) >= -M < c^b(y) b_a(x) + \text{ nonlinear terms} > \quad (5.4.11)$$

Diagrammatically for incoming momentum k' at the bottom after truncation

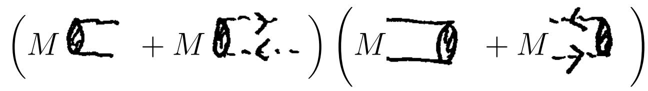
(5.4.12)

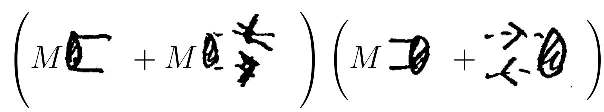
Substituting these results into the graphs in (5.4.10), we find

graph (C) = 

(5.4.13)

The factor M^{-2} in $-k_\mu k_\nu/M^2$ has cancelled the M dependence. For graph (F) and graph (G) we find using the same Ward identity (5.4.12)

graph(F) = $-\frac{1}{M^2}$ 

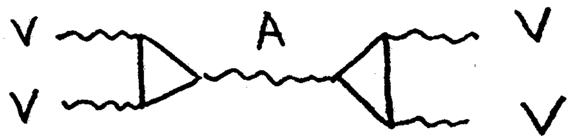
graph(G) = $-\frac{1}{M^2}$ 

(5.4.14)

The minus sign is due to the fact that in the Ward identity momenta are incoming, but in the cut graph, one is incoming and the other is outgoing. There are two cross terms with ghost lines in (5.4.13) where the arrows on the left do not match with the arrows on the right. These cancel against similar diagrams in (F) and (G). Next there are products of a $\chi\chi$ diagram times a ghost diagram. These also cancel in the sum of (C), (F) and (G). The products of two $\chi\chi$ diagrams in (C), (F) and (G) cancel with (H). Finally, the two cross terms with ghosts in (C) where the arrows on the ghosts do match cancel against (I) and (J). The reason we need in (I) and (J) an extra minus sign was explained before, namely, closed ghosts loops acquire an extra minus sign.

Hence, also in spontaneously broken Yang-Mills theory, perturbative unitarity holds. We have checked this at the level of two-particle cuts not involving the Higgs scalars, but one can give a general proof by induction.

One comment (at this place) about anomalies. When one considers elastic scattering of two vector bosons via two fermion triangles connected by a single axial vector boson, breakdown of the conservation of the axial current connected to a triangle graph leads to a new propagating longitudinal mode for the axial vector bosons at the quantum level.

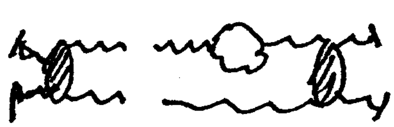


(5.4.15)

This implies a breakdown of the local Ward identity $<\delta b_a>=<-\partial^\mu A_\mu>=0$ (the Jacobian in the path integral derivation of the Ward identity no longer vanishes) and hence unitarity is violated at the perturbative level. In Euclidean field theories, if the spacetime integral of this local anomaly does not vanish, the nonconservation of the axial charge means that there is a background of Yang-Mills instantons; then the number of left-handed fermions is different from the number of right-handed fermions. Instantons play an important role in QCD, but since there are no chiral fermions in QCD, these instantons do not lead to a violation of perturbative unitarity. In the electroweak sector the gauge fields are massive, except the photon, but a $U(1)$ gauge field does not lead to winding and instantons. At high temperature where vector bosons become massless and instantons become relevant, unitarity is still preserved. [18]

5 Unitarity and renormalizability

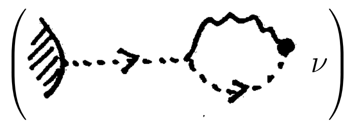
The unitarity proofs of the preceding sections were all based on the unregularized theory. However, when lines are cut and fields acquire on-shell momenta, selfenergies on external lines become divergent and one should carefully define the S matrix in order to deal with such divergences. An example is the following graph



(5.5.1)

In this section we shall therefore reanalyze the proofs of unitarity, taking properly account of the definition of the S matrix in terms of Feynman graphs. One can prove unitarity of regularized theory before or after renormalization. We now discuss the renormalized theory.

We shall begin with spin 0 and $\frac{1}{2}$, and then turn to spin 1. In the case of spin 1, we shall take care of the strange diagrams in (5.3.19) which we neglected in the proof of unitarity, namely the diagrams of the following form



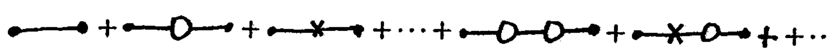
$$ip_\nu p^2 \tag{5.5.2}$$

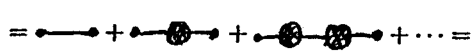
For spin 0, unitarity amounts to proving that the product of S matrix elements $\sum_n < f|T^\dagger|n >< n|T|i >$ is equal to the result one obtains from the cutting relations. Let us recall that one obtains an S matrix element in the regularized theory from a connected Green's function in the regularized theory by the following steps

(i) amputate external legs (remove all selfenergies from external legs)

(ii) multiply each external leg by a factor$\sqrt{R}$, where R is the residue of the connected propagator. In the regularized but not renormalized theory, this residue is divergent and for that reason we consider the renormalized theory.

In the renormalized theory, all proper selfenergies are finite, and thus the complete connected propagator is finite. One may sum the terms in the connected propagator as follows





$$\frac{-i}{k^2 + m^2 + \Pi(k^2) - i\epsilon} \tag{5.5.3}$$

Doubly hatched blobs denote proper selfenergies which are finite due to renormalization. Thus, after renormalization, $\Pi(k^2)$ is finite. Further, m^2 denotes the renormalized mass, **not** the physical mass which is by definition the mass at the pole of the connected propagator. To specify the finite parts in the Z factors, one must define the renormalization conditions. For the sake of the argument we shall assume that we have used minimal subtraction, although the arguments go equally well through for other more physical renormalization conditions. The pole of the propagator occurs at that value of k^2 for which $k^2 + m^2 + \Pi(k^2) = 0$. That value of k^2 defines the physical mass m_p^2, namely $-m_p^2 + m^2 + \Pi(-m_p^2) = 0$. Subtracting both expressions and considering values of k^2 near $-m_p^2$ we obtain

$$\frac{-i}{k^2 + m^2 + \Pi(k^2)} = \frac{-i}{k^2 + m_p^2 + \Pi(k^2) - \Pi(-m_p^2)} = \frac{-i}{(k^2 + m_p^2)\{1 + \Pi'(-m_p^2)\}} + \cdots \tag{5.5.4}$$

Hence the residue of the propagator at the pole is $R = [1+\Pi'(-m_p^2)]^{-1}$. This residue is finite, but not equal to unity if one uses minimal subtraction to renormalize the theory. We now prove that the S matrix is obtained by completely removing the selfenergies from all external legs in connected Green's functions and multiplying the results by $R^{1/2}$ for each external leg. (Note that removing the external legs (amputating) does not lead to a one-particle irreducible graph. For 3-point functions one finds a proper vertex, but already from 4-point functions one finds a sum of proper graphs and reducible graphs, as we showed in the appendix to chapter III).

Returning to the proof of unitarity for scalar fields, we note that if the momenta of an internal line go on shell, $k^2 + m_p^2 = 0$, then the whole internal line with all its selfenergies and counterterm insertions becomes equal to $R/(k^2 + m_p^2 - i\epsilon)$. The cutting equation can then be applied to the single factor $(k^2 + m_p^2 - i\epsilon)^{-1}$, and we can split R into $\sqrt{R}\sqrt{R}$ and associate one $\sqrt{R}$ to the left-hand part of the cut line, and another $\sqrt{R}$ to the right-hand part of the cut line. The result is then precisely what one needs for two S matrix elements. Hence, **the requirement of unitarity yields the proper definition of the S matrix**. (Usually, the S matrix is defined by an analysis based on the LSZ formalism. Because the matrix T in the unitarity relation appears once in $< f|T|i >$ and once in $< f|T^\dagger|n >< n|T|i >$, and in the latter case T must be multiplied by the factors $\sqrt{R}$ from the cut lines, one also needs the same factors $\sqrt{R}$ for the external lines, because the definition of T should be the same for external lines and cut lines. If one would forget the extra factors of $R^{1/2}$ for external legs, one would find in gauge theories that the S-matrix is no longer unitary or gauge-choice independent.

At this point we have taken care of renormalization in the proofs of unitarity for scalar fields. The same analysis holds step by step for fermions. We now turn to gauge fields.

In the case of gauge fields there are several new complications in the proof of unitarity if one considers the renormalized theory instead of the regularized but unrenormalized theory. We would, of course, like to use the same definition of S matrix as in the scalar case, but a first problem is that the propagator can be split into a longitudinal and a transversal part, and only the latter renormalizes because all selfenergies and counterterms are transversal. (Recall that we proved renormalizability by introducing the effective action $\hat{\Gamma} = \Gamma - S$ (fix) and requiring (later proven by induction) that S (fix) does not renormalize). Hence the complete connected propagator for massless gauge fields is given by[15]

$$\frac{-ik_\mu k_\nu/k^2}{k^2 - i\epsilon} + \frac{-i(\eta_{\mu\nu} - k_\mu k_\nu/k^2)}{k^2 + \Pi(k^2) - i\epsilon} = \frac{-ik_\mu k_\nu/k^2}{k^2 - i\epsilon} + \frac{-i(\eta_{\mu\nu} - k_\mu k_\nu/k^2)}{k^2(1 + \Pi'(0)) - i\epsilon} + \cdots \tag{5.5.5}$$

This makes only sense for QED, where the $k_\mu k_\nu$ terms cancel in S matrix elements. For QCD the concept of an S matrix does not make sense because due to confinement one cannot define in-and out-states. For spontaneously broken gauge theories unitarity makes perfect sense, but then one must take into account m^2 terms in the denominator.

Physical external lines are contracted with polarization vectors, and then the longitudinal terms cancel. Hence, for the definition of an S matrix element, we must

[15] We use here that $\Pi(k^2)$ is proportional to k^2, i.e., that gauge fields remain massless if one includes quantum corrections. We do not try to prove this here but note that one can impose renormalization conditions which achieve this. The same remarks apply to the ghost propagator. Of course the Ward identities from which unitarity follows require that gauge fields and ghosts are both massless or have the same mass.

multiply by $\sqrt{R}$ where R is now the residue of the transversal part of the propagator. Since in the cutting relations the physical part of $\eta_{\mu\nu}$ is given by $\sum_{m=1}^{2} \epsilon_\mu^m \epsilon_\nu^m$ for massless theories, or $\sum_{m=1}^{3} \epsilon_\mu^m \epsilon_\nu^m$ for massive theories, the same analysis as for scalar fields now applies to the terms with physical polarization vectors, and unitarity is proven in so far as the S matrix elements in the unitarity equation are indeed equal to a sum of graphs with cut lines.

However, there was a second part in the proof of unitarity for gauge fields, namely one must show that all contributions from the quartet of unphysical particles cancel, and here new complications arise. We needed Ward identities, and these we derived for the unrenormalized theory. Going over to renormalized fields and parameters

$$A_\mu{}^a = \sqrt{Z_3} A_{\mu,\text{ren}}{}^a; c^a = \sqrt{Z_3^{\text{gh}}} c_{\text{ren}}{}^a; b_a = \sqrt{Z_3^{\text{gh}}} b_a{}^{\text{ren}}; \xi = Z_3 \xi_{\text{ren}} \tag{5.5.6}$$

we see that the Ward identity $< \partial^\mu A_\mu{}^a >= 0$ in (5.3.14) becomes $< \partial^\mu A_{\mu,\text{ren}}{}^a >= 0$ since the factor $\sqrt{Z_3}$ is an overall factor which can be dropped. Hence the cut graphs with only one unphysical polarization still vanish as before. However, the Ward identity (5.3.17)

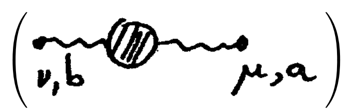
$$< A_\nu^b(y) \frac{1}{\xi} \partial^\mu A_\mu{}^a(x) > + < D_\nu c^b(y) b_a(x) > = 0 \tag{5.5.7}$$

becomes after renormalization

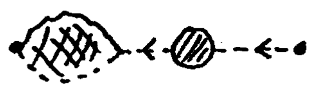
$$\begin{aligned} 0 =& < A_{\nu,\text{ren}}{}^b(y) \partial^\mu A_{\mu,\text{ren}}{}^a(x) > \\ &+ \sqrt{Z_3^{\text{gh}}} < \partial_\nu c_{\text{ren}}{}^a(y) b_a{}^{\text{ren}}(x) > \sqrt{Z_3^{\text{gh}}} + Z_1^{\text{gh}} u f^b{}_{pq} < A^p_{\nu,\text{ren}}(y) c^q_{\text{ren}}(y) b_a{}^{\text{ren}}(x) > \end{aligned} \tag{5.5.8}$$

where $Z_1^{\text{gh}} = (Z_1/Z_3) Z_{\text{gh}}$, see chapter III. Clearly, the Z factors in this relation do not cancel. Furthermore, the last term leads to the strange diagrams we omitted from the unitarity equation. (Recall that we have suppressed writing that there are external physical fields in addition to the two unphysical cut lines. Recall also that u denotes the renormalized coupling constant, and the ghost vertex renormalizes by a factor $Z_1^{\rm gh}$. The gauge fixing parameter ξ renormalizes into $\xi = Z_3\xi_{\rm ren}$, as we have explained in chapter III, and we are using the Feynman gauge with $\xi_{\rm ren} = 1$).

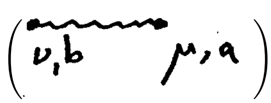
To understand the roles of the various Z factors in this relation, and their interplay with the last term, we first consider a simpler case, namely the case without external physical fields. In this case we are dealing with connected propagators. We shall first consider the Ward identity with off-shell momenta, and only later come back to on-shell momenta and truncation. The Ward identity for the connected propagator in (5.5.8) reads in graphical notation



$(ip^\mu) = Z_3^{\rm gh}(ip^\nu)$


$+Z_1^{\rm gh}$

(5.5.9)

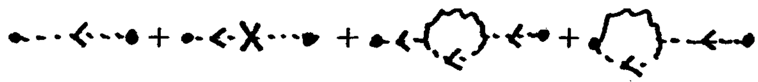
At the point x on the right-hand side a gauge field or ghost is emitted with momentum p. As usual, hatched blobs denote connected graphs and doubly hatched blobs denote proper graphs. We now note a first tremendous simplification: because the selfenergy of gauge fields is transversal, only the tree graph survives on the left-hand side

$ip^\mu =$


$$ip^\mu = \frac{p_\nu}{p^2 - i\epsilon}\delta^{ab} \tag{5.5.10}$$

This result should agree with the tree graph on the right-hand side, and it does since the ghost propagator $< c^a(y)b_b(x) >$ is $-i/p^2\delta^a_b$ (in particular, the signs come out correctly). Hence, all loop effects on the right-hand side should cancel by themselves. In other words, the strange left-over graphs play a crucial role: they must cancel all ghost selfenergies.

To see how this works out in detail consider the one-loop case. One finds on the right-hand side the following four contributions: the prefactor $(Z_3^{\rm gh} - 1)$ times the tree propagator, a counterterm insertion proportional to $(Z_3^{\rm gh} - 1)$, a regular ghost selfenergy, and finally the strange graph which looks also very much like a ghost selfenergy except that the antighost factor $\partial_\mu b_a$ is lacking on the left-hand side of this diagram. The strange graph is proportional to p_ν since the vector index of A^p_ν in $< A^{p,\mathrm{ren}}_\nu(y) c^q_{\mathrm{ren}}(y) b^{\mathrm{ren}}_a(x) >$ is not contracted while the graph depends only on the momentum p. Omitting a common factor p_ν one then has the following four contributions.

$(Z_3^{\mathrm{gh}} - 1)$  $= 0$ (5.5.11)

It is easy to check that the first two terms (which are both proportional to $(Z_3^{\mathrm{gh}} - 1)$) cancel. However, also the sum of the two diagrams with a loop graph cancels as we now show. These two diagrams have the following structure

$$\partial_\nu c(-\partial^\rho b\, A_\rho c)(-\partial b A c) b + (A_\nu c)(-\partial b A c)\ \ b \tag{5.5.12}$$

The first term reads more explicitly

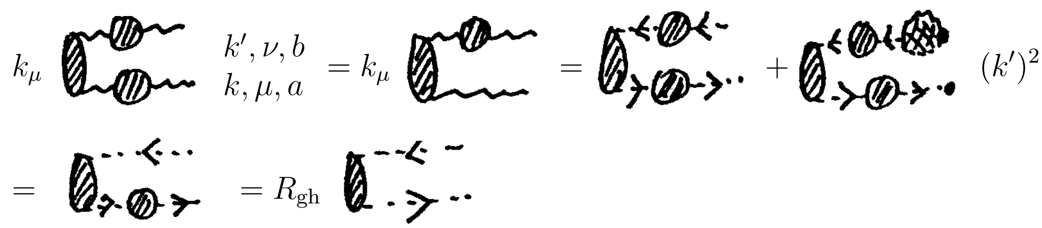
$$\begin{aligned}
&\frac{(i)^2}{2!}\partial_\nu c(y)\left[\int -\partial^\rho b A_\rho c\right]\left[\int -\partial^\sigma b A_\sigma c\right] b(x) \\
&= i^2(ip_\nu)\int(-ip^\rho)\eta_{\rho\sigma}(-iq^\sigma)\frac{-i}{p^2}\frac{-i}{q^2}\frac{-i}{(q-p)^2}\frac{-i}{p^2}d^4q \\
&= ip_\nu\frac{1}{p^2}\int\frac{p\cdot q}{q^2(q-p)^2}\frac{1}{p^2}d^4q
\end{aligned} \tag{5.5.13}$$

The second term reads

$$\begin{aligned}
&A_\nu(y)c(y)\left[i\int -\partial^\rho b A_\rho c\right] b(x) \\
&= \eta_{\nu\rho}\int(-q^\rho)\frac{-i}{(q-p^2}\frac{-i}{q^2}\frac{-i}{p^2}d^4q \\
&= -\frac{i}{p^2}\int\frac{q_\nu}{q^2(q-p)^2}d^4q
\end{aligned} \tag{5.5.14}$$

Combining $[q^2(q-p)^2]^{-1} = \int_0^1 dx[(q-px)^2 + p^2x(1-x)]^{-2}$, we set $q - px = \kappa$ and replace $p\cdot q$ in the first term by p^2x, and q_ν in the second term by $p_\nu x$. Then it becomes clear that the sum of both terms cancels. At the two-loop level one finds similar cancellations between graphs with the same number of loops and Z factors, and it is a good and amusing exercise to check this again graphically. In fact, the cancellation of loop graphs happens separately from cancellations of Z factors. Thus, already at the unrenormalized level, the strange graphs cancel the ghost selfenergies.

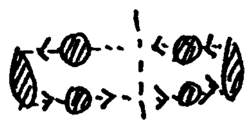
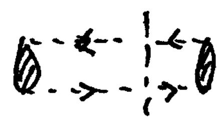
Having verified the Ward identity for connected off-shell propagators, we now return to the Ward identity with one or more external physical fields and two cut lines. The momenta of the incoming and outgoing fields on one side of the cut are now no longer equal, but the gauge boson selfenergy is still transversal, and the cancellation of ghost selfenergies and the strange graphs on the outgoing line still holds. However, there is a difference between the cases of scalars and physical gauge bosons: for the graphs with ghosts we cannot absorb factors $\sqrt{R}$ into the S matrix because there are no S matrix elements with unphysical external particles. Rather, the graphs with ghosts due to applying the Ward identity and cycling momenta around, must cancel the graphs with cut ghost lines. From the graph with two gauge fields one finds on one side of the cut the following set of identities



(5.5.15)

where $R_{\rm gh}$ denotes the residue of the connected ghost propagator. At the bottom a gauge field with momentum k_μ is incoming, and only the tree graph remains due to transversality. At the top only the tree graph ghost line is outgoing because all other corrections cancel, as we have explicitly checked for the ghost propagator before. We are left at the bottom with a full connected ghost propagator which for on-shell momentum becomes equal to $R_{\rm gh}$ times the tree propagator for the ghost. Multiplication by inverse propagators (factors k^2 and k'^2) removes the tree propagators at the bottom and at the top. The net result is that the two-particle cut through two gauge boson lines is equal to a two-particle cut through two tree propagators of the ghosts, multiplied by a factor R_{gh}.

On the other hand, the graphs with cut ghost lines are of the same kind as the graphs with scalars, hence on-shell we can associate with each part of a cut ghost line a factor $(R_{\mathrm{gh}})^{1/2}$, where R_{gh} is the (finite!) residue of the connected ghost propagator. This leaves four factors $(R_{\mathrm{gh}})^{1/2}$ in each of the two diagrams with two cut ghost lines.

 $= R_{\mathrm{gh}}$  R_{gh} (5.5.16)

Taking into account the extra factor -1 for a closed loop, all remaining ghost contributions clearly cancel because they are all products of graphs with cut tree lines multiplied by the same factors R_{gh}^2. This concludes the proof of unitarity for unbroken or broken renormalized gauge theories.

6 Locality of counter terms, causality and statistics

As a second application of the largest time equation we shall investigate under which conditions the divergences of proper graphs are spacetime integrals of local polynomials in the fields and derivatives thereof. This is clearly necessary for multiplicative renormalization. Assume that all proper graphs with up to $(n-1)$ loops have been made finite by the renormalization procedure. Then we shall show that the divergences of proper graphs with n-loops are local polynomials under certain conditions. The proof will use an expression for any given Feynman diagram as a dispersion integral with an integration variable p_0, whose integrand is a sum of cut diagrams, each cut diagram being a product of two Green's functions with k loops and ℓ loops, respectively, where $k+\ell<n$. By assumption each of these Green's functions is a

finite (in general nonlocal) expression in terms of the incoming, outgoing and intermediate momenta and the integration variable p_0. Since the integration region of the intermediate momenta is finite for finite total incoming or outgoing energy, the only divergences can come from the final integration over p_0. Applying this identity to proper diagrams, we can make the p_0 integral finite by expanding the integrand in terms of the external momenta and subtracting a few times. These subtraction "constants" are polynomials in the momenta and correspond to the local polynomials which constitute the counter terms. This proves then that the divergences are indeed spacetime integrals of polynomials in fields and derivatives of fields. We now give the proof.

Consider an arbitrary Feynman diagram with N vertices, and isolate two vertices with coordinates x_i and x_j such that x_j is later than x_i

$$x_j{}^0 > x_i{}^0 \tag{5.6.1}$$

Then the sum of all graphs with all vertices circled in all possible ways except that x_i is never circled still cancels pairwise, because x_i can never have the largest time. Hence, these $\frac{1}{2}2^N$ graphs satisfy

$$\sum_{\text{all circles except } x_i} F\left(x_1, \ldots, x_N\right) = 0 \text{ if } x_j{}^0 > x_i{}^0 \tag{5.6.2}$$

Similarly, for $x_i{}^0 > x_j{}^0$ we have

$$\sum_{\text{all circles except } x_j} F\left(x_1, \ldots, x_N\right) = 0 \text{ if } x_i{}^0 > x_j{}^0 \tag{5.6.3}$$

Adding (5.6.2) and (5.6.3), and separating off the term with no circles we find

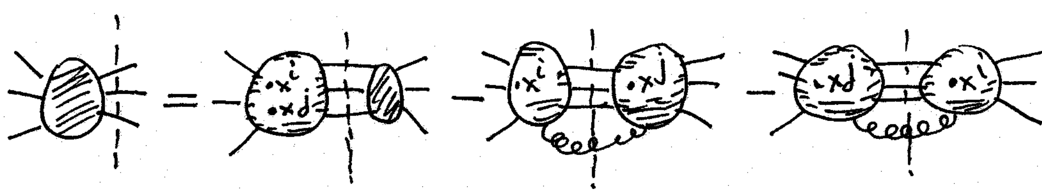
$$\begin{aligned} F\left(x_1, \ldots, x_N\right) &= - \sum_{\substack{\text{all circles except } x_i \\ \text{but at least one circle}}} \theta(x_j{}^0 - x_i{}^0) F(x_1, \ldots, x_N) \\ &\quad - \sum_{\substack{\text{all circles except } x_j, \\ \text{but at least one circle}}} \theta\left(x_i{}^0 - x_j{}^0\right) F\left(x_1, \ldots, x_N\right) \end{aligned} \tag{5.6.4}$$

The $\left(\frac{1}{4}2^N - 1\right)$ graphs on the right-hand side with neither x_i nor x_j circled have as coefficient $-\theta({x_j}^0 - {x_i}^0) - \theta({x_i}^0 - {x_j}^0) = -1$. Hence

$$\begin{aligned} F(x_1, \ldots, x_N) = &- \sum_{\substack{\text{all circles except } x_i \text{and } x_j, \\ \text{but at least one circle}}} F(x_1, \ldots, x_N) \\ &-\theta({x_j}^0 - {x_i}^0) \sum_{x_j \textit{ circled, } x_i \textit{ not circled}} F(x_1, \ldots, x_N) \\ &-\theta({x_i}^0 - {x_j}^0) \sum_{x_i \textit{ circled, } x_j \textit{ not circled}} F(x_1, \ldots, x_N) \end{aligned} \tag{5.6.5}$$

The last two terms contain each $\frac{1}{4}2^N$ graphs.

The first term on the right-hand side represents the set of cut diagrams where x_i and x_j both lie to the left of the cut (the region with uncircled vertices). The second term corresponds to the set of cut diagrams with x_j on the right-hand side and x_i on the left-hand side, and in the last term x_i and x_j are interchanged. Diagrammatically, this "two largest times equation" can be depicted as follows



(5.6.6)

The wiggly lines denote the θ functions, $\theta(x_j^0 - x_i^0)$ in the third graph and $\theta(x_i^0 - x_j^0)$ in the last (fourth) graph. Note that in the last two graphs the later time always lies to the right of the cut. This will be important.

The θ function can be written as

$$\theta(z^0) = \int_{-\infty}^{\infty} \frac{e^{ip_0 z^0}}{p_0 - i\epsilon} \frac{dp_0}{2\pi i} = \int \frac{e^{ipz}\delta^3(\vec{p})}{2\pi i(p_0 - i\epsilon)} d^4p \tag{5.6.7}$$

with $pz = \vec{p} \cdot \vec{z} + p_0 z^0$, as follows easily by closing the contour in the upper (lower) complex p_0 plane if z^0 is positive (negative). This suggests to view the θ function as one extra propagator in momentum space, given by $\frac{1}{2\pi i} \frac{\delta^3(\vec{p})}{p_0 - i\epsilon}$. We have indicated this propagator by a wiggly line in the figure. In the graphs with a θ-propagator, energy $-p_0$ flows from left to right.

As a check on the two-largest time equation, and to become familiar with the Feynman graphs with these new kinds of propagators, we consider the case of $N = 2$, i.e., two points $x_1 = x$ and $x_2 = y$. Using

$$\begin{aligned}
\Delta_F(x-y) &= \frac{1}{(2\pi)^4}\int \frac{-i}{k^2+m^2-i\epsilon}\, e^{ik(x-y)}\, d^4k \\
\Delta^-(x-y) &= \frac{1}{(2\pi)^3}\int \frac{e^{-i\vec{k}(\vec{x}-\vec{y})+i\omega(x^0-y^0)}}{2\omega} d^3k \\
&= \frac{1}{(2\pi)^3}\int e^{-ik(x-y)}\delta(k^2+m^2)\theta(-k_0)d^4k \\
&= \frac{1}{(2\pi)^3}\int e^{ik(x-y)}\delta(k^2+m^2)\theta(k_0)d^4k
\end{aligned} \tag{5.6.8}$$

one finds the following result for the two-largest time equation

$$i\Delta_F(x-y)i = -i\Delta^-(x-y)\theta(y^o-x^o)(-i) - i\Delta^-(y-x)\theta(x^o-y^o)(-i) \tag{5.6.9}$$

where the factors $(-i)$ are due to circled vertices. The first term on the right-hand side of (5.6.6) is absent since there are only two points in this example, so there is no propagator which can be cut in this term. Using $\Delta^+(x-y) = \Delta^-(y-x)$, we see that (5.6.9) is just the decomposition of Δ_F into Δ^+ and Δ^-. It is instructive to work this relation out in momentum space

$$\begin{aligned}
(i^2)\Delta_F(x-y) = &-\left(\int \frac{1}{(2\pi)^3} e^{-iq(x-y)}\delta(q^2+m^2)\theta(-q_0)d^4q\right)\left(\int_{-\infty}^{\infty} \frac{e^{ip(y-x)}}{p_0-i\epsilon}\delta^3(\vec{p})\frac{d^4p}{2\pi i}\right) \\
&-\left(\frac{1}{(2\pi)^3} e^{iq(x-y)}\delta(q^2+m^2)\theta(-q_0)d^4q\right)\left(\int_{-\infty}^{\infty} \frac{e^{ip(x-y)}}{p_0-i\epsilon}\delta^3(\vec{p})\frac{d^4p}{2\pi i}\right)
\end{aligned} \tag{5.6.10}$$

Defining $-q-p = k$ and $d^4q = d^4k$ in the first term on the right-hand side, and $q+p = k$ and $d^4q = d^4k$ in the second term, we obtain after integration over $\vec{p}$ and suppressing an overall factor $i\int d^4k/(2\pi)^4 e^{ik(x-y)}$

$$\frac{1}{k^2+m^2-i\epsilon} = \int_{-\infty}^{\infty}\left[\delta((p+k)^2+m^2)\frac{\theta(k_0+p_0)}{p_0-i\epsilon} + \delta((k-p)^2+m^2)\frac{\theta(p_0-k_0)}{p_0-i\epsilon}\right] dp_0 \tag{5.6.11}$$

where $\vec{p} = 0$. In the first term on the right-hand side we make the substitution $p_0 \to -p_0$. Then we get

$$\frac{1}{k^2+m^2-i\epsilon} = \int_{-\infty}^{\infty}\delta((k-p)^2+m^2)\left[\frac{\theta(k_0-p_0)}{-p_0-i\epsilon} + \frac{\theta(p_0-k_0)}{p_0-i\epsilon}\right] dp_0 \tag{5.6.12}$$

The first term on the right-hand side vanishes if $p_0 > k_0$ while the second term vanishes if $p_0 < k_0$. In more complicated examples, one integrates over the position of the vertices, and finds then a function $f(p_i, q_j, p_0)$ instead of $(\delta(k-p)^2+m^2)$ where p_i and q_j are the incoming and outgoing momenta. However, the factor in square brackets involving the θ functions and the integral over p_0 is still the same in the general case.

In the simple case of (5.6.12) the integration over p_0 can be performed using the delta function, and yields $k_0 - p_0 = \pm\omega$ where $\omega = (\vec{k}^2 + m^2)^{1/2}$. Then agreement is found

$$\frac{1}{k^2+m^2-i\epsilon} = \frac{1}{2\omega}\left(\frac{1}{-k_0+\omega-i\epsilon} + \frac{1}{k_0+\omega-i\epsilon}\right) = \frac{1}{(\omega-i\epsilon)^2-{k_0}^2} \tag{5.6.13}$$

where $\omega^2 = \vec{k}^{\,2} + m^2$.

Having acquired some experience with these new diagrams, we return to the two-largest time equation. Integration over $dx_1 \dots dx_N$ leads to energy (and momentum conservation) at each vertex. In particular, at the vertices x_i and x_j the total energy is conserved but in the graphs with the θ functions this involves also p_0. As illustrated by (5.6.12), the two-largest-times equation becomes in momentum space a dispersion integral

$$F(p_i, q_j) = f(p_i, q_j) - \int_{-\infty} \frac{f^+(p_i, q_j, p_0)}{-p_0 - i\epsilon}\,\frac{dp_0}{2\pi i} - \int^{+\infty} \frac{f^-(p_i, q_j, p_0)}{p_0 - i\epsilon}\,\frac{dp_0}{2\pi i} \tag{5.6.14}$$

where $f(p_i, q_j)$ denotes the cut graphs without θ functions. These are finite expressions in terms of the incoming momenta p_i and outgoing momenta q_j. The range of integration of p_0 in the first integral is from $-\infty$ up to some value, whereas in the second integral p_0 ranges from this value to $+\infty$, just as in (5.6.12).[16] Integrals over the 3-momenta of the intermediate states are finite, as their phase space is finite for given total incoming energy. (Even for massless particles there are only a finite number of

[16]In (5.6.6) the energy $E \geq 0$ of the cut propagators and the energy p_0 of the θ-propagator flow from left to right and add up to the total incoming energy E_{in}. Hence $p_0 = E_{in} - E$.

intermediate particles since we consider only n-loop graphs. Since the energy of each massless particle is still bounded by the total incoming energy, the phase space for integration over intermediate states is also in this case finite). Similarly, f^+ and f^- are finite functions.

For a general cut graph without θ functions, energy flows from left to right across the cut. At each vertex, energy (and of course momentum) is conserved. At the vertices with a theta function, "energy" p_0 is emitted from right to left. This means that p_0 cannot be too large and positive in the term with f^+, while the term with f^- vanishes for large negative p_0. This is already clear from (5.6.12), and indicated by the boundary values on the p_o integrals in (5.6.14).

If all cut lines correspond to massive particles, and the integrands of Δ^- are regular (certainly the case for the relativistic gauges $\partial^\mu A_\mu{}^a = 0$), the only divergences in $F(p_i, q_j)$ can come from the dispersion integral over p_0. A finite number of subtractions in terms of the external momenta make this dispersion integral for the proper graph convergent. (The proof of this plausible fact requires the rather complicated "Weinberg theorem" [24]). As explained before, this proves that the counter terms are polynomials. Note that this proof does not need Zimmerman's forest formula; all we needed were a few elementary identities for graphs with Feynman propagators, and Weinberg's theorem.

The dispersion integral representation of proper graphs can be used to explicitly compute loop corrections. By combining the two noncovariant integrals over p_0, one obtains a relativistically invariant dispersion relation [3]. This has been used to evaluate certain Feynman graphs [3].

We now discuss causality in perturbative field theory. Causality is one of the least-understood general principles of a quantum field theory. For example, which experiment can show up violation of causality? For unitarity the situation is much better: if the sum of probabilities adds up to unity, unitarity is preserved in such an experiment.

One definition of causality is that fields with timelike separations must commute. Related is the requirement that the propagator must vanish inside the light-cone. Another property which is familiar from Feynman's analysis of QED, but which actually goes back to Bogoliubov, is the condition that Green's functions for two points x_1 and x_2 with x_1 inside the future light cone of x_2, can be written only in terms of functions with positive energy flow from x_2 to x_1.

This last condition can easily be proven to be satisfied once the cutting rules hold, i.e., once the propagator is Feynman's propagator (with the decomposition into $\theta(x^0 - y^0)\Delta^+(x-y) + \theta(y^0 - x^0)\Delta^-(x-y)$). To demonstrate this, we return to the two largest times equation (5.6.6), and observe that the last two terms already satisfy Bogoliubov's criterion. The term with cut graphs without theta functions contains a product of a lower-loop Green's function with x_i and x_j and the complex conjugate of another Green's function. Apply now again the two largest times equation to the first of these two Green's function. Repeating this procedure, one arrives after a finite number of steps at the case where no term without θ function is left. Then the original Green's function is a sum of two sets of terms, one set containing an overall factor $\theta({x_1}^0 - {x_2}^0)$ and propagators $\Delta^-(x_2 - x_1)$ while the other set contains an overall factor $\theta({x_2}^0 - {x_1}^0)$ and propagators $\Delta^-(x_1 - x_2)$. Since $\Delta^-(x-y)$ describes positive energy flow from x to y, the original graph is causal in the sense of Bogoliubov.

Another definition of causality, sometimes called microcausality, is that the commutator or anticommutator of two fields vanishes if these fields are at points x and y which have a spacelike separation. We shall study both commutators and anticommutators for bosons and fermions.

$$[A(x), A(y)] = 0 \text{ or } \{\psi(x), \psi^\dagger(y)\} = 0 \text{ if } (x-y)^2 > 0 \tag{5.6.15}$$

The spin-statistics connection is a consequence of microcausality as we now show for bosons and fermions.

We begin with scalar fields. Assuming that the real scalar fields satisfy commutation relations, one finds

$$[\varphi(x), \varphi(y)] = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega} \left\{ \left[a(\vec{k}), a^\dagger(\vec{k}) \right] e^{ik(x-y)} + \left[a^\dagger(\vec{k}), a(\vec{k}) \right] e^{-ik(x-y)} \right\} \tag{5.6.16}$$

If the scalar fields are anticommuting, one must consider complex fields in order that the action not vanish. The result is proportional to

$$\int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega} \left(e^{ik(x-y)} \mp e^{-ik(x-y)} \right) \tag{5.6.17}$$

where the upper (lower) sum refers to commutators (anticommutators). To regulate these commutators we construct small wave packets

$$\bar{\varphi}(\vec{x}_0, t) \equiv \int \frac{e^{-(\vec{x}-\vec{x}_0)^2/b^2}}{(\pi b^2)^{3/2}} \, \varphi(\vec{x}, t) d^3x \tag{5.6.18}$$

This leads to an extra factor $\exp(-\frac{1}{2}b^2 \; \vec{k}^2)$ in (5.6.17). One finds then for $b \to 0$ that $[\bar{\varphi}(x_0), \bar{\varphi}(y_0)]$ is proportional to $\delta(\Delta x - \Delta t) - \delta(\Delta x + \Delta t) \sim \delta((x-y)^2)$ where $\Delta x = |\vec{x}_0 - \vec{y}_0|$ and $\Delta t = x^0 - y^0$. However, for the case of anticommutation rules one finds if one lets $b \to 0$ at the end of the calculation

$$\{\bar{\varphi}(x), \bar{\varphi}(y)\} = \frac{1}{(\vec{x}-\vec{y})^2 - (x^0 - y^0)^2} \tag{5.6.19}$$

This is clearly not a causal result.

For photons, there is a small subtlety. Imposing the radiation gauge $\partial^k A_k = 0$, and eliminating A_0 from its field equation, one finds (by eliminating A_3 in terms of A_1 and A_2 from the Maxwell equation) that the equal-time canonical commutation relations read $[A_i(x,t), \dot{A}_j(y,t)] = (\delta_{ij} - \partial_i \partial_j / \partial_k^2) \delta(\vec{x} - \vec{y})$, in agreement with $\partial^j A_j = 0$. Expanding the photon field as usual

$$A_i(\vec{x}, t) = \sum_{\vec{k}} \sum_{r=1}^{2} \left(\frac{\hbar c^2 4\pi}{2\omega V} \right)^{1/2} \left[a^r(\vec{k}) \epsilon_i^r(\vec{k}) e^{ikx} + h.c. \right] \tag{5.6.20}$$

where $\vec{k} \cdot \vec{\epsilon}^{\,r} = 0$ and $\sum_{i=1}^{2} \epsilon_i^r(\vec{k})\epsilon_i^s(\vec{k}) = \delta^{rs} - k^r k^s/(\vec{k})^2$, one obtains $[a^r(\vec{k}), a^s(\vec{k}')^\dagger]$ $= \delta^{rs}\delta(\vec{k} - \vec{k}')$. For the regulated commutators one finds

$$[\bar{A}_i(x), \bar{A}_j(y)] = \hbar c\, 4\pi \left(\delta_{ij} - \partial_i\partial_j/\partial_k^2\right) D(x-y;b) \tag{5.6.21}$$

where the function $D(x-y;b)$ for $b \to 0$ tends to $\delta((x-y)^2)$. Due to the nonlocality, this result is also noncausal. However, this noncausality is a gauge artefact. For the gauge-invariant electric and magnetic fields one finds local, and hence causal, commutators

$$\begin{aligned}
[E_i(x), E_j(y)] &= (\delta_{ij}\partial_k^2 - \partial_i\partial_j)\delta(x-y)^2 \\
[E_i(x), B_j(y)] &= -\epsilon_{ijk}\partial_k\partial_t\delta(x-y)^2 \\
[B_i(x), B_j(y)] &= (\delta_{ij}\partial_k^2 - \partial_i\partial_j)\delta(x-y)^2
\end{aligned} \tag{5.6.22}$$

Finally we consider fermions. For a Dirac field with $\mathcal{L} = -\alpha\hbar c\bar{\psi}(\partial^\mu\partial_\mu + \frac{mc}{\hbar})\psi$ where α is a free constant which will be studied below, the Hamiltonian density[17] is $\mathcal{H} = \dot{\psi}\pi - \mathcal{L}$

$$\mathcal{H} = \alpha\, \hbar c\, \bar{\psi}\left(\partial^k\partial_k + \frac{mc}{\hbar}\right)\psi = \alpha i\hbar\, \psi^\dagger\dot{\psi} \tag{5.6.23}$$

Expanding $\psi^\alpha(\vec{x}, t)$ in terms of a complete set of solutions

$$\psi^\alpha(\vec{x}, t) = \frac{1}{\sqrt{V}} \sum_{\vec{k}} \sum_{r=1}^{2} \left\{ c^r(\vec{k}) u_r^\dagger(\vec{k}) e^{ikx} + d^r(\vec{k})^\dagger u_r^-(-\vec{k}) e^{ikx} \right\} \tag{5.6.24}$$

one finds

$$H = \alpha \sum_{\vec{k},r} \hbar\omega \left(c^r(\vec{k})^\dagger c^r(\vec{k}) - d^r(\vec{k}) d^r(\vec{k})^\dagger \right) \tag{5.6.25}$$

The canonical momentum is

$$\pi_\alpha(\vec{x}, t) = \frac{\partial}{\partial\dot{\psi}^\alpha} \int \mathcal{L} = \pm\alpha i\hbar\psi^\dagger \tag{5.6.26}$$

[17]We fix our conventions by defining momenta as left-derivatives, $\pi = \frac{\partial}{\partial\dot{\psi}} L$. The ordering of $\dot{\psi}$ and π in $\dot{\psi}\pi$ is then fixed by requiring that δH be independent of $\delta\dot{\psi}$. The commutator $[p, q] = -i\hbar$ would suggest $\{\pi, \psi\} = -i\hbar$, whereas $[q, p] = i\hbar$ would suggest $\{\psi, \pi\} = i\hbar$. The correct choice is $\{\pi, \psi\} = -i\hbar$; this follows from compatibility with the Heisenberg equations $[H, \psi] = i\hbar\dot{\psi}$.

where the upper (lower) sign refers to commuting (anticommuting fields).

Imposing canonical anticommutators, one finds $\alpha\left\{\psi(\vec{x},t),\psi^\dagger(\vec{y},t\right\} = \delta(\vec{x}-\vec{y})$. This implies $\alpha\left\{c,c^\dagger\right\} = \alpha\left\{d,d^\dagger\right\} = 1$. Both the states $c^\dagger|0>$ and the states $c|0>$ have positive norms if and only if $\alpha > 0$. We can then scale ψ such that $\alpha = 1$, the usual case. However, the Hamiltonian $H = \sum \hbar\omega(c^\dagger c - \frac{1}{2} + \sum \hbar\omega(d^\dagger d - \frac{1}{2})$ is only positive definite if $c^\dagger$ and $d^\dagger$ create one-particle states, and c and d annihilate the vacuum. (The zero point energies are $-\frac{1}{2}\hbar\omega$ and have opposite signs from those for bosons. In supersymmetric theories, all these $+\frac{1}{2}\hbar\omega$ and $-\frac{1}{2}\hbar\omega$ cancel, leading to a vanishing cosmological constant).

Let us now see what conclusions one can reach if one imposes canonical commutation relations for Dirac fields. Now the basic relation $[\pi,\psi] = -i\hbar\delta(\vec{x}-\vec{y})$ leads to $\alpha[\psi^\dagger,\psi] = -\delta(\vec{x}-\vec{y})$. So now $\alpha[c^\dagger,c] = -1$ and $\alpha[d,d^\dagger] = -1$. The states $c^\dagger|0>$ have positive norm if $\alpha > 0$, whereas the states $c|0>$ have positive norm for $\alpha < 0$. However, the Hamiltonian is only positive definite if $c^r(\overrightarrow{k})$ are the operators which annihilate the vacuum. However, in the d sector one finds that (for $\alpha > 0$) only states $d|0>$ have positive norm, hence $d^\dagger|0>$ defines the vacuum. Then the Hamiltonian $H \sim -\alpha dd^\dagger$ (with $\alpha > 0$) is not positive definite. Thus the requirements of positive norms and positive energy for physical states rule out commutation relations for fermions: the spin-statistics connection follows from positivity of norms and energy.

Instead of requiring positive norms and positive energies, one can impose microcausality for fermions, just as we did before for scalars and photons. One finds then that the requirement of microcausality selects anticommutation relations for fermions. The reason is that the conjugate momentum for bosons is $\dot{\varphi}$ and the $\frac{\partial}{\partial t}$ in $\frac{\partial}{\partial t}\varphi$ leads to a minus sign in $-i\omega\varphi$, but for fermions there is, of course, no $\frac{\partial}{\partial t}$ in the definition of canonical momentum, and hence the bracket must provide a different sign.

7 Gauge-choice independence of the S-matrix

Renormalizability and unitarity are two crucial properties of a quantum gauge field theory. There is a third, also important property, the independence of S matrix elements from the choice of gauge fixing terms. It should not be confused with what is sometimes called gauge invariance of the S-matrix. The latter is the statement that in QED S-matrices vanish when one replaces in one or more external photons the polarization vector by its 4-momentum k^μ [?]. In QCD this symmetry property is still true but graphs with ghosts must be added [?]. One may use the BRST Ward identities for connected graphs to prove these relations, but one still refers to these Ward identities as gauge invariance of the S matrix. In this chapter we do not discuss gauge invariance of the S-matrix, but rather the gauge-choice independence of the S-matrix.

One can use the independence of the choice of gauge fixing term to go from a renormalizable gauge to a unitary gauge, or vice-versa. Thus one need not explicitly prove unitarity if one already knows that the S matrix is renormalizable and gauge-choice independent.[18] However, it gives great insight to construct explicit proofs both of renormalizability and of unitarity. We proved the renormalizability of nonabelian (and abelian) gauge theories by using Ward identities for proper vertices, and the unitarity by using cutting rules and Ward identities for connected graphs. Both kinds of Ward identities followed from the BRST invariance of the quantum action. Similarly we shall prove the gauge-choice independence of the S-matrix using Ward identities for connected graphs, but these will be derived from an extension of the usual BRST symmetry. We shall use path integrals to derive the required Ward identities for the dependence of S matrices on different gauge-fixing functions, but we shall also use Feynman graphs to clarify the abstract identities.

[18] Applications of this procedure appeared in [11] in a proof of unitarity, where unitarity-violating poles in the S-matrix were encountered, but shown to actually be absent because under a variation of parameters in the gauge fixing functions these poles moved.

In the literature one often restricts oneself to proofs that the S-matrix is independent of the value of the gauge-fixing parameters ξ in front of the Lorentz gauge-fixing term $\mathcal{L}(\text{fix}) = -\frac{1}{2}(\partial^\mu A_\mu)^2$. For spontaneously broken gauge theories one considers so-called $R(\xi)$ gauges, where R denotes that these gauges are renormalizable. (This means that the theory is power-counting renormalizable. Complete renormalizability must then still be proven, and can be proven as we have shown). For a spontaneously broken gauge theory the action has the generic form

$$\mathcal{L} = -\frac{1}{4}(G^a_{\mu\nu})^2 - \frac{1}{2\xi}\left(\partial^\mu A^a_\mu + \xi(\cdots)\right)^2 - \frac{1}{2}m^2\,(A^a_\mu)^2 + \cdots \tag{5.7.1}$$

and the propagator in the $R(\xi)$ gauge is obtained by inverting the kinetic terms. It reads

$$G^{ab}_{\mu\nu} = \frac{-i\delta^{ab}}{k^2 + M^2 - i\epsilon}\left(\eta_{\mu\nu} - (1-\xi)\,\frac{k_\mu k_\nu}{k^2 + \xi M^2 - i\epsilon}\right) \tag{5.7.2}$$

There are off-diagonal kinetic terms in the classical action which are cancelled by the cross terms in the gauge-fixing term, and these cross terms are ξ-independent as we indicated. Such gauge-fixing terms were invented by 't Hooft [?], and we call them 't Hooft gauge fixing terms. For $\xi = 1$ one has the simplest propagator, with numerator $\eta_{\mu\nu}$, while for any finite ξ the theory is power-counting renormalizable (a renormalizable gauge). However, for $\xi \to \infty$ the propagator becomes

$$G^{ab}_{\mu\nu} = \frac{-i\delta^{ab}}{k^2 + M^2}\left(\eta_{\mu\nu} + \frac{k_\mu k_\nu}{M^2}\right) \tag{5.7.3}$$

and since $\eta_{\mu\nu} + k_\mu k_\nu/M^2 = \sum_{j=1}^{3}\epsilon^j_\mu\,\epsilon^j_\nu$ where ϵ^j_μ are the 3 polarization vectors of a massive vector field, this propagator propagates only physical degrees of freedom. The theory with $\xi \to \infty$ is said to be in the unitary gauge, and is not power counting renormalizable because of the term $k_\mu k_\nu/M^2$ instead of $k_\mu k_\nu/k^2$.

Instead of taking the limit $\xi \to \infty$ in a renormalizable gauge, one may set the would-be Goldstone bosons to zero in the classical action. This can still be done in two ways: for an SU(2) Higgs doublet one may decompose the Higgs fields as follows

$$\begin{pmatrix} \varphi^1 \\ \varphi^2 \end{pmatrix} = \frac{1}{\sqrt{2}}\,(\vec{\sigma}\cdot\vec{\chi} + h_0\,) \tag{5.7.4}$$

and then set $\vec{\chi} = 0$, or one may decompose

$$\begin{pmatrix} \varphi^1 \\ \varphi^2 \end{pmatrix} = e^{\frac{i\vec{\sigma}\cdot\vec{Z}}{2}} \begin{pmatrix} 0 \\ \frac{v+\eta(x)}{\sqrt{2}} \end{pmatrix} \tag{5.7.5}$$

(which is a gauge transform of the gauge with $\vec{Z} = 0$) and then set $\vec{Z} = 0$. Of course these procedures are equivalent because (5) can be rewritten as (4) by writing the exponent as a sum of a cosine and a sine.

There are other gauges which are often used (for example axial gauges for instanton physics and in certain QCD applications), hence it is not sufficient to restrict ourselves to ξ-parameter independence of the S-matrix. We shall give a proof that the S-matrix with gauge fixing function F^A is the same as the S matrix with gauge-fixing function $F^A + \Delta F^A$, where ΔF^A is completely arbitrary. A particular complication we must deal with is that the polarization vectors change when one changes the gauge. This point is frequently overlooked in the literature. Of course, we must first carefully define the S-matrix, and in its definition residue factors $R^{1/2}$ will appear which also depend on the gauge.

It is clear that the proof of gauge-choice independence requires a rather general formalism in order not to get entangled in all details. Such a general formalism is, of course, BRST symmetry. However, the usual BRST symmetry which we used for the proofs of renormalizability and unitarity is insufficient, since it does not yield Ward identities which describe a change of gauge. We shall therefore use an extension of the BRST symmetry which yields information on the change in gauges. We shall use BRST-Ward identities for connected graphs and not for one-particle irreducible graphs (proper graphs). The former yield directly information about the S matrix. The gauge-choice dependence of proper graphs has been studied in [**?**]. Since an S matrix is a tree graph with maximal proper vertices, one can also construct a proof of the gauge-choice independence of the S matrix by analyzing the proper graphs.

Before BRST symmetry was discovered, one proved gauge-parameter independence in Lorentz covariant gauges in QED as follows. Under a variation of ξ, the gauge-fixing function $\mathcal{L}(\text{fix}) = -\frac{1}{2\xi}(\partial \cdot A)^2$ changed into $\delta\xi\,\xi^{-2}(\partial \cdot A)^2$. The same expression is obtained by subjecting $\mathcal{L}(\text{fix})$ to a particular nonlocal gauge transformation

$$\delta_\Lambda A_\mu = \partial_\mu \Lambda; \quad \Lambda(x) = -\delta\xi/\xi \int \Delta(x-y)\,\partial \cdot A\, d^4y \tag{5.7.6}$$

In QCD and electroweak theory, one can follow the same approach, but then one must replace $\Delta(x-y)$ by $[\partial^\mu D_\mu(x-y)]^{-1}$, which is the full nonlinear ghost propagator. Since the product of the measure factor $[\,dA_\mu\,]$ in the path integral and the Faddeev-Popov determinant is invariant under gauge transformations of this kind [4], one arrived at the following Ward identity

$$\sum_j \langle 0|T\phi_1(x_1)\;\cdots\;\delta_\Lambda\,\phi_j(x_j)\;\cdots\;\phi_n(x_n)\,|0\rangle$$
$$= \delta_\Lambda G \;\equiv\; \int \langle 0|\,\phi_1(x_1)\;\cdots\;\phi_n(x_n)\,\delta_\Lambda\,\mathcal{L}\,(\text{fix}, y)\,|0\rangle\; d^4y \tag{5.7.7}$$

Because changing the gauge parameter ξ still leads to gauges with $(\partial^\mu A_\mu)^2$ as gauge fixing function, the polarization tensors do not change in this case. The BRST formalism replaces the expression $\int \Delta(x-y)\,\partial \cdot A\; d^4y$ by the ghost field $c(x)$, and in nonabelian theories it replaces $(\partial^\mu D_\mu)^{-1}\;\partial \cdot A^a$ by c^a. This is clearly an enormous simplification. We shall therefore also base our more general proof of gauge-choice independence of the S-matrix on BRST symmetry.

One can construct a gauge fixing term which interpolates between different gauges F_1 and F_2 as follows

$$\mathcal{L}(\text{fix}) = [(1-p)F_1 + pF_2]^2 \equiv [F^A]^2 \tag{5.7.8}$$

If one can prove that the S matrix is independent of the constant real parameter p, one has proven that it is independent of the functional form of the gauge fixing function. The crucial property on which the proof of gauge-choice independence rests is that the sum of the gauge fixing function and the ghost action is BRST exact

$$\int (\mathcal{L}_{\text{fix}} + \mathcal{L}_{\text{ghosts}} = \int \;\delta_B\;[b_A F^A] \tag{5.7.9}$$

where b_A denotes the antighosts and F^A the gauge fixing function. This simple but important relation is often used to give an oversimplified proof of gauge-choice independence of the S-matrix. One considers

$$\delta_F \langle i|f\rangle = \int \langle\, f \,|\, \delta_B \left(b_A \delta F^A\right)(y) \,|\, i \,\rangle \, d^4y \tag{5.7.10}$$

where $\langle i|$ and $|f\rangle$ are physical states at $t = \infty$ and $t = -\infty$, respectively. By replacing $\delta_B(b_A \delta F^A)$ by $[Q_B, b_A \delta F^A]$ and using that physical states are BRST invariant

$$Q_B|i\rangle = 0, \qquad \langle f|Q_B = 0 \tag{5.7.11}$$

one concludes that $\langle i|f\rangle$ is independent of the choice of F^A. This proof contains the essence of the argument but it misses the complications due the residue factors and the gauge-choice dependence of the polarization vectors.

We introduce an extended BRST symmetry [?] which also acts on the parameter p as follows. The transformation rules $\delta_B A^a_\mu,\ \delta_B c^a,\ \delta_B b_a,\ \delta_B d^a$ of the gauge field, ghost, antighost and auxiliary field are as before, but in addition one defines $\delta_B p = z\Lambda,\ \delta_B z = 0$. Here Λ is the usual anticommuting constant imaginary BRST parameter, but z is a new constant real anticommuting Grasman variable

$$z^2 = 0$$

The pair (p, z) forms a BRST doublet. Clearly this extended BRST symmetry is still nilpotent.

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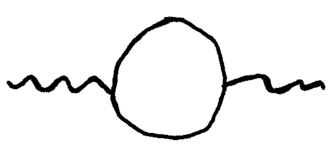
Chapter 6

Anomalies

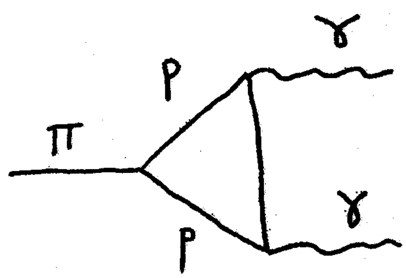
When physicists tried to compute radiative corrections to processes in QED in the 1930's, they of course stumbled on divergences and other inconsistencies. Even the simplest one-loop diagrams presented enormous difficulties, and some physicists (Heisenberg and Pauli at one time or another, and also Dirac and Oppenheimer) blamed QED itself for these difficulties. In the 1940's the problems became more focused. A diagram which exhibited very clearly some difficulties was the photon selfenergy diagram due to an electron loop (we use of course modern terminology)



$$\partial^\mu \langle 0 \mid T j_\mu^{em}(x) j_\nu^{em}(y) \mid 0 \rangle = 0 \ ? \qquad (6.0.1)$$

Gauge invariance required that this diagram be transversal, and on-shell it should vanish because the photon should remain massless, but Tomonaga and collaborators found it to be infinite, as well as not gauge invariant [1]. They studied the e^2 corrections to the Klein-Nishina formula for Compton scattering and reported that "there is an infinity containing [the] electromagnetic potential bilinearly . . . in . . . the vacuum polarization effect. [It] cannot be subtracted by amalgamation [removal by renormalization] as in the case of mass-type and charge-type infinities". This divergence could be identified as a photon mass, but unlike the mass divergence of the electron which could be "amalgamated" (more precisely: multiplicatively renormalized) into an already existing electron mass, the photon mass divergence could not be dealt with in the same way because there is of course no photon mass in Maxwell's equations.[1] Oppenheimer commented in a note attached to this article: "As ... Schwinger and others have shown, the very greatest care must be taken in evaluating such selfenergies lest, instead of the zero value they should have, they give non-gauge covariant, noncovariant [i.e., not Lorentz covariant], in general infinite results .... I would conclude ... [that] ... the difficulties ... result from ... an inadequate identification, of light quantum self-energies." [1]

Motivated by this problem, two of Tomonaga's collaborators, Fukuda and Miyamoto [2], examined the next simplest diagram, namely the triangle diagram.



(6.0.2)

It was supposed to describe the decay $\pi \to p\bar{p} \to \gamma + \gamma$. They considered the cases that the neutral meson (π^0, Yukawa's U particle) was a scalar, pseudoscalar or pseudovector, with couplings $fU\bar{\psi}\psi$, $fU\bar{\psi}\gamma_5\psi$ and $\frac{1}{2}(f/m)\bar{\psi}\gamma_5\gamma^\mu\psi U_\mu$, respectively, where m is the proton mass. They found two problems

1) the results were not gauge invariant since bare A_μ appeared

2) the results for the decay into two photons of a pseudovector U_μ and a pseudoscalar U particle were not the same when they set $U_\mu = \partial_\mu U$, even though the interactions seemed to be the same if one used the Dirac equation. (On-shell $\partial_\mu(\bar{\psi}\gamma_5\gamma^\mu\psi) = -2m\bar{\psi}\gamma_5\psi$).

They concluded: "Evidently these inconsistent results arise from the mathematical

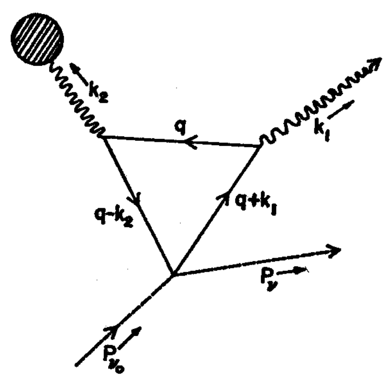
[1]According to the more recent algebraic renormalization approach, one can renormalize masses additively, but that was not known at the time).

difficulty of obtaining [a] definite expression using the singular function of Jordan and Pauli. At present we know [of] no appropriate prescription which makes one free from ambiguities of this kind". The singular function in question was $D(x) = \int \frac{d^3k}{(2\pi)^3} \frac{\sin(kx-\omega t)}{2\omega}$ which appears in the equal-time canonical commutation relations.

Steinberger [3], then a theorist at Princeton, heard from Yukawa (who was visiting Princeton) about the work of Fukuda and Miyamoto (see footnote 11 of his article) and he applied the brand new Pauli-Villars regularization scheme [4] to the triangle graph and an array of other problems. Tomonaga was of course also quite interested in these consistency problems, and with Fukuda, Miyamoto and Miyazima he also applied the Pauli-Villars regularization scheme to the calculation of the triangle graph [5]. The conclusion of these studies was a partial success: the scheme seemed to maintain gauge invariance and Lorentz covariance, and it led to a finite result for the AVV vertex, but the actual value for this finite result seemed to depend on how the calculations were performed, and the equivalence of pseudovector and pseudoscalar couplings was still not established [3]. In modern terms: there was a chiral anomaly! However, this was not yet fully understood at that time. Rather, it seemed to lead to the perplexing conclusion that the lifetime of the neutral pion was ambiguous: "We see that there remains still some ambiguity how to use the regulator, and this ambiguity would be solved only by some experiment which could detect the γ- decay of [the] neutretto" [5]. (Neutretto was another name for π^0).

Schwinger made in 1951 a fresh attack on the problem of gauge invariance of the photon selfenergy and the triangle diagrams. He introduced a regularization scheme (point splitting) which preserved gauge invariance at all intermediate stages. As he wrote in "On gauge invariance and vacuum polarization" [6]: "This paper is based on the elementary remark that the extraction of gauge invariant results from a formally gauge invariant theory is ensured if one employs methods of solution that involve only gauge covariant quantities". He found that the photon selfenergy did vanish on-shell, so gauge invariance was preserved. Moreover, he found that for the triangle graphs the pseudoscalar field gave the same result as the axial vector field namely $\mathcal{L}_{eff} = (\alpha/\pi)(f/m)\pi^0 \vec{E} \cdot \vec{B}$. Since this seems to contradict the existence of a chiral anomaly, we shall briefly elaborate on this point at the end of this subsection.

Despite the problems caused by the chiral anomaly in the triangle graph, perturbative quantum field theory was used for the weak interactions. Here, of course, one encountered divergences, but a way to eliminate these divergences was somewhat accidentally discovered by Rosenfeld in 1963 [7]. He considered electromagnetic properties of neutrinos in the V-A theory, and considered the process $\nu+$ nucleus $\rightarrow \nu+$ nucleus + photon. This led him to a triangle graph with an electron in the loop and two external photons and a V-A vertex.



(6.0.3)

He expanded the amplitude into form factors, some of which were divergent while others were convergent. By imposing electromagnetic gauge invariance (current conservation: replacing A_μ by k_μ one should get zero), he was able to express the divergent form factors in terms of convergent ones. His result contained the chiral anomaly implicitly, but he did not study the implications of his work on the (non) conservation of the axial vector current in π decay.

In the 1950's and 1960's field theory fell from favor, and alternative physical theories took the limelight: Regge theory, the S-matrix program of Chew, and current algebra. Although the first two alternatives were meant to replace field theory, current algebra was intended to describe field theory at the nonperturbative level. It was natural to try to build field theoretical models which gave a representation of current algebras and in which the consistency of current algebra could be tested at the perturbative level. In fact, several of the physicists who worked on current algebras in those days later helped create modern quantum gauge field theory.

One such attempt was a beautiful little article in 1960 by Gell-Mann and Levy on the linear sigma model [8], in which PCAC (the partially conserved axial-vector current relation) was satisfied: $\partial_\mu j_5^\mu = f_\pi m_\pi^2 \pi(x)$ where f_π is the π-decay constant ($93\ MeV$). The PCAC relation was derived from the action at the classical level, but in current algebra applications it was taken as a nonperturbative quantum relation between Heisenberg fields. At that time it was not yet known that one should regularize such expressions (for example by point-splitting) and that one then finds an extra term in this relation, due to the axial anomaly. The model contained, in addition to the nucleons, three massless pions $\pi^\pm, \pi^0$ and a massive scalar meson σ, with an $SO(4)$ symmetry which was spontaneously broken, giving the nucleons a mass. They added a Yukawa coupling which preserved the $SO(4)$ symmetry (realized on ψ and $\bar{\psi}$ as an $SU(2)_L \otimes SU(2)_R$ rigid symmetry). If a term linear in σ was added to the action, this explicit symmetry breaking also gave the pions a mass. This model became obligatory reading for graduate students at the author's Utrecht University. In Stony Brook, B. Lee started studying the renormalization program of spontaneously broken field theories and wrote an influential small book [9] on the renormalization of this model.[2] A graduate student, G. 't Hooft, heard B. Lee at the Cargèse summer school lecture on this topic, and upon returning to Utrecht, he decided to apply these ideas to gauge theories for his Ph.D., with well-known consequences.

In 1969 two important articles were submitted for publication within two weeks from each other, one by Bell and Jackiw [11], and the other by Adler [13]. Bell and

[2]Because there were no direct axial-vector couplings in this model, no problems with the chiral anomaly were encountered. (However, the chiral symmetry between pions and σ meson allowed one to define an axial vector current, and its renormalization was also studied [10]).

Jackiw noted that the amplitude for $\pi^0 \to \gamma\gamma$ could be parametrized as follows

$$T^{\mu\nu}(p,q) = \epsilon^{\mu\nu\alpha\beta} p_\alpha q_\beta T(k^2) \tag{6.0.4}$$

where p and q were the on-shell photon momenta, and $k = p + q$ was the pion momentum. They considered both the case with k^2 off-shell as well as the case with $k^2 + m_\pi^2 = 0$ for an on-shell pion. The amplitude satisfied gauge invariance ($p_\mu T^{\mu\nu}(p,q) = q_\nu T^{\mu\nu}(p,q) = 0$) as well as Bose symmetry ($T^{\mu\nu}(p,q) = T^{\nu\mu}(q,p)$). They noted that Steinberger had already calculated $T(k^2)$ and had found a nonzero result ($T(0) = g4\pi^2/m$). On the other hand Veltman and Sutherland [14] had found from current algebra that $T(0) = 0$ if one used an off-mass-shell pion field that was equal to the divergence of the axial current (PCAC, the partially conserved axial vector current). The puzzle that $T(0)$ should on the one hand be nonvanishing and on the other hand vanish was the problem Bell and Jackiw decided to tackle. They used as a model the linear sigma model which Gell-Mann and Levy had studied before [8]. They noted that the problem was "in the same tradition as that of the photon mass, noncanonical terms in commutators - Schwinger terms - and violations of the Jacobi identity." They claimed that this "demonstrates in a very simple example [the linear σ model] the unreliability of the formal manipulations common to current-algebra calculations", but then they went on to "develop a variation which respects PCAC, as well as Lorentz and gauge invariance, and find that indeed the explicit perturbation calculation also then yields $T(0) = 0$".[3] So, it seemed there was no axial anomaly in their work, although in their appendix they noted the hallmark of an anomaly: "Since the integral is linearly divergent a shift of variable picks up a surface term". The occurrence of a surface term was well-known at that time from a widely used textbook (**??**). It was later realized that the regularization procedure which yielded $T(0) = 0$ amounted to adding a Wess-Zumino term with a pion-photon coupling to

[3] This variation was the old Pauli-Villars regularization scheme, applied to the Steinberger calculation, but with mass-dependent coupling constants for the extra regulator-fermions.

the action which canceled the anomaly[4]. Their work was important because it tried to clarify the issue of ambiguities in quantum field theory at a time when the popularity of quantum field theory was at a low point.

Adler just studied the AVV triangle graph in spinor QED. He eliminated ambiguities by following the method Rosenberg had used, and took the results as they came: " . . . we demonstrate the uniqueness of the triangle diagrams [by imposing vector gauge invariance] . . . and discuss a possible connection between our results and the $\pi^0 \to 2\gamma$ and $\eta \to 2\gamma$ decays . . . [The] partial conservation of the axial-vector current ... must be modified in a well-defined manner, which completely alters the PCAC predictions for the π^0 and the η two-photon decays". Here is the axial anomaly in all its glory: its unavoidability could not be clearer. For further comments by him see (**??**).

There remains of course one historical question: what about Schwinger's treatment of the triangle graph using point-splitting as regularization scheme, and his claim that an axial vector field A_μ gives the same result as a pseudoscalar field P if A_μ is of the form $\partial_\mu P$? Shortly after the discovery of the "ABJ anomaly", Jackiw and Johnson remarked [18] that "Historically, the first derivation of [the anomaly] for **external** electromagnetic fields was given by Schwinger". Adler added a note in proof with the statement "Field-theoretic derivations [of the anomaly equation] have been given by C.R. Hagen (to be published), R. Jackiw and K. Johnson (to be published), and R.A. Brandt (to be published). Jackiw and Johnson point out that the essential features of the field-theoretic derivation, in the case of external electromagnetic fields, are contained in J. Schwinger [our [6]]".

Let us now discuss Schwinger's approach. The matrix element of the divergence of

[4]The regulator masses are due to the coupling of the fields of the original sigma model to regulator fermion fields, and the regulator masses can only become large if the regulator coupling constants become large. The pions then no longer decouple as the regulator fermion masses get large, and this generates new local interactions, coupling the pions to photons. [15]

the axial current is naively (before regularization) given by $\partial_\mu tr\gamma_5\gamma^\mu G(x)$ where $G(x)$ is the full fermion Green function. He used "point-splitting", with gauge-invariant derivatives (see section 5 of [6])

$$\frac{1}{2}\lim_{x',x''\to x}[(\partial'_\mu - ieA_\mu(x')) + (\partial''_\mu + ieA_\mu(x''))tr\gamma_5\gamma^\mu G(x',x'') \tag{6.0.5}$$

because "[this] structure is dictated by the requirement that only gauge covariant quantities be employed" [6]. Nowadays one proceeds in a way which seems equivalent (but is not equivalent as we shall show), namely one adds a Wilson loop which yields a gauge-invariant **current**

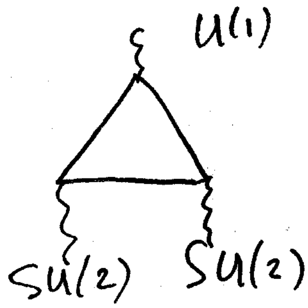
$$\bar{\psi}(x+\frac{1}{2}\epsilon)\gamma_5\gamma_\mu\left(\exp ie\int\limits_{x-1/2\epsilon}^{x+1/2\epsilon} A_\mu dx^\mu\right)\psi\left(x-\frac{1}{2}\epsilon\right) \tag{6.0.6}$$

Because this regulated current is gauge invariant one only needs to use here the ordinary derivatives. As Adler has recently observed [16], the difference between the covariant derivatives and ordinary derivatives in (6.0.5) is **minus** the anomaly, while regulating $G(x',x'')$ in a way to preserve gauge invariance (for example by using a Wilson loop) yields **plus** the anomaly. Thus Schwinger obtained the naive divergence of the axial current **without** anomaly[5]; having understood this point, it becomes clear why he found agreement between the calculation with the axial vector field and the calculation with the pseudoscalar field.

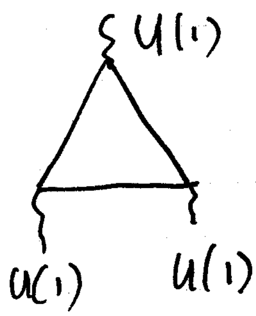
With the demonstration by 't Hooft in 1971 that nonabelian pure gauge theories are renormalizable, it was realized that anomalies would spoil renormalizability and unitarity. [19,20] Thus, one had to make sure that anomalies (more precisely anomalies in the gauge transformations of the effective action with chiral spin-$\frac{1}{2}$ fields, the quarks and leptons) would cancel. In the Standard Model the gauge group $SU(3)$ has no anomalies because it does not couple to chiral quarks, while $SU(2)$ has no anomalies

[5]Phrased differently: he computed the full divergence with anomaly and then subtracted the anomaly. In yet other words, he transported the anomaly from the right-hand side to the left-hand side of the divergence equation.

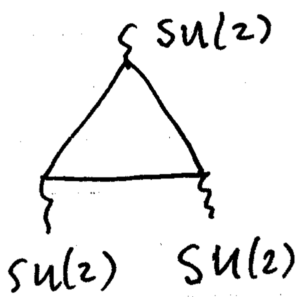
because all of its representations are real or pseudoreal. Only the $U(1)$ hypercharge gauge symmetry is potentially anomalous, but its anomalies cancel because the sum of electric charges of all quarks and leptons in a given family cancels. Thus, the threat of anomalies in the Standard Model was averted.



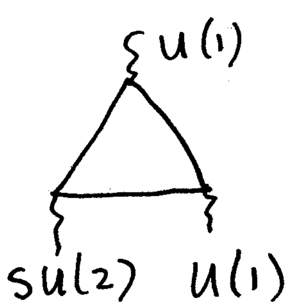
$$An \sim \textstyle\sum_{doublets} q_L^i tr\sigma_a\sigma_b$$
$$\sim \tfrac{1}{6} \times 3 \times 2 + \left(-\tfrac{1}{2}\right) \times 2 = 0$$



$$An \sim \textstyle\sum_{doublets} (q_L^i)^3 - \sum_{singlets} (q_R^i)^3$$
$$= \left(\tfrac{1}{6}\right)^3 \times 6 + \left(-\tfrac{1}{2}\right)^3 \times 2 - \left(-\tfrac{2}{3}\right)^3 \times 3 - \left(\tfrac{1}{3}\right)^3 \times 3 - (1)^3 = 0$$



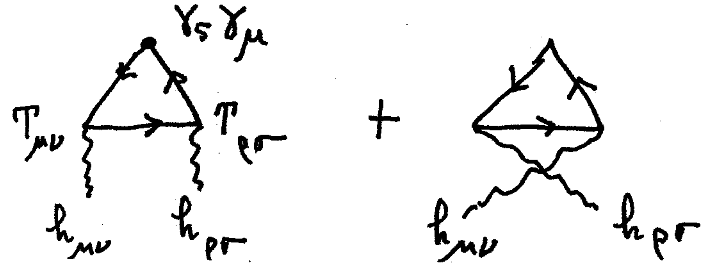
$$An \sim tr\sigma_{(a}\sigma_b\sigma_{c)} = 0$$
The **2** of $SU(2)$ is pseudoreal



$$An \sim \textstyle\sum_{doublets} (q_L^i)^2 tr\sigma_a = 0$$

Figure caption: Triangle graphs with one $U(1)$ gauge field and two $SU(2)$ gauge fields are proportional to the sum of the hypercharges of the left-handed doublets. This sum vanishes for each family: $\frac{1}{6} \times 3 \times 2 + (-\frac{1}{2}) \times 2 = 0$. Furthermore, triangle graphs with three $U(1)$ gauge fields are proportional to the sum of the cubes of the hypercharges of all fermions (rewriting right-handed fermions as charge conjugates of left-handed fermions), which also vanishes for each family: $(\frac{1}{6})^3 \times 6 + (-\frac{2}{3})^3 \times 3 + (\frac{1}{3})^3 \times 3 + (-\frac{1}{2})^3 \times 2 + (1)^3 = 0$. Triangle graphs coupled to three $SU(2)$ gauge fields yield no anomaly because for $SU(2)$ the d-symbol $Tr\sigma^a\{\sigma^b, \sigma^c\}$ vanishes.

Having settled the issue of the chiral anomalies in nongravitational theories, it was realized first by Kimura, and later by Delbourgo and Salam, and then by Eguchi and Freund (who corrected a factor of 2 in the paper by Delbourgo and Salam) that one could also encounter anomalies if one couples spin 1/2 fermions to external gravity instead of external electromagnetism. [21] These authors considered triangle graphs in four dimensions with nonchiral (Dirac) fermions in the loop, with one vertex given by the axial current $\bar{\psi}\gamma_5\gamma_\mu\psi$ and the other two vertices given by $h^{\mu\nu}T_{\mu\nu}$, where $T_{\mu\nu}$ is the stress tensor for fermions.



(6.0.7)

They indeed found anomalies of the form $\epsilon^{\mu\nu\rho\sigma} R_{\mu\nu}{}^{mn} R_{\rho\sigma mn}$ if one sets the metric $g_{\mu\nu}$ equal to $\eta_{\mu\nu} + \kappa h_{\mu\nu}$ and retains the terms quadratic in $h_{\mu\nu}$. A generalization of the gravitational γ_5 anomaly for spin $\frac{3}{2}$ was given in [22, 23].

This chapter is devoted to the properties and implications of the chiral anomaly in renormalizable quantum gauge field theories. Other anomalies (trace anomalies, conformal supersymmetry anomalies, gravitational anomalies, and anomalies in composite operators) are not discussed, nor anomalies in dimensions other than four. For other books on anomalies see [24].

1 The V-A basis and the chiral basis

The existence of anomalies in quantum gauge field theory reveals itself already at the level of simple one-loop graphs, with chiral fermions in the loop coupled to external gauge fields, or ordinary nonchiral Dirac fermions coupled to axial vector gauge fields. Since in the literature either chiral fermions are used, or nonchiral fermions coupled to vector and axial vector fields, we begin be disucssing the relations between these two choices of basis.

Consider left-handed Dirac fermions ψ_L^i and right-handed Dirac fermions ψ_R^j coupled to gauge fields ${W_{\mu,L}}^a$ and ${W_{\mu,R}}^a$, respectively. The action and the transformation rules in the chiral basis read

$$\begin{aligned}
\mathcal{L} &= -\frac{1}{4}\frac{1}{g_L^2}(F^{(L)a}_{\mu\nu})^2 - \frac{1}{4}\frac{1}{g_R^2}(F^{(R)a}_{\mu\nu})^2 - \bar\psi_{i,L}\gamma^\mu {(D_{\mu,L})^i}_j {\psi_L}^j - \bar\psi_{i,R}\gamma^\mu {(D_{\mu,R})^i}_j {\psi_R}^j \\
&- (\bar\psi_{i,L}{M_{LR}{}^i}_j{\psi_R}^j + \bar\psi_{j,R}{M_{RL}{}^j}_i{\psi_L}^i) \\
F^{(L)a}_{\mu\nu} &= \partial_\mu W^{(L)a}_\nu - \partial_\nu W^{(L)a}_\mu + {f^a}_{bc} W^{(L)b}_\mu W^{(L)c}_\nu ; \text{idem for } W^{(R)a}_\mu \\
\bar\psi_{i,L} &= ({\psi_L}^i)^\dagger i\gamma^0, {(D_{\mu,L})^i}_j = \partial_\mu {\delta^i}_j + {W_{\mu,L}}^a {(T_a)^i}_j ; \text{idem } \bar\psi_{i,R} \text{ and } {(D_{\mu,R})^i}_j \qquad (6.1.1)
\end{aligned}$$

where the matrices T_a are antihermitian and satisfy $[T_a, T_b] = {f_{ab}}^c T_c$. We denote the complete set of generators which couple to ψ_L and to ψ_R by T_a. Some generators may couple both to ψ_L and ψ_R as in QED and QCD, and then the corresponding ${W_{\mu,L}}^a$ are equal to the ${W_{\mu,R}}^a$. Other generators may only couple to ψ_L as in the case of the $SU(2) \times U(1)$ electroweak symmetry group. We have rescaled the gauge fields such that the coupling constants only appear as an overall factor in front of the gauge action. This will be useful when we compare with the nonchiral basis. For $M = 0$, this action is invariant under

$$\begin{aligned}
\delta {W_{\mu,L}}^a &= \partial_\mu {\lambda^a}_L + {f^a}_{bc} {W_{\mu,L}}^b {\lambda_L}^c, \text{ idem } \delta {W_{\mu,R}}^a \\
\delta {\psi^i}_L &= -{\lambda^a}_L {(T_a)^i}_j {\psi^j}_L \text{ , idem } \delta {\psi^i}_R \qquad (6.1.2)
\end{aligned}$$

For nonzero $M_{LR} = M_{RL}$, the action is only invariant under the diagonal subgroup with vector gauge symmetry (generated by those generators T_a which couple both to ψ_L^i and ψ_R^i, and for which $W_{\mu,L}{}^a = W_{\mu,R}{}^a$ and $g_L = g_R$).

We can also consider Dirac fermions (complex nonchiral fermions) ψ^i coupled to vector and axial vector gauge fields. Also in this case we scale the gauge fields such that in the Dirac action no coupling constants appear. (The usual gauge fields V and A are thus replaced by V/g_V and A/g_A). The action on this V-A basis reads

$$\begin{aligned}\mathcal{L} &= -\bar{\psi}_i\gamma^\mu(\partial_\mu + V_\mu{}^a T_a^{(V)} + A_\mu{}^a T_a^{(A)}\gamma_5)\psi^i - M\bar{\psi}_i\psi^i \\ &- \frac{1}{4}\frac{1}{g_V^2}(F^a_{\mu\nu}(V))^2 - \frac{1}{4}\frac{1}{g_A^2}(F^a_{\mu\nu}(A))^2; \bar{\psi}_i = (\psi^i)^\dagger i\gamma^0 \end{aligned} \tag{6.1.3}$$

and is hermitian in Minkowski space if V^a_μ and A^a_μ are real. (We recall that γ_5 is hermitian and satisfies $\gamma_5^2 = 1$, while γ^k with $k = 1, 2, 3$ is also hermitian, but γ^0 with $(\gamma^0)^2 = -1$ is antihermitian). We shall define the curvatures $F_{\mu\nu}(V)$ and $F_{\mu\nu}(A)$ later in (6.1.7).

The left- and right- handed fermions in (6.1.1) can be decomposed into Dirac fermions and Dirac fermions multiplied by γ_5 as follows

$$\psi_L = \frac{(1+\gamma_5)}{2}\psi \ , \ \psi_R = \frac{(1-\gamma_5)}{2}\psi, \text{ with } \gamma_5^2 = 1 \tag{6.1.4}$$

More precisely, we can build a Dirac spinor ψ from a chiral two-component spinor ψ_L and an antichiral two-component spinor ψ_R as $\psi = \binom{\psi_L}{\psi_R}$. The action for the chiral fermions can then be rewritten on the V-A basis as follows

$$\mathcal{L} = -\bar{\psi}\gamma^\mu\left(\partial_\mu + V_\mu + A_\mu\gamma_5\right)\psi - M\bar{\psi}\psi \tag{6.1.5}$$

if $M_{LR} = M_{RL} = M$, where

$$\begin{aligned} V_\mu &= \frac{1}{2}(W^L + W^R_\mu) \ , \ A_\mu = \frac{1}{2}(W^L - W^R_\mu) \\ W^L_\mu &= V_\mu + A_\mu \ ; \ W^R_\mu = V_\mu - A_\mu \end{aligned} \tag{6.1.6}$$

Substituting these expressions, the gauge action for the gauge fields in (6.1.1) can be rewritten in terms of the following curvatures for the vector and axial-vector gauge fields

$$\begin{aligned} F^{(V)}_{\mu\nu} &= \partial_\mu V_\nu - \partial_\nu V_\mu + [V_\mu, V_\nu] + [A_\mu, A_\nu] \\ F^{(A)}_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu + [V_\mu, A_\nu] - [V_\nu, A_\mu] \\ &= D_\mu(V)A_\nu - D_\nu(V)A_\mu \end{aligned} \tag{6.1.7}$$

In form notation $F^{(V)} = dV + V^2 + A^2$ and $F^{(A)} = dA + [V, A]$. One easily may check the following relations

$$\begin{aligned} F^{(V)}_{\mu\nu} = \frac{1}{2}F^{(L)}_{\mu\nu} + \frac{1}{2}F^{(R)}_{\mu\nu} \quad &; \quad F^{(A)}_{\mu\nu} = \frac{1}{2}F^{(L)}_{\mu\nu} - \frac{1}{2}F^{(R)}_{\mu\nu} \\ F^{(L)}_{\mu\nu} = F^{(V)}_{\mu\nu} + F^{(A)}_{\mu\nu} \quad &; \quad F^{(R)}_{\mu\nu} = F^{(V)}_{\mu\nu} - F^{(A)}_{\mu\nu} \end{aligned} \tag{6.1.8}$$

Thus the gauge action (6.1.1) can also be written as

$$-\frac{1}{4}\left(\frac{1}{g_L^2} + \frac{1}{g_R^2}\right)\left[\left(F^{(V)}_{\mu\nu}\right)^2 + \left(F^{(A)}_{\mu\nu}\right)^2\right] - \frac{1}{2}\left(\frac{1}{g_L^2} - \frac{1}{g_R^2}\right) F^{(V)}_{\mu\nu} F^{(A),\mu\nu} \tag{6.1.9}$$

The chiral transformation laws in (6.1.2) decompose in the same way

$$\begin{aligned} \delta V_\mu &= D_\mu(V)\lambda_V + [A_\mu, \lambda_A] \; ; \; \delta A_\mu = D_\mu(V)\lambda_A + [A_\mu, \lambda_V] \\ \delta\psi^i &= -(\lambda_V)^i{}_j \psi^j - (\lambda_A)^i{}_j \gamma_5 \psi^j \end{aligned} \tag{6.1.10}$$

where $\lambda^V = \frac{1}{2}(\lambda_L + \lambda_R)$ and $\lambda^A = \frac{1}{2}(\lambda_L - \lambda_R)$, while $D_\mu(V)\lambda = \partial_\mu\lambda + [V, \lambda]$ both for $\lambda = \lambda_V$ and $\lambda = \lambda_A$. These gauge transformations form a closed algebra

$$\begin{aligned} [\delta_V(\lambda_V^{(1)}), \delta_V(\lambda_V^{(2)})] &= \delta_V([\lambda_V^1, \lambda_V^2]) \\ [\delta_A(\lambda_A), \delta_V(\lambda_V)] &= \delta_A([\lambda_A, \lambda_V]) \\ [\delta_A(\lambda_A^{(1)}), \delta_A(\lambda_A^{(2)})] &= \delta_V([\lambda_A^{(1)}, \lambda_A^{(2)}]) \end{aligned} \tag{6.1.11}$$

For QCD and QED, the gauge groups and coupling constants for left-handed and right-handed fermions are equal and $W_{\mu,L} = W_{\mu,R}$, hence for these gauge groups the action is given by (6.1.3) with $A_\mu = 0$. It can be written in the form (6.1.1) with $W_\mu^L = W_\mu^R = V_\mu$ and $g_L = g_R$ with $\frac{1}{g_L^2} + \frac{1}{g_R^2} = \frac{1}{g_V^2}$. For the electroweak $SU(2)$, there is no right-handed sector and the natural way to write the action is (6.1.1), although one can also use (6.1.3) with $g_V = g_A, \frac{1}{2}\left(\frac{1}{g_V^2} + \frac{1}{g_A^2}\right) = \frac{1}{g_L^2}, V_\mu = A_\mu = \frac{1}{2}W_L$, and $F^V_{\mu\nu} = F^A_{\mu\nu} = \frac{1}{2}F^L_{\mu\nu}$.

One can always transform from the chiral basis to the V- and A-basis, and vice-versa. When one discusses for example the decay of a pion into two photons, the V-A basis is more appropriate (since the VVA triangle anomaly is reponsible for this decay), but for general theoretical discussions the chiral basis is simpler. We shall begin by computing the fundamental triangle, box and pentagon anomalies in the V-A basis, but later relate them to the anomalies on the chiral basis.

Consider the hermitian axial vector current

$$j^{5,S}_\mu = -i\bar{\psi}\gamma_5\gamma_\mu S\psi \tag{6.1.12}$$

where S can be unity or the hermitian group generator iT_a. In the first case we call the current a singlet current, in the latter case a nonabelian current. Note that the singlet current can still be defined for a theory with nonabelian gauge fields. The divergence of the current can be computed using the field equations of the action. Note that we are deriving at this point the classical conservation equation using fields which satisfy the field equations; later we shall use this conservation equation as a vertex in graphs and then the fields will be off-shell. Using the nonchiral formulation one finds

$$\begin{aligned}
\partial^\mu j^{5,S}_\mu &= -i\bar{\psi}\gamma_5 S(\not{\partial}\psi) + i(\bar{\psi}\overleftarrow{\not{\partial}})\gamma_5 S\psi \\
&= -i\bar{\psi}\gamma_5 S(-\not{V} - \not{A}\gamma_5 - M)\psi \\
&+ \; i\bar{\psi}(\not{V} + \not{A}\gamma_5 + M)\gamma_5 S\psi \\
&= i\bar{\psi}\gamma_5\gamma^\mu[S, T_a]({V_\mu}^a + {A_\mu}^a\gamma_5)\psi + 2iM\bar{\psi}\gamma_5 S\psi
\end{aligned} \tag{6.1.13}$$

The classical conservation equation thus reads $\mathcal{O} = 0$ where

$$\begin{aligned}
\mathcal{O} &\equiv \partial^\mu j^{5,S}_\mu - i\bar{\psi}\gamma_5\gamma^\mu[S, T_a]({V_\mu}^a + {A_\mu}^a\gamma_5)\psi - 2iM\bar{\psi}\gamma_5 S\psi \\
&\equiv D^\mu(A, V) j^{5,S}_\mu - 2iM\bar{\psi}\gamma_5 S\psi
\end{aligned} \tag{6.1.14}$$

Suppose now that one uses $\mathcal{O}$ as a vertex, and computes a Green function, for example a one-loop proper triangle graph with one vertex given by $\mathcal{O}$. If there were no ultraviolet divergences, one would expect this graph to be zero, because one could first calculate its imaginary part using cutting equations, and then use a dispersion integral to obtain the real part. Since the intermediate particles in the cutting equations are always on-shell, we should be able to use the classical conservation equation, so $\mathcal{O}$ would vanish, and hence the graph would vanish. However, if there are divergences, the dispersion integral will need subtractions, and in this way the matrix element of $\mathcal{O}$ could become nonzero. If it is nonzero, one has an anomaly. Because the anomaly is due to subtractions, it will be local (a polynomical in fields and derivatives) and purely real or purely imaginary (with the definition in (6.1.12) it will be purely imaginary. In gauge theories the axial vector gauge field couples to (6.1.12), so in gauge theories the anomaly is purely imaginary). Strictly speaking, one has only the possibility of an anomaly, because in some cases one can add a local finite counter term $\Delta\mathcal{L}$ to the action whose gauge variation contributes an extra local finite term to the divergence of the current such that the matrix element of $\mathcal{O} + \delta\Delta\mathcal{L}$ vanishes. Then there is no anomaly. We shall prove later that the chiral anomalies in gauge theories are genuine anomalies that cannot be removed by local counter terms that depend only on gauge fields.

Another way to understand how anomalies might arise is to note that inserting $\mathcal{O}$ in a loop graph, the zero of $\mathcal{O}$ which would be present if the fields in $\mathcal{O}$ were on-shell is compensated by a divergence due to the loop integral, and "$0 \times \infty =$ anomaly".

We shall now work these ideas out in a series of examples, which together cover all cases of chiral anomalies in renormalizable quantum field theories in four dimensions. There are further anomalies, for example rigid scale anomalies which are present if and only if the β function is nonzero. In supersymmetric theories one has the possibility of conformal supersymmetry anomalies. These will not be discussed in this chapter. There are also local scale anomalies (Weyl anomalies) and anomalies in general coordinate transformations (Einstein anomalies) or in local Lorentz transformations, but these appear in theories with gravity and will also not be discussed. There are also chiral anomalies in other (even) dimensions. In particular in 2 and 10 dimensions, the cancellation of anomalies is an important subject in string theory [25], but we shall not discuss these issues either. They have been discussed in a textbook [26].

For the singlet anomaly with $S = 1$, the anomalies are due to the vertex

$$\mathcal{O}(\text{singlet}) = -i\partial^\mu(\bar\psi\gamma_5\gamma_\mu\psi) - 2iM\bar\psi\gamma_5\psi \tag{6.1.15}$$

For nonabelian anomalies we use the vertex with $S = iT_a$

$$\begin{aligned}\mathcal{O}(\text{nonabelian}) &= \partial^\mu(\bar\psi\gamma_5\gamma_\mu T_a\psi) + \bar\psi\gamma_5\gamma^\mu f_{ab}{}^c T_c({V_\mu}^b + {A_\mu}^b\gamma_5)\psi + 2M\bar\psi\gamma_5 T_a\psi \\ &= -D_\mu(V+A){j_{aL}}^\mu + D_\mu(V-A){j_{aR}}^\mu + 2M(-\bar\psi_L T_a\psi_R + \bar\psi_R T_a\psi_L)\end{aligned} \tag{6.1.16}$$

where ${j_L}^\mu = \bar\psi_L\gamma_\mu T_a\psi_L$ and $\psi_L = \frac{1+\gamma_5}{2}\psi$. We can rewrite this result as

$$\begin{aligned}&\mathcal{O}(\text{nonabelian}) = D_\mu(V)j^\mu_{5,a} - f_{ab}{}^c A^b_\mu j^{\mu c} + 2M(\bar\psi\gamma_5 T_a\psi) \\ &j^{5\mu}_a = \bar\psi\gamma_5\gamma^\mu T_a\psi; j^\mu_a = \bar\psi\gamma^\mu T_a\psi\end{aligned} \tag{6.1.17}$$

2 Anomalies in triangle, box and pentagon graphs

In this section we calculate anomalies in one-loop graphs, using dimensional regularization, with Dirac fermions in the loop and external vector and axial vector fields. There are many other ways to compute anomalies, for example by using point splitting, or Pauli-Villars regularization, or heat kernel methods, or quantum mechanics in one dimension, or descent equations. Some of these other methods will be discussed in later sections, but the direct calculation we present in this section exhibits the anomaly in the clearest way.

We use ordinary dimensional regularization [26] and this requires a careful definition of γ_5. In the Feynman graphs we are going to calculate, the term $\partial^\mu j^5_\mu$ with the ordinary derivative will be rewritten as a sum of terms which also involve the commutator $[S, T_a]$, and terms which are of the form $M\bar{\psi}\gamma_5\psi$, and finally a left-over which is proportional to a vector in $n-4$ dimensions (thus with $n-4$ components with $n > 4$). We shall then see that all seagull terms (terms with $[S, T_a]$) and all terms with M cancel. The remainder, the true anomaly, is mass-independent; it comes from the term with $n-4$ components and is therefore finite. We distinguish between singlet anomalies for which $S = 1$, and nonabelian anomalies for which $S = iT_a$. A singlet anomaly may occur if one couples fermions at the vertex at the top of the triangle to an abelian axial vector field, and a nonabelian anomaly if one couples to a nonabelian axial vector field (for example the weak $SU(2)$ gauge field). At the two vertices at the bottom of the triangle one may couple to abelian or nonabelian vector or axial vector gauge fields. Consider now the case that one couples to two nonabelian vector gauge fields at the bottom. Then the singlet anomalies cancel if the sum of the charges of the fermions vanishes, and the nonabelian anomaly is proportional to d_{abc} where

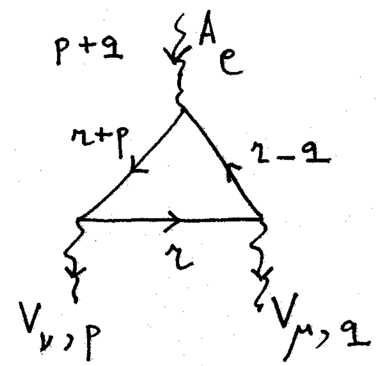
$$d_{abc} = TrT_a \{T_b, T_c\} \tag{6.2.1}$$

is a totally symmetric tensor. In some representations R this trace vanishes, in others it may be nonzero. We shall later discuss for which groups and which representations $d_{abc}(R)$ vanishes.

One can also use a matrix γ_5 which anticommutes with **all** Dirac matrices γ^μ. This approach was used to calculate some static quantities in electroweak theory [28]. A regularization scheme in which $n < 4$ instead of $n > 4$ was developed by Siegel [29]. It is called dimensional regularization by dimensional reduction. It keeps the number of field components as in 4 dimensions, but it only continues the loop momenta into $n < 4$ dimensions. The anomaly comes then from so-called evanescent counter terms.

The AVV singlet anomaly.

As a first simple example, we consider a triangle graph with a vector field V_μ and an axial vector field A_μ coupled to a massive fermion. The axial vector couples at the top, the two vectors at the bottom, and $S = 1$ by definition for singlet anomalies.



(6.2.2)

The Feynman amplitude for $\partial_\rho j_5^\rho$ is proportional to the integral of

$$-Tr\frac{\gamma_5(\not{p}+\not{q})(\not{r}-\not{q}+iM)\gamma_\mu(\not{r}+iM)\gamma_\nu(\not{r}+\not{p}+iM)}{[(r-q)^2+M^2][r^2+M^2][(r+p)^2+M^2]} \tag{6.2.3}$$

We shall later fix the overall normalization.

We choose ordinary dimensional regularization to evaluate this graph. This means that we let the loop momentum r become n-dimensional ($n > 4$) and denote by $\bar{r}^\mu$ the 4-dimensional part of r^μ, and by s^μ the extra part

$$r^\mu = \bar{r}^\mu + s^\mu. \tag{6.2.4}$$

The external momenta p^μ and q^μ remain 4-dimensional. (If this triangle graph is part of a larger Feynman graph, the momenta p and q will in general also become n dimensional.) There are n matrices γ_μ satisfying by definition $\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}$. We denote again the first four of them by $\bar{\gamma}_\mu$ and the remaining $n-4$ of them by $\hat{\gamma}_\mu$. Defining $\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4$ with $\gamma_4 = i\gamma^0$ so that $\gamma_5^2 = +1$, it follows that for $\mu = 0, 1, 2, 3$ γ_5 anticommutes

$$\{\gamma_5, \bar{\gamma}_\mu\} = 0, \tag{6.2.5}$$

while for $\mu > 4$ γ_5 commutes

$$[\gamma_5, \hat{\gamma}_\mu] = 0. \tag{6.2.6}$$

This prescription of γ_5 in dimensional regularization is due to 't Hooft and Veltman. [27] Later we shall consider "dimensional reduction" where $n \leq 4$; then always $\{\gamma_5, \gamma_\mu\} = 0$ and the anomaly comes then from the counter terms which are needed to satisfy the renormalization conditions.

In ordinary dimensional regularization where $n \geq 4$, we have the following identity

$$-\gamma_5(\not{p} + \not{q}) = \gamma_5(\not{r} - \not{q} - iM) + (\not{r} + \not{p} - iM)\gamma_5 + 2iM\gamma_5 - 2\gamma_5\not{s}. \tag{6.2.7}$$

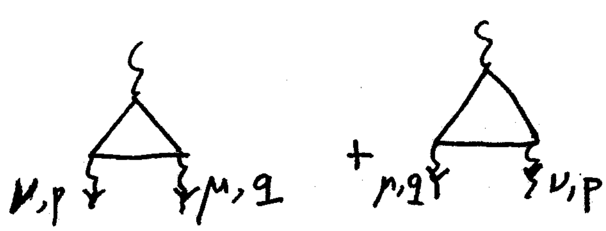
We added the r-dependent terms on the right-hand side, using $\gamma_5\not{r} + \not{r}\gamma_5 = 2\gamma_5\not{s}$. The classical conservation equation in (6.1.15) corresponds to $-\gamma_5(\not{p} + \not{q}) - 2iM\gamma_5 = 0$, but the extra term $-2\gamma_5\not{s}$ in (6.2.7) will produce the anomaly while the first two terms on the right-hand side produce seagull graphs which cancel when $S = 1$. It is easy to see that in this simple example the seagull graphs do not contribute. Indeed, replacing in (6.2.3) the vertex $-\gamma_5(\not{p}+\not{q})$ by the right-hand side of (6.2.7), contracting the term $\gamma_5(\not{r} - \not{q} - iM)$ with $(\not{r} - \not{q} + iM)/[(r-q)^2 + M^2]$ cancels one propagator, and integrating over the loop momenta yields

$$\int d^n r \;\; Tr\left\{\gamma_5\gamma_\mu \frac{(\not{r} + iM)}{r^2 + M^2}\; \gamma_\nu \; \frac{(\not{r} + \not{p} + iM)}{(r+p)^2 + M^2}\right\}, \tag{6.2.8}$$

The term $(\not{r} + \not{p} - iM)\gamma_5$ yields in a similar manner, assuming that the trace in n dimensions remains cyclic,

$$\int d^n r \; Tr\left\{\gamma_5 \frac{(\not{r} - \not{q} + iM)}{(r-q)^2 + M^2}\; \gamma_\mu \; \frac{(\not{r} + iM)}{r^2 + M^2}\gamma_\nu\right\} \tag{6.2.9}$$

Bose symmetry in the two vector fields at the two lower vertices leads to a second graph with the two external vector fields crossed, and the corresponding amplitude is obtained from (6.2.3) by interchanging p and q, and μ and ν.



(6.2.10)

This leads to two further seagull graphs

$$\int d^n r \; Tr \left\{ \gamma_5 \gamma_\nu \frac{\not r + iM}{r^2 + M^2} \gamma_\mu \frac{\not r + \not q + iM}{(r+q)^2 + M^2} \right\}$$
$$+ \quad \int d^n r \; Tr \left\{ \gamma_5 \frac{\not r - \not p + iM}{(r-p)^2 + M^2} \gamma_\nu \frac{\not r + iM}{r^2 + M^2} \gamma_\mu \right\} \tag{6.2.11}$$

One might be inclined to argue that all these expressions vanish as there are not enough independent external momenta to saturate the ϵ-symbol coming from the γ_5 in the trace. However, we have not yet specified how to take the trace in n dimensions of an expression with γ_5. We can avoid this problem by noting that the two expressions which depend on q cancel after using that $\{\gamma_5, \gamma_\nu\} = 0$ and shifting[6] in (6.2.9) the integration variable r to $r + q$. (We treat γ_μ and γ_ν thus as purely 4-dimensional, which makes sense if they are coupled to physical external fields. One can redo the analysis with n-dimensional gamma matrices. Then one obtains extra factors of $n-4$, and since the anomaly itself is already finite, the extra terms do not contribute). In a similar way one may show that the two seagull graphs which depend on p cancel.

The anomaly thus only comes from the $\gamma_5 \not s$ term. Since p, q are 4-dimensional, and the denominators contain only s^2 but no terms linear in s, we need in the numerator an even number of factors of s: terms with 4 factors of s (one of which comes from $\gamma_5 \not s$) or 2 factors of s (there is always $\gamma_5 \not s$). However, 4 factors of $\not s$ give zero since each s^2 gives a factor $n - 4$ as we shall see. Thus we only need the terms with s^2 in the numerator. We get[7] (dropping the prefactor -2, and combining the three denominators)

$$\int d^4 k d^{n-4} s \; Tr\{\gamma_5 \not s [\not s + \bar{\not r} - \not q + iM] \gamma_\mu [\not s + \bar{\not r} + iM] \gamma_\nu [\not s + \bar{\not r} + \not p + iM]\}$$

[6]We do assume that one may shift the integration variable in dimensional regularization. This can be proven by using partial integration to rewrite the integral as a finite integral, but then one assumes that one may drop boundary terms. Rather than try to prove the rules of dimensional regularization, we take the point of view that this set of rules defines this regularization scheme. Of course, one then should study whether it is a consistent scheme, but we refer at this point to the literature.

[7]Use $\frac{1}{ABC} = 2 \int_0^1 dx \int_0^x dy [(1-x)A + (x-y)B + yC]^{-3}$. Completing squares yields the square of $r - (1-x)q + yp$.

$$\times 2\int_0^1 dx \int_0^x dy \frac{1}{[s^2+k^2+\mu^2]^3}; \bar{r}^\mu = k^\mu + (1-x)q^\mu - yp^\mu \tag{6.2.12}$$

where we have shifted $\bar{r}^\mu$ to k^μ to diagonalize the denominator. The term μ^2 depends on the external momenta p and q, on x and y, and on M. There are only 3 terms which contribute a term proportional to s^2, and since they all involve only 4-dimensional Dirac matrices we know how to take the trace in the numerator. No terms with M can contribute because they have not enough Dirac matrices to saturate the trace with γ_5. Up to an overall factor $4s^2$ we find for the trace

$$\epsilon_{\mu\nu\rho\sigma}[-\bar{r}^\rho(\bar{r}+p)^\sigma + (\bar{r}-q)^\rho\ (\bar{r}+p)^\sigma - (\bar{r}-q)^\rho \bar{r}^\sigma] \tag{6.2.13}$$

In fact, this expression is even $\bar{r}$-independent, so that all infinities cancel and the momentum integrals become very simple. (This is a peculiarily of the AVV anomaly; for example it does not happen for the AAA anomaly). Using

$$\int \frac{d^n r r^\mu r^\nu}{[r^2+\mu^2]^3} = \frac{1}{n}\eta^{\mu\nu}\int \frac{r^2 d^n r}{[r^2+\mu^2]^3} = \frac{c}{n}\eta^{\mu\nu}\frac{1}{n-4}\ + \text{ finite terms} \tag{6.2.14}$$

with c a constant[8] we obtain

$$\int \frac{d^4k d^{n-4}s\ s^2}{[s^2+k^2+\mu^2]^3} = \frac{c}{n}(n-4)\frac{1}{n-4} = \frac{c}{4} + \mathcal{O}(\epsilon) \tag{6.2.15}$$

As mentioned earlier, we see that a factor s^2 indeed produces a factor $n-4$ after integration. Hence the anomaly is proportional to

$$\epsilon^{\mu\nu\rho\sigma}\epsilon_\mu(q)\epsilon_\nu(p)q_\rho p_\sigma \sim \epsilon^{\mu\nu\rho\sigma}{F_{\mu\nu}}^{\text{lin}}(V){F_{\rho\sigma}}^{\text{lin}}(V), \tag{6.2.16}$$

where ${F_{\mu\nu}}^{\text{lin}} = \partial_\mu V_\nu - \partial_\nu V_\mu$ and $\epsilon_\mu(q)$ is the polarization vector of the external vector field with momentum q. Putting all overall factors back and adding the usual $(2\pi)^{-4}$ for a one-loop graph, we obtain for the sum of the two triangle graphs

$$An\ (\text{singlet})\ = \frac{ie^2}{16\pi^2}\epsilon^{\mu\nu\rho\sigma}{F_{\mu\nu}}^{\text{lin}}{F_{\rho\sigma}}^{\text{lin}} \tag{6.2.17}$$

[8]This constant c follows from $(\int d^n r/(r^2+\mu^2)^2 = c/(n-4)$ with $c = -2i\pi^2$. The factor i in c makes the anomaly purely imaginary. We explain later why the anomaly is always purely imaginary.

The normalization factor $\frac{1}{16\pi^2}$ yields the correct decay rate for neutral pion decay into two photons (if one takes into account that the quarks in the loop have 3 colors. If one computes the matrix element for this process, there are two ways to choose which $F_{\mu\nu}$ corresponds to a particular photon). The factor i is expected because anomalies in hermitian currents are purely imaginary (they can only change the phase of the path integral as we explain in the next paragraph). We have rescaled the electromagnetic fields such that their action becomes $-\frac{1}{4\pi}{F_{\mu\nu}}^2$ while the anomaly acquires a factor e^2. Note that the anomaly does not depend on the mass of the fermions, so it is the same for massless as for massive fermions.

The anomaly is proportional to i. This is a general feature of anomalies as we now explain. [31] The effective action Γ is the sum of proper graphs. For external axial and vector gauge fields A_μ and V_μ coupled to j^5_μ and j_μ, at the one-loop level Γ consists of the loops we have calculated. Thus $\delta_{\text{gauge}}\Gamma(A,V) = An(A,V)$. Let us now make a Wick rotation to Euclidean space. Then all loop integrals are real (one can omit the $-i\epsilon$ in the propagators in Euclidean space). If the fermions are in a real representation R of the gauge group, Γ is real (one can then decompose the complex Dirac spinor into two real (Majorana) spinors). But if R is complex, we obtain a real representation if we add a second set of fermions in the complex conjugate representation: $R \oplus \bar{R}$ is real, and thus $\Gamma(A,V,R) + \Gamma(A,V,\bar{R})$ is real. For a real representation there are no anomalies (when this real representation is $R \oplus \bar{R}$ this is almost obvious, because then one can combine the left-handed fermions in R with the right-handed fermions in $\bar{R}$ to obtain a nonchiral theory). Phrased differently: for a real representation the effective action is gauge invariant. Further, note that $\Gamma(A,V,\bar{R}) = \Gamma(A,V,R)^*$ because Γ is real for $R\oplus\bar{R}$. Thus the real part of the effective action is gauge invariant; conversely, anomalies are purely imaginary. (Of course, the overall sign of the anomaly depends on the sign-convention for $\epsilon^{\mu\nu\rho\sigma}$). The one-loop graphs we have calculated, with external V_μ and A_μ and only quantized fermion fields, also yield the path integral Z (or W) for connected graphs, and $Z = e^{-\Gamma}$ in Euclidean space. Hence, the modulus of W is gauge invariant, and only the phase of W can be gauge dependent. When one makes an inverse Wick rotation which brings one back to Minkowski space, one gets two factors of i, one from $d^4x \to id^4x$ and another because due to the ϵ symbol there is always one index $\mu = 0$. Hence also in Minkowski space the anomaly is purely imaginary, as we indeed found from our explicit calculation.

The AAA singlet anomaly.

If one couples the fermions to three axial vector fields instead of one axial vector field, one obtains a triangle graph with three γ_5 vertices. We get the same expression as in (6.2.12), but with γ_μ and γ_ν replaced by $\gamma_\mu\gamma_5$ and $\gamma_\nu\gamma_5$. Bringing the two γ_5 matrices together, using the n-dimensional (anti)commutation relations of the Dirac matrices, the term $\not{s} + \bar{\not{r}} + iM$ is replaced by $-\not{s} + \bar{\not{r}} - iM$. The net result is that the second term in (6.2.13) gets an extra minus sign. Now the terms with $\bar{r}^\mu$ no longer cancel. Substituting

$$\bar{r}^\mu = k^\mu + (1-x)q^\mu - yp^\mu \tag{6.2.18}$$

into (6.2.13), the terms with k^μ do not contribute. Instead of $\epsilon^{\mu\nu\rho\sigma}(-q^\rho p^\sigma)$ in (6.2.13) one now finds $\epsilon^{\mu\nu\rho\sigma}(2(1-x)+2y-1)p^\rho q^\sigma$, which equals $\frac{1}{3}\epsilon^{\mu\nu\rho\sigma}$ after integration over x and y. Thus one finds a result that is a factor $\frac{1}{3}$ smaller than the AVV abelian anomaly. The sum of the AVV and AAA anomaly is thus proportional to

$$\epsilon^{\mu\nu\rho\sigma}\left[F_{\mu\nu}{}^{\text{lin}}(V)F_{\rho\sigma}{}^{\text{lin}}(V) + \frac{1}{3}F^{\text{lin}}_{\mu\nu}(A)F_{\rho\sigma}{}^{\text{lin}}(A)\right] \tag{6.2.19}$$

where $F_{\mu\nu}{}^{\text{lin}}(A) = \partial_\mu A_\nu - \partial_\nu A_\mu$. It seems as if the anomaly in the AAA case has been distributed equally over all three vertices. This interpretation is, however, incorrect for the box and pentagon graphs, as we shall see.

If the two gauge fields at the lower vertices are Yang-Mills fields, one obtains an extra factor $Tr(T_bT_c)$. Since this factor is by itself symmetric under exchange of the matrices T_b and T_c, the Bose symmetrized graphs gives the same result as before, multiplied by $Tr\ T_bT_c$. We recall that since we have $S = 1$ at the top, these AVV and

AAA anomalies are called singlet anomalies (for abelian or nonabelian gauge fields). The total singlet triangle anomaly for fermions in a set of representations T_a^j is thus proportional to

$$\text{Anomaly } \sim \sum_j e^j Tr\, T_b^j T_c^j \tag{6.2.20}$$

where e_j is the coupling constant of the abelian axial-vector gauge field at the top of the triangle to the fermions.

The nonabelian triangle anomaly.

With $S = iT_a$, the top vertex contains terms which depend on V and A, see (6.1.16). If one constructs a matrix element from these terms with two external V or A fields, one obtains seagull graphs in addition to triangle graphs.

(6.2.21)

However, as in the case of the singlet AVV or AAA anomalies, these seagull graphs do not contribute to the anomaly since the loops depend only on p or q so that there are not enough momenta to saturate the ϵ symbol. One would then need to make the rather plausible assumption that the contraction of an ϵ symbol with two n-dimensional factors r^μ and r^ν vanishes, and cancel terms with one r^μ by symmetric integration. However, one can do better. The cancellation of seagull graphs is purely algebraic: the seagull graphs which came from the triangle graph now no longer cancel pairwise but are proportional to $TrT_a[T_b, T_c]$. They are now canceled by the new seagull graphs which are due to the V and A terms in $\mathcal{O}$. One is left with the same result as for the singlet AVV anomaly, except that there is now a factor $Tr\, T_a T_b T_c$. Bose symmetry in the two vector fields at the bottom implies that the anomaly does not change under the simultaneous interchange of p, ν, b and q, μ, c. Since $\epsilon^{\mu\nu\rho\sigma} p_\rho q_\sigma$ is symmetric under this exchange, the anomaly is proportional to $Tr\, T_a(T_b T_c + T_c T_b)$. This expression is totally symmetric in a, b, c and is called "the d symbol" for the representation R of the fermions

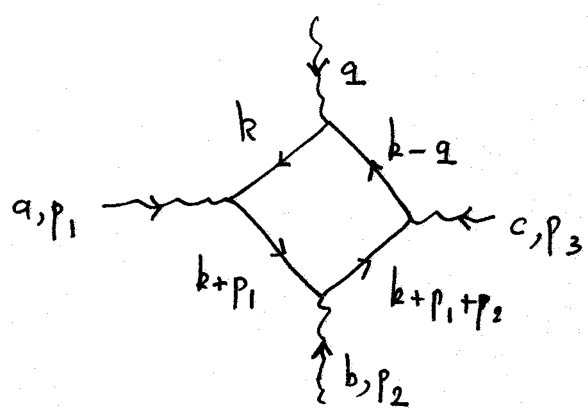
$$\text{Anomaly} \sim \frac{1}{2} Tr T_a \{T_b, T_c\} = d_{abc}(R) \tag{6.2.22}$$

Only a few simple groups have some representations for which d_{abc} is nonzero. We shall discuss these groups later. For now we record that the nonabelian triangle anomaly vanishes if and only if $d_{abc}(R)$ vanishes.

The singlet and nonabelian box anomalies.

Consider first the $AVVV$ anomaly. With $j_\mu{}^S = -i\bar{\psi}\gamma_5\gamma_\mu S\psi$ we obtain $\partial_\mu j^\mu_S = -\bar{\psi}\gamma_5 \not{q} S\psi$ where $q = -p_1 - p_2 - p_3$. (The incoming and outgoing fermion at the top vertex carry minus the incoming momentum of the gauge field at the top vertex). Putting as before

$$\gamma_5\not{q} = -\gamma_5(\not{k} - \not{q} - iM) - (\not{k} - iM)\gamma_5 - 2iM\gamma_5 + 2\gamma_5\not{s}$$

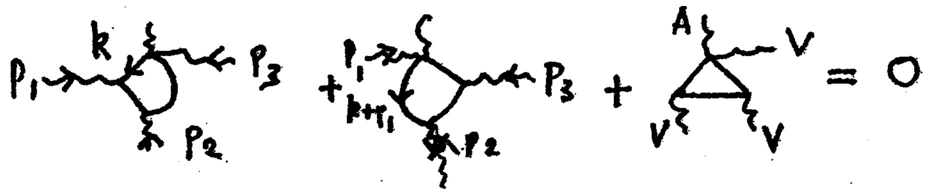


(6.2.23)

we obtain for the graph with the vertex (6.1.14) at the top of the box graph

$$\begin{aligned} &[\gamma_5(\not{k} - \not{q} - iM) + (\not{k} - iM)\gamma_5 + 2iM\gamma_5 - 2\gamma_5\not{s}]\, S \\ + \quad &[i\gamma_5\gamma^\mu(V_\mu{}^a + A_\mu{}^a\gamma_5)\,[T_a, S] - 2iM\gamma_5 S] \end{aligned} \tag{6.2.24}$$

where we recall that $q = -p_1 - p_2 - p_3$. Clearly, the M terms cancel, but in addition the two triangle graphs, which are generated when the vertices $\gamma_5(\not{k} - \not{q} - iM)$ and $(\not{k} - iM)\gamma_5$ combine with propagators to contract the box graph to a triangle graph, cancel with the triangle graph which has the vertex $i\gamma_5\gamma^\mu[T_a, S]$ in the top.



(6.2.25)

(When the fermion propagator with momentum $k-q$ collapses to a point, one obtains a factor $\gamma_5 S\not{V}(p_3)$, whereas the collapse of the propagator with momentum k produces a factor $\not{V}(p_1)\gamma_5 S$. The two triangle graphs which are thus produced have equal momentum integrals (shift in the latter $k \to k - p_1$). After Bose-symmetrization they yield together the commutator $\gamma_5[S, T_c]\not{V}^c$). So the situation is identical to the case of the nonabelian triangle graph.

The anomaly is thus again proportional to a single trace with a factor $\gamma_5\not{s}$

$$\int d^n k Tr(\gamma_5\not{s}S)(\not{k}+\not{p}_1+\not{p}_2+\not{p}_3+iM)\not{V}(p_3)(\not{k}+\not{p}_1+\not{p}_2+iM)$$
$$\not{V}(p_2)((\not{k}+\not{p}_1+iM)\not{V}(p_1)(\not{k}+iM) \qquad (6.2.26)$$

where $\not{V} = \gamma^\mu V_\mu{}^a\lambda_a$. Only the terms with s^2 contribute, since those with s^4 yield a factor $(n-4)^2$ which leads to overkill of the $(n-4)^{-1}$ from the loop integration, as we already discussed. In the remainder one can take all Dirac matrices to be 4-dimensional, since the difference with n-dimensional Dirac matrices would lead to an extra factor $n-4$ which again leads to overkill of the pole $(n-4)^{-1}$. To obtain a divergent loop integral (needed to compensate the factor $n-4$ which is due to $\not{s}$) we need the terms with four-loop momenta, i.e., two factors $\not{s}$ and two factors $\not{\kappa}$ where κ is the 4-dimensional shifted loop momentum. These terms produce a logarithmic divergence while terms in the trace with factors M are either convergent, or have an odd number of Dirac matrices. Hence we need the trace over γ_5 times 6 Dirac matrices. A typical term in the trace is $Tr(\gamma_5 s^2 S\not{V}\not{\kappa}\not{V}\not{\kappa}\not{V}\not{p}_1)$. All 6 Dirac matrices, as well as the matrix γ_5, are 4-dimensional matrices. Using cyclicity of the trace, one may cycle γ_{μ_1} around and derive that $Tr\gamma_5\gamma_{\mu_1}\dots\gamma_{\mu_k} = 4\sum_{j\neq 1}\eta_{\mu_1\mu_j}(-)^j\ \epsilon_{\mu_2\dots\mu_{j-1}\mu_{j+1}\dots\mu_k}$

for $k = 6$. In this way the trace over spinor indices can be evaluated. After combining denominators and shifting the loop momentum, the result for the loop integral is of the form,

$$\int d^n k \frac{s^2 \kappa^2}{[\kappa^2 + s^2 + \mu^2]^4} \tag{6.2.27}$$

where μ^2 is a polynomial in the mass M and the external momenta. No terms in the numerator with M contribute to the anomaly because they correspond to convergent momentum integrals. One is then left with one factor of an external momentum. The final result is proportional to the terms cubic in V in the following expression

$$\begin{aligned} \text{Anomaly} &\sim \epsilon^{\mu\nu\rho\sigma} V_\mu{}^a V_\nu{}^b V_\rho{}^c p_\sigma Tr(ST_a T_b T_c) \\ &\sim \epsilon^{\mu\nu\rho\sigma} Tr(SF^{(V)}_{\mu\nu} F^{(V)}_{\rho\sigma}) \\ &\sim \epsilon^{\mu\nu\rho\sigma} TrS(F^{\text{lin}}_{\mu\nu}[V_\rho, V_\sigma] + [V^{\rho\sigma}_\mu, V_\nu] F^{\text{lin}}_{\rho\sigma}) \end{aligned} \tag{6.2.28}$$

Bose symmetry allows only this combination as we shall later derive from the consistency conditions.

It is clear that the singlet anomaly for Yang-Mills fields is proportional to f_{abc}. Thus the final result for the singlet anomaly of the box graph is of the form

$$\text{Anomaly} \sim \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}{}^a V_\rho{}^b V_\sigma{}^c f_{bc}{}^d \left(\sum_j Tre_j T_a{}^{(j)} T_d^{(j)} \right) \tag{6.2.29}$$

where the sum runs over all fermions in the loop and e_j denotes the coupling constant of the fermions to the abelian axial vector field at the top. Since the same combination in parentheses did appear in the triangle graph, we conclude that the abelian box anomaly vanishes when the abelian triangle anomaly vanishes, and vice-versa.

For the nonabelian $AVVV$ box anomaly with $S = iT_d$ we find a result proportional to

$$TrT_d\{T_a, [T_b, T_c]\} \simeq f_{bc}{}^e d_{dae}(R) \tag{6.2.30}$$

Hence, also in the nonabelian case the box anomaly vanishes if the triangle anomaly vanishes, namely if $d_{abc}(R) = 0$. For the $AAAV$ anomaly one finds a similar result, see the general expression in (6.6.9). The $AAVV$ and $VVVV$ and $AAAA$ box graphs have an even number of γ_5 matrices, and do not contain anomalies.

The pentagon anomaly.

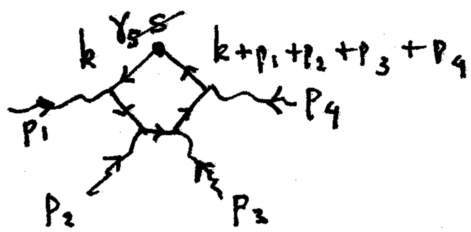
The $AVVVV$ graph yields no singlet anomaly ($S = 1$), but it yields a nonabelian anomaly (with $S = iT_e$). At first sight this statement seems paradoxical since the pentagon graph contracted with the external momentum is convergent by power counting ($\int d^n k$ times 5 propagators). However, the anomaly is not the matrix element of the operator $\partial_\mu j_5^\mu$ but rather of $D_\mu(A,V)j_5^\mu - 2iM\bar{\psi}\gamma_5 S\psi$ as we have derived in (6.1.14), and there are terms in the vertex $\mathcal{O}$ proportional to $[V_\mu, S]$ which appear in box graphs and these box graphs are logarithmically divergent by power counting.

$$\mathcal{O} = -\gamma_5 \not{q} S - i\gamma_5[S, \not{V}] - 2iM\gamma_5 S \tag{6.2.31}$$

where $q = -p_1 - p_2 - p_3 - p_4$. By rewriting the terms with $\partial^\mu j_\mu^5$ as before, we can rearrange the algebra such that the box graphs cancel, but then a term with $\gamma_5 \not\partial$ is left, and power counting yields a logarithmically divergent result. So the anomaly is possible because $\gamma_5 \not{q}$ is replaced by $\gamma_5 \not\partial$, and not because the box graphs are divergent (the box graphs cancel). In the abelian case the group theory factor vanishes as we shall show, but in the nonabelian case there is a genuine anomaly. The spinor trace of these box graphs is proportional to $n-4$, and thus we have here the same situation as before: there is a nonabelian pentagon anomaly due to the identity $(n-4)/(n-4) = 1$.

Using dimensional regularization and rewriting $\gamma_5 \not{q}$, there are two box graphs generated (due to the cancellation of the propagator on the left or on the right of the axial vertex). These box graphs cancel each other in the singlet case, whereas for $S \neq 1$ they cancel the contributions from the terms $[S, \not{V}]$ in $\mathcal{O}$. The mass terms cancel as before. Only $2\gamma_5 \not\partial$ remains and power counting shows that the graph with this vertex $\gamma_5 \not\partial$ is logarithmically divergent. Thus there is a possibility of an anomaly.

Using dimensional regularization, the pentagon anomaly is due to the following trace in spinor space



(6.2.32)

$$Tr[2\gamma_5 \not{s}(\not{k}+\not{p}_1+\not{p}_2+\not{p}_3+\not{p}_4+iM)\gamma_\sigma(\not{k}+\not{p}_1+\not{p}_2+\not{p}_3+iM)\gamma_\rho$$
$$(\not{k}+\not{p}_1+\not{p}_2+iM)\gamma_\nu(\not{k}+\not{p}_1+iM)\gamma_\mu(\not{k}+iM)] \tag{6.2.33}$$

To this result one should add all other expressions obtained from Bose symmetry. To produce a nonvanishing result, we must again extract an overall factor s^2 (recall that s^4 leads to overkill) and the remaining momentum integral must be divergent, hence all propagators must contribute loop-momentum factors. This leads to the following momentum integral

$$\int d^{n-4}s d^4\bar{k}\frac{[s^2\bar{k}^\alpha\bar{k}^\beta\bar{k}^\gamma\bar{k}^\delta]}{[s^2+\bar{k}^2+\mu^2]^5} \sim (\delta^{\alpha\beta}\delta^{\gamma\delta}+\delta^{\alpha\gamma}\delta^{\beta\delta}+\delta^{\alpha\delta}\delta^{\beta\gamma}) \tag{6.2.34}$$

Hence, the $AVVVV$ anomaly is of the form

$$An \sim Tr(\gamma_5\gamma_\sigma\gamma_\rho\gamma_\nu\gamma_\mu)Tr(ST_dT_cT_bT_a) \tag{6.2.35}$$

Due to Bose symmetry in the external four V fields, only the totally antisymmetric part $T_{[d}T_cT_bT_{a]}$ contributes, hence the group theory factor is proportional to $f_{[ab}{}^e f_{cd]}{}^f TrS\{T_e, T_f\}$. For the singlet case one finds a vanishing result due to the Jacobi identities

$$\text{Anomaly } \sim f_{[ab}{}^e f_{cd]}{}^f \delta_{ef} = 0 \text{ for } S=1 \tag{6.2.36}$$

For the nonabelian case, one obtains a result proportional to the d symbol

$$\text{Anomaly } \sim f_{[ab}{}^e f_{cd]}{}^f d_{efg} \text{ for } S=iT_g \tag{6.2.37}$$

Hence the abelian pentagon anomaly always vanishes, and the nonabelian pentagon anomaly vanishes if the nonabelian triangle anomaly vanishes (and vice-versa).

Our main conclusion is that all one-loop anomalies cancel if the triangle anomalies cancel. By adding the results for the triangle, box and pentagon anomalies, one finds the anomaly in the one-loop effective action. Since we shall derive it later from descent equations, we do not produce these results here.

The dimensional regularization we used in this section does not lead to vector anomalies, because if the top vertex contains a coupling γ^μ instead of $\gamma^\mu\gamma_5$ decomposition of $\not{p}+\not{q}$ into $(\not{k}+\not{p}-iM)-(\not{k}-\not{q}-iM)$ only leads to two seagull graphs. These seagull graphs cancel by themselves in the abelian case, whereas in the nonabelian case they cancel the contribution from the vertex $\bar{\psi}[\not{V}+\not{A}\gamma_5, S]\psi$ which is present in the classical conservation equation $D_\mu(A+V)j^\mu = 0$ for the vector current.

We make some further comments

(i) there are ambiguities: if one first moves γ_5 past some other Dirac matrices in 4 dimensions, and then goes to n dimensions one obtains a different result (differing by a finite local term), since in n dimensions $\gamma_5\gamma_\mu = \pm\gamma_\mu\gamma_5$ depending on whether the index μ is 4 dimensional or n dimensional. This explains why the AAA anomaly is not equal to the AVV anomaly: one cannot bring the two matrices γ_5 at the two lower vertices together without changing the momentum factor in the propagator between them. Keeping γ_5 in its natural position preserves vector gauge invariance. So the ambiguities are fixed by imposing that vector symmetries are without anomalies.

(ii) one cannot use ordinary dimensional regularization and maintain supersymmetry at the same time. If one uses "dimensional reduction" where $n < 4$, one can maintain supersymmetry, but then BRST symmetry is violated.

(iii) a quick way to compute anomalies is to expand the propagators about vanishing external momenta, and to retain only the terms which are power-counting divergent. Then the $(n-4)/(n-4)$ mechanism leads directly to the anomaly. For example, for the triangle anomaly, the matrix element of the chiral current with two external vector fields is given by

$$Tr\gamma_5\gamma_\tau \frac{1}{\not{k} - \not{q} + iM}\gamma_\mu \frac{1}{\not{k} + iM}\gamma_\nu \frac{1}{\not{k} + \not{p} + iM} \tag{6.2.38}$$

and one expands either the first propagator, or the last propagator. This yields two terms

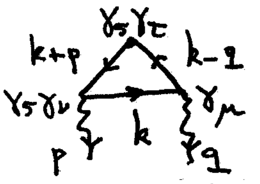
$$Tr\gamma_5\gamma_\tau \left(\frac{1}{\not{k}}\not{q}\frac{1}{\not{k}}\gamma_\mu \frac{1}{\not{k}}\gamma_\nu \frac{1}{\not{k}} - \frac{1}{\not{k}}\gamma_\mu \frac{1}{\not{k}}\gamma_\nu \frac{1}{\not{k}}\not{p}\frac{1}{\not{k}} \right) \tag{6.2.39}$$

The further terms in the expansion yield finite nonlocal contributions to the current, but we are not interested in these. Using only the Clifford algebra but not any properties of γ_5 yields

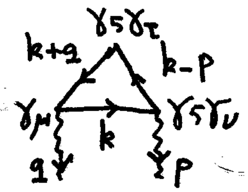
$$\begin{aligned} & Tr\gamma_5\gamma_\tau \left(-\frac{\not{q}}{k^2} + \frac{2q\cdot k\not{k}}{k^4}\right)\gamma_\mu \left(-\frac{\gamma_\nu}{k^2} + \frac{2k_\nu \not{k}}{k^4}\right) \\ - \quad & Tr\gamma_5\gamma_\tau \left(-\frac{\gamma_\mu}{k^2} + \frac{2k_\mu \not{k}}{k^4}\right)\gamma_\nu \left(-\frac{\not{p}}{k^2} + \frac{2p\cdot k\not{k}}{k^4}\right) \end{aligned} \tag{6.2.40}$$

Using n-dimensional regularization for the k integrals, we replace $k_\mu k_\nu$ by $\frac{1}{n}\eta_{\mu\nu}k^2$ and $k_\mu k_\nu k_\rho k_\sigma$ by $(n^2+2n)^{-1}(\eta_{\mu\nu}\eta_{\rho\sigma} + \eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho})(k^2)^2$. The contribution from the last term vanishes due to Bose symmetry while the rest is proportional to $(1 - \frac{4}{n})Tr\gamma_5\gamma_\tau(\not{q}\gamma_\mu\gamma_\nu - \gamma_\mu\gamma_\nu\not{p})/k^4$. Integration over k and contraction with $(p+q)^\tau$ yields then the anomaly. The advantage of this calculation, beyond being algebraically simple, is that it uses no properties of γ_5, but only the Clifford algebra and n dimensional regularization for k integrals. Thus it holds both for $n \geq 4$ and $n \leq 4$.
(iv) One expects that in a loop with an even number of γ_5 matrices, there is no genuine anomaly, but using a particluar regularization scheme the result of the calculation might be nonvanishing. In that case there should exist a local counter term whose variation cancels the anomaly.

To work this last point further out, we consider the AAV triangle. For simplicity we consider an abelian theory without masses. Then the spinor traces for the triangle graph and the Bose symmetrized graph read



$$\begin{aligned} &Tr\gamma_5\gamma_\tau(\not{k}-\not{q})\gamma_\mu\not{k}\gamma_5\gamma_\nu(\not{k}+\not{p})+ \\ &+Tr\gamma_5\gamma_\tau(\not{k}-\not{p})\gamma_5\gamma_\nu\not{k}\gamma_\mu(\not{k}+\not{q}) \end{aligned} \tag{6.2.41}$$



(Each trace should still be divided by factors $(k-q)^2, k^2$ etc., and an integral over the loop momentum k should be taken). Moving in the first trace the second γ_5 to the right, and in the second trace the second γ_5 to the left, the two matrices γ_5 cancel, and one finds in n dimensions

$$Tr\gamma_\tau(\not{k}-\not{q})\gamma_\mu\not{k}\gamma_\nu(-\not{s}+\bar{\not{k}}+\not{p})+Tr\gamma_\tau(-\not{s}+\bar{\not{k}}-\not{p})\gamma_\nu\not{k}\gamma_\mu(\not{k}+\not{q}) \tag{6.2.42}$$

Contracting γ_τ with $(p+q)^\tau$ and rewriting the result as $(\not{k}+\not{p})-(\not{k}-\not{q})$ in the first term, and as $(\not{k}+\not{q})-(\not{k}-\not{p})$ in the second term, one obtains four traces. If one writes $-\not{s}+\bar{\not{k}}$ as $-2\not{s}+\not{k}$, all terms without $-2\not{s}$ cancel. (As before one must shift the loop momentum in two of the four graphs in order that the four graphs cancel pairwise. Shifting loop momenta is allowed in dimensional regularization).

One is left with two traces which contain wach a factor $-2\not{s}$

$$\begin{aligned} Tr[&(-2\not{s})(\not{p}+\not{q})(\not{k}-\not{q})\gamma_\mu\not{k}\gamma_\nu \\ &+(-2\not{s})\gamma_\nu\not{k}\gamma_\mu(\not{k}+\not{q})(\not{p}+\not{q})] \end{aligned} \tag{6.2.43}$$

Since only terms proportional to s^2 can contribute, we decompose all $\not{k}$ in these two traces into $\bar{\not{k}}+\not{s}$ and collect all terms with s^2. One finds four terms, with the following numerators

$$\begin{aligned} &(\not{p}+\not{q})\gamma_\mu\bar{\not{k}}\gamma_\nu; (\not{p}+\not{q})(\bar{\not{k}}-\not{q})\gamma_\mu\gamma_\nu \\ &(\not{p}+\not{q})\gamma_\nu\bar{\not{k}}\gamma_\mu; (\not{p}+\not{q})\gamma_\nu\gamma_\mu(\bar{\not{k}}+\not{q}) \end{aligned} \tag{6.2.44}$$

Taking the trace of the transpose of the third term, using $\gamma_\mu^T = -C\gamma_\mu C^{-1}$, and replacing k by $-k$ cancels the first term. Similarly, the transpose of the last term and replacing k by $-k$ cancels the second term. Because we extracted a factor s^2, all Dirac matrices are 4-dimensional and the traces can be taken without having to deal with n-dimensional ambiguities.

Hence, there are no parity-conserving vector-gauge invariant counter terms needed to cancel a spurious anomaly: the anomaly cancels already by itself. This is only true for the abelian anomaly in abelian theories; for the general case one needs counter terms, see (6.9.38). If we had moved the γ_5's together in 4 dimensions rather than in n dimensions, and then had moved into n dimensions, the only difference would have been that the terms with $\hat{\not s}$ in (6.2.42) would have appeared with a plus sign. The usual cancellation of propagators takes then place and the anomaly would still have canceled.

3 Gauge anomalies ruin renormalizability and unitarity

The proofs we have given that certain quantum gauge field theories are renormalizable and unitary were based on Ward identities which follow from BRST invariance. When there are anomalies, these Ward identities are violated, and it seems very likely that then also renormalizability and unitarity are violated. In this section we study two models with anomalies, and explicitly demonstrate that renormalizability and unitarity indeed are broken by anomalies.

The first model we consider breaks unitarity. It is a toy model for the Standard Model, with masses generated by the Higgs effect and a chiral gauge group. It has a $U(1)$ vector gauge invariance with gauge field V_μ and a $U(1)$ axial vector gauge invariance with gauge field A_μ, coupled to a complex Dirac spinor ψ. In order that the axial vector gauge field can be an intermediate state in the unitarity relation with nondegenerate kinematics, we give it a mass via the Higgs mechanism. We introduce therefore also a complex scalar field $\varphi = \frac{1}{\sqrt{2}}(\varphi_1 + i\varphi_2)$ which couples to the axial vector gauge field, but not to the vector gauge field, and which decomposes into a real Higgs boson $\sigma = \varphi_1 - < \varphi_1 >$ and a would-be Goldstone boson φ_2. We couple the scalar also to the fermion because we want to make the fermion very massive by the Higgs mechanism so that it does not contribute in the unitarity cutting rules. We choose an 't Hooft off-diagonal gauge fixing term to diagonalize the kinetic terms of A_μ and φ_2, and as a consequence the ghosts associated with A_μ become also massive. The ghosts associated with V_μ remain massless.

The classical action which meets all these requirements is given by

$$\begin{aligned}\mathcal{L} \;=\; & -\frac{1}{4}(\partial_\mu V_\nu - \partial_\nu V_\mu)^2 - \frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 - \sqrt{2}G(\bar{\psi}_L\psi_R\varphi + \bar{\psi}_R\psi_L\varphi^*) \\ & -\bar{\psi}\gamma^\mu(\partial_\mu - ieV_\mu - igA_\mu\gamma_5)\psi \;- \mid (\partial^\mu - 2igA^\mu)\varphi \mid^2 + \mu^2\varphi^*\varphi - \lambda(\varphi^*\varphi)^2\end{aligned} \tag{6.3.1}$$

where $\psi_L = \frac{1}{2}(1+\gamma_5)\psi$ and $\psi_R = \frac{1}{2}(1-\gamma_5)\psi$ while $\bar{\psi} = \psi^\dagger i\gamma^0$ and $\bar{\psi}_L = \bar{\psi}(1-\gamma_5)/2, \bar{\psi}_R = \bar{\psi}(1+\psi_5)/2$. The vector gauge symmetry of this model leaves $\varphi = \frac{1}{\sqrt{2}}(\sigma + v + i\varphi_2)$ inert (where $v =< \varphi_1 >$ and $< \varphi_2 >= 0$) because φ does not couple to V_μ

$$\delta V_\mu = \partial_\mu\theta, \delta A_\mu = 0, \delta\varphi = 0, \delta\psi = ie\theta\psi \tag{6.3.2}$$

In particular the Yukawa term is vector-gauge invariant. The axial vector gauge transformations rotate φ such that the Yukawa term remains invariant. For this to happen the chiral weight of φ has to be twice that of ψ. Then also $|(\partial^\mu - 2igA^\mu)\varphi|^2$ is invariant due to the factor 2 in the covariant derivative

$$\delta V_\mu = 0, \delta A_\mu = \partial_\mu\eta, \delta\varphi = 2ig\eta\varphi, \delta\psi = ig\eta\gamma_5\psi \tag{6.3.3}$$

The invariance of this classical action under the two gauge invariances is now obvious.

The terms with φ can be written in a real basis as follows

$$\mathcal{L}(\varphi) \;=\; -G\bar{\psi}\psi(\sigma + v) + iG(\bar{\psi}\gamma_5\psi)\varphi_2 - \frac{1}{2}(\partial_\mu\sigma)^2 - \frac{1}{2}(\partial_\mu\varphi_2)^2$$

$$
\begin{aligned}
&+ \quad 2gv(\partial_\mu \varphi_2)A^\mu + 2g(\sigma \overleftrightarrow{\partial_\mu} \varphi_2)A^\mu - 2g^2 {A_\mu}^2[(\sigma+v)^2 + {\varphi_2}^2] - V[(\sigma+v)^2 + \varphi_2^2] \\
\delta\sigma &= -2g\eta\varphi_2, \delta\varphi_2 = 2g\eta(\sigma+v) \qquad (6.3.4)
\end{aligned}
$$

where $V = \lambda|\varphi|^4 - \mu^2|\varphi|^2$. Because there is a field-independent term in the transformation law of φ_2, this field is a Goldstone boson which can be gauged away, in which case A_μ becomes massive with 3 polarizations. This is called a unitary gauge because all unphysical particles have been eliminated, and unitarity is thus manifest. However, the propagator for A_μ contains then a term $k_\mu k_\nu/m^2$ and loop calculations become hopelessly complicated. Therefore we choose a so-called renormalizable gauge and can then study unitarity at loop levels.

To remove the off-diagonal term in the classical kinetic action we choose as gauge fixing terms

$$
\mathcal{L}(\text{fix}) = -\frac{1}{2\alpha}(\partial^\mu V_\mu)^2 - \frac{1}{2\beta}(\partial^\mu A_\mu - 2\beta g v \varphi_2)^2 \qquad (6.3.5)
$$

The first term fixes the vector gauge and the second one the axial vector gauge. The kinetic terms of A_μ and φ_2 are now diagonal. The corresponding ghost action follows then at once from $\delta\varphi_2 = 2g\eta(\sigma+v)$ and reads

$$
\mathcal{L}(\text{ghost}) = b_V \partial^\mu \partial_\mu c_V + b_A(\partial^\mu \partial_\mu c_A - 4\beta g^2 v(v+\sigma)c_A) \qquad (6.3.6)
$$

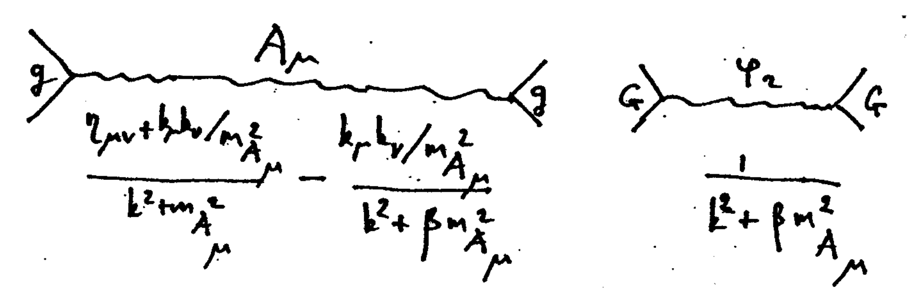
where b_V, c_V are the ghosts for the vector gauge symmetry and b_A, c_A are the ghosts for the axial vector gauge symmetry.

The mass of σ follows from $V(\varphi_1) = -\frac{1}{2}\mu^2{\varphi_1}^2 + \frac{\lambda}{4}{\varphi_1}^4 = -\frac{1}{2}\mu^2(\sigma+v)^2 + \frac{\lambda}{4}(\sigma+v)^4$; terms linear in σ cancel if $\lambda v^2 = \mu^2$, and ${m_\sigma}^2 = 2\lambda v^2$. The mass of φ_2 comes only from $\mathcal{L}(\text{fix})$ and is ${m_{\varphi_2}}^2 = 4\beta g^2 v^2$. (The potential never contributes to the mass of Goldstone bosons because at the bottom of the valley of the Mexican hat there is no curvature in the direction of the valley.) The masses of the various particles are thus

$$
\begin{aligned}
m_\psi = Gv, {m_V}^2 &= 0\ ;\ m^2_{\partial\cdot A} = 4\beta g^2 v^2; m^2_\sigma = 2\lambda v^2\ ;\ m^2_{\varphi_2} = 4\beta g^2 v^2 \\
m^2_{b_V} = m^2_{c_V} &= 0\ ;\ m^2_{b_A} = m^2_{c_A} = 4\beta g^2 v^2; m^2_{A_\mu} = 4g^2v^2 \qquad (6.3.7)
\end{aligned}
$$

We recognize the Kugo-Ojima quartet $(-\Box)^{-1/2}\partial^\mu A_\mu, \varphi_2, b_A$ and c_A with a common β-dependent mass. Furthermore, the mass m_{A_μ} of the massive physical axial vector field $A_\mu - \partial_\mu \frac{1}{\Box}\partial \cdot A$ is, of course, independent of the gauge parameter β.

As a warming up exercise we consider fermion scattering at tree level due to the exchange of V_μ, A_μ, σ and φ_2 fields. The contributions of the unphysical polarization of the massive axial vector field A_μ should cancel in this S matrix element against the contribution of the would-be Goldstone boson φ_2. It is a simple but useful exercise to check that this cancellation indeed occurs.[9] (Use that $k_\mu j^\mu_A = 2im_\psi\bar{\psi}\gamma_5\psi$ and $2m_\psi/m_{A_\mu} = G/g$).



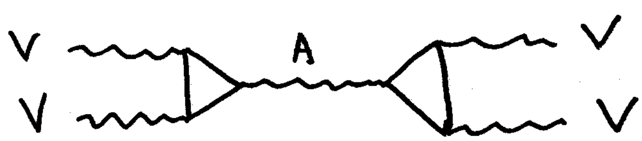
(6.3.8)

Unitarity requires, loosely speaking, that at the pole of the propagator the S matrix element factorizes into a product of two S-matrix elements with only physical intermediary states. (More precisely, $S = I + iT$ and $ImT = TT^\dagger$. Taking matrix elements as $< f|ImT|i >= \sum_n < f|T|n >< n|T^\dagger|i >$, the states $|n >$ should only be physical states. In Fock space, there are also unphysical states: (anti) ghosts, unphysical polarizations of gauge fields, and (in the case of spontaneous symmetry breaking) would-be Goldstone bosons. Unitarity means that the sum of all contributions of unphysical states to the unitarity relation should cancel. This can be proven by suitable Ward identities, which are derived from BRST symmetry). In our example, at the pole where $k^2 + m^2_{A_\mu} = 0$ there are 3 physical intermediary states, but the second term in (6.3.10) corresponds at $k^2\beta^2_A = 0$ to a state with an unphysical

[9] The propagator for A_μ is found by inverting its field operator $(k^2 + m^2_A)\eta_{\mu\nu} + \left(\frac{1}{\beta} - 1\right) k_\mu k_\nu$. It is "power counting renormalizable": for large k^2 it falls off as $[\eta_{\mu\nu} + (\beta - 1)k_\mu k_\nu/k^2]/k^2$.

polarization of A_μ. If unitarity holds, the contribution of this unphysical state must be completely canceled by another unphysical state, the would-be Goldstone boson φ_2 at the same $k^2 + \beta m_A^2 = 0$. We shall show that all nonanomalous terms indeed cancel, but the anomaly produces a term that is left over, and this isolates unitarity.

Let us now demonstrate that in this model unitarity is broken as a result of the triangle anomaly. Consider elastic $V - V$ scattering. At the one-loop level, there is only a box diagram with fermions in the loop (because V couples only to fermions), and since in this diagram there are no γ_5 matrices, this box diagram is not anomalous and unitarity holds as usual. At the two-loop level, there are several graphs, one of which will lead to a breakdown of unitarity. The graphs which are obtained from the one-loop graph by adding another loop to it are all without anomalies. For example, adding a virtual A_μ line in the fermion box, one has an even number of $\gamma_\mu\gamma_5$ vertices (two) so that no anomaly results. However, there is also a graph with two fermion triangles and a massive axial vector field in between.



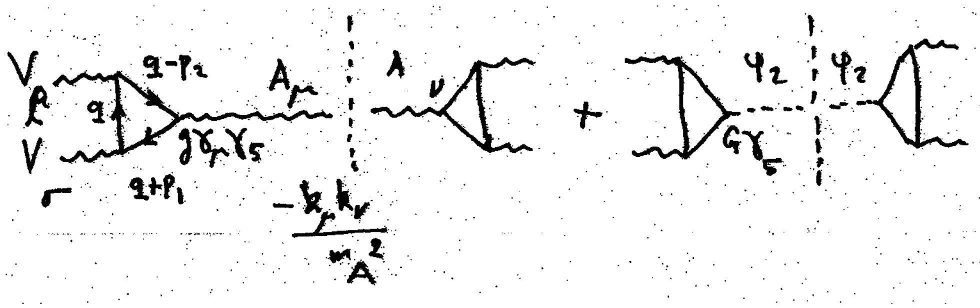
(6.3.9)

The A_μ propagator has a spin one and a spin zero part

$$\langle A_\mu A_\nu \rangle = \frac{\eta_{\mu\nu} + k_\mu k_\nu / m_{A_\mu}^2}{k^2 + m_{A_\mu}^2} - \frac{k_\mu k_\nu / m_{A_\mu}^2}{k^2 + \beta m_{A_\mu}^2} \ . \tag{6.3.10}$$

The numerator of the first term is orthogonal to the momentum k_μ and can be written as a sum over the three physical polarizations of a massive vector boson, $\sum_{m=1}^{3} \epsilon_\mu^m \epsilon_\nu^m$. The second term describes the unphysical polarization; in a frame where $\vec{k}$ vanishes it corresponds to the time component of A_μ.

Unitarity requires that the contribution of the unphysical part of A_μ is canceled by the contribution of the unphysical would-be Goldstone boson φ_2



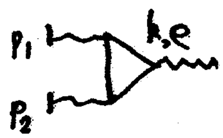
(6.3.11)

(At this loop level, ghosts do not yet enter the stage and the Higgs scalar σ is physical. Furthermore, no cuts through the fermions in the triangle need be considered if we make the fermion masses heavier than the total incoming energy. This is possible since $m_\psi = Gv$ and G is a free parameter).

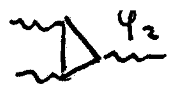
We consider the Ward identity based on $\delta_B \langle V_\mu(x) V_\nu(y) b_a(z) \rangle = 0$ and recall that antighosts vary into the gauge fixing term. The naive Ward identity (i.e., the Ward identity without the anomaly, thus the wrong Ward identity) yields after truncating the external lines with on-shell momenta and contracting with physical polarization vectors

$$
\begin{aligned}
&\epsilon^\mu(p_1)\epsilon^\nu(p_2)\delta_B < V_\mu(p_1)V_\nu(p_2)b_A(k) >= 0 \\
&= \epsilon^\mu(p_1)\epsilon^\nu(p_2) < V_\mu(p_1)V_\nu(p_2)\left\{\frac{1}{\beta}\partial\cdot A(k) - 2gv\varphi_2(k)\right\} >
\end{aligned}
\tag{6.3.12}
$$

The first term corresponds to the left-hand side of the graph with A_μ exchange while the second graph corresponds to φ_2 exchange. The Ward identity says that these terms cancel each other, but this is false as we shall show: the anomaly in the AVV triangle is the cause that (6.3.12) breaks down. Rewriting the term $\not k\gamma_5$ in the triangle on the left as usual as $(\not q + \not p_1 - im_\psi)\gamma_5 + \gamma_5(\not q - \not p_2 - im_\psi) - 2\hat{\not q}\gamma_5 + 2im_\psi\gamma_5$, where q is the loop momentum and $k = p_1 + p_2$, the contributions from the first two terms cancel each other, and the last term cancels the contribution from the graph with the would-be Goldstone boson. Recalling that $m_\psi = Gv$ and $m_{A_\mu} = 2gv$ we see that the sum of the contributions of the unphysical particles cancels for any β. (The term with $\frac{1}{\beta}\partial \cdot A$ yields $\frac{1}{\beta}g(2m_\psi\gamma_5)$ at the $A\bar{\psi}\psi$ vertex, while the term with ψ_2 yields $-2gv(G\gamma_5)$. There is also a factor β from the term $k_\mu k_\nu/m_A^2 = \beta k_\mu k_\nu/m^2\partial\cdot A$. These terms indeed cancel.) However, the correct Ward identity has an anomaly since $\partial^\mu A_\mu$ corresponds to the divergence of the axial vector current in a triangle diagram



$\times\ (ik_\rho) =$



$$+\, c\epsilon_{\mu\nu\rho\sigma}{p_1}^\rho {p_2}^\sigma \qquad (6.3.13)$$

Squaring the right-hand side we get terms with c^2 and c which violate the unitarity relation.

The Ward identity in (6.3.12) was derived from the path integral by making a change of integration variables which corresponds to a BRST transformation. A change of integration variables is always permitted so one may wonder where the anomaly resides in the path integral. (In Feynman graphs we found it to be due to the term $-2\hat{\not{k}}\gamma_5$, produced only by careful regularization). The answer is that in the path integral the anomaly comes from the Jacobian; we discuss this in section 9.

It is clear that by adding further fermions to the theory such that the sum of their c's vanishes, unitarity will be restored. This is, of course, the anomaly cancellation mechanism in the Standard Model. Note that even if these extra fermions have different masses, the anomaly cancels since it is mass independent.

The anomaly which caused unitarity violation in this example is the AVV abelian triangle anomaly. The reader may construct other processes in which other anomalies lead to a violation of unitarity. Or one may consider a chiral fermion coupled to one gauge field W_μ; the same conclusions hold. For more complicated loop graphs one should use cutting relations to directly analyze unitarity. For example, one could consider at the 3-loop level the following graph

(6.3.14)

with one or two axial vector bosons in the middle. At some time it was conjectured that this graph might contain a new kind of axial anomaly [43], but closer study showed this not to be the case [44]. In this case these axial vector bosons could even be massless. The cutting relations for such graphs are again based on Ward identities, and if there are anomalies, there are extra terms in these Ward identities which lead to a break down of unitarity. We needed triangles in (6.3.11) both on the left and the right; if we had only an $A_\mu \bar{\psi}\gamma_5\gamma_\mu\psi$ coupling on the right with physical outgoing fermion and no loop, the k^μ contraction with this vertex would have vanished and unitarily would have been satisfied for this graph. We stop here our analysis of unitarity and turn to renormalizability.

The first model with anomalies in which we study renormalizability contains a Dirac fermion ψ coupled both to a vector V_μ and an axial vector A_μ,

$$\mathcal{L} = -\frac{1}{4}(\partial_\mu V_\nu - \partial_\nu V_\mu)^2 - \frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 - \bar{\psi}\gamma^\mu(\partial_\mu - ieV_\mu - igA_\mu\gamma_5)\psi \quad (6.3.15)$$

The ghosts for the abelian vector and axial-vector symmetry are now all free. We could again add a Higgs sector to make the fermion and gauge fields massive, but this is not necessary for our arguments. We shall demonstrate that the AA selfenergy is no longer transversal but contains a divergence proportional to $p_\mu p_\nu$. This violates a Ward identity as we shall show. This in itself does not yet violate renormalizability; it simply means that the gauge fixing term renormalizes, so $Z_\xi \neq Z_3$. However, at higher loops this new divergence leads to nonrenormalizable divergences (as we explicitly show in a second model).

Consider the selfenergy of two vector fields (VV, AA or AV). At the one-loop level there is only a fermion loop and since there is no anomaly in this graph, the selfenergy is transversal. It should be transversal due to the Ward identity $\delta_{\mathrm{BRST}}\langle\partial\cdot V(x)b_V(y)\rangle = \langle\partial\cdot V(x)\partial\cdot V(y)\rangle + \langle\Box c_V(x)b_V(y)\rangle = 0$ and similarly for $\langle\partial\cdot A(x)b_A(y)\rangle$. At tree graph level this identity holds $(k^2 + (-k^2) = 0)$ and at loop levels $\langle\Box c(x)b(y)\rangle$ does not contribute because the ghosts are free. It is easy to check transversality by contracting the graph with $p_\mu p_\nu$

$$\sim \int d^n k \quad Tr \frac{[M\gamma_\mu(\not{k}+\not{p})N\gamma_\nu \not{k}]}{k^2(k+p)^2} \times p^\mu p^\nu \tag{6.3.16}$$

where M and N are either unity or γ_5. Using only the Clifford algebra one finds for $N = 1$ the following expression for the numerator

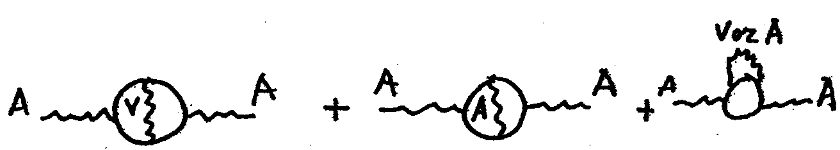
$$\begin{aligned} &\sim TrM\not{p}(\not{k}+\not{p})\not{p}\not{k} = TrM[-p^2(\not{k}+\not{p})\not{k} + 2p\cdot(k+p)\not{p}\not{k}] \\ &= TrM[-p^2(k^2+\not{p}\not{k}) + 2p\cdot(k+p)\not{p}\not{k}] \end{aligned} \tag{6.3.17}$$

We may replace $2p\cdot k$ by $-p^2$ (the difference is $2p\cdot k + p^2 = (k+p)^2 - k^2$, which yields two tadpole integrals $\int d^4k/k^2$ which vanish in dimensional regularization) and then all terms cancel. Hence, no violation of transversality occurs at the one-loop level for $N = 1$. If both M and N are equal to γ_5 one may use $\gamma_5\not{p}(\not{k}+\not{p})\gamma_5 = \not{p}(\cdot\not{k}+\not{p}-2\not{s})$, and then the term with $\not{p}(\not{k}+\not{p})$ cancels as before, but the term $\not{p}(-2\not{s})$ yields a result proportional to

$$p^2 \int d^n k \frac{s^2}{k^2(k+p)^2} \tag{6.3.18}$$

where s_μ is the $(n-4)$ dimensional part of the loop momentum. One obtains then a finite, nonzero and local contribution for $p^\mu \Pi_{\mu\nu}(p)p^\nu$, namely a term proportional to $(\partial^\mu A_\mu)^2$. One can remove this term by a finite renormalization of the gauge fixing parameter ξ but we shall use minimal subtraction and have then $Z_3 = Z_\xi$ at the 1-loop level. So we view the finite nonvanishing result we obtained for $p^\mu \pi_{\mu\nu} p^\nu$ at the one-loop level as due to an imperfect regularization scheme.

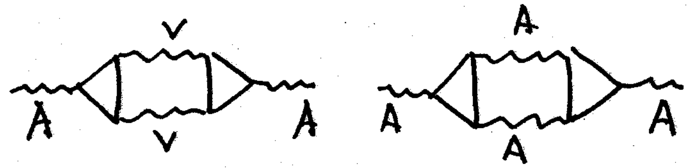
At the two-loop level one has only a dressed version of the one-loop graph.



(6.3.19)

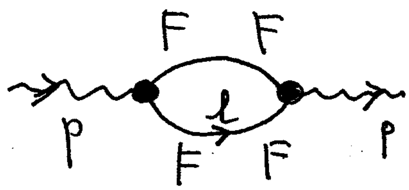
Since all one-loop subgraphs satisfy the Ward identity (there are no closed fermion triangle one-loop subgraphs), we could again add a counter term such that transversality also holds here.

At the three-loop level, however, there is a graph which violates transversality of the AA selfenergy by an infinite amount and this time one cannot find a counter term which restores transversality. It consists of two triangles back to back connected by two vector gauge fields or two axial-vector gauge fields



(6.3.20)

Contracting with the external momenta $p_\mu p_\nu$, at each vertex the triangle graph collapses to a point and only the anomaly survives (the contracted graphs with only two fermion propagators vanish). Hence we find



$$\sim \int d^n l \frac{F_{\mu\nu}(l-p)^* F^{\mu\nu}(l)(F_{\rho\sigma}(l-p)^* F^{\rho\sigma}(l)}{(l-p)^2 l^2} \tag{6.3.21}$$

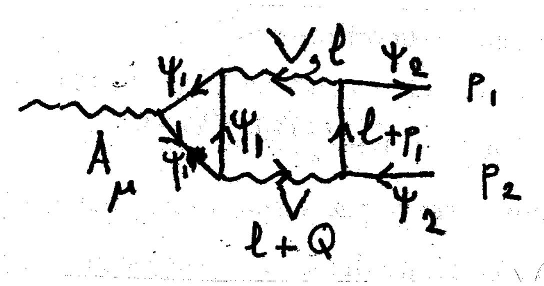
which is proportional to p^4 and logarithmically divergent. (Quartic and quadratic divergences correspond to tadpole graphs and vanish). Thus now Z_ξ and Z_3 differ by an infinite amount: the gauge fixing term renormalizes already at the level of minimal subtraction. Since the theory with $Z_3 = Z_\xi$ is renormalizable, the fact that $Z_3 \neq Z_\xi$ means that there are extra counter terms proportional to the gauge fixing term. One could fix both the infinite and the finite parts of Z_ξ by requiring that the gauge field selfenergy be transversal, but the extra counter term would contribute also to other proper vertices. The Ward identity for the divergent part of the effective action, $Q\hat{\Gamma}(\mathrm{div}) = 0$, must be modified because $\hat{\Gamma} = \Gamma - S_{\mathrm{fix}}$ is no longer finite. One would expect that at higher loops nonrenormalizable divergences would show up. In the 3-loop example we have given, there are subtleties how to define the product of two ϵ-tensors in n-dimensions, but we are only interested in the leading term for which we can safely use the 4-dimensional results.

We now consider a model in which we directly show that a divergence is introduced by an anomaly which destroys renormalizability. Consider two complex fermions, ψ_1 and ψ_2, in which ψ_1 couples both to a vector gauge field V_μ and an axial vector gauge field A_μ, but ψ_2 only to the vector gauge field. Moreover ψ_2, but not ψ_1, is massive

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}(\partial_\mu V_\nu - \partial_\nu V_\mu)^2 - \frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 \\ &\quad -\bar{\psi}_1\gamma^\mu(\partial_\mu - ieV_\mu - igA_\mu\gamma_5)\psi_1 - \bar{\psi}_2\gamma^\mu(\partial_\mu - ieV_\mu)\psi_2 - m\bar{\psi}_2\psi_2\end{aligned} \tag{6.3.22}$$

This model has a vector and an axial vector gauge invariance. The fermion ψ_1 transforms both under vector and axial vector gauge transformations and is for that reason massless, but ψ_2 only transforms under vector gauge invariance and this allows a mass term for ψ_2. (We could again use the Higgs mechanism to also make ψ_1 massive, but this is not necessary for our purposes). We shall show that the AVV triangle anomaly induces a divergence proportional to $\bar{\psi}_2 \not{A}\psi_2$, hence it produces a divergence which is not proportional to a term in the action. A similar phenomenon occurs in scalar field theory with only a Yukawa interaction $\bar{\psi}\psi\varphi$; there divergences proportional to φ^4 are generated, but adding a φ^4 term to the action, the model becomes renormalizable. In our case, we could not similarly add couplings of ψ_2 to A_μ in the classical action because that would destroy the axial-vector gauge invariance.

Consider the following diagrams for the $\bar{\psi}_2\psi_2 A_\mu$ 3-point function



(6.3.23)

By power counting the triangle graph is linearly divergent. Because there are no proper one-loop graphs contributing to this process, there will only be a divergence with a simple pole in $\epsilon = 4 - n$.[10] Furthermore, the term with the $1/\epsilon$ divergence should be local, hence proportional to $\bar{\psi}_2 \not{A} \gamma_5 \psi_2$. The full two-loop calculation is tedious, but we can simplify the calculation by considering the contribution from longitudinal A_μ, setting $A_\mu = \partial_\mu \varphi$. We have so far not made any approximations because we want to clearly exhibit which assumptions one must make in dimensional regularization. We assume that $tr\gamma_5\gamma_\mu\gamma_\nu$, and a fortiori $tr\gamma_5\gamma_{\hat{\mu}}\gamma_\nu$, will vanish as n tends to 4, no matter how one defines these traces in n dimensions. Then the loop integral due to the triangle is proportional to the anomaly

$$M \sim \epsilon^{\mu\nu\rho\sigma} Q_\rho l_\sigma, Q = p_1 - p_2 \tag{6.3.24}$$

where μ and ν are the vector indices of the two internal V fields. The second loop integral reduces then to

$$\int \frac{d^n l \epsilon^{\mu\nu\rho\sigma} Q_\rho l_\sigma (l+p_1)_\tau}{l^2 (l+p_1)^2 (l+Q)^2} \bar{\psi}(p_2)\gamma_\nu\gamma^\tau\gamma_\mu\psi(p_1) = \frac{c\epsilon^{\mu\nu\rho\sigma}}{n-4} Q_\rho \bar{\psi}(p_2)\gamma_\nu\gamma_\sigma\gamma_\mu\psi(p_1) \tag{6.3.25}$$

with c a constant. The crossed-box graph contributes the same divergence because the ψ_2-propagator contains $\not{p}_2 - \not{l}$ instead of $\not{p}_1 + \not{l}$, whereas the order of γ_μ and γ_ν in the ψ_2 line is reversed. Of course, $\epsilon^{\mu\nu\rho\sigma}\gamma_\nu\gamma_\sigma\gamma_\mu$ is proportional to $\gamma_5\gamma^\rho$. Hence there is indeed an overall divergence proportional to

$$\frac{1}{n-4}(\bar{\psi}_2\gamma_5\gamma^\rho\psi_2)A_\rho \tag{6.3.26}$$

We cannot add a counter term to the action which removes this divergence, because such a counter term would violate the axial vector gauge symmetry. (If such a counter term could have been added, we would have needed to impose a suitable renormalization condition to fix its finite part). If one were trying to save this model at all

[10]If there were a second-order pole, expansion into ϵ of the integrand would produce a first-order pole multiplied by a logarithm containing momenta. Such nonlocal divergences cannot occur in power-counting renormalizable field theories. For processes where graphs with counter terms contribute, the contributions from these graphs with counter terms lead to similar first-order poles with nonlocal logarithms such that in the sum all nonlocalities cancel. A typical example is massless φ^4 theory where the 2-loop self energy diagram has only a first-order pole.

costs, one might just remove the divergence by minimal subtraction, just dropping the divergent term. That would save the model at two loops, but at three loops one would find nonlocal divergences. At this point we give up our efforts to save this model, and conclude that the presence of an axial anomaly has rendered the model nonrenormalizable.

If one would add another fermion in the triangle loop such that the anomaly cancels, there would be no contribution from the triangle (the anomaly is mass independent), and hence the model would become renormalizable at this level. This clearly illustrates the connection between anomalies and nonrenormalizability.

4 When do anomalies cancel, and when should they cancel?

We saw that the nonabelian triangle anomalies for the AVV case or for left-handed chiral fermions ψ_L are proportional to the d-symbol, defined by

$$d^L_{abc} = \frac{1}{2} Tr(T^L_a \{T^L_b, T^L_c\}); \quad [T^L_a, T^L_b] = f^{L,c}_{ab} T^L_c \tag{6.4.1}$$

Furthermore we saw that if these triangle anomalies canceled, all other one-loop nonabelian chiral anomalies also cancel. [32] We shall now study when the triangle anomalies cancel. This is a purely group-theoretical problem. Later we shall distinguish between anomalies in the local gauge symmetry which in general lead to an inconsistent quantum field theory, and anomalies in axial vector currents which are the Noether currents of rigid symmetries. The latter are in some cases welcome because they explain certain experimental facts, such as the fact that neutral pions decay into two photons, without rendering the quantum field theory inconsistent.

If there are, in addition to left-handed fermions ψ_L with a gauge algebra with structure constant ${f_{ab}}^{L,c}$, also right-handed fermions ψ_R with a gauge algebra with structure constants ${f_{ab}}^{R,c}$, there are also anomalies due to triangle loops with ψ_R, proportional to

$$d^R_{abc} = \frac{1}{2} Tr\left(T^R_a \left\{T^R_b, T^R_c\right\}\right); \quad \left[T^R_a, T^R_b\right] = f^{R\ c}_{ab} T^R_c \tag{6.4.2}$$

One can always rewrite right-handed fermions as charge-conjugate left-handed fermions

$$\begin{aligned} &\psi^c_L = C^{-1}\bar{\psi}^T_R; \Big(\text{ or } \psi^{c,\alpha}_L = (C^{-1})^{\alpha\beta}\bar{\psi}_{R,\beta} \text{ with } \bar{\psi}_{R,\beta} = (\psi^\dagger)_{\beta'}(i\gamma^0)^{\ \beta'}{}_{\beta}\Big) \\ &\bar{\psi}^c_L \equiv \psi^{c,\dagger}_L i\gamma^0 = \bar{\psi}^*_R C i\gamma^0 = -\psi^T_R C \end{aligned} \tag{6.4.3}$$

If the right-handed fermions belong to a representation R of the gauge group, these charge-conjugate fermions belong to the complex conjugate representation R^*.[11] Because we can always take the generators of a compact Lie algebra to be antihermitian (if need be, by making a similarity transformation), it is clear that the anomaly of the charge-conjugate fermions is minus the anomaly of right-handed fermions. This is also clear from the fact that in the V-A basis the anomaly involves a trace with γ_5, and the left-handed fermions are proportional to $\frac{1}{2}(1+\gamma_5)$ while the right-handed fermions are proportional to $\frac{1}{2}(1-\gamma_5)$.

Hence the total anomaly due to triangle graphs of left-handed fermions in representations R_L, and right-handed fermions in representations R_R, is proportional to

$$A = d^L_{abc} - d^R_{abc} = \sum_{R_L} \frac{1}{2} Tr\left(T(R_L)_a \left\{T(R_L)_b, T(R_L)_c\right\}\right) - \text{ idem for } R_R \tag{6.4.4}$$

From this simple formula it is clear that anomalies will cancel either if $d^L_{abc} = d^R_{abc}$, or (a special but important case) if both d^L_{abc} and d^R_{abc} vanish. If we rewrite the right-handed fermions as charge-conjugate left-handed fermions, then anomalies cancel if

[11] The current $\bar{\psi}_R\gamma^\mu\psi_R$ can be rewritten as $-\psi^T_R(-C\gamma^\mu C^{-1})\bar{\psi}^T_R = -\bar{\psi}^c_L\gamma^\mu\psi^c_L$ where we transposed the spinors and used that $C\gamma^\mu C^{-1} = -\gamma^{\mu,T}$. So the current of the conjugate spinors has an extra minis sign. The ψ^c_L couple to color matrices $T^T_a = -T^*_a$ instead of T_a due to the transposition of the spinors. Because traces are invariant under transposition, the d symbol of the complex conjugate representation R^* is equal to minus the d symbol of the representation R. The two minus signs cancel, so the charge-conjugate fermions couple the same way as the original fermions, but with matrices T^*_a.

the contributions from the various representations cancel in the sum

$$A = \sum_{R_L, R_L^c} \frac{1}{2} Tr\left(T_a, \{T_b, T_c\}\right) \tag{6.4.5}$$

It is clear that the symbols d_{abc} are totally symmetric. Since the trace is invariant under similarily transformations with group elements, the d_{abc} are also invariant tensors in the adjoint representation. Finally, the d symbols as defined in (6.4.1) are purely imaginary, as follows from the fact that the trace is invariant under transposition and the generators are antihermitian. It is customary to extract an overall factor i, so that then the d symbols become real. We shall normalize the generators of simple Lie algebras by the condition $Tr T_a T_b = -\frac{1}{2}\delta_{ab}$ for the defining $N \times N$ matrix representation for $SO(N), SU(N)$ and $Sp(N)$.

The d symbols are closely related to Casimir operators, which are operators which commute with every generator. For any simple Lie algebra with rank l there are l Casimir operators. A Casimir operator which is built out of k generators is said to be of rank k. There is always the quadratic Casimir operator $C_2(G) = g^{ab} T_a T_b$ where g^{ab} is the inverse of the Killing metric $g_{ab} \equiv -f_{pa}{}^q f_{qb}{}^p$. So for $SU(2)$ which has rank 1, this is the only Casimir operator. Other Casimir operators are of the form

$$C_k(G) = d^{a_1 \ldots a_k} T_{a_1} \ldots T_{a_k} \tag{6.4.6}$$

where $d^{a_1 \ldots a_k}$ is a totally symmetric real irreducible invariant tensor. (By irreducible we mean that it is not the product of invariant tensors of lower rank or that it is not the sum of two or more Casimir operators). Thus, our d symbols d_{abc} (or d^{abc} after raising indices which is possible in semisimple Lie algebra) correspond to the existence of a third rank Casimir operator.

If a Lie algebra has no Casimir operator $C_3(G)$, then none of its representations can give rise to an anomaly. The corresponding groups are called safe groups. **Safe groups never lead to anomalies**. It is easy to find the safe groups. The ranks of the Casimir operators $C_k(G)$ for the various Lie algebras are given by the following list

$$\begin{aligned}
A(n-1) = SU(n) \quad &: \quad C_2, C_3, \dots, C_n \\
B(n) = SO(2n+1) \quad &: \quad C_2, C_4, \dots, C_{2n} \\
C(n) = Sp(2n) \quad &: \quad C_2, C_4, \dots, C_{2n} \\
D(n) = SO(2n) \quad &: \quad C_2, C_4, \dots, C_{2n-2}, C_n \\
G_2 \quad &: \quad C_2, C_6 \\
F_4 \quad &: \quad C_2, C_6, C_8, C_{12} \\
E_6 \quad &: \quad C_2, C_5, C_6, C_8, C_9, C_{12} \\
E_7 \quad &: \quad C_2, C_6, C_8, C_{10}, C_{12}, C_{14}, C_{18} \\
E_8 \quad &: \quad C_2, C_8, C_{12}, C_{14}, C_{18}, C_{20}, C_{24}, C_{30}
\end{aligned} \tag{6.4.7}$$

Hence, the only algebras with anomalies are $SU(n)$ with $n \geq 3$ and $SO(6)$. The algebra of $SO(3)$ is isomorphic to the algebra of $SU(2)$, while $SO(4) \sim SO(3) \times SO(3)$ and $SO(6) \sim SU(4)$. Thus the appearance of $SO(6)$ in the list of unsafe algebras is already included in the list of $SU(n)$ algebras.

Since invariant tensors are Clebsch-Gordon coupling constants for the product of irreducible representations, a nonvanishing tensor d_{abc} means that in the symmetric product of two adjoint representations one finds the adjoint representation back. Since there is only one C_3 in the list of Casimir operators of $SU(n)$, the symbol d_{abc} is unique[12] (up to an overall scale) . One may check by using Young-tableau methods that one finds the adjoint representation indeed only once if one decomposes the symmetric product of two adjoint representations into irreducible representations.

Even though some groups are not safe, some of their representations may be safe. Namely, for some groups, d_{abc} may vanish for some representations while for other

[12]Of course, the structure constants are a totally antisymmetric invariant tensor, and one finds always the adjoint representation at least once in the antisymmetric product of two adjoint representations.

representations it may be nonvanishing. In fact, the following representations are safe

(i) **real representations.** (More precisely representations for which all matrices T_a are real and thus antisymmetric). It is clear that for these representations d_{abc} vanishes because the trace goes over into minus itself after transposition.

(ii) **pseudoreal representations.** These are representations for which $T_a^* = ST_aS^{-1}$ for some matrix S (not necessarily in the Lie algebra or group). It is clear that the matrices T_a^* satisfy the same commutation relations as the T_a, and when $T_a^* = ST_aS^{-1}$ this is certainly the case (the relation $T_a^* = -ST_aS^{-1}$ is inconsistent with the commutation relations). Since T_a are antihermitian, $TrT_a\{T_b, T_c\} = -Tr\{T_c^*, T_b^*\}T_a^*$, so when $T_a^* = ST_aS^{-1}$, then d_{abc} will vanish.

If a representation R is neither real nor pseudoreal, it is called complex. The unsafe groups are $SU(N)$ with $N \geq 3$; they have thus complex representations, for example the **N** which is unsafe. Not every complex representation has anomalies (for example the spinor representations of $SO(10)$ are complex, but $SO(10)$ is safe), but all complex representations of the unsafe groups $SU(N)$ carry anomalies. So, safe groups and safe representations of nonsafe groups are without anomalies. However, a third possibility may occur: the nonvanishing anomalies of several unsafe representations may sum up to zero. To study this further, we must first determine how the d_{abc} of different unsafe representations are related to each other.

Since for simple groups d_{abc} is unique (if it exists) up to an overall scale, the d symbol of representation R is related to the d symbol of the fundamental representation R_f by

$$d_{abc}^{(R)} = d(R)d_{abc}^{(f)} \tag{6.4.8}$$

where $d(R)$ is a positive integer. To compute the coefficients $d(R)$ for $SU(N)$, we take a diagonal traceless purely imaginary generator A of $SU(N)$ in the $N \times N$ fundamental representation (for example, the diagonal matrix with entrices $(i, i \dots, (1-N)i)$ but we shall not need its explicit form). Since every irreducible representation of $SU(N)$ can be obtained from the fundamental representation by tensoring and reducing, we see that the $d(R)$ are positive or negative **integers** (or vanish). The following rules are obvious but useful

(1) For a direct sum of two representations R and R' one has $d(R \oplus R') = d(R) + d(R')$. To prove this note that $A^{R\oplus R'} = \begin{pmatrix} A^R & 0 \\ 0 & A^{R'} \end{pmatrix}$

(2) For a direct product of two representations R and R' one has $d(R \otimes R') = \dim R\, d(R') + \dim R'\, d(R)$. This follows from $A^{R\otimes R'} = A^R \otimes I^{R'} + I^R \otimes A^{R'}$.

(3) Under complex conjugation one has $d(R^*) = -d(R)$. This follows from $T_a^\dagger = -T_a$ as we already discussed.

Let us now compute the $d(R)$ for some frequently used irreducible representations of $SU(N)$.

Antisymmetric tensors $t^{\mu\nu}$. The components $t^{\mu\nu}$ with $\mu < \nu$ form a $\frac{1}{2}N(N-1)$ dimensional linear vector space in which the generators T_a act according to

$$\delta t^{ij} = (T_a)^i{}_{i'} t^{i'j} + (T_a)^j{}_{j'} t^{ij'} \tag{6.4.9}$$

where $(T_a)^i{}_j$ are the generators of the fundamental representation. Hence, denoting the irreducible representation $t^{ij} = -t^{ji}$ by $f \wedge f$, we find for the generators A

$$d^{f\wedge f}_{AAA} = \sum_{i<j} \left(A_i^f + A_j^f\right)^3 = \frac{1}{2}\left(\sum_{i,j} - \sum_{i=j}\right)\left(A_i^f + A_j^f\right)^3 \tag{6.4.10}$$

where A_i^f denotes the i-th diagonal entry of the generator A in the fundamental representation. Since $\sum_i A_i^{(f)} = 0$, one find

$$d^{f\wedge f}_{AAA} = N\sum_i \left(A_i^f\right)^3 - \frac{1}{2}\sum_i \left(2A_i^f\right)^3 = (N-4)\sum_i \left(A_i^f\right)^3 \tag{6.4.11}$$

In other words

$$d^{(f\wedge f)} = (N-4)d^{(f)} \tag{6.4.12}$$

As a check, note that for $SU(4)$ the anomaly cancels because the **6** representation $t^{ij} = -t^{ji}$ is real[13] (it is the vector representation of $SO(6)$). Note also that d is an integer, but negative for $N = 3$. For $N = 2$ $d(f)$ vanishes because $SU(2)$ is pseudoreal, and also $\sum_i (A_i^f)^3$ in (6.4.11) vanishes in that case because $A_1^f = -A_2^f$.

Similarly, for $R = f \wedge f \wedge f$ the $d(R)$ follows from a sum over $i < j < k$, and one finds

$$\sum_{i<j<k} (A_i^f + A_j^f + A_k^f)^3 = \frac{1}{6}\left(\sum_{i,j,k} -3\sum_{i=j,k} +2\sum_{i=j=k}\right)(A_i^f + A_j^f + A_k^f)^3$$
$$d(f \wedge f \wedge f) = \frac{1}{2}(N-3)(N-6)d(f) \tag{6.4.13}$$

As a check note that the totally antisymmetric representation t^{ijk} is pseudoreal for $N = 6$ and a singlet for $N = 3$. (For $SU(6)$ one has $(t^{ijk})^* = \epsilon_{ijklmn}t^{lmn}$ and for $SU(3)$ one has $t^{ijk} \sim \epsilon^{ijk}$).

Symmetric tensors: the symmetric tensor $t^{ij} = t^{ji}$ forms a representation R which we denote by $f \vee f$. Then

$$\begin{aligned} d_{AAA}^{f\vee f} &= \sum_{i\leq j}\left(A_i^f + A_j^f\right)^2 = \frac{1}{2}\left(\sum_{i,j} + \sum_{i=j}\right)\left(A_i^f + A_j^f\right)^3 \\ &= N\sum_i \left(A_i^f\right)^3 + \frac{1}{2}\sum\left(2A_i^f\right)^3 = (N+4)\sum_i\left(A_i^f\right)^3 \end{aligned} \tag{6.4.14}$$

Hence

$$d(f \vee f) = (N+4)d(f) \tag{6.4.15}$$

As a check, note that

$$d(f \otimes f) = d(f \wedge f) + d(f \vee f) = 2Nd(f) \tag{6.4.16}$$

[13] Actually, the situation is a bit more subtle for $SU(4)$. The complex representation t^{ij} is reducible, and can be decomposed into a selfdual and an anti selfdual part, $t^{ij} \pm \frac{1}{2}\epsilon^{ijkl}t^*_{kl}$, where $t^*_{kl} \equiv (t^{kl})^*$. One can solve $t^*_{k_4}$ in terms of t^{ij} with $1 \leq i,j \leq 3$, but the solution for $t^*_{k_4}$ transforms under $SU(4)$ the same way as if it were independent. (This is possible because ϵ^{ijkl} is an invariant tensor of $SU(4)$). One should evaluate the trace only on t^{ij} with $i,j \neq 4$, but the terms with t^{i4} give the minus the same contribution (because for example $A_1^f + A_2^f = -A_3^f - A_4^f$).

Similarly one finds for a totally symmetric t^{ijk}

$$d(f \vee f \vee f) = \frac{1}{2}(N+3)(N+6)d(f) \tag{6.4.17}$$

Mixed tensor products: For mixed tensors, the rules for direct sums and direct products of representations can be used. For example, for $N \geq 3$ the representation $t_k^{ij} = -t_k^{ji}$ is given by $(f \wedge f) \otimes f^* - f$. (Since $t_k^{ij}\delta_i^k = 0$, one must subtract for $N \geq 3$ the fundamental representation once). One finds then

$$\begin{aligned} d\left(t_k^{ij}\right) &= d(f \wedge f) \dim f^* + \dim(f \wedge f) d(f^*) - d(f) \\ &= [(N-4)N + \frac{1}{2}N(N-1)(-1) - 1]d(f) \\ &= \frac{1}{2}(N^2 - 7N - 2)d(f) \text{ for } N \geq 3 \end{aligned} \tag{6.4.18}$$

which is, of course, always an integer. As a check note that for $N = 3$ the t_k^{ij} transform like the $t_{(kl)}$ (because $\epsilon^{ijl}t_{(kl)}$ transforms like t_k^{ij}). The results of (6.4.15) and (6.4.18) indeed agree for $N = 3$.

As another application, consider the representation in the form of a gun, $\begin{smallmatrix}\square&\square\\\square&\end{smallmatrix}$. Using

$$d(f \otimes f \otimes f) = 3N^2 = d(f \vee f \vee f) + d(f \wedge f \wedge f) + 2d\left(\begin{smallmatrix}\square&\square\\\square&\end{smallmatrix}\right) \tag{6.4.19}$$

one finds $d\left(\begin{smallmatrix}\square&\square\\\square&\end{smallmatrix}\right) = N^2 - 9$ which makes sense because for $N = 3$ this is the adjoint representation.

As a final check we consider the direct product of three fundamental representations of $SU(N)$. We illustrate the reduction into irreducible representations by Young tableaux, and write the corresponding d symbols below the diagrams. For $SU(3)$ this corresponds to $3 \otimes 3 \otimes 3 = (6 \oplus 3^*) \otimes 3 = 10 \oplus 8 \oplus 8 \oplus 1$. We need that t_j^i corresponds to $t^{[ij],k}$ minus a trace t_i^i which is a singlet, hence $t_j^i \sim (f \wedge f) \otimes f - f \wedge f \wedge f - 1$.

$$\begin{aligned} d(t_j^i) &= d(f \wedge f) \dim f + \dim(f \wedge f) d(f) - d(f \wedge f \wedge f) \\ &= (N-4)Nd(f) + \frac{1}{2}N(N-1)d(f) - \frac{1}{2}(N-3)(N-6)d(f) \\ &= N^2 - 9 \end{aligned} \tag{6.4.20}$$

For $SU(3)$ this is the adjoint representation which has no anomaly as it is real.

$$\begin{aligned}
\square \otimes \square \otimes \square &= \ydiagram{1,1} + 2\,\ydiagram{2,1} + \ydiagram{3} \\
d(f \otimes f \otimes f) &= d(f \wedge f \wedge f) + 2d\left(\ydiagram{2,1}\right) + d(f \vee f \vee f) \\
3N^2 &= \frac{1}{2}(N-3)(N-6) + 2(N^2-9) + \frac{1}{2}(N+3)(N+6)
\end{aligned} \tag{6.4.21}$$

We now study in more detail which representations are real, pseudoreal or complex. (Recall that a representation which is not real nor pseudoreal is called complex: it is not equivalent to its complex conjugate).

A real or pseudoreal representation R satisfies by definition $T_a^* = ST_aS^{-1}$. Clearly, such a matrix S, if it exists, is unique (up to an overall scale). Furthermore, by taking the complex conjugate of this relation, we see that S^*S is proportional to the unit matrix. For a unitary representation this condition can be rewritten as

$$ST + T_a^T S = 0 \tag{6.4.22}$$

This shows that S is an invariant tensor in the representation $R \times R$. Taking once more the transpose of $-(T_a)^T = ST_aS^{-1}$, we find that $S^{-1,T}S$ is proportional to the unit matrix. Hence S^T is proportional to S, and thus ***S* is either symmetric or antisymmetric**. Together with the information that $S^*S = \alpha I$, we conclude that $S^\dagger S = \beta I$, hence (after rescaling) ***S* is unitary**. We claim that if and only if S is symmetric, then R is similar to a real representation, whereas if and only if S is antisymmetric, then R is pseudoreal but not similar to a real representation. To prove this, first assume that R is equivalent to a real antisymmetric representation R'

$$T'_a = VT_aV^{-1}, (T'_a)^T = -T'_a \tag{6.4.23}$$

Then

$$\begin{aligned}
(T'_a)^T &= V^{-1,T}T_a^TV^T = -V^{-1,T}T_a^*V^T \\
&= -(T'_a) = -VT_aV^{-1}
\end{aligned} \tag{6.4.24}$$

Hence, $V^TV = S$ and S is symmetric. Conversely, if $T_a^* = ST_aS^{-1}$ and S is symmetric (and unitary, see above) we can take a unitary symmetric square root $\sqrt{S}$ (because a real symmetric matrix can be diagonalized by a real orthogonal matrix), and then $T_a' = \sqrt{S}T_a(\sqrt{S})^{-1}$ is not only antihermitian, but also antisymmetric, and thus real.

Applying this knowledge to the 3-index antisymmetric tensor representation of $SU(6)$ denoted by the Young tableau $\begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline\end{array}$, we conclude from the existence of an antisymmetric invariant tensor $S^{IJ} = \epsilon^{\mu\nu\rho\alpha\beta\gamma}$ (with $I = (\mu,\nu,\rho)$ and $J = (\alpha,\beta,\gamma)$) that this representation is pseudoreal. We can also check that the $\mathbf{6}$ of $SU(4)$ is real because $S^{IJ} = \epsilon^{\mu\nu\rho\sigma}$ is a symmetric invariant tensor.

The representations of $SU(2)$ are pseudoreal for odd spin, and real for even spin. This is clear for the fundamental $\mathbf{2}$ representation (because $(i\tau_a)^* = S(i\tau_a)S^{-1}$ where $S = \tau_2$ is indeed antisymmetric) and for the adjoint representation (because the latter is always real, namely $(T_a)^b{}_c = f^b{}_{ac}$). All representations of $Sp(2n)$ are pseudoreal or real because the fundamental representation is pseudoreal (it satisfies $\Omega T_a + T_a^T\Omega = 0$, hence $T_a^* = ST_aS^{-1}$ with $S = \Omega$, where the symplectic metric Ω is antisymmetric). The same holds for $SO(5)$ because it is equivalent to $Sp(4)$.

In fact, all tensor representations of $SO(n)$ are real, but the spinor representations need not be real. For even n there are two inequivalent fundamental spinor representations, but in odd dimensions only one. The reality properties of the fundamental spinor representations of $SO(n)$ are as follows

$$\begin{array}{ll} SO(8n) & \text{: both complex} \\ SO(8n+1) & \text{: real} \\ SO(8n+2) & \text{: both pseudoreal} \\ SO(8n+3) & \text{: pseudoreal} \\ SO(8n+4) & \text{: both pseudoreal} \\ SO(8n+5) & \text{: pseudoreal} \\ SO(8n+6) & \text{: both complex} \\ SO(8n+7) & \text{: real} \end{array}$$

Since $SO(3) \sim SU(2)$ we understand the pseudoreality for $SO(8k+3)$. Further, $SO(4) = SO(3) \otimes SO(3)$, so also here the pseudoreality is clear. For $SO(5)$ we use that $SO(5) \sim Sp(4)$ where the fundamental representation $\mathbf{4}$ of $Sp(4)$ is pseudoreal.[14] For $SO(6)$ the spinor representations are complex, as one may note from the isomorphy of $SO(6)$ and $SU(4)$. (The generators are the 15 antihermitian matrices $[\gamma^\mu, \gamma^\nu], i\gamma^\mu, \gamma^\mu\gamma^5, i\gamma^5$ of 4 space dimensions). Thus $SO(8k+6)$ has complex spinor representations. For $SO(8k+7)$ with $k=0$ we use that there exists a purely imaginary 8×8 representation of the Dirac matrices $\Gamma^\mu_{(7)}$ in 7 space dimensions, hence the spinor representation of $SO(7)$ is real. Taking for 8 space dimensions $\Gamma^\mu_{(7)} \otimes \tau_2$ and $\mathbf{1} \otimes \tau_1$ as real symmetric Dirac matrices, it is also clear that both spinor representations of $SO(8)$ are real (and inequivalent. One denotes them by $\mathbf{8}_s$ and $\mathbf{8}_c$, and together with the vector representation $\mathbf{8}_v$ they satisfy triality relations). For 9 space dimensions one may then take as Dirac matrices $\Gamma^\mu_{(8)}$ and $\Gamma^9_{(8)}$ where $\Gamma^9_{(8)} \equiv \Gamma^1_{(8)} \dots \Gamma^8_{(8)}$ is real and symmetric and satisfies $\left(\Gamma^9_{(8)}\right)^2 = I$. Thus the spinor representation of $SO(9)$ is also real. The charge conjugation matrix C satisfies $C\Gamma^M C^{-1} = -\gamma^{M,T}$ hence $C[\Gamma^M, \Gamma^N]C^{-1} = -[\Gamma^M, \Gamma^N]^T = [\Gamma^M, \Gamma^N]^*$. It is given by $C = I \otimes \tau_2$, hence it maps one spinor representation into the other, and vice-versa, and so it is **not** an invariant tensor of either of the spinor representations of $SO(10)$. (Rather, it couples a spinor and a conjugate spinor to a scalar). The charge conjugation matrix C in $d=9$ satisfies $C\Gamma^\mu C^{-1} = \sigma\Gamma^{\mu,T}$ and is given by $C = I$ and $\sigma = +1$. In 10 dimensions the charge conjugation matrix is either $C_+ = I \otimes \tau_1$ or $C_- = I \otimes \tau_2$, where $C_\pm \Gamma^M C_\pm^{-1} = \pm(\Gamma^M)^\Gamma$ with $M = 1, \dots, 10$.

The situation is different for $SO(10)$. As Dirac matrices in 10 Euclidean space dimensions we take $\Gamma^\mu_{(9)} \otimes \tau_1$ and $I_{8\times 8} \otimes \tau_2$. The generators of the two spinor representations in $d=10$ are then block-diagonal and given by $[\Gamma^\mu_{(9)}, \Gamma^\nu_{(9)}]$ and $\pm i\Gamma^\mu_{(9)}$. We denote them by L_I^{MN} and L_{II}^{MN}. They are complex conjugates of each other and antihermitian. They are block off-diagonal, and thus cannot play the role of S in

[14]The generators of the spinor representation of $SO(5)$ are $[\Gamma^M, \Gamma^N]$ with hermitian $\Gamma^M = \{i\gamma^\mu\gamma^5, \gamma^5\}$ where γ^μ, γ^5 are hermitian Dirac matrices in 4 dimensions. The Lorentz generators are then $[\gamma^\mu, \gamma^\nu]$ and $i\gamma^\mu$. They satisfy $C\Gamma^{MN}C^{-1} = -(\Gamma^{MN})^T$ and $(\Gamma^{MN})^T = -(\Gamma^{MN})^*$. Since C is antisymmetric, the spinor representation of $SO(5)$ is pseudoreal.

the reality condition for the block-diagonal Lorentz generator. So, the change conjugation matrices are not invariant tensors of either of the spinor representation, but, rather, C_+ and C_- couple the two spinor representations to a scalar. In fact, there is no matrix S such that $SL_I^{MN}S^{-1} = (L_I^{MN})^* = L_{II}^{MN}$ because for $L_I^{MN} = i\Gamma^\mu_{(9)}$ this would imply that $S\Gamma^\mu_{(9)}S^{-1} = -\Gamma^\mu_{(9)}$ but if this were true for $\mu = 1, \dots 8$, it would be false for $\mu = 9$ (recall $\Gamma^9_{(9)} = \Gamma^1_{(9)} \dots \Gamma^8_{(9)}$). Thus the spinor representations of $SO(10)$ are complex. (For $SO(9,1)$ the reality properties are, of course, very different. The two spinor representations are both real, as is clear by taking $I_{8\times 8} \otimes i\tau_2$ instead of $I_{8\times 8} \otimes \tau_2$ as tenth Dirac matrix.)

If a group is unsafe, it must have at least one complex representation. However, not every complex representation leads to an anomaly (is unsafe): $SO(10)$ is safe but has complex spinor representations. The unsafe groups are $SU(N)$ with $N \geq 3$; the representations with Dynkin labels $(m_1, \dots, m_{N-1})$ are real if the sequence $m_1, \dots m_{N-1}$ is equal to the sequence $m_{N-1}, \dots, m_1$. The other representations are complex, and all complex representations of $SU(N)$ with $N \geq 3$ are unsafe.

As a first and very important application of these general results, we discuss the cancellation of chiral anomalies in the Standard Model. The cancellation takes place within each of the three families. The color group $SU(3)$ does not couple to chiral fermions, so it does not lead to anomalies. Triangle graphs with three $SU(2)$ gauge fields do not contain anomalies either since the fermions are in the **2** of $SU(2)$ which is pseudoreal. The only triangle graphs which could possibly contain anomalies are either graphs with three $U(1)$ gauge fields, or graphs with one $U(1)$ gauge field and two $SU(2)$ gauge fields. Since all fermions which couple to $SU(2)$ form doublets, the anomalies in triangles with one $U(1)$ gauge field and two $SU(2)$ gauge fields are proportional to $\sum e_j Tr T_a T_b \sim \sum e_j$, and these anomalies cancel because the sum of the hypercharges vanishes. (It is equal to the sum of the electric charges because the average electric charge in a multiplet is the hypercharge of that multiplet). The triangle graphs with three $U(1)$ gauge fields do not contain anomalies either because the electromagnetic group $U(1)$ only couples to nonchiral fermions. (The $U(1)$ gauge field of the $U(1)$ in $SU(2) \otimes U(1)$ is the hypercharge gauge field. It couples with the hypercharge e_Y, but also $\sum_j (e_Y^j)^3 = 0$ for each family: from the left-handed quarks $6(\frac{1}{6})^3$, from the right-handed quarks $-3(\frac{2}{3})^3 - 3(-\frac{1}{3})^3$, from left-handed leptons $2(-\frac{1}{2})^3$ and from the right-handed lepton $-(-1)^3$, adding up to zero).

For another application we consider the grand unified $SU(5)$ theory. The 15 left-handed fermions of one family fit into the **10** dimensional antisymmetric-tensor representation $t^{\mu\nu} = -t^{\nu\mu}$ and the complex conjugate of the **5**-dimensional representation.

$$\psi_{10} = \begin{pmatrix} \epsilon^{ijk} d_k^c & u^i & d^i \\ -u^j & 0 & e^- \\ -d^j & -e^- & 0 \end{pmatrix}; \quad \psi_{5^*} = (u_k^c, \nu, e^c) \tag{6.4.25}$$

Since for $SU(5)$ according to (6.4.12)

$$\begin{aligned} d(\mathbf{10}) &= (N-4)d(5) = d(5) \\ d(\mathbf{5}^*) &= -d(\mathbf{5}) \end{aligned} \tag{6.4.26}$$

it is clear that the anomalies cancel.

Another interesting grand unified model is $SO(10)$. The 15 fermions of one family, plus an extra hypothetical right-handed neutrino, form the **16** dimensional spinor representation of $SO(10)$. But because $SO(10)$ is safe there is no anomaly due to these fermions. In fact, one can decompose $SO(10)$ representations with respect to the subgroup $SU(5)$, and then $\mathbf{16} \to \mathbf{10} + \mathbf{5}^* + \mathbf{1}$. (The spinor generators of $SO(10)$ are the 16×16 matrices $M^{\mu\nu} = \{\gamma^\mu\gamma^\nu, i\gamma^\mu\}$ with $1 \leq \mu < \nu \leq 10$, where γ^μ are the nine real 16×16 Dirac matrices for $SO(9)$). This demonstrates that the **16** of $SO(10)$ has no $SU(5)$ anomalies; in fact, it does not even have $SO(10)$ anomalies as we already explained.

5 $\pi^0 \to 2\gamma$: a good anomaly

The neutral pion π^0 decays into two photons[15] (branching ratio 98.8%) with a lifetime of $(8.4 \pm 0.6)10^{-17}$ seconds (corresponding to a width of $\Gamma = 7.4eV$). As we shall discuss in this section, the dominant contribution to this decay comes from a diagram with a pion propagator connected to the AVV triangle graph. Taking only this contribution into account predicts that there are 3.01 ± 0.08 colors in QCD. This is one of the triumphs of quantum field theory, relating ultraviolet divergences (which produce the anomaly) to low energy hadron dynamics. It shows that anomalies not always merely contribute small corrections to processes; sometimes they yield the main contribution.

In the introduction to this chapter we recalled the confusion to which the self-energies and triangle diagrams led in the 1940's. A regularization scheme was needed to evaluate the divergent triangle graph. One used the Pauli-Villars subtraction method to obtain finite Lorentz- and gauge-invariant results, and also unitarity was preserved (for large enough regulator masses[16]). However, like any other regularization scheme, this method contained ambiguities which should be fixed by proper renormalization conditions. For example, the selfenergy of a neutron due to a loop with a neutron and a meson could be made finite by requiring that $\sum C_i m_i = 0$ where $C_i = \pm 1$ for commuting or anticommuting fields and $m_i (i \neq 0)$ denotes the masses of the regulator fields. However, a finite term proportional to $\sum C_i m_i \ln m_i$ was left and would diverge as m_i for $i \neq 0$ was made large. Thus Steinberger [3] imposed the additional requirement $\sum C_i m_i \ln m_i =$ constant, but this constant was not fixed by any physical requirement and introduced an ambiguity into the regularization scheme.

[15] The experiments measure π^0 production by a high energy photon in the Coulomb field of a nucleus (the Primakoff effect). See H.W. Atherton et al., *Phys. Lett.* **458B** (1985) 81 and A. Browman et al., *Phys. Rev. Lett.* **33** (1974) 1400.

[16] So large, in fact, that, for given incoming energy, no cuts through the regulator fermions could satisfy energy-momentum conservation.

He fixed this ambiguity by requiring this constant to vanish.

For the triangle graph of a pseudscalar π^0 meson decaying into two photons due to a triangle loop with a proton in the loop which coupled to the pions with a Yukawa coupling, Steinberger obtained a lifetime $\tau = \frac{1}{1.8}\frac{1}{g^2}10^{-14}\,\mathrm{sec}$, which agrees with experiment if one were to take the strong Yukawa coupling constant to be $g = 10$. However, there was absolutely no reason to limit one's attention to this single diagram in strong interaction physics, and since nobody knew how to sum all other diagrams with strong interactions, the value $g = 10$ was not convincing. One clearly needed a method to deal with the strong interactions in a nonperturbative way.

In the 1960's "current algebra" was developed for precisely this purpose. Hadronic and leptonic vector and axial vector currents were introduced to describe the weak interactions. The V-A interaction was written as

$$\begin{aligned}
\mathcal{L} &= \frac{G}{\sqrt{2}}(\bar{j}^\rho_{\mathrm{lep}} + \bar{j}^\rho_{\mathrm{had}})(j_{\mathrm{lep},\rho} + j_{\mathrm{had},\rho}) \\
j^\rho_{\mathrm{lep}} &= \bar{\psi}_\mu \gamma^\rho (1+\gamma^5)\psi_{\nu_\mu} + \bar{\psi}_e \gamma^\rho (1+\gamma^5)\psi_{\nu_e} \\
j^\rho_{\mathrm{had}} &= j^{\rho,\Delta S=0}_{\mathrm{had}} \cos\theta_C + j^{\rho,\Delta S\neq 0}_{\mathrm{had}} \sin\theta_C
\end{aligned} \tag{6.5.1}$$

where ΔS denotes the change in strangeness, and $\theta_C \sim 0.25$ is the Cabibbo angle. Each of these two hadronic currents decomposed into a vector current and an axial vector current[17], and according to the CVC (conserved vector current) hypothesis, the two charged weak vector currents with $\Delta S = 0$ (and without the factor $\cos\theta_C$) formed a $SU(2)$ triplet with the isospin-one part of the electromagnetic current. The Standard Model explains this: all these currents are conserved so they do not renormalize (the coupling constants at zero momentum transfer are simply the renormalized tree level coupling constants), and at tree level the electroweak interactions of the quarks are $-i\bar{\psi}\gamma^\mu[g_2 \vec{W}_\mu \cdot \frac{\vec{\tau}}{2} + g_1 Y]\psi$ with $\bar{\psi}\gamma^\mu \frac{\vec{\tau}}{2}\psi$ being just the triplet of currents of the CVC hypothesis.

[17] In the quark model, j_μ (had) $= \bar{u}\gamma^\mu(1+\gamma^5)d' + \bar{c}\gamma^\mu(1+\gamma^5)s'$ with $d' = d\cos\theta_C + s\sin\theta_C$ and $s' = s\cos\theta_C - d\sin\theta_C$.

For our discussion of π^0 decay into two photons we need an axial vector current, rather than a vector current. It is described by another famous current algebra relation, this time a relation which will allow us to control all hadronic effects in π^0 decay, namely the PCAC relation, where PCAC stands for partially conserved axial vector current. This time the hypothesis was

$$\partial^\mu j^{5,a}_{\mu,\mathrm{had}} = f_\pi m_\pi^2 \pi^a \,; a = 1,2,3 \tag{6.5.2}$$

where f_π is the pion decay constant (to be explained and computed below). This is an operator relation: it states that in any purely hadronic matrix element one may replace $f_\pi m_\pi^2$ times the Heisenberg operator for the pion by the divergence of the composite operator $j^{5,a}_{\mu,\mathrm{had}}$. The $\pm$ components are the charged weak axial vector currents in the V-A action with $\Delta S = 0$, while the meaning of the third component was initially not clear, but now it is recognized as the isovector axial-vector part of the weak neutral current.

In the quark model, $\partial^\mu j^5_\mu = 2im_q(\bar{q}\gamma^5 q)$, and the claim is that by hadronization this relation turns into the operator equality $\partial^\mu j^5_\mu = m_\pi^2 f_\pi \pi$. Dimensional analysis explains the presence of m_π^2 (f_π has the dimension of a mass, and could have been written as m_π times a dimensionless constant). Behind this claim is the idea that $\bar{q}\gamma_5 q$ is dominated by graphs in which a pion is exchanged, as in the graph in (6.5.3). Hence instead of the name PCAC for partially conserved axial vector current, one might have called the relation $\partial^\mu j^5_\mu = m_\pi^2 f_\pi \pi$ the PPDDAC, the pion-pole dominated divergence of the axial vector current. We shall later discuss a model (the linear sigma model) which produces this relation, but for the moment we just assume it to be true.

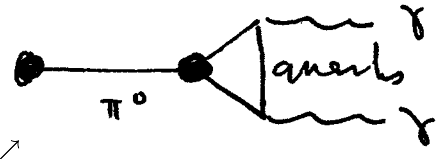
The crucial observation which explains π^0 decay, is that the PCAC relation, though correct for the $\pm$ components, is incorrect for the third decomponent because the anomaly is lacking. In section 2 we explicity demonstrated in concrete models that chiral anomalies in gauge symmetries make a quantum gauge field theory inconsistent:

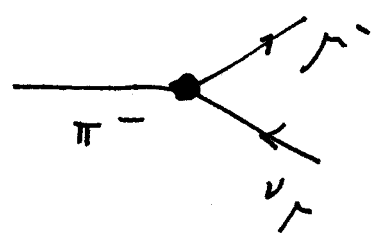
they violate unitarity and renormalizability. For π^0 decay one considers the chiral anomaly in a rigid symmetry; these anomalies do not lead to inconsistencies, and are actually needed to obtain agreement with experiment.

There are other examples of "good anomalies". We mention a few
(i) the "$U(1)$ problem". The QCD action with massless u, d, s quarks has a rigid $U(3)_L \times U(3)_R$ flavor symmetry. The diagonal part of this group contains the vector symmetries $SU(3)_V$ and $U(1)_V$. The former corresponds to the $SU(3)$ symmetry between the up, down and strange quarks proposed by Gell-Mann and Neeman and others. It leads to various phenomenological relations, for example, the Gell-Mann-Nishijina mass relation which was used to predict the mass of the spin 3/2 Ω^- baryon, a prediction which was completely confirmed by experiment. The $U(1)_V$ corresponds to baryon number, and as far as is known, this symmetry is exactly conserved. The axial-vector symmetries $SU(3)_A$ and $U(1)_A$ do not satisfy similar properties. If they were equally conserved, the hadron spectrum would have to display "parity-doublets": for each baryon with one parity there should exist another with opposite parity. Rather, it is believed that the $SU(3)_A$ symmetry is spontaneously broken, with the octet of pseudoscalar mesons $(\pi^\pm, \pi^0, K^+, K^0, \bar{K}^0, K^-, \eta)$ as Goldstone bosons. Goldstone bosons should be strictly massless, but explicit symmetry breaking either by small quark masses or by nonperturbative effects (due to instantons for example) could lead to small masses (small with respect to the other hadrons such as nucleons). The same argument would predict a very light $SU(3)$-singlet pseudoscalar boson η'. There exists in nature such a meson but its mass is not at all very light: it is of the order of the mass of nucleons, namely 900 MeV. The resolution to this problem of the absence of a nineth Goldstone boson is the axial anomaly: the $SU(3)$-singlet axial current has a nonvanishing divergence proportional to $\epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}$ (in Euclidean space, the x-integral of this expression is nonvanishing if the gauge fields form an instanton solution).
(ii) Another area where anomalies have been useful is the low-energy physics of pions

and kaons. The low-energy physics of pseudoscalar mesons is well described by a phenomenological effective action, the $WZWN$ term, which has a clear theoretical meaning: it is due to integrating the chiral anomaly. One can prove that the symmetries of the QCD action forbid certain processes, such as $K^+ + K^- \to \pi^+ + \pi^0 + \pi^-$ [34], but adding the $WZWN$ action (which breaks some of the symmetries of the QCD action) these processes can be described; in fact, the predictions are quite good, with errors typically of the order of 10%. By "gauging" the $WZWN$ model, one can also well describe processes involving both QED and QCD, for example $\gamma\gamma \to \pi^+\pi^-\pi^0$.
(iii) In certain weak decays, such as $K^+ \to \pi^+\pi^0\gamma$ and $K^+ \to \pi^+\pi^- e^+\nu_e$, there is a contribution from the chiral anomaly.

Let us now discuss π^0 decay. We shall relate it to $\pi^\pm$ decay. We consider both processes to all orders in the strong interactions, but π^0 decay to first order in electromagnetism, and $\pi^\pm$ decay to first order in weak interactions. Since pions are pseudoscalars we need axial vector currents in the description of these processes. The main idea is that the following two diagrams (or, rather, infinite set of quark diagrams) gives the dominant contributions to the decays





(6.5.3)

matrix element of $j_\mu^{5,3}$ — $\underbrace{\qquad\qquad}_{\pi^0 \text{ decay amplitude}}$

↑ matrix element of $j_\mu^{5,-}$

The theoretical current algebra relation on which the whole calculation rests, is the old PCAC relation of the early 1960's, corrected for electromagnetic processes by adding the anomaly term

$$\begin{aligned}
\partial^\mu j_\mu^{A,\pm} &= 2 f_\pi m_\pi^2 \pi^\pm(x) \\
\partial_\mu j_\mu^{A,3} &= 2 f_\pi m_\pi^2 \pi^3(x) + c\frac{e^2}{16\pi^2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}
\end{aligned} \tag{6.5.4}$$

The constant c is unity if one has one electron in the loop, but we consider colored quarks in the loop, and then c depends on the number N_c of colors, as well as on the electric charges of quarks. We derived the anomaly term in section 2 from the triangle diagram. There are, of course, infinitely many more diagrams contributing to the matrix elements of $j_\mu^{A,3}$, but according to the Adler-Bardeen [**?**] theorem in QED, only the lowest-order triangle graph contributes to $\partial^\mu j_\mu^{A,3}$. Thus (6.5.6) is supposed to be correct to all orders in strong interactions and to first order in α_{QED}.

The current in the modified PCAC relation is normalized to $\bar{\psi}\gamma^5\gamma^\mu\psi$ for one electron in the loop, and for one electron $c = 1$. However, we consider quarks in the loop with interaction $\sim \bar{\psi}\gamma_5\frac{\vec{\tau}}{2}\psi \cdot \vec{\pi}$ which couple to π^0 to the doublet of up and down quarks, $\psi = \binom{u}{d}$, with a matrix $\frac{1}{2}\tau_3$ where $\tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ in the Standard Model. In flavor $SU(3)$, one has $\tau_3 = \begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix}$ which yields the same results. Then $c = tr\frac{\tau^3}{2}Q^2$ hence[18]

$$c = \frac{1}{2}N_c\left[\left(\frac{2}{3}\right)^2 - \left(\frac{1}{3}\right)^2\right] = \frac{1}{6}N_c \tag{6.5.5}$$

where N_c is the number of colors of the quarks.

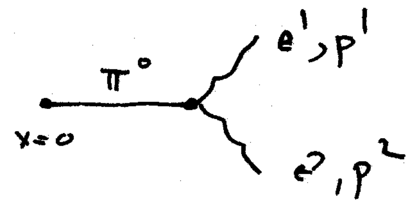
We shall now first derive a formula for the lifetime of π^0 setting everywhere $m_\pi = 0$. Afterwards we shall redo the derivation, keeping $m_\pi \neq 0$ this time. Often either the first or the second derivation is used, but since there appear pion pole terms proportional to $(q^2 + m_\pi^2)^{-1}$ in the derivation, setting m_π to zero and taking the limit $q \to 0$ requires some discussion, and for this discussion it is useful to compare the massive and massless cases.

With $m_\pi^2 = 0$, the divergence of the axial current is equal to the anomaly term. Taking matrix elements of this operator equation at $x = 0$ between the vacuum and

[18] In approaches which used physical particles instead of quarks, one considered loops with nucleons (p and n) instead of loops with quarks. This yielded the same result for three colors since $3 \times \left[\left(\frac{2}{3}\right)^2 - \left(\frac{1}{3}\right)^2\right] = 1^2 - 0^2$.

the two-photon final state yields the matrix element $\langle \epsilon^1, p^1; \epsilon^2, p^2 | \partial^\mu j_\mu^{A,3} | 0 \rangle$ on the left-hand side of (6.5.4). The claim is that the main contribution to this matrix element comes from the one-pion intermediate state

$$\langle e^1 p^1; \epsilon^2 p^2 | j_\mu^{A,3}(0) | 0 \rangle =$$



$$= \sum_{\vec{q}} \langle e^1 p^1; \epsilon^2 p^2 | \pi^0, \vec{q} \rangle \, \langle \pi^0, \vec{q} | j_\mu^{A,3}(0) | 0 \rangle \tag{6.5.6}$$

In the second matrix element in this result we use PCAC without any electromagnetic corrections because the latter appear in the first matrix element and we work to first order in α

$$\langle \pi^0, \vec{q} | j_\mu^{A,3} | 0 \rangle = q_\mu f_\pi \frac{1}{\sqrt{2\omega_q}} \tag{6.5.7}$$

We used here that for nonvanishing m_π^2 one has $\partial^\mu j_\mu^{A,3} = m_\pi^2 f_\pi \pi^0$, and the factor $(2\omega_q)^{-1/2}$ is the usual wave function normalization for second-quantized fields

$$\langle \pi^0, \vec{q} | \pi^0(0) | 0 \rangle = \frac{1}{\sqrt{2\omega_q}} \; . \tag{6.5.8}$$

The first matrix element in (6.5.6) yields the amplitude for π^0 decay and is parametrized by only one constant

$$\langle \epsilon' p'; \epsilon^2 p^2 | \pi^0, \vec{q} \rangle = \frac{1}{\sqrt{2\omega_1}} \frac{1}{\sqrt{2\omega_2}} A(\epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}) \frac{1}{\sqrt{2\omega_q}} \tag{6.5.9}$$

The integral over $\vec{q}$ of $\frac{1}{\sqrt{2\omega_q}}$ yields the Feynman propagator: $\Delta_F(x) = \Delta^+ \theta(x^0) + \Delta^- \theta(-x^0)$ with $\Delta^+(x) = \int d^3q (2\omega_q)^{-1} e^{iqx}$. Note now that the factor q^2 which is due to contracting the matrix element of j_μ with q^μ, yielding the matrix element of $\partial^\mu j_\mu$, is canceled by the propagator $1/q^2$. The net result is that, with $m_\pi^2 = 0$, the limit $q^2 \to 0$ does not vanish! Thus only diagrams with a pole term at $q^2 = 0$ can contribute, and this is the reason that one only considers the graph in (6.5.6).

The total pion-pole dominated matrix element in (6.5.6) is thus given by

$$\langle \epsilon' p'; \epsilon^2 p^2 | \partial^\mu j_\mu^{A,3}(0)|0\rangle = \frac{1}{\sqrt{2\omega_1}} \frac{1}{\sqrt{2\omega_2}} f_\pi A \epsilon^{\mu\nu\rho\sigma}(F_{\mu\nu}F_{\rho\sigma}) \tag{6.5.10}$$

The invariant matrix element of the anomaly term on the right-hand side of the PCAC relation in (6.5.4) is easily evaluated

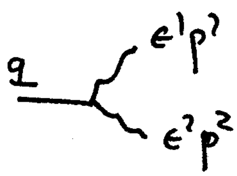
$$\begin{aligned} &\langle \epsilon' p'; \epsilon^2 p^2 | c\frac{e^2}{16\pi^2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}|0\rangle = \\ &\qquad \frac{1}{\sqrt{2\omega_1}}\frac{1}{\sqrt{2\omega_2}} c\frac{e^2}{16\pi^2}(8\epsilon^{\mu\nu\rho\sigma}\epsilon_\mu^1\epsilon_\nu^2 p_\rho^1 p_\sigma^2) \end{aligned} \tag{6.5.11}$$

Equating both matrix elements yields A in terms of the pion decay constant f_π and the number of colors of quarks, N_c,

$$A = \frac{\alpha}{\pi}\frac{1}{f_\pi}\left(\frac{1}{3}N_c\right) \tag{6.5.12}$$

The invariant matrix element for $\pi^0 \to 2\gamma$ is then given by

$$M(\pi^0 \to 2\gamma) =$$



$$= \frac{\alpha}{\pi} f_\pi \left(\frac{1}{3}N_c\right)\epsilon^{\mu\nu\rho\sigma}\epsilon_\mu^1\epsilon_\nu^2 p_\rho^1 p_\sigma^2 \tag{6.5.13}$$

The decay rate of π^0 is now easily computed

$$\begin{aligned} \Gamma(\pi^0 \to 2\gamma) &= \frac{1}{2m_{\pi^0}}\int \frac{d^3p^1}{(2\pi)^3 2\omega_1}\frac{d^3p^2}{(2\pi)^3 2\omega_2} \\ &\frac{1}{2}\left(\sum_{\text{pols}}|M|^2\right)(2\pi)^4\delta^4(q-p^1-p^2) \end{aligned} \tag{6.5.14}$$

The factor 1/2 is added to implement Bose statistics.

Replacing the sums over the two transversal polarizations by $\eta_{\mu\nu}$ tensors as usual, one obtains

$$\Gamma(\pi^0 \to 2\gamma) = \frac{1}{32\pi}A^2(m_{\pi^0})^3; A = \frac{\alpha}{\pi}\frac{1}{f_\pi}\left(\frac{1}{3}N_c\right) \tag{6.5.15}$$

We must now determine the pion decay constant f_π. We shall fix it by the π^- decay $\pi^- \rightarrow \mu^- + \bar{\nu}_\mu$. We intorduced f_π in (6.5.7). Since the strong interactions preserve isospin, we have also

$$\langle 0|j_\mu^{5,-}|\pi^-\rangle = q_\mu f_\pi \frac{1}{\sqrt{2m_{\pi^-}}} \ . \tag{6.5.16}$$

Namely, $j_\mu^{5,-}(\text{hadronic}) = f_\pi \partial_\rho \pi^-$, and we use again the normalization

$$\langle 0|\pi^-(0)|\pi^-\rangle = \sqrt{\frac{1}{2m_{\pi^-}}} \tag{6.5.17}$$

The interaction Lagrangian for π^- decay is then given by the V-A theory as

$$\mathcal{L}\ (\text{weak})\ = \frac{G_F}{\sqrt{2}}[\bar{\psi}_\mu \gamma^\rho (1+\gamma^5)\psi_{\nu_\mu}][j_\rho^{V,-} + j_\rho^{A,-}] \tag{6.5.18}$$

The decay rate for $\pi^- \rightarrow \mu^- + \bar{\nu}_\mu$ is given by

$$\Gamma(\pi^- \rightarrow \mu^- + \bar{\nu}_\mu) = \frac{1}{2m_\pi}\int \frac{d^3p_\mu}{(2\pi)^3}\frac{d^3p_\nu}{(2\pi)^3}$$
$$\sum_{\text{pols}} |\ \frac{G_F}{\sqrt{2}} f_\pi m_\mu \bar{u}^+(p_\mu)(1+\gamma^5)u^-(-p_\nu)\ |^2\ (2\pi)^4\delta^4(q-p_\mu-p_\nu) \tag{6.5.19}$$

where q^ρ is the pion momentum. We used the Dirac equation. Standard manipulations lead to

$$\Gamma = \frac{1}{2m_\pi}\int \frac{d^3p_\mu}{(2\pi)^3 2E_\mu}\frac{d^3p_\nu}{(2\pi)^3 2E_\nu}\left(\frac{G_F^2}{2}f_\pi^2 m_\mu^2\right)$$
$$[Tr(-i\not{p}_\mu + m_\mu)(1+\gamma_5)(-i\not{p}_\nu)(1-\gamma_5)](2\pi)^4\delta^4(q-p_\mu-p_\nu) \tag{6.5.20}$$

The trace inside the square brackets yields

$$2Tr(-i\not{p}_\mu + m_\mu)(-i\not{p}_\nu) = -8p_\mu \cdot p_\nu = 4(m_\pi^2 - m_\mu^2) \tag{6.5.21}$$

and the integrals over momenta lead to

$$\int d^3p_\mu\ \delta(m_\mu - E_\mu - p_\mu) = \frac{4\pi p_\mu^2}{p_\mu/E_\mu + 1} = \frac{4\pi p_\mu^2 E_\mu}{m^\mu} \tag{6.5.22}$$

where p_μ follows from $m_\mu = E_\mu + p_\mu$ and is given by $p_\mu = (m_\pi^2 - m_\mu^2)/(2m_\pi)$. This yields

$$\Gamma = \frac{1}{\tau(\pi^\pm)} = \frac{1}{8\pi}\left(\frac{m_\pi^2 - m_\mu^2}{m_\pi^2}\right)^2 (G_F m_\mu m_\pi)^2 \left(\frac{f\pi}{m_\pi}\right)^2 m_\pi \tag{6.5.23}$$

Substituting the experimental value of the lifetime of charged pions $\tau(\pi^{\pm}) = 2.510^{-8}$ sec or $m_\pi \tau(\pi^{\pm}) = \frac{1}{2}10^{16}$, and the value of the Fermi constant[19] $G_F m_p^2 = 1.026\ 10^{-5}$ or $G_F m_\mu m_\pi = \frac{1}{6}10^{-6}$, one finds[20] for the pion decay constant $f_\pi = 130.7 \pm 0.4 f_\pi = 0.931 m_\pi = 93\,\text{MeV}$.

With this result for f_π, one finds excellent agreement between (6.5.15) and experiment for 3 colors; the quantitative details were given above.

6 Consistency conditions and Bardeen anomaly

The anomalies in the Ward identity must satisfy certain consistency conditions. These consistency conditions determine the anomalies completely, once the leading term is known (for example the AVV anomaly). The origin of these consistency conditions is that the anomalies are the result of a gauge variation of the effective action (if the effective action is gauge invariant there is no anomaly). Making two successive gauge variations on the effective action is equivalent to a gauge transformation of the anomaly, and from this information the consistency conditions follow. We shall consider the effective action (proper graphs) at the one-loop level with only external gauge fields. These are the only proper graphs at the one-loop level that can carry anomalies, but they can also appear inside higher-order loop graphs. Because for gauge fields a BRST transformation is equal to a gauge transformation (with Λ^a replaced by $c^a\Lambda$), making a gauge variation of the one-loop effective action with external gauge fields is equivalent to making a BRST transformation.

It is not possible to construct a local counter term depending only on the gauge fields whose variation cancels the anomalies. We prove this in the next section.

[19] Experimentally, $\sqrt{2} f_{\pi^+} = 130.7 \pm 0.4\,\text{MeV}$ and $\sqrt{2} f_{\pi^0} = 130 \pm 5\,\text{MeV}$, which shows that isospin symmetry is well satisfied, (see Review of Particle Physics in *The European Physical Journal* **C15** (2000) 1-878).

[20] Strictly speaking we should have used the Calibbo corrected coupling constant $G_F \cos\theta_C$ instead of G_F for the strangeness conserving weak current, but $\cos^2\theta_C$ is almost unity ($\theta_C = 0.25$).

However, if one adds further fields to the effective action, in particular spin 0 fields, it is possible to construct such a local counter term. One can then decompose the effective action into a sum of one term which is local but anomalous, and another term which is nonlocal but without anomaly. The local anomalous term is called the Wess-Zumino (WZ) term [33,34], and plays a role in models which are used to describe low-energy phenomenology in terms of mesons (pions, kaons, etc.). For certain processes the WZ term is the only term that can contribute to these processes, and this explains why it plays a crucial role in describing nonperturbative low-energy hadronic physics. We begin this section by studying the consistency conditions on anomalies. Next we construct the general solution of the consistency condition. Finally we construct the WZ term.

To derive the consistency conditions, we note that the anomaly is the response of the effective action under a local gauge transformation. We shall first discuss the theory on the basis of vector fields V and axial vector fields A; later we repeat the discussion for the chiral theory with gauge fields W_L. Let the one-loop effective action due to loops with fermions in the loop and external vector fields ${V_\mu}^a$ and axial vector fields ${A_\mu}^a$ be denoted by $\Gamma(V, A)$. (For one-loop graphs with external gauge fields there is no difference between connected or proper graphs, hence $\Gamma(V, A)$ is the same functional as $W(V, A)$). As we have seen in (6.1.10), the vector and axial vector gauge transformations are given by

$$\begin{aligned} \delta V_\mu &= D_\mu(V)\Lambda_V + [A_\mu, \Lambda_A] \\ \delta A_\mu &= D_\mu(V)\Lambda_A + [A_\mu, \Lambda_V] \end{aligned} \tag{6.6.1}$$

where $D_\mu(V)\Lambda = \partial_\mu\Lambda + [\, V_\mu, \Lambda]$. We introduce operators $X(\Lambda_V)$ and $Y(\Lambda_A)$ whose action on V_μ, and A_μ are these gauge transformations

$$\begin{aligned} X(\Lambda_V) &= (D_\mu(V)\Lambda_V)\frac{\partial}{\partial V_\mu} + [A_\mu, \Lambda_V]\frac{\partial}{\partial A_\mu} \\ Y(\Lambda_A) &= (D_\mu(V)\Lambda_A)\frac{\partial}{\partial A_\mu} + [A_\mu, \Lambda_A]\frac{\partial}{\partial V_\mu} \end{aligned} \tag{6.6.2}$$

(On the right-hand sides there is an integral over all points x, but we did not write this to make the notation simple. If $\Lambda_V(x)$ and $\Lambda_A(x)$ are only nonzero in a small region, these x-integrals are finite). Let us assume that, if necessary, a suitable counter term has been added to the effective action such that no vector gauge anomalies are left. (This can always be done, and we shall say more about this later). Then

$$\begin{aligned} X(\Lambda_V)\Gamma(A,V) &= 0 \\ Y(\Lambda_A)\Gamma(A,V) &= \int {\Lambda_A}^a G_a(A,V) d^4x \end{aligned} \tag{6.6.3}$$

where $G_a(A,V)$ is by definition the axial-vector anomaly. As any anomaly, it is a local polynomial in the gauge fields and their derivatives. We shall show that it is equal to a particular expression (called the Bardeen anomaly) which is completely fixed by consistency up to an overall constant.

The algebra of vector and axial vector transformations closes on ${V_\mu}^a$ and ${A_\mu}^a$. In particular, the commutator of a vector and axial vector gauge transformation of ${V_\mu}^a$ is another axial vector gauge transformation of V_μ whose parameter is the commutator of the two original gauge parameters

$$\begin{aligned} &[\delta_A(\Lambda_A), \delta_V(\Lambda_V)]V_\mu = \delta_A(\Lambda_A)D_\mu(V)\Lambda_V - \delta_V(\Lambda_V)[A_\mu, \Lambda_A] \\ &= [[A_\mu, \Lambda_A], \Lambda_V] - [[A_\mu, \Lambda_V], \Lambda_A] = [A_\mu, [\Lambda_A, \Lambda_V]] = \delta_A([\Lambda_A, \Lambda_V])V_\mu \end{aligned} \tag{6.6.4}$$

The same result is found for A_μ and hence for any function of V_μ and A_μ. In terms of the operators X and Y we therefore obtain

$$[Y(\Lambda_A), X(\Lambda_V)] = Y([\Lambda_A, \Lambda_V]) \tag{6.6.5}$$

Similarly one finds for the commutator of two vector or two axial vector gauge transformations

$$\begin{aligned} [X(\Lambda_V^{(1)}), X(\Lambda_V^{(2)})] &= X([\Lambda_V^{(1)}, \Lambda_V^{(2)}]) \\ [Y(\Lambda_A^{(1)}), Y(\Lambda_A^{(2)})] &= X([\Lambda_A^{(1)}, \Lambda_A^{(2)}]) \end{aligned} \tag{6.6.6}$$

These brackets satisfy the Jacobi identities because X and Y are explicit representations of the generators.

The consistency relations follow now by acting with X or Y on the anomaly, and using the commutation relations and the fact that $X(\Lambda_V)\Gamma = 0$. The first consistency condition is due to the action of X on G_a

$$\begin{aligned}
&\delta_V(\Lambda_V)\int \Lambda_A{}^a G_a d^4x = X(\Lambda_V)\int \Lambda_A{}^a G_a d^4x = X(\Lambda_V)Y(\Lambda_A)\Gamma(A,V) \\
&= [X(\Lambda_V), Y(\Lambda_A)]\Gamma(A,V) = Y([\Lambda_V,\Lambda_A])\Gamma(A,V) = \int [\Lambda_V,\Lambda_A]^a G_a d^4x \\
&= f_{bc}{}^a \int \Lambda_V{}^b \Lambda_A{}^c G_a d^4x
\end{aligned} \tag{6.6.7}$$

Hence the axial vector anomaly G_a transforms under vector gauge transformations as a vector in the adjoint representation

$$\delta_V(\Lambda_V)G_a = f_{ab}{}^c G^b \Lambda_{V,c} \tag{6.6.8}$$

where $f_{ab}{}^c$ comes from $[\Lambda_V,\Lambda_A] = \Lambda_V{}^a \Lambda_A{}^b f_{ab}{}^c T_c$.

Before solving the second consistency condition we discuss the general form of the chiral anomaly. The general solution of the consistency equations for the chiral anomaly can be obtained in two steps: first one writes down the most general expression which has the correct dimension and parity and is invariant under local V-gauge transformations. This expression contains arbitrary constants. Then one requires that it satisfies the consistency conditions for local A-gauge transformations. This fixes all coefficients up to an overall constant. As we shall show, the general solution of the consistency conditions on the V-A basis is the Bardeen anomaly [36]

$$\begin{aligned}
&G_a = \frac{1}{16\pi^2}\epsilon^{\mu\nu\rho\sigma} tr T_a [F_{\mu\nu}(V)F_{\rho\sigma}(V) + \frac{1}{3}F_{\mu\nu}(A)F_{\rho\sigma}(A) \\
&-\frac{8}{3}\{A_\mu A_\nu F_{\rho\sigma}(V) + A_\mu F_{\nu\rho}(V)A_\sigma + F_{\mu\nu}(V)A_\rho A_\sigma\} + \frac{32}{3}A_\mu A_\nu A_\rho A_\sigma] \\
&F_{\mu\nu}(V) = (\partial_\mu V_\nu - \partial_\nu V_\mu + [V_\mu, V_\nu]) + [A_\mu, A_\nu] \\
&F_{\mu\nu}(A) = D_\mu(V)A_\nu - D_\nu(V)A_\mu \,.
\end{aligned} \tag{6.6.9}$$

It is indeed covariant under vector gauge transformations: no bare V_μ fields appear but only curvatures $F_{\mu\nu}(V)$ or covariant derivatives $D_\mu(V)$. Under vector gauge transformations, $\delta_V(\Lambda_V)F_{\mu\nu} = [F_{\mu\nu}, \Lambda_V]$ both for $F_{\mu\nu}(V)$ and $F_{\mu\nu}(A)$, and also $\delta_V(\Lambda_V)A_\mu = [A_\mu, \Lambda_V]$. Then each term yields a commutator $[\Lambda_V, T_a]$ if one uses cyclicity of the trace, and since the trace of a commutator vanishes, (6.6.8) follows. It is also easy to check that no other structures than those which appear in the Bardeen anomaly are allowed by dimension and parity. However, the coefficients of these terms are not determined by vector covariance.

The expression for the nonabelian anomaly (the anomaly with T_a) can also be used to construct the abelian anomaly: one just has to replace T_a by i. The abelian anomaly for nonabelian anomalies has no $AAAA$ terms because $trAAAA$ vanishes (moving one A from the left to the right produces a minus sign). Of course the abelian anomaly for abelian gauge theories contains only contributions from triangle graphs proportional to squares of Maxwell curvatures.

The second consistency condition is less obvious, and fixes all coefficients in the Bardeen anomaly up to an overall constant. It is due to acting with Y on G_a

$$\delta_A(\Lambda_A^{(1)}) \int \Lambda_A^{(2),a} G_a d^4x = Y(\Lambda_A^{(1)})Y(\Lambda_A^{(2)})\Gamma \tag{6.6.10}$$

Antisymmetrizing in $\Lambda_A^{(1)}$ and $\Lambda_A^{(2)}$, we obtain $X\Gamma$, and hence zero. Thus, the axial vector gauge variation with parameter $\Lambda_A^{(1)}$ of the axial vector anomaly vanishes if one contracts the latter with $\Lambda_A^{(2)}$ and antisymmetrizes the result in $\Lambda_A^{(1)}$ and $\Lambda_A^{(2)}$. The reader may verify that the Bardeen anomaly passes this consistency test, and that one indeed finds the coefficients $1, \frac{1}{3}, -\frac{8}{3}$ and $\frac{32}{3}$ in this anomaly in this way. (To prove this one may use that $\delta(\Lambda_A)A_\mu = D_\mu(V)\Lambda_A$, and further $\delta(\Lambda_A)F_{\mu\nu}(V) = [F_{\mu\nu}(A), \Lambda_A]$ and $\delta(\Lambda_A)F_{\mu\nu}(A) = [F_{\mu\nu}(V), \Lambda_A]$. It helps to use form notation, but the algebra is nevertheless tedious. The least laborious is to check the terms with a minimal number of fields, or the terms with a maximal number of fields).

Let us now derive the general form for the axial anomaly on the chiral basis. We repeat the steps of the derivation of the consistency requirements for the axial anomaly for chiral gauge transformations. Now the generator of gauge transformations on the gauge fields ${W_\mu}^a$ which couple to left-handed fermions is given by

$$\begin{aligned} \delta(\Lambda_L){W_\mu}^a &= D_\mu(W)\Lambda_L = \partial_\mu\Lambda_L + [W_\mu, \Lambda_L] \\ X_L(\Lambda_L) &= (\partial_\mu\Lambda_L + [W_\mu, \Lambda_L])^a \frac{\partial}{\partial {W_\mu}^a} \end{aligned} \tag{6.6.11}$$

The gauge algebra now only consists of

$$[X_L({\Lambda_L}^{(1)}), X_L({\Lambda_L}^{(2)})] = X_L([{\Lambda_L}^{(1)}, {\Lambda_L}^{(2)}]) \tag{6.6.12}$$

and the anomaly is given by

$$X(\Lambda_L)\Gamma(W) = \int {\Lambda_L}^a G_a(W) d^4x \tag{6.6.13}$$

Hence, the anomaly must satisfy

$$X_L({\Lambda_L}^{(1)}) \int {\Lambda_L}^{(2)a} G_a(W) d^4x - 1 \leftrightarrow 2 = \int [{\Lambda_L}^{(1)}, {\Lambda_L}^{(2)}]^a G_a d^4x \tag{6.6.14}$$

(We shall in the next section show that this equation can be rewritten as $s \int c^a G_a d^4x = 0$, where s denotes a BRST variation).

It is not too difficult to check that

$$G_a(W) = trT_a d(WdW + \alpha WWW) \tag{6.6.15}$$

satisfies this consistency requirement for $\alpha = 1/2$. Partially integrating the overall d such that it acts on $\Lambda_L^{(1)}$, we must show that

$$\begin{aligned} & \int tr(d{\Lambda_L}^{(1)})\delta({\Lambda_L}^{(2)}) \left\{ WdW + \frac{1}{2}WWW \right\} d^4x - 1 \leftrightarrow 2 \\ = & \int tr(d[{\Lambda_L}^{(2)}, {\Lambda_L}^{(1)}]) \left\{ WdW + \frac{1}{2}WWW \right\} d^4x \end{aligned} \tag{6.6.16}$$

Using

$$\begin{aligned} \delta({\Lambda_L}^{(2)})WdW &= (d{\Lambda_L}^{(2)})dW + [W, {\Lambda_L}^{(2)}]dW + Wd[W, {\Lambda_L}^{(2)}] \\ \delta({\Lambda_L}^{(2)})WWW &= (d{\Lambda_L}^{(2)})WW + Wd{\Lambda_L}^{(2)}W \\ &+ WWd{\Lambda_L}^{(2)} + [WWW, {\Lambda_L}^{(2)}] \end{aligned} \tag{6.6.17}$$

one finds

$$\begin{aligned}
&tr(d\Lambda_L{}^{(1)})\delta(\Lambda_L{}^{(2)})\{WdW+\alpha WWW\} = \\
&tr(d\Lambda_L{}^{(1)})\{(d\Lambda_L{}^{(2)})dW+\alpha(d\Lambda_L{}^{(2)})WW-(1-\alpha)WWd\Lambda_L{}^{(2)} \\
&-(1-\alpha)W(d\Lambda_L{}^{(2)})W+[WdW,\Lambda_L{}^{(2)}]+\alpha[WWW,\Lambda_L{}^{(2)}]\}
\end{aligned} \tag{6.6.18}$$

The term $d\Lambda_\Lambda{}^{(1)}d\Lambda_L{}^{(2)}dW$ vanishes as it is a total derivative, while $trd\Lambda_L{}^{(1)}Wd\Lambda_L{}^{(2)}W$ vanishes after antisymmetrization in 1 and 2. The last two terms satisfy already the consistency condition for any α, hence in order that the consistency condition be satisfied, the two remaining terms

$$tr(\alpha d\Lambda_L{}^{(1)}d\Lambda_L{}^{(2)}WW-(1-\alpha)d\Lambda_L{}^{(1)}WWd\Lambda_L{}^{(2)}) \tag{6.6.19}$$

must cancel after antisymmetrization in $\Lambda_L{}^{(1)}$ and $\Lambda_L{}^{(2)}$. This happens if, and only if, $\alpha = 1/2$. Hence the consistent nonabelian anomaly for chiral fermions is given by

$$G_a = D_\mu(W)\frac{\delta\Gamma}{\delta W^a_\mu} = \frac{i}{24\pi^2}\epsilon^{\mu\nu\rho\sigma}trT_a\partial_\mu\left\{W_\nu\partial_\rho W_\sigma+\frac{1}{2}W_\nu W_\rho W_\sigma\right\} \tag{6.6.20}$$

The normalization constant $\frac{i}{24\pi^2}$ can be obtained by computing the triangle graph for chiral fermions, see section 2. It is a factor 1/6 smaller than the AVV anomaly for nonchiral fermions,[21] as can be understood as follows. The AVV anomaly is proportional to $\int TrT_a dVdVd^4x$. Chiral fields contain at each vertex a factor $\frac{1}{2}(1+\gamma_5)$. Expressing the chiral triangle into the triangle on the V-A basis yields

$$j^Lj^Lj^L = \frac{1}{8}(j+j_5)^3 = \frac{1}{8}(j_5jj+jj_5j+jjj_5+j_5j_5j_5)+\cdots \tag{6.6.21}$$

where the dots denote terms with no or two j_5. Since j_μ is conserved but $j_{\mu,5}$ is not, only the first and last term contribute. The contribution of the AVV anomaly to the chiral anomaly is then a factor $\frac{1}{8}$ smaller, and the AAA anomaly a factor $\frac{1}{24}$ smaller (recall that the AAA anomaly was $\frac{1}{3}$ times the AVV anomaly). Together this reduces

[21] The rewriting of $\partial_\mu(W_\nu\partial_\rho W_\sigma)$ into $(\partial_\mu W_\nu - \partial_v W_\mu)(\partial_\rho W_\sigma - \partial_\sigma W_\rho)$ requires a factor $\frac{1}{4}$ which leads to a factor $\frac{1}{4}\times\frac{1}{24}$, which is indeed a factor $\frac{1}{6}$ smaller than the $\frac{1}{16}$ of the Bardeen anomaly.

the anomaly for chiral gauge fields by a factor $\frac{1}{8}(1+\frac{1}{3}) = \frac{1}{6}$ as compared to the AVV anomaly.[22]

Note that there are no W^4 terms in the nonabelian anomaly on the chiral basis. On the other hand, there certainly are terms with four gauge fields if one uses the A, V basis (the Bardeen anomaly). In fact, we did explicitly show that for the nonabelian case pentagon anomalies were present. The origin of these 4-gauge fields terms in the Bardeen anomaly is that on the V-A basis one removes the anomalies in the vector symmetry by suitable local counter terms, and these counter terms produce the terms with four gauge fields (see (6.9.37)).

7 The Wess Zumino term

We now turn to the Wess-Zumino term. [33] The idea is to split the full effective action into one part which is free from anomalies but nonlocal (and complicated), and another part which contains the anomaly but is local

$$\Gamma = \Gamma_{\text{non}} + \Gamma_{\text{an}} \tag{6.7.1}$$

One may check by direct but laborious algebra that no local expression exists which only depends on V_μ and A_μ and whose axial vector gauge variation yields the Bardeen anomaly. (A simple proof will be given in the next section.) Therefore we must introduce other fields besides V_μ and A_μ, and try to construct a local functional Γ_{an} which depends on V_μ, A_μ and these extra fields, and whose axial vector gauge transformation yields the anomaly. This is possible if one introduces extra spin 0 fields. These spin 0 fields must be pseudoscalars if parity is preserved, and the natural

[22] The reader may wonder whether one needs finite local counter terms in the action in order to remove spurious anomalies in the V-gauge symmetry. Such counter terms would also in general contribute to the axial anomaly in the V-A basis, and this would invalidate our explanation of the factor 1/6. The answer is that there are indeed such counter terms, see (6.9.38), but they contribute only to the term with 3 and more V and A fields.

choice is to interpret them as pions and kaons. One may consider these mesons ξ^a as coordinates of group elements g, namely $g = \exp \xi^a(x) T_a$. Such group elements can be multiplied by other group elements either from the left or from the right. Then the natural choice of gauge group would be $SU(3)_L \otimes SU(3)_R$ acting on the three lightest quarks (up, down and strange), with the diagonal $SU(3)$ corresponding to the flavor group $SU(3)$ of Gell-Mann and Ne'eman. However, other interpretations of the pseudo scalars and gauge group are also possible.

We now construct a local (but nonlinear and nonrenormalizable) term in the effective action $\Gamma_{an}(V, A, \xi)$ which depends on these pseudoscalars, and whose axial vector gauge variation produces the anomaly $G_a(V, A)$. (Note that the anomaly itself does not depend on the pseudoscalars). Consider a group element $g = e^\xi$ where $\xi = \xi^a(x) T_a$ and $\xi^a(x)$ are the pseudo scalars. We can couple the pseudo scalars to Dirac fermions in the $SU(3)_L \otimes SU(3)_R$ invariant way $\bar{\psi}_L e^\xi \psi_R$. If $\psi'_R = U\psi_R$ and $\psi'_L = V\psi_L$ then $\exp \xi' = V(\exp \xi) U^{-1}$ with U in $SU(3)_R$ and V in $SU(3)_L$. The kinetic terms $tr(\partial_\mu e^\xi \partial^\mu e^{-\xi})$ and $\bar{\psi}_L \psi_L$ and $\bar{\psi}_R \partial\!\!\!/ \psi_R$ are separately invariant, and as invariant mass term[23] for ξ one can take $\det e^\xi$. Under vector gauge transformations the pseudo scalars transform as

$$e^{\xi'} = e^u e^\xi e^{-u} = \exp(e^u \xi e^{-u}), u = \Lambda_V^a T_a \tag{6.7.2}$$

while under axial vector gauge transformations

$$e^{\xi'} = e^v e^\xi e^v, v = \Lambda_A{}^a T_a \tag{6.7.3}$$

One may verify that the corresponding infinitesimal transformation rules satisfy the same gauge algebra as those for $V_\mu{}^a$ and $A_\mu{}^a$ in (6.6.5) and (6.6.6). (To perform this check, it is best not to expand the exponents and use $(\exp A)(\exp B) = \exp(A + B) \exp \frac{1}{2}[A, B] + \ldots)$.

[23]In the case of $U(2)_L \otimes U(2)_R$ one often uses $\sigma + i\vec{\pi} \cdot \vec{\tau}$ instead of $\exp(i\sigma + i\vec{\pi} \cdot \vec{\tau})$. The mass term is then $\det(\sigma + i\vec{\pi} \cdot \vec{\tau}) = \sigma^2 + \vec{\pi}^2$.

Let the generator of axial vector gauge transformations of the pseudo scalars be denoted by $Z(\Lambda_A) = \Lambda_A{}^a Z_a$. The anomaly is by definition the response of the full effective action under variation of the external fields V_μ, A_μ and ξ^a

$$(Y + Z)\Gamma(V, A, \xi) = G(V, A) = \int \Lambda_A{}^a G_a(V, A) d^4x \tag{6.7.4}$$

Furthermore, we assume that the effective action has been made invariant under vector gauge transformations (if necessary by adding finite local counter terms).

$$X\Gamma[V, A, \xi] = 0 \; . \tag{6.7.5}$$

Since Γ_{non} is annihilated by $Y_a + Z_a$, we find an equation for Γ_{an}

$$(Y + Z)\Gamma_{\text{an}}(V, A, \xi) = G(V, A) \tag{6.7.6}$$

One may view this as a first order differential equation for Γ_{an}. As boundary condition we impose

$$\Gamma_{\text{an}}(V, A, \xi = 0) = 0 \tag{6.7.7}$$

because there is no local counter term depending only on V, A whose variation yields the anomaly. (Even if there would have been such a counter term, this boundary condition does not imply a loss of generality because the difference of two solutions of the inhomogeneous differential equation is a solution of the homogeneous equation, and the latter ($(Y + Z)\Gamma_{\text{an}} = 0$) has as solution the most general vector-and axial-vector gauge invariant local polynomial. This polynomial can be moved to the nonanomalous part Γ_{non}).

Using $(Y + Z)^n\Gamma_{\text{an}} = (Y + Z)^{n-1}G = Y^{n-1}G$ since G does not depend on ξ, dividing by $n!$ and summing over n, and adding Γ_{an} on both sides of this equation, yields

$$e^{Y+Z}\Gamma_{\text{an}} = \Gamma_{\text{an}} + \frac{e^Y - 1}{Y} G \tag{6.7.8}$$

The left-hand side is equal to $\Gamma_{\text{an}}(V', A', \xi')$ where the primed variables are the result of a finite gauge transformation with Λ_A. In general ξ' is a complicated expression in terms of ξ and Λ_A, but there exists a particular Λ_A for which $\xi' = 0$, namely $v = \Lambda_A^a T_a = -\frac{1}{2}\xi$ because

$$e^{\xi'} = e^v e^\xi e^v = 1 \text{ if } v = -\frac{1}{2}\xi \tag{6.7.9}$$

Then, using that $\Gamma_{\text{an}}(V', A', 0) = 0$, the left-hand side of (6.7.8) vanishes. Using furthermore that $(e^Y - 1)/Y$ equals $\int_0^1 e^{tY} dt$, we obtain

$$\begin{aligned}\Gamma_{\text{an}}(V, A, \xi) &= -\int_0^1 dt e^{tY} G(V, A) \\ &= -\int_0^1 dt e^{t(-\frac{1}{2}\xi^a Y_a)} \int (-\frac{1}{2}\xi^a) G_a(V, A) d^4x \end{aligned}\tag{6.7.10}$$

The functional

$$\Gamma_{\text{an}}(V, A, \xi) = \frac{1}{2}\int_0^1 dt e^{-\frac{1}{2}t\xi^a Y_a} \int \xi^a G_a(V, A) d^4x \tag{6.7.11}$$

is the Wess-Zumino term. It is the integrated axial anomaly because its axial gauge variation is the axial anomaly, see (6.7.6). Note that the operator Y_a in this expression does not act on the pseudoscalars ξ^a. One may check by a direct calculation that $Y + Z$ acting on (6.7.11) indeed yields G (use $\Gamma_{an} = (1 - expY)Y^{-1}G$ and $Z\xi = -\xi$).

In general the fields A_μ and V_μ are not gauge fields which are part of a consistent quantum field theory, because, as we have seen, anomalies ruin unitarity and renormalizability. Rather, in first instance one should view the A_μ and V_μ are auxiliary gauge fields which couple to the axial vector and vector currents of rigid symmetries. However, if a subgroup H of the group G if free from anomalies (which is the case when $d^{(L)}_{abc} - d^{(R)}_{abc} = 0$) then the corresponding gauge fields can be physical. Typical examples are $H = U(1)$ of QED, or $H = SU(2)_L$ of electroweak theory. The coupling of the Goldstone bosons to these H-gauge fields is then already contained in $G(V, A, \xi)$, although one can also derive it from scratch by applying the Noether method to $G(V, A, 0)$ [34].

Expanding $\exp -\frac{1}{2}t\xi^a Y_a$, the operator $\xi^a Y_a$ transforms V_μ and A_μ into themselves and into scalars. For example, the term $(\xi Y)^4$ acting on $tr\xi A^4$ yields a five-pseudoscalar interaction

$$\mathcal{L} \sim \epsilon^{\mu\nu\rho\sigma} tr\xi\partial_\mu\xi\partial_\nu\xi\partial_\rho\xi\partial_\sigma\xi \tag{6.7.12}$$

This is, of course, not a renormalizable term, and the main use of the Wess-Zumino term has been as a phenomenological framework for low-energy hadronic interactions. There is also a term with two vector fields and one pseudo scalar which describes $\pi^0 \to \gamma + \gamma$ decay. For further discussions see [34].

One can gain a deeper understanding of the topological aspects of the chiral anomaly by writing the Wess-Zumino term as a five-dimensional integral. First consider the case that $V_\mu = A_\mu = 0$. Then $e^{t(-\frac{1}{2}\xi^a)Y_a}G_a(V,A)$ is equal to $G_a(V',A')$ where V',A' are the finite axial gauge transformations of $A_\mu = V_\mu = 0$. The result for A'_μ is obtained from $A_\mu = \frac{1}{2}(W_{\mu,L} - W_{\mu,R})$, by using $W'_{\mu,L} = (\exp -\Lambda_L)\partial_\mu \exp\Lambda_L$ and $W'_{\mu,R} = (\exp -\Lambda_R)\partial_\mu \exp\Lambda_R$ with $\Lambda_L = \Lambda_A$ and $\Lambda_R = -\Lambda_A$. In our case $\Lambda_A^a = -\frac{1}{2}t\xi^a$, so

$$\begin{aligned} A'_\mu &= \frac{1}{2}(e^{-\Lambda_A}\partial_\mu e^{\Lambda_A} - e^{\Lambda_A}\partial_\mu e^{-\Lambda_A}) \\ &= \frac{1}{2}e^{\Lambda_A}e^{-2\Lambda_A}(\partial_\mu e^{2\Lambda_A})e^{-\Lambda_A}; \Lambda_A = -\frac{1}{2}t\xi \end{aligned} \tag{6.7.13}$$

When $V_\mu = A_\mu = 0$ also $F_{\mu\nu}(V') = F_{\mu\nu}(A') = 0$, hence only the term with four A's in the Bardeen anomaly contributes. One finds then using (6.6.9)

$$\Gamma(0,0,\xi) = \int \left(\frac{1}{16\pi^2}\frac{32}{3}\epsilon^{\mu\nu\rho\sigma} tr\frac{1}{2}\xi^a T_a A'_\mu A'_\nu A'_\rho A'_\sigma\right) dt d^4x \tag{6.7.14}$$

(Note that $\Gamma(0,0,\xi)$ is purely imaginery since all our T_a are always antihermitian.) Defining $\xi^a(x,t) \equiv t\xi^a(x)$, we can write ξ^a as $-e^{-2\Lambda_A}\partial_t e^{2\Lambda_A}$, while the factors $\exp \pm\Lambda_A$ in A' cancel in the trace. Then, writing $e^{2\Lambda_A} = e^{-t\xi^a T_a} = g^{-1}$, we find

$$\begin{aligned} \Gamma_{\text{an}}(0,0,\xi) &= \frac{-\epsilon^{\mu\nu\rho\sigma}}{48\pi^2}\int trg\partial_\mu g^{-1}g\partial_\nu g^{-1}g\partial_\rho g^{-1}g\partial_\sigma g^{-1}g\partial_t g^{-1} \\ &= \frac{1}{240\pi^2}\epsilon^{\mu\nu\rho\sigma\tau}\int_{B_5} tr(g^{-1}\partial_\mu g\cdots g^{-1}\partial_\tau g)d^5x \end{aligned} \tag{6.7.15}$$

where one integrates over the five-dimensional ball $0 \le t \le 1$, and $g = \exp t\xi^a T_a$. (The factor $\frac{1}{240}$ is obtained because there are 5 ways to choose which of the indices μ,ν,ρ,σ,τ is t).

One can view B_5 as half of the sphere S_5; to draw an analogy, if S_5 is the surface of the earth, B_5 is the northern hemisphere, and the equator corresponds to our 4-dimensional world. This analogy shows that one could also have constructed another ball B_5' (corresponding to the southern hemisphere) [34]. In our derivation of Γ_{an} the t-dependence of the function $\xi^a(x,t) = t\xi^a(x)$ was fixed, but if one would have used any other $\xi^a(x,t)$ which is smooth in B_5 and still equals $\xi^a(x)$ at the boundary $t=1$, one would have obtained the same value. Namely, if one varies g into $g + \delta g$, the factor $g^{-1}\partial_\mu g$ varies into

$$-g^{-1}\delta g g^{-1}\delta_\mu g + g^{-1}\partial_\mu \delta g = g^{-1}(\partial_\mu[\delta g g^{-1}])g \tag{6.7.16}$$

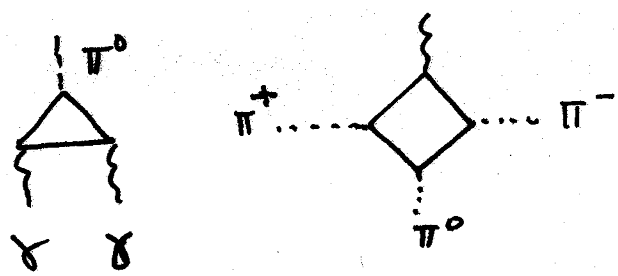
Substituting this expression into (6.7.15), partial integration of the derivative ∂_μ yields a total derivative (when ∂_μ hits $\partial_\nu g$ one gets zero, but when ∂_μ hits the four factors g^{-1} one gets equal contributions but with alternating signs, hence their sum vanishes, too). So the value of the integral of Γ_{an} over B_5 only depends on the values of ξ^a on the boundary, i.e., the values of the pseudoscalars in Minkowski spacetime. On the other hand, if one computes the integral over B_5' with the same boundary values for $\xi^a(x)$, one obtains minus the result for B_5, because the orientation of B_5' is opposite to that of B_5.

Suppose one would like to add a term $S_{WZ} = \alpha \int_{B_5}(g^{-1}dg)^5 d^5x$ to the action, but one did not know whether one should integrate over B_5 or B_5'. (This is **not** the case we studied above, because in that case we knew that $\xi^a(x,t) = t\xi^a(x)$, and that selected which B_5 one has to use). Then the difference between S_{WZ} evaluated on B_5 and on B_5' should be equal to $2\pi\hbar$ (or a multiple thereof), because in that case the path integral with $\exp \frac{i}{\hbar} S_{WZ}$ would not depend on the choices of five-ball. Since B_5 and B_5' together form S_5, the condition is

$$\frac{\alpha}{\hbar}\int_{S_5}(g^{-1}dg)^5 d^5x = 2\pi n\ . \tag{6.7.17}$$

For $\alpha = \frac{1}{240\pi^2}$ the value of this integral over S_5 equals 2π. This is just the value we got when we integrated the axial anomaly.

One can "gauge" a subgroup H of the group G. The most interesting case is $G = SU(3)_L \times SU(3)_R$ spontaneously broken down to $SU(3)_V$, or H equal to the $U(1)$ of electromagnetism. One finds then terms in the effective action proportional to $\alpha\pi^0\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}$ and $\beta\epsilon^{\mu\nu\rho\sigma}A_\mu\partial_\nu\pi^+\partial_\rho\pi^-\partial_\sigma\pi^0$ which describe $\pi^0 \to 2\gamma$ and $\gamma \to \pi^+\pi^-\pi^0$, respectively. The corresponding one-loop graphs with quarks in the loops are given by



The coefficients α and β are proportional to n, the number of colors. There is one other free parameter, F_π, which relates ξ^a to the physical pseudoscalars by $\xi^a = \pi^a/F_\pi$. The value of the parameter F_π is fixed by the pion decay[24] $\pi^+ \to \mu^+\nu_\mu$, and one finds then perfect agreement with $\pi^0 \to 2\gamma$ if one takes $n = 3$.

8 Consistent and covariant anomalies. Descent equations

The nonabelian (and singlet) anomalies for Dirac fermions coupled to nonabelian gauge fields we obtained in previous sections are not covariant (invariant) under axial vector gauge transformations. Similarly, the nonabelian (and singlet) anomalies for chiral fermions are not covariant (invariant) under chiral gauge transformations. Rather, these anomalies satisfy the consistency conditions which are a consequence of the fact that these anomalies are the gauge variation of the effective action. In

[24]The kinetic term $\mathcal{L} = \frac{1}{16\pi^2}F_\pi^2 tr\partial_\mu U^{-1}\partial^\mu U$ and $\Gamma_{\rm an}$ can be gauged w.r.t. subgroup $SU(2)_L$, and the kinetic term yields then a term $FW_\mu^a\partial^\mu\pi_a$ which can describe $\pi \to \mu\nu$ with a virtual W boson.

this section we show that there also exist nonabelian covariant anomalies.[25] These nonabelian covariant anomalies do not satisfy the consistency conditions, but they are useful tools for proving general theorems on consistent anomalies. Both the consistent and the covariant anomalies appear as terms in the so-called descent equations, which start from a Chern form TrF^{n+1} in $2n+2$ dimensions, and which contain also the Chern-Simons form and other forms.

For chiral fermions in four dimensions, the consistent nonabelian anomaly is given by

$$G_a = c\, trT_a d\left(WdW + \frac{1}{2}WWW\right) \tag{6.8.1}$$

with c a constant, and is the covariant divergence of a **nonlocal** current $J_a{}^\mu = -\frac{\delta}{\delta W_\mu{}^a}\Gamma$

$$\begin{aligned}\delta(\text{gauge},\Lambda)\Gamma &= \int D_\mu \Lambda^a(x)\frac{\delta}{\delta W_\mu{}^a(x)}\Gamma d^4x \\ &= \int \Lambda^a(x) D_\mu J^\mu{}_a dx = \int \Lambda^a G_a d^4x;\quad J_a{}^\mu = -\frac{\delta}{\delta W_\mu{}^a}\Gamma\end{aligned} \tag{6.8.2}$$

Since the anomaly G_a is not covariant, the current $J_a{}^\mu$ cannot be covariant either (as we shall prove). The anomaly G_a and the current $J_a{}^\mu$ are called the consistent anomaly and consistent current, respectively, since the latter yields the former, and the former satisfies consistency conditions.

One may, however, add a **local** term $X_a{}^\mu$ to the nonlocal consistent current

$$J_a{}^\mu + X_a{}^\mu = \tilde{J}_a{}^\mu \tag{6.8.3}$$

such that the resulting nonlocal current $\tilde{J}_a{}^\mu$ is covariant (i.e., transforms as a vector under chiral gauge transformations). The covariant divergence of this covariant current is then also covariant (and local), and is called the covariant anomaly $\tilde{G}_a$. Similarly, $\tilde{J}_a{}^\mu$ will be called the covariant current.

The consistent anomaly is the anomaly with a direct physical meaning. On the other hand, the covariant anomaly is a useful mathematical object, and allows one

[25] For abelian gauge theories, there is no difference between covariant and consistent anomalies, as is clear from (6.6.9).

in some cases to prove properties of the consistent anomaly in a simpler way, by first proving corresponding properties for the covariant anomaly and then transforming back to the consistent anomaly. We shall later in this section discuss the construction and properties of ${X_a}^\mu$, but assuming that it exists, we can already illustrate the usefulness of the covariant anomaly by giving a simple proof that the consistent anomaly is not the chiral gauge variation of a local counter term which only depends on gauge fields. The consistent anomaly (in particularly the Bardeen anomaly on the V_μ, A_μ basis) is a rather complicated object, and the most general local counter term one would have to vary to compare with the consistent anomaly would also be rather complicated. Knowing that the covariant anomaly

$$\tilde{G}_a = D_\mu {J_a}^\mu + D_\mu {X_a}^\mu = D_\mu {\tilde{J}^\mu}_{\ a} = G_a + D_\mu {X_a}^\mu \tag{6.8.4}$$

is a covariant local object, it is sufficient to prove that **no local covariant current exists whose covariant divergence gives the covariant anomaly**.[26] It is rather easy to write down the most general local covariant current, and to show that its covariant divergence can never produce the covariant anomaly. We shall give a few examples.

Given $\tilde{G}_a$ and G_a, the local object ${X_a}^\mu$ is unique, because no local covariant object constructed from W_μ exists whose covariant derivative vanishes and which has the correct dimension and parity (i.e., contains an ϵ symbol). This statement is easy to check in 4 dimensions. Of course, $D^\nu G_{\nu\mu}$ is a local covariant object whose covariant divergence vanishes, but it contains no ϵ symbol. On the other hand the local covariant object $\epsilon^{\mu\nu\rho\sigma} D_\nu G_{\rho\sigma}$ vanishes.

In examples, it is easy to find ${X_a}^\mu$. For example, in two dimensions, the consistent

[26]The argument goes as follows. Suppose that the consistent anomaly G_a would have been the variation of a local counter term $\Delta S, G_a = -D_\mu \frac{\delta}{\delta {W_\mu}^a} \Delta S$. Then ${J_a}^\mu = -\frac{\delta}{\delta {W^a}_\mu} \Delta S$ would have been local, so ${J_a}^\mu + {X_a}^\mu$ would also have been local as well as covariant (from the definition of ${X_a}^\mu$), and its covariant divergence would have been the covariant anomaly, $D_\mu({J_a}^\mu + {X_a}^\mu) = G_a + D_\mu {X^\mu}_a = \tilde{G}_a$.

anomaly is $G^a = c\partial_\mu A_\nu{}^a \epsilon^{\mu\nu}$, with c a constant. It satisfies the consistency condition

$$\int [\Lambda_2{}^a \epsilon^{\mu\nu} \partial_\mu D_\nu \Lambda_1{}^a - 1 \leftrightarrow 2] d^2x = \int [\Lambda_1, \Lambda_2]^a \epsilon^{\mu\nu} \partial_\mu A_\nu{}^a d^2x \tag{6.8.5}$$

as one easily checks by substituting $D\Lambda_1 = d\Lambda_1 + [A, \Lambda_1]$. The local current $X_a{}^\mu$ is in this case

$$X_a{}^\mu = c\epsilon^{\mu\nu} A_{\nu,a} \tag{6.8.6}$$

The covariant divergence of the covariant current is then

$$\begin{aligned} D_\mu \tilde{J}^\mu &= D_\mu J^\mu + D_\mu X^\mu = c\partial_\mu A_\nu \epsilon^{\mu\nu} + c\epsilon^{\mu\nu} D_\mu A_\nu \\ &= c(\partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu])\epsilon^{\mu\nu} = c\epsilon^{\mu\nu} F_{\mu\nu} = \tilde{G} \end{aligned} \tag{6.8.7}$$

which is clearly covariant. It is easy to check that there is indeed no local covariant current with the appropriate dimension and parity (only $\epsilon^{\mu\nu} A_\nu$ is available, but it is not covariant).

In four dimensions, the consistent anomaly is the one given in (6.6.20).

$$G_a = ctrT_a d(WdW + \frac{1}{2}WWW) \tag{6.8.8}$$

In this case, the local addition X^μ to the consistent current J^μ is given by

$$X_a = c\; trT_a \left\{ dWW + WdW + \frac{3}{2}WWW \right\} \tag{6.8.9}$$

It is straightforward to verify that $D_\mu \tilde{J}_a{}^\mu$ is covariant

$$G_a + DX_a = c\; trT_a(3FF) = \tilde{G}_a \tag{6.8.10}$$

To obtain this result, we used that $DX_a = dX_a + cf_{abc}W^b trT^c\{\cdots\} = dX_a - ctr[W, T_a]\{\cdots\}$. As we already noted, in this case the most general local covariant current with the correct dimension and parity would be $\epsilon^{\mu\nu\rho\sigma} D_\nu G_{\rho\sigma}$, but it vanishes due to the Bianchi identity.

One can, of course, always add local counter terms to the action whose variation adds local terms to the anomalies and currents. For example, in two dimensions one may add $\Delta S = \frac{1}{2}c' \int A_\mu{}^a A_a{}^\mu d^2x$; for $c' = \pm c$ the consistent anomaly becomes then $\partial_- A_+$ or $\partial_+ A_-$ where $A_\pm = A_0 \pm A_1$. In four dimensions there are many local counter terms; we shall use them to remove vector anomalies, or to go from the chiral anomaly to the Bardeen anomaly.

We now wish to show explicitly that the consistent anomaly is not covariant. To prove this we need to know $\delta(\text{gauge})J_a{}^\mu = -\delta(\text{gauge})\frac{\delta}{\delta W_\mu{}^a}\Gamma$. Using that $\delta(\text{gauge})\Gamma$ is the consistent anomaly, it seems promising to consider the commutator $[\delta(\text{gauge}), \frac{\delta}{\delta W_\mu{}^a}]$. Since it is easier to work with operators without indices, we consider instead of $\delta/\delta W_\mu{}^a$ the operator $\hat{\delta}(B)$

$$\hat{\delta}(B) = \int B_\mu{}^a(x) \frac{\delta}{\delta W_\mu{}^a(x)} dx \tag{6.8.11}$$

Clearly, $\hat{\delta}(B) B_\mu{}^a = 0$. We also define that B_μ^a does not transform under gauge transformations. (We shall relax this condition later). The commutator of $\hat{\delta}(B)$ and $\delta(\text{gauge}, \Lambda)$ closes on $W_\mu{}^a$

$$[\hat{\delta}(B), \delta(\text{gauge}, \Lambda)] W_\mu{}^a(y) = g f^a{}_{bc} B_\mu{}^b \Lambda^c = \hat{\delta}(g f^a{}_{bc} B_\mu{}^b \Lambda^c) W_\mu{}^a(y) \tag{6.8.12}$$

It then also closes on any function of $W_\mu{}^a$. For example, on the curvature two-form F one has $\hat{\delta}(B)F = D(W)B$ and $\delta(\text{gauge}, \Lambda)F = [F, \Lambda]$, so that

$$\begin{aligned} [\tilde{\delta}(B), \delta(\text{gauge}, \Lambda)]F &= [D(W)B, \Lambda] - [D(W)\Lambda, B] \\ &= D(W)[B, \Lambda] = \hat{\delta}([B, \Lambda])F \end{aligned} \tag{6.8.13}$$

Clearly (6.8.12) holds. Acting with (6.8.12) on $\Gamma(W)$, we obtain

$$\begin{aligned} [\hat{\delta}(B), \delta(\text{gauge})]\Gamma &= \hat{\delta}(B) \int \Lambda^a G_a(\text{cons}) dx + \delta(\text{gauge}) \int B_\mu{}^a J_a{}^\mu dx \\ &= - \int g f^a{}_{bc} B_\mu{}^b \Lambda^c J_a{}^\mu dx \end{aligned} \tag{6.8.14}$$

Extracting the field $B_\mu{}^a$, one is left with

$$\delta(\text{gauge}) J_a{}^\mu = g f_a{}^{bc} J_b{}^\mu \Lambda^c - \frac{\delta}{\delta W_\mu{}^a} \int \Lambda^a G_a(\text{cons}) dx \tag{6.8.15}$$

The first term on the right-hand side is the covariant transformation law for a current, but the last term shows that the consistent current is not covariant if the consistent anomaly is nonvanishing.

We now consider the construction of the covariant current, i.e., of the local object ${X_a}^\mu$ in (6.8.4). Clearly, we must find a local object ${X_a}^\mu$ which satisfies

$$\delta(\text{gauge}, \Lambda){X_a}^\mu = g{f_a}^{bc}{X_b}^\mu\Lambda^c + \frac{\delta}{\delta {W_\mu}^a}\int \Lambda^a G_a(\text{cons})dx \tag{6.8.16}$$

because then $\delta(\text{gauge})\tilde{J}_a{}^\mu$ is just $g{f_a}^{bc}\tilde{J}_b{}^\mu\Lambda^c$, as follows from adding the last two equations. The construction of ${X_a}^\mu$ proceeds via the so-called descent equations, which are equations which connect the exterior derivative d and the BRST transformations with a kind of inverse of the exterior derivative. We shall now first give an account of the descent equations, and come back to the construction of ${X_a}^\mu$ in (6.8.49).

Rather than introducing at this point a series of definitions and then proceed to derive theorems, we prefer first to give an example where the reason for these definitions becomes clear. Consider the Chern form in 6 dimensions (we will later give the corresponding formulas for 4 and 2 dimensions). It is proportional to an ϵ symbol contracted with 3 curvatures, and will be denoted by $\omega_{2n+2} = \omega_6$

$$\omega_6 = trF^3\ ;\ F = dW + WW \tag{6.8.17}$$

So ω_6 is proportional to $\epsilon^{\mu_1\cdots\mu_6}trF_{\mu_1\mu_2}\cdots F_{\mu_5\mu_6}$. Of course, in four dimensions this expression would vanish, as both the coordinates and the gauge fields have indices which range from 0 to 3. We will nevertheless descend from 6 to 4 dimensions, but this is achieved by using forms and factoring out factors of d where $d = dx^\mu\frac{\partial}{\partial x^\mu}$. This will reduce 6-dimensional ϵ tensors to 4-dimensional ϵ tensors.

It is clear that $d\omega_{2n+2} = 0$ since $d(trF^{n+1}) = (n+1)tr(dF)F^n$ and $tr(dF)F^n = tr(DF)F^n$ because $DF = dF + [W, F]$ and $tr[W, F]F^n = 0$ (due to the cyclicity of the trace). Since $DF = 0$ due to the Bianchi identity ($D_{[\mu}F_{\nu\rho]} = 0$), it is clear that $d\omega_{2n+2} = 0$. Then, $\omega_{2n+2} = d\omega_{2n+1}$. (Usually one can only conclude from $dA = 0$

that $A = dB$ locally, but for polynomials this holds globally as well). This $(2n+1)$ form ω_{2n+1} is the Chern-Simons Lagrangian in $2n+1$ dimensions. In the example

$$\begin{aligned} \omega_6 = d\omega_5, \omega_5 &= tr\left(F^2W - \frac{1}{2}FW^3 + \frac{1}{10}W^5\right) \\ &= tr\left(dWdWW + \frac{3}{2}dWW^3 + \frac{3}{5}W^5\right) \end{aligned} \tag{6.8.18}$$

It is straightforward to check that $d\omega_5 = \omega_6$ by acting with d on the first line; one just has to use $dF = -[W, F]$ and the definition $F = dW + WW$. One can also directly act with d on the second line. (Note that $trdWWdWW = 0$ and $trW^6 = 0$. More generally, for any odd form A one has $trAA = 0$.)

Of course, ω_{2n+1} can only be determined up to a total derivative dX because $d(\omega_{2n+1}) = d(\omega_{2n+1} + dX)$. So (6.8.18) is only one expression of the Chern-Simons form. We shall discuss these ambiguities below in more detail.

Under a general variation $W \to W + \delta W$, F varies as $F \to F + D\delta W$, and the Chern form $\omega_{2n+2} = trF^{n+1}$ varies into a total derivative

$$\delta\omega_{2n+2} = (n+1)\ tr(D\delta W)F^n = (n+1)\ trD(\delta WF^n) = (n+1)\ d\ tr\delta WF^n \tag{6.8.19}$$

Since $\delta\omega_{2n+2} = \delta d\omega_{2n+1} = d(\delta\omega_{2n+1})$, we see that

$$\delta\omega_{2n+1} = (n+1)tr\delta WF^n + d(\ldots). \tag{6.8.20}$$

We shall determine the total derivative term in a moment. The field equation which is obtained from the Chern-Simons action[27] $\int \omega_{2n+1}$ is covariant, $F^n = 0$, and ω_{2n+1} can be obtained by integrating the δW in $tr\delta WF^n$ from zero to W. Setting $\delta W = Wdt$ we find

$$\omega_{2n+1} = (n+1)\int_0^1 dt\ trW(tdW + t^2W^2)^n dt \tag{6.8.21}$$

[27]The spacetime integrals $\int \omega_{2n+1}$ can also be written as integrals $\int d^nx$ with an ϵ symbol.

This is a simple useful formula to get the Chern-Simons action. For example, it easily reproduces ω_5 in (6.8.18).

The Chern-Simons action is gauge invariant. One can prove this by replacing δW by $D\Lambda$ in the general variation of ω_{2n+1} in (6.8.19); one finds then indeed a vanishing result

$$\delta(\text{gauge}) \int \omega_{2n+1} = (n+1) \int tr(D\Lambda F^n) = (n+1) \int d(tr\Lambda F^n) = 0 \qquad (6.8.22)$$

(Because Λ is local there is no boundary term). The Chern character $\omega = trF^{n+1}$ has a physical meaning: it is proportional to the singlet anomaly for chiral fermions in $d = 2n+2$ dimensions. (On the V-A basis, the singlet anomaly is proportional to $(F_{\mu\nu}^{(V)})^2 + \frac{1}{d-1}(F_{\mu\nu}^{(A)})^2$ which is vector-gauge invariant. The other terms vanish; for example, in 4 dimensions $-\frac{8}{3}tr(AAF(V) + AF(V)A + F(V)AA)$ and $\frac{32}{3}trAAAA$ vanish, as is clear by moving one A from the left to the right, using cyclicity of the trace. This agrees with the triangle result in 4 dimensions). Since the Chern-Simons action is gauge invariant, and its field equations are gauge covariant, one has $\delta(\text{gauge})\omega_{2n+1} = dY$ where $\delta(\text{gauge})W = d\Lambda + [W, \Lambda]$. Moreover, Y must be proportional to $d\Lambda$ since ω_{2n+1} is invariant under rigid gauge transformations (transformations with a constant Λ). In fact, in the example one finds by direct evaluation[28]

$$\begin{aligned} \delta(\text{gauge})\omega_5 &= tr\left((d\Lambda)d\left(WdW + \frac{1}{2}WWW\right)\right) \\ &= d\, tr\Lambda d\left(WdW + \frac{1}{2}W^3\right) \\ \hat{\delta}(B) \int \omega_5 &= \int trB3FF \end{aligned} \qquad (6.8.23)$$

Recall now that the 4-form $trT_a d(WdW + \frac{1}{2}W^3)$ is the consistent nonabelian anomaly G_a in 4 dimensions, while $trT_a 3FF$ is the covariant nonabelian anomaly $\tilde{G}_a$ in 4 dimensions. So, the gauge variation of the Chern-Simons Lagrangian yields the con-

[28] If one makes a gauge transformation of (6.8.18), the terms due to $\delta W = [W, \Lambda]$ and $\delta F = [F, \Lambda]$ all cancel because the trace is cyclic. Hence one only needs to use $\delta F = 0$ and $\delta W = d\Lambda$. This already shows that the result must be proportional to $d\Lambda$.

sistent nonabelian anomaly, and the field equation of the Chern-Simons action yields the covariant nonabelian anomaly.

Extrapolating from these examples we see a general scheme emerging: ω_{2n+2} is the singlet anomaly in $2n+2$ dimensions, $\omega_{2n+2} = d\omega_{2n+1}$ with ω_{2n+1} the Chern-Simons Lagrangian in $2n+1$ dimensions, $\delta(\text{gauge}, \Lambda)\omega_{2n+1} = (d\Lambda^a)G_a(\text{cons})$ with $G_a(\text{cons}) = trT_a d(\cdots)$ the consistent nonabelian anomaly in $2n$ dimensions, and $trT_a \frac{\delta}{\delta W} \int \omega_{2n+1}$ is the covariant nonabelian anomaly in $2n$ dimensions. Furthermore, both the singlet anomaly and the consistent nonabelian anomaly for chiral fermions are total derivatives. (The latter follows by acting with d on the relation $\delta(\text{gauge}, \Lambda)\omega_{2n+1} = d\Lambda^a G_a(\text{cons})$ and using that $\delta(\text{gauge}, \Lambda)\omega_{2n+1} = dY$). In order to prove these results, and derive further properties, it is useful to introduce BRST symmetry at this point.

The consistency condition of the consistent anomaly reads

$$\int [\delta(\text{gauge}, \Lambda_1)(\Lambda_2{}^a G_a) - 1 \leftrightarrow 2] d^4x = \int [\Lambda_1, \Lambda_2]^a G_a d^4x \tag{6.8.24}$$

This can be rewritten in a simpler form using the BRST formalism. Recalling that $\delta(\text{gauge}, \Lambda_1 = c\Lambda)$ is equal to the BRST variation when acting on W or F, the left-hand side is twice the BRST variation of G_a. Recalling furthermore that $\delta_B c^a = \frac{1}{2} f^a{}_{bc} c^b c^c \Lambda$, the right-hand side can be written as minus twice $(\delta_B c^a)G_a$. Bringing the expression on the right-hand side of (6.8.27) to the left-hand side, we obtain that (twice) the BRST variation of $\int c^a G_a$ vanishes.

At this point we must settle whether ghost fields commute or anticommute with the one-forms $dx^\mu W_\mu$. Suppose one views ghosts themselves as one-forms $c^a = dg^i c_i{}^a(x)$ with $dg^i dx^\mu = -dx^\mu dg^i$. The dg^i form a basis for one-forms in the space of the Lie algebra, just as the dx^μ form a basis for one-forms in spacetime. There are as many independent $c^a(x)$ at a given point x as there are dg^i, namely as many as the number of generators of the gauge group. If one considers two points in spacetime, there are more $c^a(x)$ and $c^a(y)$ than dg^i, so then writing ghosts as ordinary commuting fields ${c_i}^a(x)$ times constant anticommuting dg^i is too restrictive. Also, derivatives of $c^a(x)$ are linearly independent of $c^a(x)$, hence also when there are derivatives of the ghosts, one cannot expand over dg^i. However, for the purposes of descent equations it is sufficient to work at one point in spacetime and no derivatives of $c^a(x)$ will appear. Then it is consistent to expand $c^a(x)$ in terms of dg^i. In any case, one may view this picture of ghosts as only a tool to get all signs defined, and one may forget it afterwards.

With this picture in mind, we adopt the definition that ghosts anticommute with the one-forms W^a but commute with the curvature two-forms, $c^a W^b = -W^b c^a$ but $c^a F^b = F^b c^a$. Since the gauge parameter Λ^a commutes of course with ${W_\mu}^b$ and also with $dx^\mu W_\mu$, while it is replaced by $c^a\Lambda$ in the BRST formalism, the constant BRST parameter Λ anticommutes with dx^μ and dg^i. In the literature on descent equations one works without this Λ. We therefore define BRST transformations s as the previous transformations δ_B but with Λ removed from the left.[29]

$$\delta_{\mathrm{BRST}}\ (\text{anything}) \ = (-\Lambda)s\ (\text{anything}) \tag{6.8.25}$$

For example, from $\delta_B W_\mu = D_\mu c\Lambda$ it follows that $sW_\mu = D_\mu c$, and from $\delta_B c^a = \frac{1}{2} f^a{}_{bc} c^b c^c \Lambda$ it follows that $sc^a = -\frac{1}{2} f^a{}_{bc} c^b c^c$ or $sc = -\frac{1}{2}\{c,c\} = -c^2$. (We have absorbed the coupling constant g into W_μ, thus $D_\mu c = \partial_\mu c + [W_\mu, c]$). The operator s acts as an antiderivation since $s(AB) = (sA)B \pm A(sB)$. The BRST nilpotency in terms of s is quite simple to prove. For example

$$\begin{aligned}
&sW_\mu = D_\mu c, s^2 W_\mu = D_\mu(sc) + \{D_\mu c, c\} = D_\mu(sc + \frac{1}{2}c^2) = 0\\
&s^2 c = s(-c^2) = -(sc)c + csc = c^3 - c^3 = 0\\
&sF = [F,c], s^2 F = \{sF, c\} + [F, sc] = 0\ .
\end{aligned} \tag{6.8.26}$$

[29] Removing Λ from the right we would get $s(AB) = AsB \pm (sA)B$. We prefer the definition which yields $s(AB) = (sA)B \pm A(sB)$.

We view the BRST transformation s as an exterior derivative in the direction of the group manifold

$$s = dg^i \frac{\partial}{\partial g^i} \tag{6.8.27}$$

and hence s and the ordinary exterior derivative anticommute

$$sd + ds = 0 \tag{6.8.28}$$

Note that s acts now as an exterior derivative: when it moves past W or c, it picks up a minus sign. From the definition of s one deduces

$$\begin{aligned} sW_\mu{}^a &= (\partial_\mu c^a + f^a{}_{bc} W_\mu{}^b c^c) \\ sW^a &= (-dc^a - f^a{}_{bc} W^b c^c) \\ sW &= (-dc - \{W, c\}) \end{aligned} \tag{6.8.29}$$

where $W^a = dx^\mu W_\mu{}^a$ and $W = W^a T_a$. The minus signs are due to s moving past the dx^μ. Similarly, from $\delta_B F_{\mu\nu} = [F_{\mu\nu}, c]\Lambda$ we obtain $sF = [F, c]$. Note that if we define BRST variations as $s = dg^i \frac{\partial}{\partial g^i}$, it is natural not to include the constant BRST parameter Λ, but one should not forget that s is now an anticommuting entity. Treating ghosts as one-forms allows us to view W and c as different components of the same extended gauge field one-form $\mathcal{A}$, as we shall see.

With these definitions, the consistency condition for the consistent anomaly can be written as

$$s \int c^a G_a = 0 \text{ or } s(c^a G_a) = d\text{ (something)} \tag{6.8.30}$$

Using $sc = -cc$, we can check this in the example[30]

$$\begin{aligned} s\, trcd\left(WdW + \frac{1}{2}W^3\right) &= -c^2 d\left(WdW + \frac{1}{2}W^s\right) + cds\left(WdW + \frac{1}{2}W^3\right) \\ &= -c^2 d\left(WdW + \frac{1}{2}W^3\right) + cd(DcdW + WdDc \begin{array}{l} +\frac{1}{2}DcWW \\ +\frac{1}{2}WDcW \\ +\frac{1}{2}WWDc \end{array}\Bigg) \end{aligned} \tag{6.8.31}$$

$$\begin{aligned} &s\, trcd\left\{WdW + \frac{1}{2}W^3\right\} = tr(sc)d\left\{WdW + \frac{1}{2}W^3\right\} + \\ &trcds\left\{WdW + \frac{1}{2}W^3\right\} = dtr\left\{(-cc)\left(WdW + \frac{1}{2}W^3\right)\right\} \\ &-dtrc\left(\left[\frac{1}{2}W^2, dc\right] - \left\{c, WdW + \frac{1}{2}W^3\right\} - \frac{1}{2}WdcW\right) \end{aligned} \tag{6.8.32}$$

Note that the BRST variation of $c^a G_a$ is indeed a total derivative.

These results suggest to replace δ (gauge) by the BRST variation s wherever this is possible. Then we can rewrite the previous results for the example as follows

$d\omega_6 = 0, \omega_6 = d\omega_5, \omega_5 =$ Chern-Simons 5-form
$s\omega_5 + d{\omega_4}^{(1)} = 0, \int {\omega_4}^{(1)} = \int c^a G_a =$ consistent anomaly
$s{\omega_4}^{(1)} + d{\omega_3}^{(2)} = 0,$ etc.

These are examples of the "descent equations". At each stage, a gauge field W is replaced by a ghost field c. Also s and d both appear in each equation, and each is an exterior derivative.

$$s = dg^i \frac{\partial}{\partial g^i}, d = dx^\mu \frac{\partial}{\partial x^\mu} \tag{6.8.33}$$

This suggests to consider the combination

$$\mathcal{A} = W + c; W = dx^\mu W_\mu \tag{6.8.34}$$

[30]The details are as follows. Use

$$s\left\{WdW + \frac{1}{2}W^3\right\} = -dcdW + \frac{1}{2}[W^2, dc] - \left\{c, WdW + \frac{1}{2}W^3\right\} - \frac{1}{2}WdcW$$

Dropping the term $-dcdW$, and then partially integrating the d in $trcds\{\ldots\}$, use $trdc\frac{1}{2}[W^2, dc] = 0$ and also $trdcWdcW = 0$. Then use

$$-trdc\left\{c, WdW + \frac{1}{2}W^3\right\} = -tr(dc^2)\left(WdW + \frac{1}{2}W^3\right)$$

where we recall $c = dg^i c_i{}^a T_a$ with $c_i{}^a$ commuting.[31] Furthermore, we introduce a generalized exterior derivative $\mathcal{D}$ which is nilpotent

$$\mathcal{D}\mathcal{D} = 0\ ,\ \mathcal{D} = d + s \tag{6.8.35}$$

as follows from $d^2 = 0, ds + sd = 0$ (see before) and $ss = 0$ (as follows from $dg^i dg^j = -dg^j dg^i$).

The extended exterior derivative $\mathcal{D} = d+s$ and the extended gauge field $\mathcal{A} = W+c$ lead to an extended curvature two form

$$\mathcal{F} = \mathcal{D}\mathcal{A} + \mathcal{A}\mathcal{A} = (d+s)(W+c) + (W+c)(W+c) \tag{6.8.36}$$

Expanding this curvature on the basis $dx^\mu dx^\nu, dx^\mu dg^i$ and $dg^i dg^j$, its components in nonhorizontal directions vanish as a consequence of the BRST transformation laws in (6.8.26) and (6.8.29)

$$\left.\begin{array}{l} sW + dc + \{W, c\} = 0 \\ sc + cc = 0 \end{array}\right\} \mathcal{F} = F \tag{6.8.37}$$

Conversely, one could have determined the BRST laws by requiring that the curvature only exists in horizontal directions.

Because $d\omega_{2n+2} = 0$ due to the Bianchi identities, and $s\omega_{2n+2} = 0$ because the trace of F^{n+1} is gauge invariant, one has $\mathcal{D}\omega_{2n+2}(\mathcal{F}) = 0$. In fact, $\omega_{2n+2}(\mathcal{F}) = \omega_{2n+2}(F)$ as we have just seen. Since the relation $d\omega_{2n+1}(A, F) = \omega_{2n+2}(F)$ only uses that $dA = F - A^2$ and $d^2 = 0$, we also have

$$\mathcal{D}\omega_{2n+1}(\mathcal{A}, \mathcal{F}) = \omega_{2n+2}(\mathcal{F}) \tag{6.8.38}$$

Using $\mathcal{F} = F$, this reduces to

$$(d+s)\omega_{2n+1}(W + c, F) = \omega_{2n+2}(F) \tag{6.8.39}$$

[31] Zumino [?] replaces W in this definition by $g^{-1}Wg + g^{-1}dg$ and c by $g^{-1}sg$. Then (6.8.37) follows, instead of having to be imposed. However, the results for the descent equations are the same.

Expanding $\omega_{2n+1}(W+c,F)$ in terms of c

$$\omega_{2n+1}(W+c,F) = \omega_{2n+1}{}^{(0)}(W,F) + \omega_{2n}{}^{(1)}(W,c,F) + \ldots \omega_0{}^{(2n+1)}(c) \qquad (6.8.40)$$

where the upper index denotes the number of ghosts and the lower index the degree of the form in terms of power of dx, we arrive at the descent equations

$$\begin{aligned} s\omega_{2n+1}{}^{(0)} + d\omega_{2n}{}^{(1)} &= 0 \\ s\omega_{2n}{}^{(1)} + d\omega_{2n-1}{}^{(2)} &= 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ s\omega_1{}^{(2n)} + d\omega_0{}^{(2n+1)} &= 0 \\ s\omega_0{}^{(2n+1)} &= 0 \end{aligned} \qquad (6.8.41)$$

To illustrate the descent equations, consider as an example the case with $n=1$ ($\omega_4 = trF^2$). One finds after some algebra

$$\begin{aligned} \omega_3(\mathcal{A},F) &= tr\left(F\mathcal{A} - \frac{1}{3}\mathcal{A}^3\right), \\ \omega_3{}^{(0)} &= tr\left(FW - \frac{1}{3}W^3\right) = tr\left(dWW + \frac{2}{3}W^3\right) \\ \omega_2{}^{(1)} &= trc(F-W^2) = trcdW \\ \omega_1{}^{(2)} &= tr - c^2W \\ \omega_0{}^{(3)} &= -\frac{1}{3}trc^3 \end{aligned} \qquad (6.8.42)$$

One may check the descent equations for these expressions, using $sc = -c^2$ and $sW = -dc - \{W,c\}$.

Let us now come back to possible ambiguities. The most general solution of $d\omega^{(0)}_{2n+1} = \omega_{2n+2}$ reads $\tilde{\omega}^{(0)}_{2n+1} = \omega^{(0)}_{2n+1} + d\omega^{(0)}_{2n}$ where $\omega^{(0)}_{2n}$ is arbitrary. The descent equations in (6.8.41) allow then the following terms to be added

$$\begin{aligned} s\left(\omega^{(0)}_{2n+1} + d\omega^{(0)}_{2n}\right) &+ d\left(\omega^{(1)}_{2n} + s\omega^{(0)}_{2n} + d\omega^{(1)}_{2n-1}\right) = 0 \\ s\left(\omega^{(1)}_{2n} + s\omega^{(0)}_{2n} + d\omega^{(1)}_{2n-1}\right) &+ d\left(\omega^{(2)}_{2n-1} + s\omega^{(1)}_{2n-1} + d\omega^{(2)}_{2n-2}\right) = 0 \\ s\left(\omega^{(2)}_{2n-1} + s\omega^{(1)}_{2n-1} + d\omega^{(2)}_{2n-2}\right) &= d\left(\omega^{(3)}_{2n-2} + s\omega^{(2)}_{2n-2} + d\omega^{(3)}_{2n-3}\right) = 0 \end{aligned} \qquad (6.8.43)$$

and so one. The extra terms cancel due to $sd + ds = 0, s^2 = 0$ and $d^2 = 0$. One can use the freedom to add d-exact and s-exact terms to simplify the expressions for the ω's. For example, for $n = 1$ one can obtain the following chain

$$\begin{aligned}
\tilde{\omega}_3^{(0)} &= \omega_3^{(0)} (\text{unchanged since } \omega_2^{(0)} \sim trW^2 \text{ vanishes}) \\
\tilde{\omega}_2^{(1)} &= \omega_2^{(1)} + d(trcW) = trdcW, \text{ so } \omega_1^{(1)} = trcW \\
\tilde{\omega}_1^{(2)} &= \omega_1^{(2)} + s(trcW) + d(c^2) = trdcc, \text{ so } \omega_0^{(2)} = trc^2 \\
\tilde{\omega}_0^{(3)} &= \omega_0^{(3)} + s(trc^2) = \omega_0^{(3)} = -\frac{1}{3}trc^3.
\end{aligned} \tag{6.8.44}$$

Hence the new chain reads as follows

$$\begin{aligned}
\tilde{\omega}_3^{(0)} &= tr(dWW) + \frac{2}{3}W^3 \\
\tilde{\omega}_2^{(1)} &= trdcW \\
\tilde{\omega}_1^{(2)} &= trcdc \text{ (independent of } W) \\
\tilde{\omega}_0^{(3)} &= -\frac{1}{3}c^3
\end{aligned} \tag{6.8.45}$$

Partial integration of $\tilde{\omega}_2^{(1)}$ would yield $\omega_2^{(1)}$ back, but these expressions are un-integrated. For $n = 2$ the direct and simplified expressions read

$$\begin{aligned}
\omega_5^{(0)} &= tr\left(W(dW)^2 + \frac{3}{2}W^3dW + \frac{3}{5}W^5\right) = \tilde{\omega}_5^{(0)} \\
\omega_4^{(1)} &= trc\left(WdW + \frac{1}{2}W^3\right); \tilde{\omega}_4^{(1)} = \frac{1}{2}trdc(WdW + dWW + W^3) \\
\omega_3^{(2)} &= -\frac{1}{2}tr(c^2W + cWc + Wc^2)dW + c^2W^3); \tilde{\omega}_3^{(2)} = tr(dc)^2W \\
\omega_2^{(3)} &= \frac{1}{2}tr(-c^3dW + WcWc^2); \qquad \tilde{\omega}_2^{(3)} = tr(dc)^2c \\
\omega_1^{(4)} &= \frac{1}{2}trc^4W; \qquad \tilde{\omega}_1^{(4)} = -\frac{1}{2}trdcc^3 \\
\omega_0^{(5)} &= \frac{1}{10}trc^5; \qquad \tilde{\omega}_0^{(5)} = \frac{1}{10}trc^5
\end{aligned} \tag{6.8.46}$$

Another issue we now mention concerns the global validity of the descent equations. Up to now all considerations were local, at one point x only. Thus our results were only valid for a trival fibre-bundle structure. The generalizations to nontrivial fibre bundles [50] involves **two** connections: a connection W and a connection W_0. The latter is invariant under BRST transformations, $sW_0 = 0$. One considers a field $W_t = tW + (1-t)W_0$ which interpolates between the background field W_0 and the quantum field W. The effective action depends on W and W_0, and the gauge variation yields the anomaly

$$s\Gamma(W, W_0) = \int c^a G_a(W, W_0) \equiv \int G \tag{6.8.47}$$

The consistency conditions read again $s\int G = 0$, and the general solution has ambiguities

$$\tilde{G} = G_{2n}^{(1)} + sG_{2n}^{(0)} + dG_{2n-1}^{(1)} \tag{6.8.48}$$

One can then again derive descent equations, which now hold globally, but which involve both W_0 and W.

Finally, we come back to the construction of a local $X_a{}^\mu$ which leads to the covariant current $\tilde{J}_a{}^\mu = J_a{}^\mu + X_a{}^\mu$. We found that it should satisfy

$$\delta(\text{gauge}, \Lambda)X_a{}^\mu = gf_a{}^{bc}X_b{}^\mu\Lambda^c + \frac{\delta}{\delta W_\mu{}^a}\int \Lambda^a G_a dx \tag{6.8.49}$$

Multiplying by $B_\mu{}^a$ and integrating over spacetime, and defining that $B_\mu{}^a$ transforms as a vector under gauge transformations, simplifies this to

$$\begin{aligned}\int \delta(\text{gauge}, \Lambda)B\cdot X &= \int \hat{\delta}(B)\Lambda^a G_a d^4x \\ \hat{\delta}(B) &= \int B_\mu^a(x)\partial/\partial W_\mu^a(x) d^4x\end{aligned} \tag{6.8.50}$$

We can rewrite this as an equation involving the BRST variation of $B\cdot X$

$$\int sB\cdot X = \int \hat{\delta}(B)c^a G_a dx \tag{6.8.51}$$

if we use that $\hat{\delta}(B)c = 0$ and define

$$sB_\mu{}^a = gf^a{}_{bc}B_\mu{}^b c^c; sB = -\{B, c\} \tag{6.8.52}$$

To solve (6.8.51), we introduce an operator ℓ satisfying

$$\ell d + d\ell = \hat{\delta}(B) \tag{6.8.53}$$

So l reduces the degree of a form by one, and then the only natural action of ℓ on the basic fields F, B, W and c is

$$\ell F = B, \ell B = 0, \ell W = 0, \ell c = 0 \tag{6.8.54}$$

In fact, (6.8.53) follows from $\ell F = B$ if one requires consistency with $F = dW + W^2$. It also follows from consistency that $\ell dc = \ell dB = 0$. Clearly ℓ is a kind of inverse of the exterior derivative d. (The alternative definition $\hat{\ell} F = W$ and $\hat{\ell} W = \hat{\ell} c = 0$ gives no useful results. For example $(d\hat{\ell} + \hat{\ell} d)F = dW + \hat{\ell}(FW - WF) = (F + WW)$, assuming that $\hat{\ell}$ anticommutes with W).

The operators d, ℓ, s and $\hat{\delta}(B)$ satisfy simple anti-commutation relations

$$d^2 = \ell^2 = s^2 = 0, ds + sd = \ell s + s\ell = 0, d\ell + \ell d = \hat{\delta}(B) \tag{6.8.55}$$

Further relations involving $\hat{\delta}(B)$ such as $d\hat{\delta}(B) = \hat{\delta}(B)d$ follow from these by replacing $\hat{\delta}(B)$ by $\ell d + d\ell$.

It follows that the equation for X can be rewritten (up to a constant) as

$$\int sB \cdot X = \int (d\ell + \ell d) {\omega_{2n}}^{(1)}(c, A, F) \tag{6.8.56}$$

Using the descent equation $d{\omega_{2n}}^{(1)} = -s{\omega_{2n+1}}^{(0)}$, and the relation $\ell s + s\ell = 0$, we obtain

$$\int sB \cdot X = \int [d\ell {\omega_{2n}}^{(1)} + s\ell {\omega_{2n+1}}^{(0)}] \tag{6.8.57}$$

Dropping the integral of the total derivative, we find a solution for ${X_a}^\mu$

$${B_\mu}^a {X_a}^\mu d^{2n}x = \ell {\omega_{2n+1}}^{(0)} \tag{6.8.58}$$

For example, in the case of $n = 1$, we already found $X_a{}^\mu$, namely $X_a{}^\mu = \epsilon^{\mu\nu} W_\nu$. Using $\omega_3{}^{(0)} = tr(FW - \frac{1}{3}W^3)$, the descent equations yield

$$\begin{aligned} B \cdot X d^2x &= \ell tr\left(FW - \frac{1}{3}W^3\right) \\ &= trBW = B_\mu W_\nu \epsilon^{\mu\nu} d^2x \end{aligned} \tag{6.8.59}$$

which agrees with $X^\mu = \epsilon^{\mu\nu} W_\nu$.

Comment: in this section we have only considered chiral anomalies in Yang-Mills theory. In this case one starts from the $(n+1)$th Chern character $trF^{(n+1)}$. For chiral gravitational anomalies (chiral fermions in a loop coupled to external gravity) a similar formalism exists, with local Lorentz symmetry replacing Yang-Mills symmetry. However, in this case one must start from more complicated $2n + 2$ forms, involving products of traces over Riemann curvatures. In these gravitational theories, there are a priori two gauge invariances: Einstein symmetry (=general coordinate invariance) and local Lorentz invariance. One can always remove the anomaly in one of these symmetries by adding a local counter term, similarly to the case of Yang-Mills theory, where a local counter term can remove the vector anomaly (or the axial vector anomaly, but not both).

9 The Pauli-Villars method

In section 1 we evaluated the one-loop anomalies in triangle, box and pentagon graphs by using ordinary dimensional regularization. In this scheme there is no vector current anomaly. (Technically the reason was that if one rewrites $\not{q}$ as a sum of two propagators and a mass term, there is no left-over $n-4$ dimensional piece with $\not{s}$, contrary to the case with $\gamma_5 \not{q}$). Although the trace $tr\gamma_5\gamma_\alpha\gamma_\beta\gamma_\gamma\gamma_\delta$ in n-dimensions is a priori undefined, we did not need to determine its value, since either all $\gamma_5, \gamma_\alpha, \gamma_\beta, \gamma_\gamma$ and γ_δ which contributed were 4-dimensional, or we needed to assume that the trace $\int tr\gamma_5\gamma_\mu(\not{r} - iM)^{-1}\gamma_\nu(\not{r} + \not{p} - iM)^{-1} d^n r$ vanishes. (This trace occurred in graphs with one contracted propagator. Since it only depends on p_μ and any reasonable definition of $\epsilon_{\mu\nu\alpha\nu}$ in n dimensions should still be totally antisymmetric in all four indices, the result must be proportional to $\epsilon_{\mu\nu\alpha\beta}p^\alpha p^\beta$ with whatever definition of $\epsilon_{\mu\nu\rho\sigma}$ in n dimensions, and therefore it vanishes). At higher loops there are several definitions of γ_5 and $\epsilon_{\mu\nu\alpha\beta}$. The most used is the 't Hooft-Veltman definition in which γ_5 (and $\epsilon_{\mu\nu\rho\sigma}$) remain 4-dimensional, while momenta and γ^μ become n-dimensional with $n > 4$. This scheme has been thoroughly studied, and it seems consistent. However, it does not preserve supersymmetry. A variant, dimensional reduction with $n < 4$, preserves supersymmetry but becomes inconsistent at 7 loops. It is clearly desirable to have a scheme for γ_5 which preserves supersymmetry and is consistent. Such a scheme is Pauli-Villars regularization [4], which works for one-loop graphs with spin 0 and spin 1/2 fields in loops and external gauge fields. For quantized gauge fields, though, so far no consistent Pauli-Villars regularization scheme has been devised, as we shall discuss later.

In its simplest form the Pauli-Villars regularization scheme uses an action with several anticommuting and commuting fermions, with various masses and various coupling constants.[32]

At the one-loop level we only need to add one heavy commuting Dirac fermion. For an abelian model the action reads

$$\begin{aligned}\mathcal{L} = \; & - \; \bar\psi\gamma^\mu(\partial_\mu - ieV_\mu - igA_\mu\gamma_5)\psi - m\bar\psi\psi \\ & - \; \bar\chi\gamma^\mu(\partial_\mu - ieV_\mu - igA_\mu\gamma_5)\chi - M\bar\chi\chi \end{aligned} \tag{6.9.1}$$

[32]The older and better known Pauli-Villars-type regularization replaced a field φ in the interactions by a sum of fields φ_i, and replaced the free part by a sum of free parts with factors $(e_i)^{-1}$ in front, satisfying relations like $\sum e_i = 0, \sum e_i {M_i}^2 = 0$ etc. For example, $\mathcal{L} = -\frac{1}{2}(\partial_\mu\varphi)^2 - \frac{1}{2}m^2\varphi^2 + \frac{1}{2}(\partial_\mu\chi)^2 + \frac{1}{2}M^2\chi^2 + \mathcal{L}_{\rm int}(\varphi+\chi)$. For each φ propagator with one sign there is a χ propagator with an opposite sign, which improves the convergence. However, this action violates BRST symmetry even for vanishing masses, and is therefore not suitable for our purposes. In our applications we have either complete φ loops, or complete φ_i loops, but never more than one species of particles in the same loop.

We define the regularized axial current by

$$j_5{}^\mu(\text{reg}) = -i\bar{\psi}\gamma_5\gamma_\mu\psi - i\bar{\chi}\gamma_5\gamma_\mu\chi \tag{6.9.2}$$

Because χ is commuting, this current is not hermitian. However in the limit $M \to \infty$ the effects of this nonhermiticity disappear. At the one-loop level one could also take χ to be anticommuting, and supply a minus sign for the loop by hand. Then the regularized current is hermitian. The current $j_5{}^\mu$ (reg) is analogous to $-i\bar{\psi}\gamma_5\gamma_\mu\psi$ with n matrices γ_μ in dimensional regularization. So we have already two regularizations of the same current, one by going into n dimensions, the other by adding another field. Any divergent expression can now be regularized in this way, but for finite expressions the contribution from χ should cancel as M tends to infinity.

The classical conservation law reads

$$\partial_\mu j_5{}^\mu(\psi) = 2im\bar{\psi}\gamma_5\psi \tag{6.9.3}$$

but the regularized current satisfies the operator equation

$$\partial_\mu j^\mu(reg) = 2im\bar{\psi}\gamma_5\psi + 2iM\bar{\chi}\gamma_5\chi \tag{6.9.4}$$

If we now evaluate the triangle graph with these terms at the top vertex, we find the *difference* between the ψ and χ loop since the commuting spinor χ does not acquire a minus sign in a closed loop. This implies that the matrix elements of j^μ (reg) are finite. (In the trace with $\gamma_5\gamma_\mu$, only two $\not{k}$ can contribute, by symmetric integration, hence the matrix element of $\bar{\psi}\gamma_5\gamma_\mu\psi$ is logarithmically divergent by power counting. The difference between the loop with ψ and the loop with χ is then finite). On the right-hand side, the triangle graphs with $\bar{\psi}\gamma_5\psi$ and $\bar{\chi}\gamma_5\chi$ in the top vertex are **separately** finite, since (i) there are 5 Dirac matrices in each trace (in addition to the matrix γ_5), but only an even number of Dirac matrices can contribute. Thus, only terms with one factor M and a trace over γ_5 and four Dirac matrices can contribute. (ii) after taking the (ordinary 4-dimensional) trace, terms with two factors of the loop momentum k_μ cancel (since $\epsilon_{\mu\nu\alpha\beta}k^\alpha k^\beta = 0$). (iii) the remaining integrals are finite, and terms with one power of k_μ cancel if we use symmetric integration. Thus, the triangle loop with $\bar\psi\gamma_5\psi$ at the top is finite by itself, and hence need not be regularized

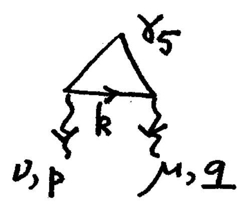
$$< \bar\psi\gamma_5\psi >_{\rm reg} = \lim_{M\to\infty} < \bar\psi\gamma_5\psi + \bar\chi\gamma_5\chi > = < \bar\psi\gamma_5\psi > \tag{6.9.5}$$

The vanishing of the matrix element $< \bar\chi\gamma_5\chi >$ when M tends to infinity is already clear by dimensional arguments: from the numerator we get a factor M, and the loop integral $\sim \int d^4k/(k^2+M)^3$ is proportional to M^{-2}. However, $M < \bar\chi\gamma^5\chi >$ does not vanish for $M\to\infty$, and thus we obtain the relation

$$< \partial^\mu j_\mu >_{\rm reg} = 2im\left(< \bar\psi\gamma_5\psi >\right)_{\rm reg} + \lim_{M\to\infty} 2iM < \bar\chi\gamma^5\chi > \tag{6.9.6}$$

Written this way, it is clear that the last term is the anomaly.

The evaluation of the anomaly is easy. The numerator is proportional to



$$2iMtr\gamma_5(\not k - \not q + iM)\begin{pmatrix}\gamma_\mu\\ \gamma_5\gamma_\mu\end{pmatrix}(\not k + iM)\begin{pmatrix}\gamma_\nu\\ \gamma_5\gamma_\nu\end{pmatrix}(\not k + \not p + iM) \tag{6.9.7}$$

where the contribution from the AVV anomaly corresponds to the matrices γ_μ and γ_ν on top, while for the AAA anomaly one should take the matrices $\gamma_5\gamma_\mu$ and $\gamma_5\gamma_\nu$ at the bottom. Taking the trace in spinor space yields a factor 4, and one finds

$$\begin{aligned} &8M^2\epsilon^{\mu\nu\rho\sigma}[k_\rho(k+p)_\sigma \mp (k-q)_\rho(k+p)_\sigma + (k-q)_\rho k_\sigma] \\ &= 8M^2\epsilon^{\mu\nu\rho\sigma}(k_\rho p_\sigma \mp (k_\rho p_\sigma - q_\rho k_\sigma - q_\rho p_\sigma) - q_\rho k_\sigma) \end{aligned} \tag{6.9.8}$$

where the upper (lower) signs refer to the $VVA(AAA)$ case. The terms with two k's cancel due to antisymmetry, hence the integral is finite, as we already announced.

Combining the propagators and shifting the loop momentum (which is now allowed as the integral is finite)

$$k = \kappa + q(1-x) - py \tag{6.9.9}$$

also the terms linear in κ cancel by symmetric integration, and one finds after elementary algebra

$$\begin{aligned}&\lim_{M\to\infty} 8M^2\epsilon^{\mu\nu\rho\sigma}\left(2\int_0^1 dx \int_0^x dy\right)\\ &\int \frac{d^4\kappa}{(2\pi)^4}\frac{[q_\rho p_\sigma \text{ or } q_\rho p_\sigma - 2(x-y)q_\rho p_\sigma]}{(\kappa^2+M^2+\mathcal{O}(M^0))^3}\end{aligned} \tag{6.9.10}$$

for the AVV or the AAA case. Since $M^2\int d^4\kappa(\kappa^2+M^2)^{-3}$ is equal to $\frac{\pi^2}{2}$, one finds a finite but nonvanishing result. Adding the crossed diagram yields a factor 2, so we obtain for the anomaly, after adding the factor i from the Wick rotation of the momentum integral in (6.9.10)

$$\begin{aligned} An &= \frac{i}{2\pi^2}\epsilon_{\mu\nu}{}^{\rho\sigma}\left(q_\rho p_\sigma \text{ or } \frac{1}{3}q_\rho p_\sigma\right) \\ &= \frac{i}{16\pi^2}\epsilon^{\mu\nu\rho\sigma}(F_{\mu\nu}(V)F_{\rho\sigma}(V) + \frac{1}{3}F_{\mu\nu}(A)F_{\rho\sigma}(A)) \end{aligned} \tag{6.9.11}$$

where $q_\rho p_s$ refers to the VVA case and $\frac{1}{3}q_\rho p_\sigma$ refers to the AAA case. (Contracting with polarization vectors $\epsilon_\mu(p)$ and $\epsilon_\nu(q)$, we replaced $\epsilon_\mu(p)p_\sigma$ by $\frac{1}{2}F_{\mu\sigma}(p)$, and $\epsilon_\nu(q)q_\rho$ by $\frac{1}{2}F_{\nu\rho}(q)$. An extra factor 1/2 is needed if one writes this as $F_{\mu\sigma}F_{\nu\rho}$ because there are then two ways to contract with external photons). Hence we find the same result as obtained from ordinary dimensional regularization.

Also with Pauli-Villars regularization there is no vector-current anomaly, since the regularized vector current is $j_V{}^\mu = \bar\psi\gamma^\mu\psi + \bar\chi\gamma^\mu\chi$, and each current is conserved, hence also $\partial_\mu j_V{}^\mu$ (reg) $= 0$. Taking the limit $M\to\infty, \partial_\mu j_V{}^\mu$ (reg) remains of course zero, hence no anomaly is found in the vector gauge invariance.

We now incorporate the Pauli-Villars method in the path integral formalism. This will yield all triangle, box and pentagon anomalies for all combinations of external

${V_\mu}^a$ and ${A_\mu}^a$ lines in one stroke. Also, this sets the stage for, and is, in fact, very closely related to, the Fujikawa method of computing anomalies from path integrals.

Consider the path integral for the anticommuting Dirac field ψ and the commuting Dirac field χ coupled to nonabelian external gauge fields. We shall work in Minkowski space, so $\bar{\psi} = \psi^\dagger i\gamma^0$.

$$
\begin{aligned}
Z[V, A] &= \int d\psi d\bar{\psi} d\chi d\bar{\chi} \exp \frac{i}{\hbar} \int d^4x \mathcal{L} \\
\mathcal{L} &= -\bar{\psi}\gamma^\mu(\partial_\mu + V_\mu + A_\mu\gamma_5)\psi - m\bar{\psi}\psi \\
&\quad -\bar{\chi}\gamma^\mu(\partial_\mu + V_\mu + A_\mu\gamma_5)\chi - M\bar{\chi}\chi; V_\mu \equiv {V_\mu}^a T_a
\end{aligned}
\tag{6.9.12}
$$

We shall first evaluate the expression for the anomaly in Minkovski spacetime, and then we shall evaluate the Gaussian momentum integral $\int d^4k$ by making a Wick rotation to Euclidean space. Under an infinitesimal chiral transformation

$$
\begin{aligned}
&\psi \to \lambda \cdot T\gamma_5\psi \quad , \chi \to \lambda \cdot T\gamma_5\chi \\
&\bar{\psi} \to \bar{\psi}\lambda \cdot T\gamma_5 \quad , \bar{\chi} \to \bar{\chi}\lambda \cdot T\gamma_5
\end{aligned}
\tag{6.9.13}
$$

the Jacobians cancel each other since ψ and χ have opposite statistics. (We assume here that the regulators of the Jacobian of ψ and χ are the same, for example $\not{D}\not{D}$ for both. More about this in the next section.) If we now make a local chiral change of integration variables, we find to first order in λ

$$
\begin{aligned}
0 = \int < &-\bar{\psi}\gamma^\mu(D_\mu\lambda)\gamma_5\psi - 2m\bar{\psi}\lambda^a T_a\gamma_5\psi \\
&-\bar{\chi}\gamma^\mu(D_\mu\lambda)\gamma_5\chi - 2M\bar{\chi}\lambda^a T_a\gamma_5\chi >_{\psi,\chi}
\end{aligned}
\tag{6.9.14}
$$

where $D_\mu\lambda = \partial_\mu\lambda + [V_\mu + A_\mu\gamma_5, \lambda]$ and the subscripts ψ, χ indicate that we are dealing with a path integral over ψ and χ. The sum of the first two terms gives the naive Ward identity for chiral symmetry, the third term regulates the first term, and the last term will yield the anomaly. Exponentiating the last two terms (which is allowed as we are working to first order in λ) we perform the path integral over χ and find

$$
c \det{}^{-1}[\gamma^\mu(\partial_\mu + V_\mu + A_\mu\gamma_5) + M + \gamma^\mu(D_\mu\lambda)\gamma_5 + 2M\lambda^a T_a\gamma_5]
\tag{6.9.15}
$$

where c is a constant which will be canceled when we re-exponentiate part of the terms. Because χ is commuting we get the inverse of the determinant. Since χ has been integrated out only a path integral average over ψ is left. Since the external fields A_μ and V_μ are arbitrary there are no fermionic zero modes so that the determinant is nonvanishing.

Next we factorize the determinant such that the term with $2M\lambda\gamma_5$ is extracted, because we already know that this term will lead to the anomaly. This yields

$$\begin{array}{l} c\,\det^{-1}[\gamma^\mu(\partial_\mu + V_\mu + A_\mu\gamma_5) + M + \gamma^\mu(D_\mu\lambda)\gamma_5] \\ \det^{-1}\left[1 + \dfrac{2M\lambda^a T_a\gamma_5}{\partial\!\!\!/ + \gamma\cdot V + \gamma\cdot A\gamma_5 + M + \gamma^\mu(D_\mu\lambda)\gamma_5}\right] \end{array} \tag{6.9.16}$$

Since the trace is cyclic, it does not matter whether one writes in the second determinant the numerator before or after the denominator, and since we work to first order in the axial gauge parameter λ, we drop the term $\gamma^\mu(D_\mu\lambda)\gamma_5$ in the denominator. The first determinant is exponentiated again, which removes the constant c as announced and brings us back to the original path integral over ψ and χ. The term $\exp(-\bar\chi\gamma^\mu(D_\mu\lambda^a)T_a\gamma_5\chi)$ combines with $-\bar\psi\gamma^\mu(D_\mu\lambda^a)T_a\psi$ into the regularized current multiplied by $D_\mu\lambda^a$. If there was no anomaly the covariant divergence of the regularized current would vanish, hence the second determinant yields the anomaly.

We can take the second determinant outside the path integral over ψ and χ, and expanding the determinant to first order in λ, we find that the anomaly An is given by a trace

$$An_a = \lim_{M\to\infty} -Tr\left(\frac{2MT_a\gamma_5}{D\!\!\!\!/ + M}\right), D\!\!\!\!/ = D\!\!\!\!/ + \gamma\cdot V + \gamma\cdot A\gamma_5 \tag{6.9.17}$$

This trace contains an integral over x and a trace in the internal space and in the spinor space. We multiply this expression by $\frac{D\!\!\!\!/-M}{D\!\!\!\!/-M}$, and find

$$An = \lim_{M\to\infty} -Tr2MT_a\gamma_5(D\!\!\!\!/ - M)\left(\frac{1}{D\!\!\!\!/D\!\!\!\!/ - M^2}\right) \tag{6.9.18}$$

Since terms with an odd number of Dirac matrices cancel in the trace, we drop the $\not{D}$ in the numerator,

$$An_a = \lim_{M\to\infty} -2M^2 TrT_a\gamma_5 \frac{1}{M^2 - \not{D}\not{D}} \tag{6.9.19}$$

To evaluate this trace, we introduce a complete set of plane waves[33].

$$TrT_a\gamma_5 \frac{1}{M^2 - \not{D}\not{D}} = \int d^4x \int \frac{d^4k}{(2\pi)^4} e^{-ikx} trT_a\gamma_5 \frac{1}{M^2 - \not{D}\not{D}} e^{ikx} \tag{6.9.20}$$

The trace tr denotes only the trace over internal indices and spinor indices, and we used completeness of x-and p-eigenstates. Using that $\not{\partial} e^{ikx} = e^{ikx}(\not{\partial} + i\not{k})$, we find for the local chiral anomaly (i.e., before integrating over x)

$$An_a(x) = \lim_{M\to\infty} \int \frac{d^4k}{(2\pi)^4} tr \left(-2M^2 T_a\gamma_5 \, \frac{1}{k^2 + M^2 - B} \right) \tag{6.9.21}$$

where $k^2 - B$ is equal to the operator $e^{-ikx}(-\not{D}\not{D})e^{ikx} = -(\not{D} + i\not{k})(\not{D} + i\not{k})$

$$\begin{aligned}
B &= ik \cdot (D + \bar{D}) + i\gamma^{\mu\nu} k_\mu (D_\nu - \bar{D}_\nu) + \not{D}\not{D} \\
&= 2ik \cdot (\partial + V) + \bar{D} \cdot D + \frac{1}{2}\gamma^{\mu\nu} \left(\hat{F}_{\mu\nu} + \hat{G}_{\mu\nu}\gamma_5 + 4ik_\mu A_\nu \gamma_5 - 2A_\mu \partial_\nu \gamma_5 \right) \\
\bar{D}_\mu &= \partial_\mu + V_\mu - A_\mu\gamma_5 \; ; \; D_\mu = \partial_\mu + V_\mu + A_\mu\gamma_5 \; ; \\
\hat{F}_{\mu\nu} &= \partial_\mu V_\nu - \partial_\nu V_\mu + [V_\mu, V_\nu] - [A_\mu, A_\nu] \\
\hat{G}_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu + \{V_\mu, A_\nu\} - \{ A_\mu, V_\nu\}
\end{aligned} \tag{6.9.22}$$

Note that the curvatures $\hat{F}$ and $\hat{G}_{\mu\nu}$ differ from the Bardeen curvatures in (6.1.7) at several places. It is now clear that expanding $(k^2 + M^2 - B)^{-1}$, the terms without B or with one B do not contribute in the trace with γ_5, while the terms with $2, 3$ and 4 B's do contribute. For example, the term with 4 B's contributes if in each factor B one uses the k-dependent terms, since this leads to an integral $M^2 \int d^4k k^4 (k^2 + M^2)^{-5}$

[33]The trace of an operator A is given by $TrA = \int \frac{d^4p}{(2\pi)^4} \langle p \mid A \mid p\rangle$ where $|p>$ denotes the complete set of p-eigenstates. Introducing a complete set of x-eigenstates, we obtain $TrA = \int d^4x \int \frac{d^4p}{(2\pi)^4} \langle p \mid A \mid x\rangle\langle x \mid p\rangle$. The x-operators become c-numbers if one pulls them to the right where they meet $\mid x\rangle$, while the p-operators become c- numbers if one pulls them to the left where they meet $< p|$.

which still limits to a finite nonzero constant if $M \to \infty$. Similarly, only the terms with two k's contribute in B^3 in the limit $M \to \infty$. For five B's no finite remainder is left. (The reader may wonder whether a single B contributes. There are terms proportional to γ_5 in $\bar{D}D$ which yield "normal-parity" contributions, which are odd in A. However, in the trace only a term $(\partial^\mu A_\mu)$ survives, and this is a total derivative which vanishes after integration over x).

To evaluate the k integrals we make a Wick rotation to Euclidean space on k_μ and x^μ simultaneously; at the end we rotate x^μ back to Minowski space. The Euclidean k integrals are easy in each case. One finds

$$\begin{aligned} \int \frac{d^4k}{(k^2+M^2)^3} &= \frac{1}{2}\frac{\pi^2}{M^2}, \int \frac{d^4k k^2}{(k^2+M^2)^4} = \frac{1}{3}\frac{\pi^2}{M^2} \\ \int \frac{d^4k (k^2)^2}{(k^2+M^2)^5} &= \frac{1}{4}\ \frac{\pi^2}{M^2} \end{aligned} \tag{6.9.23}$$

So the evaluation of the complete anomaly has been reduced to a trivial, though laborious, exercise. We postpone this exercise, because we first want to repeat the Pauli-Villars path integral approach for chiral fermions.

For the derivation of the axial anomaly using the Pauli-Villars method for chiral fermions, one introduces commuting left-handed fermions χ_L and $\bar{\chi}_L$ in addition to the anticommuting left-handed fermions ψ_L and $\bar{\psi}_L$. However, in order to construct a mass term for these fields, one must add further right-handed fermions χ_R and $\bar{\chi}_R$. Since the χ loops should be the same as the ψ loops (up to a sign, in order to cancel divergences) the left-handed χ_L fields should couple in exactly the same was as the fields ψ_L while the right-handed fermions χ_R should not couple at all to external gauge fields. Hence one is led to the following action

$$\begin{aligned} \mathcal{L} &= -\bar{\psi}_L \not{D}_L \psi_L - \bar{\chi}_L \not{D}_L \chi_L - \bar{\chi}_R \not{\partial} \chi_R - M\left(\bar{\chi}_R \chi_L + \bar{\chi}_L \chi_R\right) \\ \not{D} &= \not{\partial} + \gamma^\mu W_\mu; W_\mu = W^a T_a; \chi_L = \frac{1}{2}(1+\gamma_5)\chi \end{aligned} \tag{6.9.24}$$

The action, except the mass term, is invariant under the following chiral gauge transformations with parameters $\Lambda_L = {\Lambda_L}^a T_a$

$$\begin{array}{ll} \delta\psi_L = -g\Lambda_L\psi_L & \delta\bar{\psi}_L = g\bar{\psi}_L\Lambda_L \\ \delta\chi_L = -g\Lambda_L\chi_L & \delta\bar{\chi}_L = g\bar{\chi}_L\Lambda_L \\ \delta\chi_R = 0 & \delta\bar{\chi}_R = 0 \end{array} \tag{6.9.25}$$

and, of course, $\delta W_{L,\mu} = \partial_\mu\Lambda_L + [W_{L,\mu}, \Lambda_L]$. The path integral is given by

$$Z(W) = \int d\psi_L d\bar{\psi}_L d\chi_L d\bar{\chi}_L d\chi_R d\bar{\chi}_R e^{\frac{1}{\hbar}\int \mathcal{L} d^4x} \tag{6.9.26}$$

As before the path integral measure is invariant, if the Jacobians of ψ_L and $\bar{\psi}_L$ are regulated with the same regulators as those of χ_L and $\bar{\chi}_L$. Thus the anomaly is again only due to the noninvariance of the mass term in the action.

Making a change of integration variables which is equal to an infinitesimal gauge transformation of the fermions, we find to first order in Λ_L

$$\begin{aligned} 0 &= \int < \bar{\psi}_L(\partial\!\!\!/\Lambda_L)\psi_L + \bar{\psi}_L\gamma^\mu[W_\mu, \Lambda_L]\psi_L \\ &+ \bar{\chi}_L(\partial\!\!\!/\Lambda_L)\chi_L + \bar{\chi}_L\gamma^\mu[W_\mu, \Lambda_L]\chi_L + M(\bar{\chi}_R\Lambda_L\chi_L - \bar{\chi}_L\Lambda_L\chi_R) > d^4x \end{aligned} \tag{6.9.27}$$

We put all terms in the exponent (we work to first order in Λ_L) and put $\chi_L = \frac{1}{2}(1+\gamma_5)\chi = P_L\chi$ with $P_L = \frac{1}{2}(1+\gamma_5)$ a projection operator, and $\chi_R = P_R\chi$ with $P_R = \frac{1}{2}(1-\gamma_5)$. Integrating over χ and $\bar{\chi}$ we obtain

$$\det{}^{-1}\left[D\!\!\!\!/\,P_L + \partial\!\!\!/ P_R + M + (\partial\!\!\!/\Lambda_L)P_L + [W_\mu, \Lambda_L]P_L + M\Lambda_L P_L - M\Lambda_L P_R\right] \tag{6.9.28}$$

This expression corresponds to (6.9.15). We factorize the last two terms out, since they are the terms which lead to the anomaly as we have seen. The other terms are re-exponentiated, and yield $-\bar{\chi}_L D\!\!\!\!/\,\chi_L - \bar{\chi}_R\partial\!\!\!/\chi_R - M(\bar{\chi}_R\chi_L + \bar{\chi}_L\chi_R) - \bar{\chi}_L(\partial\!\!\!/\Lambda_L)\chi_L - \bar{\chi}_L[W_\mu, \Lambda_L]\chi_L$. The last two terms combine with $-\bar{\psi}_L\partial\!\!\!/\Lambda_L\psi_L - \bar{\psi}_L[W_\mu, \Lambda_L]\psi_L$ to yield the regularized covariant divergence of the current $\bar{\psi}_L\gamma^\mu T_a\psi_L$. This is equal to the anomaly as we have discussed in section 1. Hence the local anomaly (before integration over x) is given by

$$An = \det{}^{-1}\left[1 + \frac{M\Lambda_L(P_L - P_R)}{D\!\!\!\!/\,P_L + \partial\!\!\!/ P_R + M + (\partial\!\!\!/\Lambda_L)P_L + [W_\mu, \Lambda_L]P_L}\right] - 1$$

$$
\begin{aligned}
&\simeq \quad Tr\frac{M\Lambda_L(P_L-P_R)}{\not{D}P_L+\not{\partial}P_R+M}=Tr\frac{M\Lambda_L(P_L-P_R)}{(\not{D}P_L+\not{\partial}P_R+M)}\,\frac{\not{D}P_L+\not{\partial}P_R-M}{\not{D}P_L+\not{\partial}P_R-M}\\
&= \quad Tr\frac{M^2\Lambda_L(P_L-P_R)}{M^2-\not{\partial}\not{D}P_L-\not{D}\not{\partial}P_R}=Tr\left(\frac{M^2\Lambda_L}{M^2-\not{\partial}\not{D}}P_L-\frac{M^2\Lambda_L}{M^2-\not{D}\not{\partial}}P_R\right)
\end{aligned}
\tag{6.9.29}
$$

Since we work to first order in Λ_L, we dropped the Λ_L dependent term in the denominator, and we dropped terms with an odd number of Dirac matrices in the trace. Finally we used that $\not{D}P_L\not{D}P_L=\not{\partial}P_R\not{\partial}P_R=0$.

The computation of the two traces is somewhat cumbersome. We use plane waves, and define $An=\int(An_I+An_{II})d^4x$ where

$$
\begin{aligned}
An_I &= \int\frac{d^4k}{(2\pi)^4}tr\frac{M^2\Lambda_L\frac{1}{2}(\gamma_5+1)}{k^2+M^2-B_L}, B_L=\partial\cdot D+2ik\cdot\partial+i\not{k}\not{W}+\gamma^{\mu\nu}\partial_\mu D_\nu\\
An_{II} &= \int\frac{d^4k}{(2\pi)^4}tr\frac{M^2\Lambda_L\frac{1}{2}(\gamma_5-1)}{k^2+M^2-B_R}, B_R=D\cdot\partial+2ik\cdot\partial+i\not{W}\not{k}+\gamma^{\mu\nu}D_\mu\partial_\nu
\end{aligned}
\tag{6.9.30}
$$

and of course $D_\mu=\partial_\mu+W_\mu$. The terms with γ_5 receive only contributions from the terms with B^2 and B^3 since the terms linear in B do not have enough Dirac matrices. The terms with B^4 need all the k factors they can get in order to survive the $M\to\infty$ limit, but, as in the case of the anomaly in terms of V_μ and A_μ, these terms vanish after symmetric integration over k.

It is straightforward to show that the ${B_L}^2$ terms only contribute a term proportional to $\epsilon^{\mu\nu\rho\sigma}\partial_\mu W_\nu\partial_\rho W_\sigma$, but the ${B_R}^2$ terms do not contribute (a bare ∂_ν on the right in the trace gives zero). Next note that the ${B_L}^3$ terms and the ${B_R}^3$ contribute each 3 terms, each of these six terms having two k's. They yield, respectively

$$
\begin{aligned}
&tr\gamma_5{B_L}^3P_L\simeq tr[-2\not{\partial}\not{D}\not{\partial}\not{D}+\not{D}\not{D}\not{\partial}\not{D}-\not{D}\not{\partial}\not{D}\not{D}-2\not{\partial}\not{D}\not{\partial}\not{D}+\not{\partial}\not{D}\not{D}\not{D}]\\
&tr\gamma_5{B_R}^3P_R\simeq tr[\not{D}\not{D}\not{\partial}\not{D}-\not{D}\not{\partial}\not{D}\not{D}-2\not{\partial}\not{D}\not{\partial}\not{D}]
\end{aligned}
\tag{6.9.31}
$$

Adding all contributions, one finds [40] the consistent anomaly for chiral fermions

$$
An(\text{left})=\frac{1}{24\pi^2}\epsilon^{\mu\nu\rho\sigma}tr\Lambda_L\partial_\mu\left({W_\nu}^L\partial_\rho {W_\sigma}^L+\frac{1}{2}{W_\nu}^L{W_\rho}^L{W_\sigma}^L\right)
\tag{6.9.32}
$$

This agrees with the result in (6.2.21) obtained from the consistency conditions. We have denoted the gauge fields which couple to the left-handed fermions by ${W_\mu}^L$, since we are also going to consider the right-handed sector. If one would have started with right-handed fermions (or added right-handed fermions to the left-handed ones), one would have found the same expression up to an overall sign, and in terms of the gauge fields which couple to the right-handed fermions.

$$An(\text{right}) = -\frac{1}{24\pi^2}\epsilon^{\mu\nu\rho\sigma} Tr\Lambda_R \partial_\mu \left({W_\nu}^R \partial_\rho {W_\sigma}^R + \frac{1}{2}{W_\nu}^R {W_\rho}^R {W_\sigma}^R \right) \tag{6.9.33}$$

Since purely vector couplings have ${W_\mu}^L = {W_\mu}^R$ and $\Lambda_L = \Lambda_R$, the anomaly for vector couplings ($An(\text{left}) + An(\text{right})$) is seen to vanish. However, if ${W_\mu}^L$ and ${W_\mu}^R$ are not equal, one can still decompose them into vector fields ${V_\mu}^a$ and axial vector fields ${A_\mu}^a$, and then the vector anomalies are associated with Λ_V, and the axial vector anomalies are associated with Λ_A, where, as derived in section 1,

$$\begin{aligned} \Lambda_L &= \Lambda_V + \Lambda_A, \Lambda_R = \Lambda_V - \Lambda_A \\ W_L &= V + A, W_R = V - A \end{aligned} \tag{6.9.34}$$

Then the vector anomaly is given by

$$\begin{aligned} &An(V) = An(W_L) + An(W_R) = \\ &\frac{1}{24\pi^2} Tr\Lambda_V \left\{ 2dVdA + 2dAdV + d(VVA + VAV + AVV + AAA) \right\} \end{aligned} \tag{6.9.35}$$

The axial anomaly is similarly given by

$$\begin{aligned} &An(A) = An(W_L) - An(W_R) \\ &= \frac{1}{24\pi^2} tr\Lambda_A \{2dVdV + 2dAdA + d\{VVV + VAA + AVA + AAV\} \end{aligned} \tag{6.9.36}$$

There are two problems. One would expect the vector currents to be conserved, and $An(A)$ is not proportional to the Bardeen anomaly. Both problems are simultaneously solved [41] by adding a local counter term to the action, whose vector gauge variation cancels the vector anomaly, and whose axial vector gauge transformations completes $An(W_L) - An(W_R)$ to the Bardeen anomaly

$$\begin{aligned} An(V) &+ \frac{\delta}{\delta\Lambda_V}\Delta S = 0 \\ An(A) &+ \frac{\delta}{\delta\Lambda_A}\Delta S = An(\text{Bardeen}) \end{aligned} \tag{6.9.37}$$

To cancel the vector anomaly, we need a counter term ΔS which has an odd number of A fields. To find this ΔS, one may write down the most general candidate which is odd in A with arbitrary coefficients. There are four nonvanishing structures

$$\Delta S = \frac{-1}{24\pi^2}\int tr[a_1 dVAV + a_2 dVVA + a_3 AAAV + a_4 AVVV]d^4x \tag{6.9.38}$$

(Others, like $dAAA$ vanish after integration and taking the trace). One may then fix $a_1, \ldots a_4$ by requiring that there be no vector anomaly. The algebra is tedious but straightforward, and the answer is

$$a_1 = 2, a_2 = -2, a_3 = 1, a_4 = 3 \tag{6.9.39}$$

Note that for abelian gauge theories ΔS vanishes.

This completes the derivation of the Bardeen anomaly from the Pauli-Villars method applied to path integrals. We now turn to the Fujikawa method, which is closely related to the Pauli-Villars method as we shall see.

10 The Fujikawa method

In the Fujikawa method, [51], one studies the path integral for ψ and $\bar{\psi}$, and since there are no compensating commuting fermions χ and $\bar{\chi}$ as in the Pauli-Villars method, the Jacobian does not vanish, and it is here that the anomaly resides. The problem then arises how to regulate this Jacobian. We shall first use as regulator $R = \not{D}\not{D}$ where $D_\mu = \partial_\mu + V_\mu + A_\mu\gamma_5$ is the operator one encounters in the action. However, then we shall use the regulator $\not{D}\not{D}$ for ψ but $\bar{\not{D}}\bar{\not{D}}$ for $\bar{\psi}$ where $\bar{D}_\mu = -D_\mu^\dagger = \partial_\mu + V_\mu - A_\mu\gamma_5$.

We shall see that the answers differ by the axial-gauge variation of a local counter term. This will prompt us to study more generally the relation between regulators and anomalies.

Some authors prefer to first Wick rotate the whole theory to Euclidean space, and then to construct there the Jacobian. This raises then the question whether one should replace A_μ by iA_μ in Euclidean space. (A pseudoscalar made from 4 scalars as $\epsilon^{\mu\nu\rho\sigma}\partial_\mu\varphi_1\partial_\nu\varphi_2\partial_\rho\varphi_3\partial_\sigma\varphi_4$, acquires a factor of i, and the product of a vector and pseudoscalar is an axial vector). We prefer to derive the Jacobian in Minkowski spacetime, and only make a Wick rotation of the momentum integrals, as is customary in Feynman graph calculations. Even in Minkowski spacetime one could use regulators with $\not\partial + \not V + i\not A\gamma_5$ instead of $\not\partial + \not V + \not A\gamma_5$, but we shall mainly consider the latter. When we discuss chiral fermions we use as regulator $(\not\partial + \not W)P_L$, and then $((\not\partial + \not W)P_L)^\dagger = -(\not\partial + \not W)P_R$.

The path integral in Minkowski space with a hermitian action is given by

$$Z[V,A] = \int d\psi d\bar\psi \exp\frac{i}{\hbar}\int d^4x[-\bar\psi(\not\partial + \not V + \not A\gamma_5 + m)\psi] \tag{6.10.1}$$

In a time-discretized approach, $d\psi d\bar\psi$ stands for N products $d\psi_j d\bar\psi_j$, and by a unitary transformation we go over to modes

$$\left.\begin{array}{l}\psi(x) = \sum a_n\varphi_n(x)\\ \bar\psi(x) = \sum \varphi_n^\dagger b_n\end{array}\right\}\begin{array}{l} d\psi d\bar\psi = \prod_n da_n db_n\\ \int \varphi_m^\dagger\varphi_n d^4x = \delta_{mn}\end{array} \tag{6.10.2}$$

In Fujikawa's approach to anomalies, one regulates the Jacobian in the path integral with a regulator $\exp R/M^2$ and takes the limit of the regulator mass tending to infinity at the end of the calculation

$$An = \lim_{M^2\to\infty} TrJe^{R/M^2} \tag{6.10.3}$$

The Jacobian for a change of variables which corresponds to a symmetry transformation is unambiguous but singular; for example for a Dirac fermion, an axial vector gauge transformation gives

$$J_\psi{}^A = \gamma_5\lambda^a T_a\delta(x-y); J_{\bar\psi}{}^A = \gamma_5\lambda^a T_a\delta(x-y) \tag{6.10.4}$$

while a vector gauge transformation gives

$$J_{\psi}{}^{V} = \lambda^a T_a \delta(x-y); J_{\bar{\psi}}{}^{V} = -\lambda^a T_a \delta(x-y) \tag{6.10.5}$$

However, different R give different anomalies, sometimes related by the gauge variation of a local counter term, sometimes not. In particular $R = \not{D}\not{D}$ for ψ and $\bar{\psi}$ yielded the consistent axial anomaly. One might perhaps at first sight have expected that $R = \not{D}\not{D}$ would give the covariant axial anomaly. The question arises: which regulator gives which anomaly? We shall first study a few examples, but then we shall present a general theory for regulators for **consistent** anomalies [1]. No corresponding results for covariant anomalies seem to exist to date.

Consider first a massless nonabelian vector gauge field coupled to Dirac fermions in Minkowski space. Making a change of variables $\delta\psi = \gamma_5 \lambda^a T_a \psi, \delta\bar{\psi} = \bar{\psi}\lambda^a T_a \gamma_5$ in the path integral while keeping V_μ fixed yields two terms: a term conatining the variation of the action and a term containing the Jacobian for this change of integration variables. Since we kept V_μ fixed, the path integral $Z[V]$ is the same before and after the change of integration variables. Thus the sum of all terms linear in λ must vanish. The variation of the action contains the covariant derivative of the axial current, where the axial current $j^\mu_{A,a} = \bar{\psi}T_a\gamma_5\gamma^\mu\psi$ is a hermitian composite operator. Thus the anomaly is given by the Jacobian

$$D_\mu(V) j_A{}^\mu = 2iTrT_a\gamma_5\delta(x-y) \tag{6.10.6}$$

Suppose one chooses the following regulator both for ψ and for $\bar{\psi}$

$$R_\psi = R_{\bar{\psi}} = \not{D}\not{D}, \not{D} = \gamma^\mu(\partial_\mu + eV_\mu) \tag{6.10.7}$$

where $V_\mu = V_\mu{}^a T_a$ with $V_\mu{}^a$ real and γ^μ hermitian for all $\mu = 0, 3$ while also γ_5 is hermitian. Then the anomaly is given by

$$An = 2i \lim_{M\to\infty} TrT_a\gamma_5 e^{\not{D}\not{D}/M^2} = \frac{ie^2}{16\pi^2} trT_a F_{\mu\nu}(V)F_{\rho\sigma}(V)\epsilon^{\mu\nu\rho\sigma} \tag{6.10.8}$$

Since there are only vector gauge fields in this example, the anomaly is (vector-gauge) covariant and satisfies the (vector-gauge) consistency conditions.

As a less trivial example, consider next a Dirac field coupled both to a nonabelian vector field and a nonabelian axial vector field, again in Minkowski space [52]

$$\mathcal{L} = -\bar{\psi}\gamma^\mu(\partial_\mu + eV_\mu + gA_\mu\gamma_5)\psi \tag{6.10.9}$$

Although the action is hermitian, the operator $\not{D} = \not{\partial} + e\not{V} + g\not{A}\gamma_5$ is **not** hermitian: $\not{D}^\dagger \neq \not{D}$. In fact, $\not{D}^\dagger$ is equal to $\not{D}$ with the sign in front of the $\not{A}\gamma_5$ term changed. We can now think of various regulators

$$\begin{array}{llll} R_\psi = \not{D}\not{D} & , \quad R_{\bar{\psi}} = \not{D}\not{D} & (a) \\ R_\psi = \not{D}^\dagger\not{D} & , \quad R_{\bar{\psi}} = \not{D}\not{D}^\dagger & (b) \\ R_\psi = \not{D}\not{D} & , \quad R_{\bar{\psi}} = \not{D}^\dagger\not{D}^\dagger & (c) \\ R_\psi = \not{D}^\dagger\not{D} + \not{D}\not{D}^\dagger & , \quad R_{\bar{\psi}} = \not{D}^\dagger\not{D} + \not{D}\not{D}^\dagger & (d) \end{array} \tag{6.10.10}$$

To this list one may add the regulator [40]

$$R = \not{\partial}\not{D}(W_L)P_L + \not{D}(W_L)\not{\partial}P_R \tag{6.10.11}$$

which yields the consistent anomaly for left-handed chiral fermions, as we already saw. Again one can think of other regulators for chiral fermions. Explicit calculation reveals that the regulators in (6.10.10) yield the following anomalies

a) yields the consistent axial vector (Bardeen) anomaly after adding a local counter term ΔS which is vector gauge invariant and whose axial vector gauge variation removes the "normal parity terms" in the axial vector anomaly (these are the terms with an odd number of A fields and without ϵ symbol, for example, a term $F_{\mu\nu}(V)F^{\mu\nu}(A)$). It clearly yields no vector gauge anomaly (either before or after adding ΔS since ΔS is vector gauge invariant).

b) yields the covariant axial vector anomaly. By writing $\not{D} = \not{D}(W_L)P_L + \not{D}(W_R)P_R$ with $W_L = V + gA$ and $W_R = eV - gA$, and $\not{D}^\dagger = \not{D}(W_L)P_R + \not{D}(W_R)P_L$, we obtain

$$\begin{aligned} \not{D}^\dagger\not{D} &= \not{D}(W_L)^2P_L + \not{D}(W_R)^2P_R \\ \not{D}\not{D}^\dagger &= \not{D}(W_L)^2P_R + \not{D}(W_R)^2P_L \end{aligned} \tag{6.10.12}$$

and hence

$$\begin{aligned} e^{\not{D}^\dagger \not{D}/M^2} &= e^{\not{D}(W_L)^2}P_L + e^{\not{D}(W_R)^2}P_R \\ e^{\not{D}\not{D}^\dagger/M^2} &= e^{\not{D}(W_L)^2}P_R + e^{\not{D}(W_R)^2}P_L \end{aligned} \tag{6.10.13}$$

The axial vector anomaly becomes then

$$\begin{aligned} An(\gamma_5) &= iTrT_a\gamma_5(e^{\not{D}(W_L)^2} + e^{\not{D}(W_R)^2}) \\ &= \frac{1}{32\pi^2} trT_a(F_{\mu\nu}(W_L)F_{\rho\sigma}(W_L) + F_{\mu\nu}(W_R)F_{\rho\sigma}(W_R))e^{\mu\nu\rho\sigma} \end{aligned} \tag{6.10.14}$$

It contains no terms with an odd number of A fields. Note that this regulator also yields a vector gauge anomaly

$$\begin{aligned} An(I) &= iTrT^a(e^{\not{D}^\dagger\not{D}/M^2} - e^{\not{D}\not{D}^\dagger/M^2}) = iTrT^a\gamma_5(e^{\not{D}^2|W_L)} - e^{\not{D}(W_R)^2}) \\ &= \frac{i}{32\pi^2} tr(F_{\mu\nu}(W_L)F_{\rho\sigma}(W_L) - F_{\mu\nu}(W_R)F_{\rho\sigma}(W_R))\epsilon^{\mu\nu\rho\sigma} \end{aligned} \tag{6.10.15}$$

which only contains terms with an odd number of A fields. Both anomalies are covariant under $\delta W_L = d\Lambda_L + [W_L, \Lambda_L]$ and $\delta W_R = d\Lambda_R + [W_R, \Lambda_L]$. In terms of the original gauge fields V_μ and A_μ and parameters $\Lambda_L = eV_\Lambda + g\Lambda_A$ and $\Lambda_R = e\Lambda_V - g\Lambda_A$ one finds

$$\begin{aligned} \delta V_\mu &= \partial_\mu \Lambda_V + e\,[V_\mu, \Lambda_V] + \frac{g^2}{e}\,[A_\mu, \Lambda_A] \\ \delta A_\mu &= \partial_\mu \Lambda_A + e\,[V_\mu, \Lambda_A] + e\,[A_\mu, \Lambda_V] \end{aligned} \tag{6.10.16}$$

and

$$\begin{aligned} F^2_{\mu\nu}(W_L) + F^2_{\mu\nu}(W_R) &= e^2 F^2_{\mu\nu}(V) + g^2 F^2_{\mu\nu}(A) \\ F_{\mu\nu}(V) &= \partial_\mu V_\nu - \partial_\nu V_\mu + e\,[V_\mu, V_\nu] + \frac{g^2}{e}\,[A_\mu, A_\nu] \\ F_{\mu\nu}(A) &= \partial_\mu A_\nu - \partial_\nu A_\mu + e\,[V_\mu, A_\nu] + e\,[A_\mu, V_\nu] \end{aligned} \tag{6.10.17}$$

(Can a local counter term remove $An(I)$ or $An(\gamma_5)$?).

c) yields the Bardeen anomaly, without having to add a local counter term to remove

terms which are odd in ${A_\mu}^a$ (such terms clearly cancel if one uses this regulator). However, this regulator yields a vector gauge anomaly which one could cancel by a local counter term which is axial-vector gauge invariant.
d) yields clearly no vector gauge anomaly. Using

$$e^{\not{D}^\dagger \not{D} + \not{D}\not{D}^\dagger} = e^{\not{D}(W_L)^2 + \not{D}(W_R)^2} \tag{6.10.18}$$

it yields a complicated result for the axial vector anomaly.

Before presenting an algorithm which selects those regulators which yield consistent anomalies, we note that if one were to use canonical quantization of fermions in Minkowski spacetime, $p(\psi)$ would be equal to $\psi^\dagger$, and hence one might wish to consider as measure

$$d\mu = d\psi d\psi^\dagger \tag{6.10.19}$$

instead of $d\psi d\bar{\psi}$. Then the kinetic operator in the action would be $\mathcal{D} = i\gamma^0 \not{D}$ and the covariant regulator (b) would become

$$R_\psi = \mathcal{D}^\dagger \mathcal{D} = \not{D}\not{D}; R_{\bar{\psi}} = \mathcal{D}\mathcal{D}^\dagger = i\gamma^0 \not{D}\not{D} i\gamma^0 \tag{6.10.20}$$

However, also the Jacobians would obtain extra factors γ^0 since $\delta\psi^\dagger = \bar{\psi} i\gamma_5 i\gamma^0 = \psi^\dagger i\gamma_0 (i\gamma_5) i\gamma^0$ and all factors $i\gamma^0$ cancel in

$$Tr i\gamma_0 i\gamma_5 i\gamma_0 e^{i\gamma^0 \not{D}\not{D} i\gamma^0} = Tr i\gamma_0 i\gamma_5 i\gamma_0 (i\gamma_0 e^{\not{D}\not{D}} i\gamma_0) \tag{6.10.21}$$

(since $i\gamma^0 = (i\gamma^0)^{-1}$, the regulator can be written as $i\gamma^0 e^{\not{D}\not{D}} i\gamma^0$ and the similarity transformation with the matrix $i\gamma^0$ cancels in the trace due to cyclicity). Thus, for the covariant regulator it makes no difference whether one uses $d\psi d\bar{\psi}$ or $d\psi d\psi^\dagger$ as measure.

We now derive an algorithm for the construction of consistent regulators. [1] A consistent regulator must in some way be related to the action, because the anomaly is directly obtained from the effective action, and the latter is directly obtained from the action. To regulate one-loop problems involving fermions, Pauli-Villars regularization leads to $(1 + R/M^2)^{-1}$ as regulator as we found before. Since

$$\frac{1}{1 + R/M^2} = \int_0^\infty e^{-\lambda(R/M^2+1)} d\lambda \tag{6.10.22}$$

we see that the Pauli-Villars regulator is a kind of average of the Fujikawa regulator.

Let the original action and the action for the Pauli-Villars field be written as

$$\mathcal{L} = -\frac{1}{2}\phi^T T\mathcal{O}\phi - \frac{1}{2}\chi^T T\mathcal{O}\chi - \frac{1}{2}M\chi^T T\chi \tag{6.10.23}$$

where $T\mathcal{O}$ and T are constant "symmetric" matrices (in the graded sense: antisymmetric for fermions). The action for ϕ is obtained from the full nonlinear action for ϕ by expanding to second order in ϕ (possibly about a background). For fermions this is not necessary since their action is (usually) quadratic in fermion fields. For fermions, ϕ is a column vector which contains both ψ and $\bar{\psi}^T$. Let the action be invariant for $M = 0$ under

$$\delta_\lambda \phi^a = K^a(\lambda, \phi), \delta_\lambda \chi^a = K'^a{}_b(\lambda, \phi)\chi^b \tag{6.10.24}$$

where the prime indicates differentiation w.r.t. ϕ. For linear symmetries the transformation law $K'^a{}_b(\lambda, \phi)\chi^b$ is equal to $K^a(\lambda, \chi)$, but for nonlinear symmetries (for example $\delta_B c^a = -\frac{1}{2} f^a{}_{bc} c^b c^c$) the transformation rules for ϕ^a and χ^a are different but give the same Jacobian.

It is not always true that these rules are a symmetry of $\mathcal{L}(\phi) + \mathcal{L}(\chi)$, and we shall later give an example where things are more complicated. The measure $d\phi d\chi$ is invariant since the regulated Jacobian of χ gets an extra minus sign w.r.t. that of ϕ, hence the anomaly is only due to the possible noninvariance of the mass term,

$$\delta\mathcal{L}_M = \frac{1}{2}M\chi^T(TK' + K'^T T)\chi \tag{6.10.25}$$

Following the same steps as in section 3 we obtain for the anomaly

$$An \;=\; \frac{1}{2}MTr(TK' + K'^T T)\frac{1}{T\mathcal{O} + TM}$$

$$
\begin{aligned}
&= \frac{1}{2} M Tr(K' + T^{-1}K'^T T)\frac{1}{M+\mathcal{O}} \\
&= \frac{1}{2} M Tr(K' + T^{-1}K'^T T)\frac{(M-\mathcal{O})}{M^2-\mathcal{O}^2} \qquad (6.10.26)
\end{aligned}
$$

For fermions, $\mathcal{O}$ is odd in the number of Dirac matrices (for example, $\mathcal{O} = \not{D}$), and one may drop the $\mathcal{O}$ in the numerator. Hence, we find as regulator

$$R = \mathcal{O}^2 \qquad (6.10.27)$$

In more general cases one should include the factor $1 - \mathcal{O}/M$ in the definition of the regularized Jacobian.

The contribution from the term $T^{-1}K'^T T$ doubles that of K' as one verifies by transposing $K'^T T/(T\mathcal{O} + TM)$ and using cyclicity of the trace. Hence, as Jacobian we may use the naively expected result

$$J = K' \qquad (6.10.28)$$

We note that this Pauli-Villars (PV) regulator has the following properties (i) the coupling of the PV fields is the same as the original fields (except for the mass term). This regulates the one-loop graphs.

(ii) The PV action with $M = 0$ should be invariant under the same set of symmetries as the original action. This guarantees that the anomaly will only be due to the noninvariance of the mass term.

(iii) The PV fields χ should have the same statistics as the fields ϕ, but χ loops require an extra minus sign. For a real commuting scalar field φ, the action $\partial_\mu\chi\partial^\mu\chi$ with real anticommuting χ is a total derivative, hence we must require χ to be commuting, but on the other hand, loops need an extra minus sign to regulate φ loops. This means that the χ integration is not the ordinary Berezin integral, rather it gives the inverse result. There is no problem with consistency since we do not couple the PV fields to external sources and then complete squares, hence we do not need translational invariance of the χ "integration" (which would require Berezinian integration. Whether or not we

call this χ prescription an integration is a matter of terminology).
(iv) The measure should be gauge- or BRST invariant. This is the fundamental principle of PV regularization in path integrals, and forces us to define $\delta\chi^a$ as the derivative of $\delta\varphi^a$.
(v) The mass term should be quadratic in quantum fields and when it is invariant under certain "preferred" symmetries, the corresponding regulator does not produce anomalies in these symmetries. If the mass term contains in addition to the two quantum fields further background fields, one should redefine the quantum fields such that the mass term contains only these two-quantum fields. For example, a mass term $m\sqrt{g}\chi\chi$ in curved space selects $\tilde{\varphi} = g^{1/4}\varphi$ and $\tilde{\chi} = g^{1/4}\chi$ as the correct path integral variables, but if the mass term is $m\chi\chi$, then χ is the correct path integral variable.
(vi) Sometimes one needs more PV fields than ϕ fields, but the couplings should be the same. For example, for chiral fermions, one needs both χ_L and χ_R, when only ψ_L is present.

Let us give a few examples. For a Dirac fermion ψ with $\mathcal{L} = -\bar{\psi}(\not{\partial} + \not{V} + \not{A}\gamma_5)\psi$ and PV mass term $M\bar{\chi}\chi$, we find on the basis $\phi = \{\psi, \bar{\psi}^T\}$

$$T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, T\mathcal{O} = \begin{pmatrix} \not{\emptyset} & -\overleftarrow{\not{\partial}}^T - \not{V}^T - \gamma_5^T \not{A}^T \\ \not{\partial} + \not{V} + \not{A}\gamma_5 & \not{\emptyset} \end{pmatrix} \tag{6.10.29}$$

where T denotes transposition in spinor and internal space, while $J_\psi{}^A = \gamma_5$ and $J_{\bar{\psi}^T} = \gamma_5{}^T$. The regulator $\mathcal{O}^2$ is then

$$R = \begin{pmatrix} \not{D}^2 & 0 \\ 0 & (\not{D}^T)^2 \end{pmatrix} \tag{6.10.30}$$

Taking the transpose in the second trace, one finds that the ψ and $\bar{\psi}$ trace yield equal contributions, and the common regulator $\not{D}\not{D}$ is indeed the consistent regulator.

Although we do not discuss curved space in detail in this book, let us make an exception at this point because this sheds further light on anomalies and regulation. The same results which one derives for vector and axial vector symmetries hold for

Einstein (= general coordinate) and local Lorentz symmetries. One can define consistent and covariant anomalies, consistency conditons and a Wess-Zumino term. It is known that one can always add a local counter term such that Einstein symmetry is preserved; then the anomaly, if at all present, resides in the local Lorentz symmetry. This corresponds to choosing as PV mass term $\sqrt{g}\chi_L^\dagger\chi_L$. This mass term is Einstein invariant but breaks local Lorentz symmetry in general, hence choosing this mass terms the corresponding Fujikawa regulator preserves Einstein symmetry. Let us call these anomalies gravitational anomalies. In 4 dimensions they contain the linearized part of $\epsilon^{\mu\nu\rho\sigma}R_{\mu\nu}{}^{ab}R_{\rho\sigma}{}^{ab}$ where $R_{\mu\nu}{}^{ab}$ is the Riemann curvature in terms of the spin connection (the gauge field for local Lorentz symmetry).

We first consider field theories in d-dimensional Minkowski space. It is known that in $d = 4k$ there are no gravitational anomalies, but in $d = 4k + 2$ there are. We can easily see why, using our PV formalism. In $d = 4 + 8k$ one can take as mass term $M\sqrt{g}\chi_L{}^T C\chi_L$ because the charge conjugation matrix is antisymmetric in $d = 4 + 8k$. Thus there are no gravitational anomalies in $d = 4 + 8k$. In $d = 8k$ dimensions C is symmetric but it is diagonal in chiralities, so one can take two PV fields (each with a factor 1/2 added to the loop) and use as mass term $M\sqrt{g}\chi_1{}^T C\chi_2$. This shows that also $d = 8k$ there are no gravitational anomalies. For chiral spinors in $d = 4k + 2$, the mass term $\sqrt{g}\chi_L^\dagger\chi_L$ breaks local Lorentz invariance, and since no Einstein and local Lorentz invariant mass term can be constructed, this suggests (as confirmed by explicit calculations) that in these dimensions gravitational anomalies do exist.

A last example concerns two dimensional gravity, in particular the ghost action

$$\mathcal{L} = b^{\alpha\beta}(c^\gamma\partial_\gamma g_{\alpha\beta} + g_{\gamma\alpha}\partial_\beta c^\gamma + g_{\gamma\beta}\partial_\alpha c^\gamma + cg_{\alpha\beta}) + d^{\alpha\beta}(g_{\alpha\beta} - g_{\alpha\beta}{}^{(0)}) \qquad (6.10.31)$$

where $c^\alpha(\alpha = 1, 2)$ is the ghost field for Einstein symmetry, c is the ghost field for local scale (Weyl) symmetry, and $b^{\alpha\beta} = b^{\beta\alpha}$ is the antighost field for both. All these fields are anticommuting. The fields $g_{\alpha\beta}{}^{(0)}$ are the background fields, and in the path integral we integrate over $dc^\alpha db^{\alpha\beta}dg_{\alpha\beta}dd^{\alpha\beta}$. The PV action is the same but with c^γ

replaced by C^γ, c by C and $b^{\alpha\beta}$ by $B^{\alpha\beta}$.

$$\mathcal{L}_{PV} = B^{\alpha\beta}\left(C^\gamma \partial_\gamma g_{\alpha\beta} + \partial_\alpha C^\gamma g_{\gamma\beta} + \partial_B C^\gamma g_{\alpha\gamma} + C g_{\alpha\beta}\right) \tag{6.10.32}$$

As mass term we choose

$$\mathcal{L}(M) = \frac{1}{2} M_1 \det g \; \epsilon_{\alpha\beta} C^\alpha C^\beta + \frac{1}{2} M_2 B^{\alpha\beta} B^{\gamma\delta} g_{\alpha\gamma} \epsilon_{\beta\delta} \tag{6.10.33}$$

Clearly C^α and $B^{\alpha\beta}$ should have the same statistics as c^α and $b^{\alpha\beta}$ in order that these mass terms do not vanish. The fields ${g_{\alpha\beta}}^{(0)}$ are the background fields (like V_μ and A_μ), and the symmetry is BRST symmetry, given by

$$\begin{aligned} \delta c^\alpha &= -c^\beta \partial_\beta c^\alpha \Lambda; \delta c = -c^\alpha \partial_\alpha c \Lambda \\ \delta b^{\alpha\beta} &= d^{\alpha\beta}\Lambda, \delta d^{\alpha\beta} = 0 \\ \delta g_{\alpha\beta} &= (c^\gamma \partial_\gamma g_{\alpha\beta} + g_{\gamma\alpha}\partial_\beta c^\gamma + g_{\gamma\beta}\partial_\alpha c^\gamma + c g_{\alpha\beta})\Lambda \end{aligned} \tag{6.10.34}$$

These transformation rules are nilpotent and leave (6.10.31) invariant.

The rules for C^α and C agree with $\delta\chi = K'\chi$ but the rule for $B^{\alpha\beta}$ is not given by $\delta\chi = K'\chi$, but rather it follows by requiring invariance of the PV action.

$$\begin{aligned} \delta C^\alpha &= -(C^\beta \partial_\beta C^\alpha + C^\beta \partial_\beta c^\alpha)\Lambda \\ \delta C &= -(c^\beta \partial_\beta C + C^\beta \partial_\beta c)\Lambda \\ \delta B^{\alpha\beta} &= (\partial_\gamma (B^{\alpha\beta} c^\gamma) + (\partial_\gamma c^\alpha) B^{\gamma\beta} + (\partial_\gamma c^\beta) B^{\alpha\gamma} - B^{\alpha\beta} c)\Lambda \end{aligned} \tag{6.10.35}$$

Also these transformation rules are nilpotent. We note that C^α and $B^{\alpha\beta}$ transform as a vector and tensor density in general relativity with parameter $\xi^\alpha = c^\alpha\Lambda$, and the Jacobians for c^α and c cancel those for C^α and C. Thus the mass term is BRST invariant (as one may also check directly). However, in the sector of antighosts, auxiliary field and metric a different cancellation mechanism of Jacobian is at work. The antighost and the auxiliary field have vanishing Jacobian (for regulators which do not mix them) while the Jacobian for $B^{\alpha\beta}$ is nonvanishing. Furthermore, there is no PV counterpart for $g_{\alpha\beta}$ since $g_{\alpha\beta}$ appears the same way in the PV ghost action as in the original ghost action. One may check, however, that the regularized Jacobians of $g_{\alpha\beta}$ and $B^{\alpha\beta}$ cancel each other.

Since the measure is invariant, we can go on and construct the consistent regulator for this system. To remove the fields $g_{\alpha\beta}$ and $\sqrt{g}$ from the mass term, we introduce vielbein fields $e_\alpha{}^a$ satisfying $e_\alpha{}^a e_\beta{}^b \eta_{ab} = g_{\alpha\beta}$. Then

$$\begin{aligned}\mathcal{L}(M) &= \frac{1}{2} M \epsilon_{\alpha\beta} \tilde{C}^\alpha \tilde{C}^\beta + \frac{1}{2} M_2 \tilde{B}^{\alpha a} \tilde{B}^{b\beta} \eta_{ab} \epsilon_{\alpha\beta} \\ \tilde{C}^\alpha &= g^{1/2} C^\alpha ; \tilde{B}^{\alpha a} = e^a{}_\beta B^{\alpha\beta}.\end{aligned} \tag{6.10.36}$$

We prefer to introduce fields with all indces flat $\tilde{C}^a = g^{1/4} e_\alpha{}^a C^\alpha$ or $\tilde{B}^{ab} = e_\alpha{}^a e_\beta{}^b g^{-1/4} B^{\alpha\beta}$).

Integrating over c and C in the path integral leads to the constraints that $b^{\alpha\beta}$ and $B^{\alpha\beta}$ are traceless fields with indices $+$ and $-$, defined by $C^\pm = C^1 \pm C^2$ in Minkowski space. The matrix T then becomes on the basis C^+, C^-, B^{++}, B^{--}

$$MT = \begin{pmatrix} 0 & M_1 & & \\ -M_1 & 0 & & \ominus \\ & \ominus & 0 & M_2 \\ & & -M_2 & 0 \end{pmatrix} \tag{6.10.37}$$

while from the action on the new basis

$$\mathcal{L} = \tilde{B}^{ab} e_a{}^\alpha e_b^\beta (g^{-1/4} e_c{}^\gamma \tilde{C}^c \partial_\gamma g_{\alpha\beta} + \ldots) \tag{6.10.38}$$

we read off what $T\mathcal{O}$ is. Then the regulators for the fields $\tilde{C}^+, \tilde{C}^-, \tilde{B}^{++}, \tilde{B}^{--}$ become then

$$\begin{aligned} R(\tilde{C}^-) &= D_+{}^\dagger D_-, R(\tilde{C}^-) = D_-{}^\dagger D_+ \\ R(\tilde{B}^{++}) &= D_- D_+{}^\dagger, R(\tilde{B}^{--}) = D_+ D_-{}^\dagger \end{aligned} \tag{6.10.39}$$

where in the conformal gauge $g_{\alpha\beta}{}^{(0)} = \eta_{\alpha\beta}\rho(x)$ one has $D_\pm = \rho^{1/2}\partial_{a\pm}\rho^{-1}$ and $D_\pm{}^\dagger = \rho^{-1}\partial_\pm \rho^{1/2}$. It has indeed been found that the anomalies computed with this regulator are consistent (check).

The consistent regulator follows in a well-defined way from the action but the examples have illustrated that the form of the consistent regulator may depend on the theory one considers. In four dimensional gauge theories, $R = \not{D}\not{D}$ but not $\not{D}^\dagger\not{D}$, while in two dimensional gravitational theories the consistent regulator involves both ${D_\pm}^\dagger$ and $D_\pm$.

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Chapter 7

The background field method

In ordinary quantum gauge field theories, the classical gauge invariance is broken when a gauge fixing term is added. Although there remains a rigid BRST symmetry, and corresponding Ward identities for the connected and proper Green's functions can be derived, the solution of these Ward identities are not gauge invariant. It would be both technically and also conceptually advantageous to keep gauge invariance at the quantum level. As far as is known, this is not possible with one gauge field. However, there is a way to keep classical gauge invariance at the quantum level, namely by splitting the gauge field into two gauge fields, a background gauge field ${A_\mu}^a$ and a quantum gauge field ${Q_\mu}^a$, and to fix the gauge of the latter but not of the former. The corresponding formalism is called the background field formalism, and was introduced forty years ago at the one-loop level by DeWitt. [1] For applications and further aspects at the one-loop level, see [2]. The multiloop case was treated in [3, 4]. We follow here the presentation of Abbott [4].

The background field formalism yields an effective action which still has background gauge invariance. This is its main technical advantage. As a consequence, as we shall discuss, the renormalization constant Z_g of the gauge coupling constant g is the inverse of the square root of the wave function renormalization constant Z_A of the background gauge field A_μ^a, namely $Z_g Z_A^{1/2} = 1$. This makes the calculation of the

β function in Yang-Mills theories much simpler because one only needs to evaluate 2-point (selfenergy) graphs but not 3-point graphs (vertex corrections). A conceptual advantage is that one can choose different gauge fixing terms for the effective action (proper graphs) and for the propagators in the tree graphs which have proper graphs as vertices. In section 6 we discuss some areas where the background field method has been used with success.

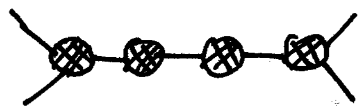


Figure caption: Amputated (and non-amputated) Green functions can be written as tree graphs with proper vertices.

The basic idea of the background field formalism is to split the classical gauge field into a sum ${A_\mu}^a + {Q_\mu}^a$ and to write the classical gauge action as $\mathcal{L}_{YM}({A_\mu}^a + {Q_\mu}^a)$.[1] Since one treats ${A_\mu}^a$ as a classical field, one has not doubled the number of degrees of freedom. With such a special classical action, one can maintain gauge invariance at the quantum level as we now show.

The classical gauge invariance for the field $A^a_\mu + Q^a_\mu$

$$\delta(\text{gauge})({A_\mu}^a + {Q_\mu}^a) = \partial_\mu \lambda^a + g {f^a}_{bc}({A_\mu}^b + {Q_\mu}^b)\lambda^c \equiv D_\mu(A+Q)\lambda^a \tag{7.0.1}$$

can be decomposed in two ways into separate transformation rules for ${A_\mu}^a$ and ${Q_\mu}^a$. First, one may define background gauge transformations

$$\left.\begin{array}{l} \delta\ (\text{back})\ {A_\mu}^a = D_\mu(A)\lambda^a \\ \delta\ (\text{back})\ {Q_\mu}^a = g{f^a}_{bc}{Q_\mu}^b\lambda^c \end{array}\right\} \begin{array}{l}\text{background} \\ \text{gauge invariance}\end{array} \tag{7.0.2}$$

[1]The starting point of taking the classical action with $A_\mu + Q_\mu$ is fairly obvious for ordinary gauge theories. However, in some cases less obvious dependence of the classical action on A_μ and Q_μ occurs. An example is superspace Yang Mills theory where one considers products of exponentials depending on A_μ and Q_μ. In such cases one may prove (and derive) the uniqueness of the starting point as follows. One begins with the usual classical action $S_{cl}(Q)$ without background fields and adds a Q- and A-dependent gauge fixing term whose precise form is left free for the moment. Then one imposes the requirements of BRST invariance ($\{\Gamma, \Gamma\} = 0$) and background gauge invariance ($W\Gamma = 0$). The solution for Γ of these equations can then be shown to be equivalent to starting with the Q and A dependent classical action after a suitable redefinition of Q. For ordinary gauge theories this means that the starting point $S_{cl}(A+Q)$ is "stable" (meaning that it is unique). [5]

Here ${Q_\mu}^a$ transforms as a vector in the adjoint representation, not as a connection, while ${A_\mu}^a$ is a connection. From this point of view one considers A^a_μ as "the" gauge field and Q^a_μ as some kind of matter field. Because the sum of a connection and a vector field is again a connection, the field $A^a_\mu + Q^a_\mu$ is a connection. Thus the Yang-Mills actions $\mathcal{L}_{cl}(A)$ and $\mathcal{L}_{cl}(A+Q)$ are both background gauge invariant. For future use we draw the reader's attention to the fact that if ${Q_\mu}^a = 0$ also $\delta(\text{back}){Q_\mu}^a = 0$.

Second, one may decompose the classical gauge transformations into $\delta {A_\mu}^a = 0$ and $\delta {Q_\mu}^a = D_\mu(A+Q)\lambda^a$. These symmetries may be called quantum gauge transformations and will be gauge fixed. From this point of view one considers ${A_\mu}^a$ as an infinite set of constants (for example by expanding ${A_\mu}^a$ into a suitable complete set of functions), on the same footing as the gauge parameter ξ. Different background fields ${A_\mu}^a$ then correspond to a different set of constants, and one is still in the realm of ordinary field theory. There is really only one quantum field, ${Q_\mu}^a$, and all standard manipulations and BRST methods can be applied to ${Q_\mu}^a$. The corresponding BRST transformations are obtained as usual, by replacing λ^a by $c^a\Lambda$, and read

$$\left.\begin{array}{l} \delta_B {A_\mu}^a = 0 \\ \delta_B {Q_\mu}^a = D_\mu(A+Q)c^a\Lambda \end{array}\right\} BRST \text{ symmetry} \tag{7.0.3}$$

The main idea of the background field formalism is that one can find gauge fixing terms which fix the quantum gauge invariance but preserve the background gauge invariance. The most used gauge fixing term for nonabelian gauge theories is[2]

$$\mathcal{L}\ (\text{fix})\ = -\frac{1}{2\xi}\left(D^\mu(A){Q_\mu}^a\right)^2 \tag{7.0.4}$$

It preserves rigid Lorentz symmetry, rigid group symmetry, is power counting renormalizable, but unlike the corresponding $(\partial^\mu {Q_\mu}^a)^2$, it is also background gauge invariant. (The covariant derivative $D_\mu(A)Q^{\mu a}$ of the vector $Q^{\mu a}$ is again a vector). Note that $D^\mu(A)Q_\mu = D^\mu(A+Q)Q_\mu$; this will be used repeatedly. One can also

[2]For gravity the best choice is the covariant De Donder gauge $\sqrt{g}F_\mu F_\nu g^{\mu\nu}$ with $F^\nu = D_\mu(g)(h^{\mu\nu} - \frac{1}{2}g^{\mu\nu}g_{\rho\sigma}h^{\rho\sigma})$ where $g_{\mu\nu}$ is the background metric and $h^{\mu\nu}$ the quantum field. [6]

use other background-gauge-invariant gauge fixing terms, for example the Lorentz covariant $m^2(Q_\mu^a)^2$ or the Lorentz-noncovariant $(n^\mu Q_\mu^a)^2$ where n^μ is a constant vector. The propagator of the former is the same as for a massive gauge field, hence not power-counting renormalizable because it contains a term $k_\mu k_\nu/m^2$ in the numerator. Furthermore, no complete proofs of renormalizability exists for these noncovariant gauges, so one is really restricted to (7.0.4). Following the usual BRST construction, the ghost action is obtained from the gauge fixing term by applying a BRST transformation to Q_μ

$$\mathcal{L}(\text{ghost}) = b_a D^\mu(A) D_\mu(A+Q) c^a \ . \tag{7.0.5}$$

Note that the ghost action contains now two covariant derivatives, while in ordinary field theory it contains one ordinary and one covariant derivative. By construction the quantum action is now BRST invariant, but it is also background gauge invariant if one defines b_a and c^a to transform as vectors in the adjoint representation

$$\delta\ (\text{back})\ b^a = g f^a{}_{bc} b^b \lambda^c\ ;\ \delta\ (\text{back})\ c^a = g f^a{}_{bc} c^b \lambda^c \tag{7.0.6}$$

Also the source terms for the BRST variations of $Q_\mu{}^a$ and c^a

$$\mathcal{L}\ (\text{extra})\ = K_a{}^\mu D_\mu(A+Q) c^a + L_a \frac{1}{2} g f^a{}_{bc} c^b c^c \tag{7.0.7}$$

are BRST and background gauge invariant if $K_a{}^\mu$ and L_a transform as vectors under background gauge transformations.[3] In this way one arrives at a path integral Z which depends on the usual sources and on a background field $A_\mu{}^a$

$$\begin{aligned} &Z[J_a{}^\mu, \beta_a, \gamma^a, K_a{}^\mu, L_a, A_\mu{}^a] = \int dQ_\mu{}^a db_a dc^a e^{\frac{i}{\hbar}(S+S\ (\text{source}))} \\ &S = S_{YM}(A+Q) + S(\text{fix}) + S(\text{ghost}) + S(\text{extra}) \\ &S(\text{source}) = \int (J_a{}^\mu Q_\mu{}^a + \beta_a c^a + b_a \gamma^a) d^4x \end{aligned} \tag{7.0.8}$$

[3]Note that also $K_a^\mu D_\mu(A) c^a$ is background gauge invariant, but we must couple K_a^μ to the BRST variation of Q_μ^a and the latter is $D_\mu(A+Q) c^a \Lambda$ and not $D_\mu(A) c^a \Lambda$.

The terms in S (source) are also background gauge invariant if the sources J_a^μ, β_a and γ^a transform as vectors. In the absence of chiral fermions or ϵ symbols, the quantum action is both background gauge invariant and BRST invariant in n dimensions. Then the renormalized effective action will be both background gauge invariant, and it will satisfy the BRST Ward identities without anomalous extra terms on the right-hand side of the $(\hat\Gamma, \hat\Gamma)$ equation (where $\hat\Gamma$ is the effective action minus S(fix).)

Note that we only couple the quantum field Q to an external source [3], as usual in quantum field theory. In earlier approaches one sometimes coupled $Q + A$ to J which may seem natural for a theory with $S_{YM}(A + Q)$ and which only differs by a term $J_a{}^\mu A_\mu{}^a$ which one can extract from the path integral. However, coupling J only to the quantum fields is the general procedure for any quantum gauge field theory, and this has become the common procedure.

In these earlier approaches to the background field formalism, not only was the source J_a coupled to $A_\mu{}^a + Q_\mu{}^a$, but in additon the field $A_\mu{}^a$ was supposed to satisfy the classical field equation, $\frac{\delta}{\delta A_\mu{}^a} S_{YM}(A) + J_a{}^\mu = 0$. Hence, in those approaches $A_\mu{}^a$ was a function of $J_\mu{}^a$. The reason for imposing the field equation of $A_\mu{}^a$ was that then the terms in S which are linear in Q canceled. We shall not impose any conditions on $A_\mu{}^a$ or $J_\mu{}^a$. We shall first compute proper Green's functions to which terms linear in Q do not contribute (except in proper one-point functions[4]), and later construct S matrix elements by building tree graphs with the proper Green functions as vertices. In this approach there is no need to get rid of the terms in the action which are linear in Q, and we shall therefore not impose any constraint on $A_\mu{}^a$: it is an independent external source. This greatly simplifies the analysis.

In the next section we begin by discussing the background field method for scalar fields for pedagogical reasons. Then we generalize to gauge theories, and derive the

[4]Proper one-point functions (proper "tadpoles") vanish for vector or spinor fields as a consequence of Lorentz invariance, but for scalar fields they are in general nonvanishing in the presence of background fields, see next section.

connection between the effective action in the background field method and the effective action of a corresponding ordinary quantum field theory. We explicitly demonstrate that the n point proper graphs of the background field method are equal to the n point functions of ordinary quantum field theory in an unusual (but very useful) gauge. [4] In section 2 we show that the S matrix in the background field formalism coincides with the S-matrix in ordinary quantum field theory [17, 18]. In particular we prove, using a Ward identity, that the S matrix is independent of the background field: the S matrix elements with external A^a_μ lines vanish. In section 3 we prove that background gauge field theory is renormalizable. [9] To achieve this, we shall extend the BRST symmetry with an anticommuting external source ${M_\mu}^a$, such that the pair $({A_\mu}^a, {M_\mu}^a)$ forms a contractible pair $(\delta_B {A_\mu}^a = {M_\mu}^a\Lambda, \delta_B {M_\mu}^a = 0)$. In section 4 we find a first application: a simple proof that the β function of gauge theories is independent of the gauge fixing parameter ξ. To obtain this result we again extend the BRST symmetry by introducing a constant anticommuting parameter N, which forms with ξ a contractible pair $(\delta_B\xi = 2\xi N\Lambda, \delta_B N = 0)$. [9–13] In section 5 we give a second application: the two-loop calculation of the beta function of pure Yang-Mills theory. [4] The β function is obtained from the $\frac{1}{\epsilon}$ terms in the two-loop AA selfenergy; no vertex corrections need be calculated in the background field method. We calculate with dimensional regularization all $\frac{1}{\epsilon}$ terms. The fact that only selfenergies but no vertex corrections need be calculated, is a great simplification over ordinary field theory where the calculation of the β function at the two-loop level [14] is quite complicated. This calculation clearly illustrates the main advantage of the background field method: it makes calculations in general much simpler. Finally, we mention in section 6 some further areas where the background field method has been applied.

1 Background gauge invariant effective actions

Consider as an introduction a scalar field. The path integral in the background field formalism is defined by

$$\tilde{Z}[J,A] = \int dQ e^{\frac{i}{\hbar}S(A+Q)+\frac{i}{\hbar}\int JQd^4x} \tag{7.1.1}$$

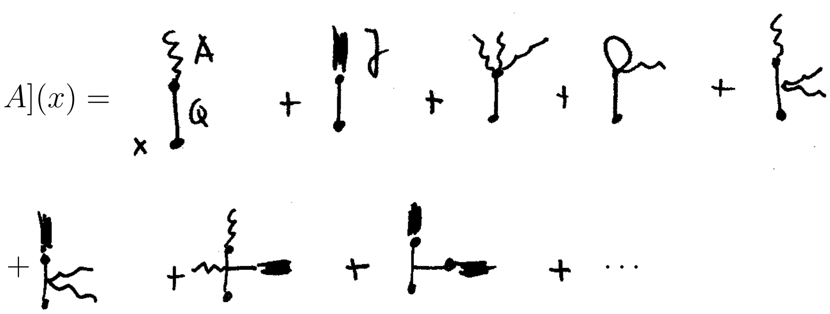
Twiddles denote in this section quantities in the background field formalism. Of course, there is no gauge fixing term or ghost action for scalar fields. We define $\tilde{Z} = \exp \frac{i}{\hbar}\tilde{W}$, and

$$\frac{\delta}{\delta J}\tilde{W}[J,A] =< Q >\equiv \tilde{Q} \tag{7.1.2}$$

Because there are terms in the action which are linear in the quantum field Q, the expression for $\tilde{Q}$ corresponds to graphs with any number of tadpoles (with 1, 2 or 3 background fields A leaving the end of the tadpole). Hence $\tilde{Q}$ is a function of both J and A.

For example, in massive $\lambda\varphi^4$ theory one finds the following diagrammatic result for $\tilde{Q}[J,A](x)$, setting $\varphi = A+Q$, taking $\mathcal{L} = -\frac{1}{2}(\partial_\mu Q)^2 - \frac{1}{2}m^2Q^2$ as kinetic terms which yield the propagator for Q, and treating $\mathcal{L} = -\partial^\mu A\partial_\mu Q - \frac{1}{2}(\partial_\mu A)^2 - m^2AQ - \frac{1}{2}m^2A^2 - \frac{\lambda}{4!}(A+Q)^4 + JQ$ as interactions

$\tilde{Q}[J,A](x) =$



(7.1.3)

Figure caption: Diagrammatic representation of $\tilde{Q} =< Q >$. Wiggly lines indicate A, straight lines Q, and bold lines J.

Note that in ordinary field theory one always imposes the renormalization condition that tadpoles vanish, $Q[J=0]=0$. In the presence of a background field, the expectation value of the quantum field no longer vanishes: $\tilde{Q}[J=0,A] \neq 0$. This is well-known from the theory of solitons where the expectation value of the quantum field acquires a time-independent but space-dependent value [15]: the fluctuations $\hat{Q}(x,t)$ around a classical soliton solution $\varphi_K(x)$ satisfy $<\hat{Q}(x,t)>= \varphi_1(x)$ where $\varphi_1(x)$ can be computed in perturbation theory and only vanishes far away from the soliton. In the background field formalism one has a general background field $A(x,t)$ instead of a time-independent soliton solution, hence there is even more reason that $\tilde{Q}[J=0,A]$ does not vanish. One could call $\tilde{Q}[J=0,A]$ the mean field, but note that it is in general time-dependent if A is time-dependent.

Later we shall set $\tilde{Q}=0$; this implies then a relation between A and J. The effective action is defined by

$$\tilde{\Gamma}[\tilde{Q},A] = \tilde{W}[J,A] - \int J\tilde{Q}d^4x \tag{7.1.4}$$

We could equally well have written everywhere Q instead of $\tilde{Q}$ since $\tilde{Q}$ is a variable, but for purposes of comparison with ordinary quantum field theory we keep the notation $\tilde{Q}$.

To establish the connection with ordinary field theory, we make the shift $Q \to Q - A$ in the integration variable of $\tilde{Z}$. In a careful treatment one should use regularization, but we only assume here that a proper regularization scheme is used at all steps, which preserves both background gauge invariance and BRST symmetry. Then

$$\begin{aligned} \tilde{Z}[J,A] &= \int dQ e^{\frac{i}{\hbar}S(Q)+\frac{i}{\hbar}\int J(Q-A)d^4x} \\ &= Z[J]e^{-\frac{i}{\hbar}\int JAd^4x} \end{aligned} \tag{7.1.5}$$

where $Z[J]$ denotes the path integral of ordinary quantum field theory

$$Z[J] = \int dQ e^{\frac{i}{\hbar}S(Q)+\frac{i}{\hbar}\int JQd^4x} \tag{7.1.6}$$

We furthermore define, as usual in quantum field theory, the generating functionals for connected and proper graphs $Z[J] = \exp\frac{i}{\hbar}W[J]$ and $\Gamma[Q] = W[J] - \int JQd^4x$.

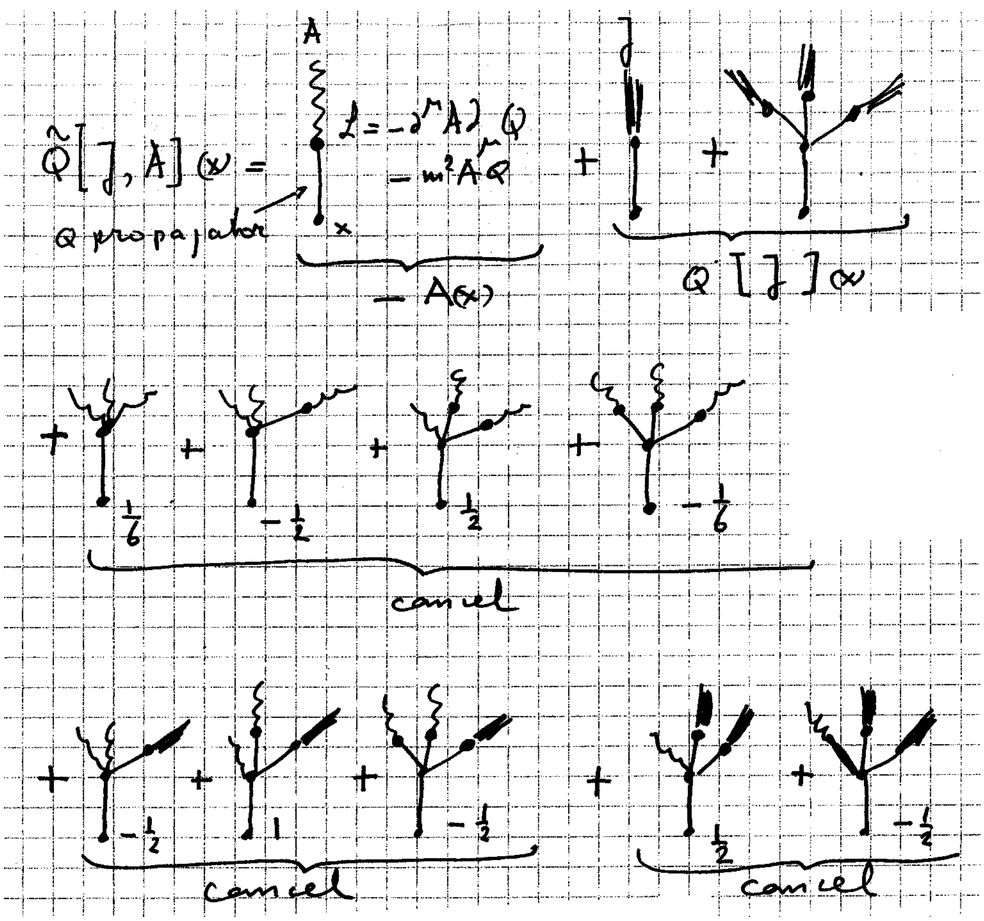
Then one has clearly

$$\tilde{W}[J,A] = W[J] - \int JAd^4x \tag{7.1.7}$$

It follows by differentiating w.r.t. to J that

$$\tilde{Q} = Q - A, Q \equiv \frac{\delta}{\delta J} W[J] \tag{7.1.8}$$

Because $\tilde{Q}$ and Q stand for infinite sets of diagrams as in (**??**), this relation may be viewed as a relation between infinite sets of diagrams. For example, on the left-hand side the one-point function with vertex $-\int \partial^\mu A \partial_\mu Q d^4x$ and propagator $(\Box)^{-1}$ reproduces the term $-A$ on the right-hand side. Each diagram on the left-hand side corresponds to a diagram on the right-hand side, and vice-versa.



(7.1.9)

Figure caption: The relation $\tilde{Q} = Q - A$ to order λ^0 and λ, as a relation between connected Feynman diagrams. At the point x a quantum field $Q(x)$ starts propagating. All A-independent graphs match, whereas A-dependent graphs cancel, except the graph with one Q propagator and one A field, which gives the term -A in the relaiton $\tilde{Q} = Q - A$. It yields the diagrammatic identity  which is used in all cancellations.

Substituting the results for $\tilde{W}$ and $\tilde{Q}$ into (7.1.4) we find

$$\begin{aligned}\tilde{\Gamma}[\tilde{Q}, A] &= (W[J] - \int JAd^4x) - \int J(Q-A)d^4x \\ &= W[J] - \int JQd^4x = \Gamma[Q] = \Gamma[\tilde{Q} + A] \end{aligned} \quad (7.1.10)$$

Hence the effective action of scalar field theory in the background formalism, $\tilde{\Gamma}[\tilde{Q}, A]$, actually only depends on $\tilde{Q} + A$. At $\tilde{Q} = 0$, we find $\tilde{\Gamma}[0, A] = \Gamma[A]$. Setting $\tilde{Q} = 0$ implies that J itself becomes a functional of A, so $J = J[A]$, but because J does not appear in Γ, the relation $\tilde{\Gamma}[0, A] = \Gamma[A]$ is a relation between two infinite sets of proper graphs, both depending on the unconstrained field A. At order $\hbar = 0$ both sets contain $-\frac{1}{2}(\partial_\mu A)^2 - \frac{\lambda}{4!}A^4$. At order $\hbar$ one finds a $\tilde{Q}$ loop with external A fields in $\tilde{\Gamma}[0, A]$ while $\Gamma[A]$ yields the same graphs with an internal Q loop and external A-fields. In both cases the vertices come from the term $\frac{1}{4!}\lambda\varphi^4$. We conclude that differentiating $\tilde{\Gamma}[0, A]$ w.r.t. A gives the proper[5] n-point functions of ordinary quantum field theory.

For gauge theories, there is a difference: $\mathcal{L}(\text{fix})$ is not a function of only the sum of ${A_\mu}^a$ and ${Q_\mu}^a$, see (7.0.4). Going through the same steps we now find

$$\tilde{Z}_{F(A,Q)}[J, A] = \int d{Q_\mu}^a db_a dc^a \exp \frac{i}{\hbar} \int \left[\begin{array}{l} \mathcal{L}_{YM}(A+Q) - \frac{1}{2\xi}(D(A)Q)^2 \\ +bD(A)D(A+Q)c + KD(A+Q)c \\ +L(\frac{1}{2}uc \times c) + JQ + \beta c + b\gamma \end{array} \right] d^4x \quad (7.1.11)$$

where $c \times c$ stands for ${f^a}_{bc}c^b c^c$. The subscript $F(A, Q)$ indicates that the gauge fixing term is $F(A, Q)$, in this case $F(A, Q) = D(A)Q$. Note that we couple again only Q_μ but not $(A_\mu + Q_\mu)$ to J_μ. We do not want to couple J_μ also to A_μ because this would violate background gauge invariance. Shifting the integration variable ${Q_\mu}^a$ to

[5]Proper with respect to $\tilde{Q}$ because we made the Legendre transformation with respect to $\tilde{Q}$. Hence, cutting one $\tilde{Q}$ propagator, the graph does not decompose into two separate graphs.

${Q_\mu}^a - {A_\mu}^a$, we find again a path integral Z for ordinary quantum gauge field theory times a factor $\exp \frac{i}{\hbar} \int(-JA)$, except that now the gauge fixing function is $F(A, Q-A)$, i.e., it is still ${A_\mu}^a$ dependent.

$$Z_{F(A,Q-A)}[J, A] = \int d{Q_\mu}^a db_a dc^a \exp \frac{i}{\hbar} \left[\begin{array}{l} \mathcal{L}_{YM}(Q) - \frac{1}{2\xi}(D(A)Q - \partial A)^2 \\ +bD(A)D(Q)c + KD(Q)c \\ +L(\frac{1}{2}uc \times c) + JQ + \beta c + b\gamma \end{array} \right] d^4x \tag{7.1.12}$$

We used that $D(A)(Q - A) = D(A)Q - \partial A$. After the shift, the field ${Q_\mu}^a$ transforms under BRST transformations as usual in ordinary quantum gauge field theories, and thus we find the derivative $D(Q)$, and not $D(A + Q)$, in the ghost action. Note that there are also terms linear in Q in this action, which contribute to connected but not to 1PI graphs. Hence

$$\begin{aligned} \tilde{Z}_{F(A,Q)}[J, A] &= Z_{F(A,Q-A)}[J, A] e^{-\frac{i}{\hbar}\int JAd^4x} \\ \tilde{W}_{F(A,Q)}[J, A] &= W_{F(A,Q-A)}[J, A] - \int JAd^4x \end{aligned} \tag{7.1.13}$$

Taking the derivative w.r.t. ${J_\mu}^a$ while keeping ${A_\mu}^a$ fixed leads again to

$$\tilde{Q} = Q - A \tag{7.1.14}$$

Of course, both $\tilde{Q}$ and Q depend on ${J_\mu}^a$ and ${A_\mu}^a$, and $\tilde{Q}$ is the expectation value of the quantum field (the set of connected 1-point functions) using $F(A, Q)$ as gauge fixing function, while Q has been calculated using $F(A, Q - A)$.

There is one detail concerning the definition of $\tilde{Z}$ and Z that we should still settle: the usual normalization constant N in front of the path integral which in ordinary quantum gauge field theory is chosen such that Z becomes unity when the external sources K, L, J, β and γ vanish. This removes the vacuum selfenergy graphs from the path integral. It might seem that in the background field formalism the path integrals (Z and $\tilde{Z}$) may still depend on the background field ${A_\mu}^a$ even when we set $K = L = J = \beta = \gamma = 0$. We now show that this is not so

$$\frac{\delta}{\delta A^b_\nu} \tilde{Z}\left[{A_\mu}^a, {K_a}^\mu = L_a = {J_a}^\mu = \beta_a = \gamma^a = 0\right] = 0 \tag{7.1.15}$$

So we can normalize the path integral by an A-independent and K, L, J, β, γ-independent normalization constant. The proof is as follows. If we vary ${A_\mu}^a$ into ${A_\mu}^a + \delta {A_\mu}^a$, and at the same time replace the integration variable ${Q_\mu}^a$ by ${Q_\mu}^a - \delta {A_\mu}^a$, the classical action $S_{YM}(Q+A)$ is clearly invariant, and from the gauge fixing term and ghost action we obtain the following two terms

$$\delta \tilde{Z}\left[{A_\mu}^a, K=L=J=\beta=\gamma=0\right] = \int \Big[\langle \frac{1}{\xi}(D_\mu(A)Q^\mu)(D_\nu(A+Q)\delta A^\nu)$$
$$+ b_a u {f^a}_{bc} \delta A^b_\nu D^\nu(A+Q)c^c\rangle\Big] d^4x \tag{7.1.16}$$

We used that $D_\mu(A)Q^\mu = D_\mu(A+Q)Q^\mu$. We shall now show that the sum of these two terms vanishes by deriving a Ward identity which follows from a change of the integration variable ${Q_\mu}^a$ in the path integral corresponding to a complicated nonlocal gauge transformation. This gauge transformation led to BRST symmetry as we discussed in chapter II. (For a first reading one may skip the following somewhat technical discussion, and move to (7.1.20)).

Let us change the integration variable Q^b_ν in $\tilde{Z}$ by the following nonlocal classical gauge transformation

$$\delta_g Q^b_\nu = D_\nu(A+Q)\zeta^b \text{ where } \zeta^b = \left[D^\rho(A)D_\rho(A+Q)\right]^{-1} D^\sigma(A+Q)\delta A^b_\sigma \tag{7.1.17}$$

If we define in addition that the background field does not transform under this gauge transformation, $\delta_g A^b_\nu = 0$, then $\mathcal{L}_{YM}(Q+A)$ is invariant, while the gauge fixing term $-\frac{1}{2\xi}[D(A)Q]^2$ varies into minus the first term in $\delta\tilde{Z}$ in (7.1.16). We now show that the second term in $\delta\tilde{Z}$ is due to the change in the integration measure $\Pi_x d{Q_\mu}^a(x)\Delta_F(x)$ under $Q^b_\nu \to Q^b_\nu + \delta_g Q^b_\nu$, where $\Delta_F = \det M_F$ denotes the ghost determinant with $M_F = D^\mu(A)D_\mu(A+Q)$. The Jacobian yields

$$J = 1 - Tr\int \Big\{D_\nu(A+Q)M_F^{-1}(\delta M_F/\delta Q^b_\nu)M_F^{-1}D^\sigma(A+Q)\delta A_\sigma\Big\}$$
$$+ D^\nu(A+Q)M_F^{-1} \times \delta A_\nu] d^4x \tag{7.1.18}$$

We used that differentiation of $D_\nu(A+Q)$ with respect to Q^b_ν yields zero because the structure constants are traceless. The variation of the Faddeev-Popov determinant yields a factor

$$\delta_g \Delta_F = \exp Tr\left\{M_F^{-1}(\delta M_F/\delta Q^b_\nu)\delta Q^b_\nu\right\} \tag{7.1.19}$$

In the appendix to this chapter it is shown that the sum of the first term in (7.1.18) and (7.1.19) cancels. Using the cyclicity of the trace, the second term in (7.1.18) becomes equal to minus the second term in (7.1.16). This is also shown in the appendix. Hence (7.1.16) can be compensated by making the change of integration variables given in (7.1.17), and thus vanishes. This proves that (7.1.16) vanishes.

The effective action in the background field formalism is defined by

$$\tilde{\Gamma}_{F(A,Q)}[\tilde{Q}, A] = \tilde{W}_{F(A,Q)}[J, A] - \int J\tilde{Q}d^4x \tag{7.1.20}$$

where from now on it is understood that by $J\tilde{Q}$ we mean all source terms, so $J\tilde{Q} + \beta c + b\gamma$.

Substituting the expressions for $\tilde{W}$ and $\tilde{Q}$ we get

$$\begin{aligned} & \tilde{\Gamma}_{F(A,Q)}[\tilde{Q}, A] = (W_{F(A,Q-A)}[J, A] - \int JAd^4x) - \int J(Q-A)d^4x \\ = \;& W_{F(A,Q-A)}[J, A] - \int JQd^4x = \Gamma_{F(A,Q-A)}[Q, A] = \Gamma_{F(A,Q-A)}[\tilde{Q}+A, A] \end{aligned} \tag{7.1.21}$$

Hence

$$\tilde{\Gamma}_{F(A,Q)}[\tilde{Q}, A] = \Gamma_{F(A,Q-A)}[\tilde{Q}+A, A] \tag{7.1.22}$$

Putting $\tilde{Q}_\mu{}^a = 0$, which means for given arbitrary $A_\mu{}^a$ that $J_\mu{}^a$ becomes a function of $A_\mu{}^a$ and K^a_μ, L_a, β_a and γ^a, we find the fundamental result

$$\tilde{\Gamma}_{F(A,Q)}[0, A] = \Gamma_{F(A,Q-A)}[A, A] \tag{7.1.23}$$

Since $\tilde{\Gamma}$ does not depend on J but only on $\tilde{Q}$ and A, we can just forget about the dependence of J on $\tilde{Q}$ and A. The effective action $\tilde{\Gamma}$ depends thus on two unconstrained fields, $\tilde{Q}$ and A. On the left-hand side of (7.1.23) we have proper graphs with internal $\tilde{Q}$ lines, external A lines and computed from (7.1.11) using $F(A, Q)$ as gauge fixing term. Since there are no external $\tilde{Q}$ lines, these graphs are vacuum selfenergy diagrams in the background of A^a_μ. There are terms in the action which are linear in Q; they yield the proper tadpole graphs which need not vanish in the presence of background fields, and since we do not require that $\partial\tilde{\Gamma}/\partial Q$ at $Q = 0$ vanishes, the background fields need not satisfy any field equations. As we mentioned in the introduction to this chapter, in earlier approaches to the background field method, the terms linear in Q were dropped on the grounds that the background fields satisfied the field equations (expanding $\mathcal{L}(A + Q)$ to first order in Q leads to the Euler-Lagrange field equations.)

On the right-hand side of (7.1.23), we find the effective action of ordinary quantum field theory, but computed using (7.1.12) with the unusual gauge fixing term $F(A, Q - A)$. So in this approach the gauge fixing term introduces an external background field A, and as a consequence Green's functions depend on A, but S matrix elements should not depend on A because in ordinary field theory A is a gauge artefact (introduced by the gauge fixing term). This we shall prove using a Ward identity. Again there are terms linear in Q, which only contribute proper tadpole graphs to the effective action. Differentiation of $\Gamma[A, A]$ w.r.t. the first field ${A_\mu}^a$ yields the n point functions of ordinary quantum field theory. ($\Gamma[Q, A]$ yields n-point functions with n external Q's. After differentiating $\Gamma[Q, A]$ w.r.t. Q one is only supposed to set the remaining Q's to zero; in this case the result still depends on A.) However, we shall soon prove that differentiation w.r.t. the second ${A_\mu}^a$ field leads to proper n-point functions which do not contribute to the S matrix. One might expect this because the background field specifies the gauge and is equivalent to an infinite set of constants which appear on a par with the gauge parameter ξ, as we already noted. Hence the claim is that differentiating $\Gamma[A, A]$ w.r.t. ${A_\mu}^a$ gives n-point functions which are not those of ordinary quantum gauge field theory, but which nevertheless give the same S matrix. Therefore, also differentiation of $\tilde{\Gamma}[0, A]$ w.r.t. A yields the n-point functions which lead to the correct S matrix.

Since $\tilde{Z}[{J_a}^\mu, {A_\mu}^a]$ is invariant under background gauge transformations if one transforms J_a^μ like a vector, also $\tilde{W}[{J_a}^\mu, {A_\mu}^a]$ is background gauge invariant. Then $\tilde{Q}_\mu{}^a = \frac{\delta}{\delta {J_a}^\mu}\tilde{W}$ transforms as a vector, and $\int J_a^\mu \tilde{Q}_\mu^a$ and this also $\tilde{\Gamma}[\tilde{Q}_\mu{}^a, {A_\mu}^a]$ are background gauge invariant. Putting $\tilde{Q}_\mu{}^a = 0$, also δ (back) $\tilde{Q}_\mu{}^a = 0$. (In technical terms, setting $\tilde{Q} = 0$ is a so-called consistent truncation). Hence $\tilde{\Gamma}[0, {A_\mu}^a]$ is gauge invariant. This is an extraordinary result: **the full effective quantum action $\tilde{\Gamma}[0, {A_\mu}^a]$, which yields proper vertices from which the correct S matrix can be constructed, is actually (background) gauge invariant.** This is, of course, due to the particular gauge fixing term employed.

To avoid misunderstanding, let us already at this point mention that $\tilde{\Gamma}$ does depend on ξ. Hence $\tilde{\Gamma}$ is background gauge invariant, but gauge-choice dependent. In the next section we discuss renormalization in the background field formalism. There we shall see that, as in ordinary quantum gauge field theories, S(fix) does not renormalize. This tells us that for multiplicatively renormalizable field theories, the following relations between Z factors will hold

$$Z_\xi = Z_Q,\ Z_g {Z_A}^{1/2} = 1 \tag{7.1.24}$$

where ${A_\mu}^a = {Z_A}^{1/2} {A_\mu}^{a,\text{ren}}, {Q_\mu}^a = {Z_Q}^{1/2} {Q_\mu}^{a,\text{ren}}$ and $g = Z_g u \mu^{\frac{1}{2}(4-n)}$ in n dimensions. The relation $Z_\xi = Z_Q$ is the same as in ordinary quantum field theory, but the relation $Z_g {Z_A}^{1/2} = 1$ states that the coupling constant renormalization follows from the wave function renormalization of ${A_\mu}^a$. This will allow us to extract the β function from two-point functions only.

Since $g{A_\mu}^a$ does not renormalize, it follows that the Yang-Mills action with ${Q_\mu}^a =$

0 is only renormalized by an overall factor Z_A

$$
\begin{aligned}
G_{\mu\nu}(A_\mu{}^a + Q_\mu{}^a, g) &= G_{\mu\nu}(Z_A{}^{1/2} A_\mu{}^{a,\mathrm{ren}} + Z_Q{}^{1/2} Q_\mu{}^{a,\mathrm{ren}}, Z_g u) \\
G_{\mu\nu}(A_\mu{}^a, g) &= Z_A{}^{1/2} G_{\mu\nu}(A_\mu{}^{a,\mathrm{ren}}, u)
\end{aligned}
\tag{7.1.25}
$$

Also, the renormalized background gauge transformations of the background gauge field have the same form as in the nonrenormalized case (no Z factors are present after redefining the local gauge parameter)

$$
\begin{aligned}
\delta A_\mu{}^a &= \partial_\mu \lambda^a + g f^a{}_{bc} A_\mu{}^b \lambda^c \\
\delta A_\mu{}^{a,\mathrm{ren}} &= \partial_\mu \tilde{\lambda}^a + u f^a{}_{bc} A_\mu{}^{b,\mathrm{ren}} \tilde{\lambda}^c \, , \tilde{\lambda}^a = \lambda^a / Z_A{}^{1/2}
\end{aligned}
\tag{7.1.26}
$$

In fact, one can turn this argument around, and use it to argue that the relation $Z_g Z_A^{1/2}$ **must** be true for consistency. If the theory is renormalizable, $\tilde{\Gamma}_{\mathrm{ren}}[0, A_{\mathrm{ren}}]$ must be finite and background gauge invariant, but then the relation $Z_g Z_A^{1/2} = 1$ must hold, because otherwise $\delta A_\mu^{\mathrm{ren},a}$ would contain Z factors and could not leave $\tilde{\Gamma}_{\mathrm{ren}}[0, A_{\mathrm{ren}}]$ invariant.

For field theories which are not multiplicatively renormalizable but only additively, such as gauge theories coupled to chiral fermions with dimensional regularization, one can still use the background field formalism. Now one must approach the problem of renormalization with the methods of cohomology, and study also the Ward identities of background gauge invariance. In fact, the possible obstructions Δ in the (Γ, Γ) Ward identity, namely $\frac{1}{2}(\Gamma, \Gamma) = \Delta$, and the obstructions Δ_a^W in the Ward identities for linearly realized symmetries (such as the background gauge invariance), namely $W_a(x)\Gamma = (\Delta^W(x))_a$, are then related by the equation

$$
(\Gamma, W_a \Gamma) = (\Gamma, \Delta_a^W) \equiv \mathcal{S} \Delta_a^W = W_a \Delta \tag{7.1.27}
$$

This equation holds for any regularization scheme. After having made the theory finite by renormalization, one should add further finite local counter terms to Γ such that both Δ and Δ_a^W are removed. Then one has restored BRST symmetry and background gauge invariance at that loop level. Again this is only then not possible when there are chiral anomalies in the theory. The net results are similar to those of multiplicative renormalization, and we refer the reader to the literature [16].

2 The S matrix

Let us now show that the background field method yields the correct S matrix. We shall continue to take pure Yang-Mills theory as a model. Due to infrared problems there is the usual problem that an S-matrix does not exist in this case, and one should really take a spontaneously broken gauge theory. However the general ideas are much simpler for pure Yang-Mills theory, and the reader may consult [16] for the spontaneously broken case.

S matrix elements are obtained by constructing tree diagrams from (dressed) propagators and proper vertices. The proper vertices are obtained by expanding $\tilde{\Gamma}$ and are gauge invariant but gauge-choice dependent. To construct A-propagators one must add a gauge fixing term for the A fields to the action. Since tree graphs for a gauge invariant classical theory with physical incoming and outgoing particles are in general independent of the gauge choosen, we may choose the gauge for the A fields to be different from the gauge fixing term used to compute the effective action (the proper graphs). S-matrix elements in the background field formalism are then independent of both gauge-fixing parameters [9, 10]. Consider first the corresponding ordinary quantum field theory. Connected and disconnected graphs are obtained from the following path integral of ordinary quantum field theory which we already discussed in (7.1.12)

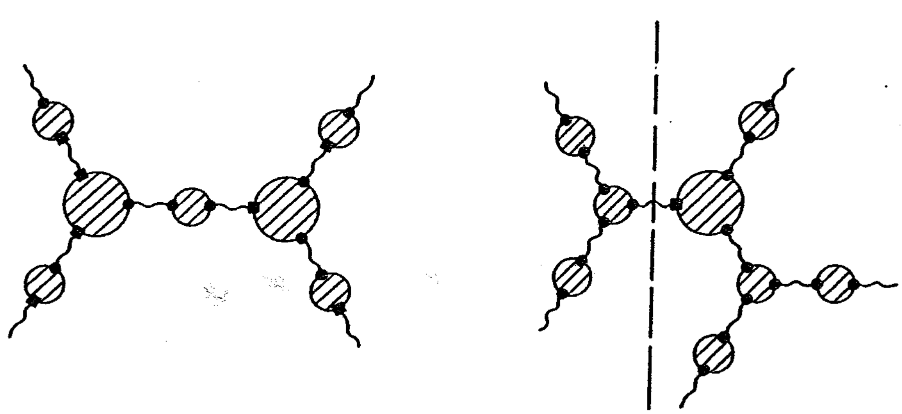
$$\begin{aligned} Z[J,A] &= \int dQ_\mu{}^a db_a dc^a \exp\frac{i}{\hbar}\left[S_{YM}(Q) - \frac{1}{2\xi}\int F(A,Q-A)^2\right. \\ &\left. +S(\text{ghost}) + S(\text{extra}) + \int J_a{}^\mu Q_\mu{}^a d^4x\right] \qquad (7.2.1) \end{aligned}$$

where $F(A, Q-A) = D^\mu(A)(Q-A) = -D^\mu(Q)A_\mu + \partial^\mu Q_\mu$. We already proved that for the effective action

$$\tilde{\Gamma}[0, A] = \Gamma[Q, A]_{|Q=A} \tag{7.2.2}$$

The S matrix elements of ordinary quantum gauge field theory are obtained by constructing proper graphs from $\Gamma[Q, A]$ by differentiating w.r.t. ${Q_\mu}^a$, contracting Q lines with Q propagators, and in the end putting some ${Q_\mu}^a$ fields on-shell and setting all other Q fields equal to zero. The result should not depend on ${A_\mu}^a$. Differentiation of $\Gamma[Q, A]$ w.r.t. ${A_\mu}^a$ clearly only involves proper graphs with at least one external A-field. In the backgound field formalism one differentiates with respect to all A fields in $\tilde{\Gamma}[0, A]$, hence one also differentiates $\Gamma[Q, A]$ w.r.t. ${A_\mu}^a$. To obtain S matrix elements in the background field formalism one must contract Q and A fields with Q and A fields, so one can have QQ, QA and AA propagators. We shall now show that S-matrix elements constructed from $\Gamma[Q, A]$ containing at least one differentiation w.r.t. ${A_\mu}^a$ vanish. So all S matrix elements with QA, AQ or AA propagators and/or external A fields vanish. This then establishes that the background field formalism gives the same S matrix as ordinary quantum field theory. So, the following proper graphs should not contribute to the S matrix

(7.2.3)



(7.2.4)

Figure caption: On the left: an S-matrix element in the background field formalism; solid circles denote emission of Q fields while solid squares denote emission of A fields. On the right: an S-matrix element with one internal A field being emitted. This contribution vanishes when all other external fields are truncated, put on-shell and are contracted with physical polarization vectors. [24]

Differentiation of $Z[J,A]$ in (7.2.1) w.r.t. ${A_\mu}^a$ yields

$$\hbar \frac{\delta}{\delta {A_\mu}^a(x)} Z =< -\frac{i}{\xi} D^\mu(Q) F(A, Q-A)^a - igf^a{}_{bc} b^b (D^\mu(Q)c)^c > \tag{7.2.5}$$

where we used that $F = D_\mu(A)Q^\mu - \partial^\mu A_\mu$ can be rewritten as $-D_\mu(Q)A^\mu + \partial^\mu Q_\mu$, and $\mathcal{L}(\text{ghost}) = b_a D^\mu(A) D_\mu(Q) c^a$. Since the ${A_\mu}^a$ dependence is only present in $S(\text{fix})$ and $S(\text{ghost})$, while $S(\text{fix}) + S(\text{ghost})$ is BRST exact, one expects that it should be possible to simplify this expression using BRST symmetry. In fact, the right-hand side of (7.2.5) is almost BRST exact, because it contains the expectation value of a BRST variation

$$\frac{\delta}{\delta {A_\mu}^a} Z =< \delta_B \left(\frac{i}{\hbar} D^\mu(Q) b_a \right) > \tag{7.2.6}$$

where we have removed the BRST parameter Λ from the right. (It is not precisely BRST exact because it is not the BRST variation of an expectation value). To rewrite this expression in a form which will prove useful, we recall that the antighost varies into $F(A, Q-A)$ and consider the Ward identity

$$\delta_B < b_a >= 0 \tag{7.2.7}$$

where $< b_a(x) >= \int dQdbdc\ b_a(x) \exp \frac{i}{\hbar}[S + \int {J_b}^\mu {Q_\mu}^b d^4y]$ and S is given in (7.2.1). This is the same Ward identity we used in the proof of unitarity, but since in this case we still have ${J_a}^\mu$ present, we obtain two terms rather than one term

$$\delta_B < b_a(x) >=< \frac{-1}{\xi} F_a(A, Q-A)(x) + b_a(x) \int \frac{i}{\hbar} {J_b}^\mu D_\mu(Q) c^b(y) d^4y >= 0 \tag{7.2.8}$$

Acting with $D^\mu(Q) = D^\mu\left(\frac{\hbar}{i}\frac{\partial}{\partial {J_a}^\mu}\right)$ on this identity produces three terms (the J-derivative in D^μ can also hit the J in the integral)

$$< -\frac{1}{\xi} D^\mu(Q) F(A, Q-A)^a + (D^\mu(Q) b^a) \int \frac{i}{\hbar} {J_b}^\nu D_\nu(Q)\ c^b\ d^4y - gf^a{}_{bc} b^b D_\mu(Q) c^c >= 0 \tag{7.2.9}$$

The first and last term are equal to $\frac{\hbar}{i}\frac{\partial}{\partial A_\mu{}^a}Z$ in (7.2.5). Hence

$$i\hbar\frac{\partial}{\partial A_\mu{}^a(x)}Z =< (D^\mu(Q)b_a(x))\int\frac{i}{\hbar}J_b{}^\nu D_\nu(Q)c^b(y)d^4y > \tag{7.2.10}$$

One obtains S matrix elements with n external Q fields by differentiating Z w.r.t. external sources $J_a{}^\mu(x_j), j = 1, \ldots, n$, next setting all remaining J equal to zero, and then amputating, putting these external lines on-shell, contracting them with transversal polarization vectors ϵ^μ, and multiplying them by the square roots of the residues of the propagators. There is then always a point x_j where one finds the operator $D_\nu(Q)c^b(x_j)$ contracted with ϵ^ν. After going over to momentum space, the operator $\partial_\nu c^b(x_j)$ yields $k_\nu\epsilon^\nu$ which vanishes. The nonlinear term $gf^b{}_{pq}Q^p_\nu b^q(x_j)$ either does not lead to a pole in k^2, in which case truncation (i.e., multiplication by k^2 at $k^2 = 0$) yields zero, or if it does lead to a pole, the resulting graph (which we called a strange graph in chapter V section 3) is proportional to k_ν and still vanishes due to $k_\nu\epsilon^\nu = 0$. We encountered this argument already in our discussion of unitarity. Hence, differentiating an S matrix element with n external Q fields w.r.t. $A_\mu{}^a(x)$ yields zero. Therefore S matrix elements obtained by extracting an A field from a proper vertex and using it on a par with the Q fields yields zero.

The preceding proof was based on a diagrammatic approach: first constructing proper vertices, and then using tree graphs to obtain the S-matrix. In a functional approach where one performs an inverse Legendre transformation to construct W from Γ, the limit $Q_\mu{}^a = A_\mu{}^a$ in the corresponding ordinary field theory requires special care [14] because Γ is gauge invariant and the gauge fixing term $[D(A)(Q - A)]^2$ vanishes for $Q = A$. One can in this case add to Γ a new gauge fixing term for the A fields, for example $-\frac{1}{2\xi}(\partial^\mu A^a_\mu)^2$, and then perform the Legendre transform. In [15] the background field A^a_μ has been multiplied by a parameter h, so that one can go continuously from ordinary field theory to the background field formalism. Note that even though $\hat{\Gamma}$ is gauge independent one still has to prove gauge-choice independence of the S-matrix.

This concludes the proof that background field theory gives the same unrenormalized but regularized S matrix as ordinary field theory. We must now discuss renormalization.

3 Renormalization of background gauge field theory

To prove the renormalizability of gauge theory in the background field formalism, one might expect that it would be sufficient to add the usual sources for BRST transformations of the quantum gauge field ${Q_\mu}^a$ and the ghost field c^a, and to renormalize the fields and constants as follows

$$\begin{aligned} {A_\mu}^a &= {Z_A}^{1/2}{A_\mu}^{a,\mathrm{ren}}; {Q_\mu}^a = {Z_Q}^{1/2}{Q_\mu}^{a,\mathrm{ren}}; g = Z_g u \mu^{\frac{1}{2}(4-n)} \\ b_a &= {Z_{gh}}^{1/2}{b_a}^{\mathrm{ren}}; c^a = {Z_{gh}}^{1/2}{c^a}_{\mathrm{ren}}; \xi = Z_\xi \xi_{\mathrm{ren}} \end{aligned} \tag{7.3.1}$$

(We mentioned before that ${A_\mu}^a$ can be considered as an infinite set of constants, and in principle one might even expect to need an infinite set of Z factors for these constants, but it would clearly be much simpler if one constant Z_A would be needed). However, one can immediately see a problem. There are now two background-gauge and BRST invariant terms possible in the divergences, namely

$$\alpha_1 \mathcal{L}_{YM}(A+Q) + \alpha_2 \mathcal{L}_{YM}(A) \tag{7.3.2}$$

but there is no extra Z factor since the factor Z_A is related to Z_g by ${Z_A}^{1/2}Z_g = 1$ as we already mentioned (and will be derived shortly). Hence, in the background field formalism there seems to be one more divergent structure than Z factors, and this would ruin renormalizability. Furthermore, since there is no term $\mathcal{L}_{YM}(A)$ at the classical level, multiplicative renormalizability could not hold, but this is not an essential problem because we could switch to additive (algebraic) renormalization. (Alternatively, we could add a term $\alpha\mathcal{L}_{YM}(A)$ to the classical action, and then prove multiplicative renormalizability without subtleties. However, this is not the classical action of the background field formalism).

We are in the same situation as when we discussed the renormalization of spontaneously broken gauge theories. In that case the problem was resolved by finding an extra Ward identity which followed from the fact that the matter sector did only depend on the sum $v+\sigma$ of the Higgs scalar and its vacuum expectation value. Here the resolution will also be given by an extra Ward identity which follows from a further symmetry. It is not difficult to spot an extra gauge symmetry: at the classical level, the gauge action depends only on the sum ${A_\mu}^a + {Q_\mu}^a$, hence any transformation

$$\delta_M {A_\mu}^a = -{\hat{M}_\mu}{}^a, \delta_M {Q_\mu}^a = {\hat{M}_\mu}{}^a \tag{7.3.3}$$

leaves $S(\text{class})$ invariant. Here ${\hat{M}_\mu}{}^a$ is a commuting gauge parameter. We now present a detailed discussion of the Ward identity which follows from this extra local symmetry. However, the reader who is willing to believe the derivation may go directly to below (7.3.26).

Persuing this idea at the quantum level, we soon discover that $S(\text{fix})$ and $S(\text{ghost})$ are not invariant under these transformations

$$\begin{aligned} \delta_M S(\text{fix}) &= \int -\frac{1}{\xi}(D^\mu(A)Q_\mu)D^\nu(A+Q)\hat{M}_\nu \\ \delta_M S(\text{ghost}) &= \int (\hat{M}^\mu \times b) D_\mu(A+Q)c \end{aligned} \tag{7.3.4}$$

(We used again that $D_\mu(A)Q^\mu = D_\mu(A+Q)Q^\mu$ and varied only the last Q^μ). To cancel these variations, we recall that $D^\mu(A)Q_\mu$ is proportional to the BRST variation of the antighost b_a. Hence, we try to view the M transformations as extensions of the usual BRST variations, and then add a new term to the action whose variation will cancel $\delta_M S(\text{fix})$.[6] So, at this point we no longer consider the M transformations as an extra gauge symmetry which by itself leaves the quantum action invariant. The

[6] There are no further such terms needed for the matter sector, because the actions in the matter sector depend only on $A^a_\mu + Q^a_\mu$.

extended BRST transformations will lead to an extended BRST operator Q which will restrict possible divergences more than the usual BRST operator, and as a result we shall reach our goal of eliminating the extra divergence proportional to $S_{YM}(A)$.

Promoting the $\hat{M}$ transformations to BRST transformations leads to

$$\delta_B A_\mu{}^a = -M_\mu{}^a \Lambda, \delta_B Q_\mu{}^a = M_\mu{}^a \Lambda \tag{7.3.5}$$

where $M_\mu{}^a$ is now an anticommuting background field with the same dimension and indices as $K_a{}^\mu$, but with opposite ghost number.[7] It transforms under background gauge transformations as a vector, just like J_a^μ and K_a^μ. The new term to be added to the action is

$$\mathcal{L}(M) = M_\mu{}^a D^\mu(A+Q) b_a \tag{7.3.6}$$

Substituting the BRST variation $\delta_B b = -\frac{1}{\xi} D^\mu(A) Q_\mu \Lambda$ into $\mathcal{L}(M)$ cancels $\delta_M S(\text{fix})$, and substituting the BRST variation $\delta_B Q_\mu = D_\mu(A+Q)c\Lambda$ into $\mathcal{L}(M)$ cancels $\delta_M S(\text{ghost})$. The new BRST transformations in (7.3.5) leave $\mathcal{L}(M)$ clearly invariant because $\mathcal{L}(M)$ only depends on $A+Q$.

It is interesting to study "the group structure", namely the commutator of a BRST transformation and a background gauge transformation. If the action is invariant under each, it should also be invariant under the commutator, so the commutator has to be a symmetry: it can be proportional to (a linear combination of) old symmetries, or be a new symmetry. One finds easily that δ_B and δ (back) commute on any field (for the ghosts one needs the Jacobi identities). The extra terms in (7.3.5) in the BRST transformations do not destroy this commutativity.

Although this action is BRST invariant, it is not yet the action we should use to prove renormalizability. Loop graphs with a vertex $\mathcal{L}(M)$ may produce divergences

[7] The source K_a^μ corresponds to the antifield for A_μ^a in the antifield formalism of Batalin and Vilkovisky and has ghost number -1, but M_μ^a cannot be identified with the antifield of A_μ^a because its ghost number is $+1$.

proportional to $M_\mu{}^a K_a{}^\mu$, as the following example shows

Hence we also add a term $M_\mu{}^a K_a{}^\mu$ to the action. We fix its normalization such that in the action only the combination $(K_a{}^\mu - D^\mu(A+Q)b_a)M_\mu{}^a$ appears. (This may seem puzzling at first: from the ghost action and the usual $K_a{}^\mu$ term one finds the combination $K_a{}^\mu - D^\mu(A)b_a$, while now for the M terms one finds the combination $K_a{}^\mu - D^\mu(A+Q)b_a$. However, we shall soon see that the difference $gQbM$ does not renormalize.) This leads to the action we shall use to prove renormalizability.

$$\begin{aligned} \mathcal{L} &= \mathcal{L}_{YM}(A+Q) - \frac{1}{2\xi}(D^\mu(A)Q_\mu)^2 - (D^\mu(A)b)D_\mu(A+Q)c \\ &+ K^\mu D_\mu(A+Q)c + L\frac{1}{2}u\, c\times c + M_\mu{}^a(D^\mu(A+Q)b_a - K_a{}^\mu) \end{aligned} \tag{7.3.7}$$

This action is invariant under the following set of extended BRST transformations.

$$\begin{aligned} \delta_B{}' A_\mu{}^a &= -M_\mu{}^a\Lambda,\ \delta'_B Q_\mu = D_\mu(A+Q)c\Lambda + M_\mu\Lambda \\ \delta'_B c &= \frac{1}{2}c\times c\Lambda,\ \delta'_B b = -\frac{1}{\xi}D^\mu(A)Q_\mu\Lambda \\ \delta_B{}' K_a{}^\mu &= \delta_B{}' L_a = \delta' M_a{}^\mu = 0 \end{aligned} \tag{7.3.8}$$

These transformations are again nilpotent (for the antighost after one adds the auxiliary field d). Note that the terms with $K_a{}^\mu$ in (7.3.7) yield the complete $\delta'_B Q_\mu{}^a$, so this confirms the normalization of the term $M_\mu{}^a K_a{}^\mu$. The term $M_\mu{}^a D^\mu(A+Q)b_a$ can be combined with S(ghost) to give the ghost action as it would be obtained if one would vary Q in the gauge fixing term w.r.t. δ'_B instead of δ_B

$$b_a\delta'_B\left[D^\mu(A)Q_\mu{}^a\right] = \left[-(D^\mu(A)b)D_\mu(A+Q)c + M_\mu{}^a D^\mu(A+Q)b_a\right]\Lambda \tag{7.3.9}$$

In fact, all gauge artefacts are BRST exact: $\mathcal{L}(\text{fix}) + \mathcal{L}(\text{ghost}) + MD(A+Q)b = \delta'_B(b_a D_\mu(A)Q_\mu{}^a)$. The action in (7.3.7) remains background gauge invariant since $M_\mu{}^a$ transforms per definition as a vector in the adjoint representation.

We shall now derive the Ward identities which follow from the extended BRST symmetry. Consider the path integral

$$\tilde{Z}[A, J, \beta, \gamma, K, L, M] = \int dQ db dc \ e^{\frac{i}{\hbar}[S + \int (J \cdot Q + \beta c + b\gamma) d^4x]} \tag{7.3.10}$$

If we make a BRST change of integration variables $Q = Q' + \delta'_B Q$ etc., and simultaneously change $A_\mu{}^a$ into $A'_\mu{}^a + \delta'_B A_\mu{}^a$, the action S is invariant. We assume that the Jacobian for (Q, b, c) is again unity; this is normally the case but one should use regularization (or cohomology methods). The source terms and the change in $A_\mu{}^a$ yield then the following Ward identity (dropping the prime on $A'_\mu{}^a$)

$$\int \left[\frac{i}{\hbar} J_a{}^\mu \frac{\partial}{\partial K_a{}^\mu} \tilde{Z} + \frac{i}{\hbar} \beta \frac{\partial}{\partial L} \tilde{Z} + \frac{i}{\hbar} \left(\frac{1}{\xi} D_\mu(A) \frac{\hbar}{i} \frac{\partial}{\partial J_\mu} \tilde{Z} \right) \gamma + M_\mu{}^a \frac{\partial}{\partial A_\mu{}^a} \tilde{Z} \right] d^4x = 0 \tag{7.3.11}$$

The same relation holds for $\tilde{W}$, and after the Legendre transformation the effective action $\tilde{\Gamma}$ satisfies (using $\frac{\partial}{\partial Q} \tilde{W} = -\frac{\partial}{\partial Q} \tilde{\Gamma}$ and $\frac{\partial}{\partial K} \tilde{W} = \frac{\partial}{\partial K} \tilde{\Gamma}$)

$$\left(\frac{\partial}{\partial Q_\mu{}^a} \tilde{\Gamma} \right) \left(\frac{\partial}{\partial K_a{}^\mu} \tilde{\Gamma} \right) - \left(\frac{\partial}{\partial c^a} \tilde{\Gamma} \right) \left(\frac{\partial}{\partial L_a} \tilde{\Gamma} \right) + \frac{1}{\xi} (D^\mu(A) Q_\mu) \frac{\partial}{\partial b} \tilde{\Gamma} - M_\mu \frac{\partial}{\partial A_\mu} \tilde{\Gamma} = 0 \tag{7.3.12}$$

As usual one may check the signs in this Ward identity by considering the $\hbar = 0$ terms, using that at $\hbar = 0$ the effective action $\tilde{\Gamma}$ reduces to the quantum action. To simplify this expression, we use again the ghost field equation. From the expression for $\tilde{Z}$ we derive by differentiation w.r.t. b

$$< D^\mu(A) D_\mu(A + Q) c + D^\mu(A + Q) M_\mu + \gamma >= 0 \tag{7.3.13}$$

Using that $D^\mu(A)[D_\mu(A + Q)c + M_\mu]\Lambda$ equals $D^\mu(A)\delta'_B Q_\mu$, the local Ward identity can be written as follows

$$D^\mu(A) \frac{\partial}{\partial K_a{}^\mu} \tilde{W} - g M_\mu \times \frac{\partial}{\partial J_\mu} \tilde{W} + \gamma = 0 \tag{7.3.14}$$

Of course the term with M_μ in this equation is new (it did not appear in ordinary quantum field theory). Hence, using $\gamma = -\frac{\partial}{\partial b} \hat{\Gamma}$,

$$D^\mu(A) \frac{\partial}{\partial K_a{}^\mu} \tilde{\Gamma} - g M^\mu \times Q_\mu - \frac{\partial}{\partial b} \tilde{\Gamma} = 0 \tag{7.3.15}$$

where we used that $\gamma = -\frac{\partial}{\partial b}\tilde{\Gamma}$ because $\tilde{\Gamma} = \tilde{W} \ldots - b\gamma$.

Substituting (7.3.15) into (7.3.12) and defining $\tilde{\Gamma} = \hat{\Gamma} + S(\text{fix})$ we arrive at

$$\int \left[\left(\frac{\partial}{\partial Q_\mu}\hat{\Gamma} \right) \left(\frac{\partial}{\partial K^\mu}\hat{\Gamma} \right) - \left(\frac{\partial}{\partial c}\hat{\Gamma} \right) \left(\frac{\partial}{\partial L}\hat{\Gamma} \right) \right.$$
$$\left. -\frac{1}{\xi} g D(A)Q \cdot (M \times Q) - M_\mu \frac{\partial}{\partial A_\mu}\hat{\Gamma} \right] d^4x = 0$$
$$D^\mu(A)\frac{\partial}{\partial K_a{}^\mu}\hat{\Gamma} - gM^\mu \times Q_\mu - \frac{\partial}{\partial b}\hat{\Gamma} = 0 \tag{7.3.16}$$

As in ordinary quantum field theory, most terms involving the gauge fixing term S_{fix} have canceled in the integrated Ward identity for $\hat{\Gamma}$, but there is a left-over: the term $(D(A)Q) \cdot (M \times Q)$. We shall soon see how to handle it.

Multiplicative renormalizability requires that the Z factors in the renormalized equations combine into an overall factor. Hence we assume in the inductive proof of renormalizability the following relations between Z factors

$$\begin{aligned} Z_Q{}^{1/2} Z_K{}^{1/2} &= Z_{gh}{}^{1/2} Z_L{}^{1/2} = \frac{1}{Z_M{}^{1/2}} Z_A{}^{1/2} = Z_g Z_M^{1/2} \\ Z_K{}^{1/2} &= Z_{gh}{}^{1/2} = (Z_g Z_M{}^{1/2} Z_Q{}^{1/2})^{-1},\ Z_g Z_A{}^{1/2} = 1 \end{aligned} \tag{7.3.17}$$

The requirement that $S(\text{fix})$ does not renormalize leads to the usual relation $Z_\xi = Z_Q$ and confirms the relation $Z_g Z_A{}^{1/2} = 1$

$$\begin{aligned} \mathcal{L}(\text{fix}) &= -\frac{1}{2\xi}(\partial^\mu Q_\mu{}^a + g f^a{}_{bc} A_\mu{}^b Q_\mu{}^c)^2 \\ &= -\frac{1}{2\xi_{\text{ren}}}\left(\partial^\mu Q_\mu{}^{a,\text{ren}} + u f^a{}_{bc} A_\mu{}^{b,\text{ren}} Q_\mu{}^{c,\text{ren}} \right)^2 \\ Z_Q &= Z_\xi, Z_g Z_A{}^{1/2} = 1 \end{aligned} \tag{7.3.18}$$

These relations are compatible, and leave only 3 independent Z factors, for example Z_A, Z_Q and Z_{gh}.

To proceed we follow induction, and assume that all $(n-1)$-loop proper graphs have been made finite. We renormalize the two Ward identities by removing in each of them an overall Z factor. One then finds that the peculiar term $-\frac{1}{\xi}g(D_\mu(A)Q^\mu) \cdot$

$(M_\nu \times Q^\nu)$ enters in the renormalized $\tilde{\Gamma}\tilde{\Gamma}$ equation without any Z-factors. So we may drop it at this point, as we are only interested in divergences. Similarly the term $-gM \times Q$ in the local Ward identity can be omitted if one is only inerested in the divergences of $\tilde{\Gamma}$. Thus, in the divergences we only need allow the combination $K^\mu - D^\mu(A)b$.

The divergences at the n-loop level produced by the $(n-1)$-loop effective action satisfy then the usual two equations

$$\begin{aligned}
&Q\hat{\Gamma}(\text{div}) = 0\ ;\ (D^\mu(A)\frac{\partial}{\partial K_a{}^\mu} - \frac{\partial}{\partial b_a})\hat{\Gamma}(\text{div}) = 0 \\
&Q = \int \left[\left(\frac{\partial}{\partial Q_\mu}\hat{S}\right)\frac{\partial}{\partial K^\mu} + \left(\frac{\partial}{\partial K^\mu}\hat{S}\right)\frac{\partial}{\partial Q_\mu}\right. \\
&\left.-\left(\frac{\partial}{\partial c}\hat{S}\right)\frac{\partial}{\partial L} - \left(\frac{\partial}{\partial L}\hat{S}\right)\frac{\partial}{\partial c} - M_\mu \frac{\partial}{\partial A_\mu}\right] d^4x
\end{aligned} \tag{7.3.19}$$

All terms in these equations are expressed in terms of renormalized quantities. One can prove that $(Q)^2 = 0$ as follows. We introduce a further anticommuting external source H^μ which couples to M_μ as $H^\mu M_\mu$, similarly to the procedure we followed when we proved nilpotency of Q in the presence of an auxiliary field d. Then $-M_\mu \partial/\partial A_\mu$ can be rewritten as $\partial S/\partial H^a_\mu\ \partial/\partial A^a_\mu$. Furthermore, using that $\hat{\Gamma}(\text{div})$ does not depend on H^μ, we can add for free the extra term $-(\frac{\partial}{\partial A}\hat{S})\frac{\partial}{\partial H}$ in Q. Let us call this operator Q'; it can be written as

$$Q' = \sum_i \left(\frac{\partial}{\partial x^i}\hat{S}\right)\frac{\partial}{\partial \theta_i} + \left(\frac{\partial}{\partial \theta_i}\hat{S}\right)\frac{\partial}{\partial x^i} \tag{7.3.20}$$

where $\{x^i, \theta_i\}$ are the pairs $(Q, K), (-L, c)$ and $(A, -H)$, and nilpotency of Q follows. The general solution of the divergences is then: gauge-invariant terms + Q'-exact terms, but since there are no divergences with H^μ_a, we may replace this by: gauge-invariant terms plus Q-exact terms.

We could at this point study the cohomology problem $QX = 0$. Noting that A_μ

and M_μ form what is called a contractible pair

$$\delta_B{}' A_\mu = -M_\mu \Lambda, \delta_B{}' M_\mu = 0 \,. \tag{7.3.21}$$

We could begin by removing the fields A_μ and M_μ from X, and then proceed as in ordinary field theory.

We shall follow here the conceptually much simpler method of power counting, even though the algebra is a bit tedious. We begin by determining all proper n-point functions which are divergent by power counting. The degree of divergence is

$$D = 4 - 2I_Q - 2I_{gh} + n_{3Q} + n_{AQQ} + n_{bAc} + n_{bQc} + 4L \tag{7.3.22}$$

(The vertices with AAQ from S (fix), and $K\partial c$, $M\partial b$ are linear in quantum fields and only contribute to proper graphs at the tree level. The vertex MK only contributes to tree graphs.) Eliminating $2I_Q + E_Q + E_A$ and $2I_{gh} + E_c + E_b$ in terms of vertices containing Q and A, and b and c fields, respectively, one finds, after also expressing the number of loops L in the number of vertices

$$D = 4 - E_Q - E_A - E_c - E_b - 2n_K - 2n_L - 2n_M \tag{7.3.23}$$

where n_K counts the vertices Kbc, n_L the vertices $L\frac{1}{2}uc \times c$, and n_M the vertices MQb and MAb. Because there are now vertices with undifferentiated b fields, we no longer have a term $-2E_b$ in this formula. New in this formula is the term $-2n_M$, but it is easy to check the need for this term by considering the graph below (7.3.6).

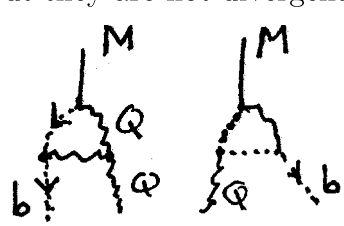
Dimensional arguments, ghost number conservation, together with background gauge invariance and the second Ward identity, leaves the following a priori possible stuctures for the divergences in $\hat{\Gamma} = \tilde{\Gamma} - S(\text{fix})$ for proper graphs

$$\begin{aligned} \hat{\Gamma}(\text{div}) &= (K^\mu - D^\mu(A)b)[\beta_1 D_\mu(A)c + \beta_2 u Q_\mu \times c + \beta_3 M_\mu] \\ &+ \beta_4 L \frac{1}{2} uc \times c + \Sigma(A, Q) \end{aligned} \tag{7.3.24}$$

New is the term with β_3. Background gauge invariance has eliminated a divergence $[K - D(A)b](Ac)$. No term $\beta_5 ubQ^\mu \times M_\mu$ is present; although it is allowed by power counting, it does not renormalize. (It is also not allowed by the ghost equation, which only allows terms with $K_a^\mu - D^\mu b_a$). We deduced this nonrenormalization before from the assumption in the iteration procedure that multiplicative renormalization is possible. These divergences should satisfy $Q\hat{\Gamma}(\text{div}) = 0$. This fixes the functional form of $\sum(A, Q)$ and some of the constants $\beta_1, \dots, \beta_4$ and those in $\sum(A, Q)$. The algebra is the same as in the proof of renormalizability of pure gauge theories, and the result is also the same: all divergences can be absorbed by rescaling the parameters. There remain indeed only 3 divergences, and the renormalized action has the following form

$$\begin{aligned}
&S^{\text{ren}} + \Delta S^{\text{ren}} = S_{YM}({Z_A}^{1/2} A^{\text{ren}} + {Z_Q}^{1/2} Q^{\text{ren}}, Z_g u) \\
&- \frac{1}{2\xi_{\text{ren}}} (D^\mu(A^{\text{ren}}) {Q_\mu}^{\text{ren}})^2 - Z_{gh} [D^\mu(A^{\text{ren}}) b_{\text{ren}} - K^\mu_{\text{ren}}][D_\mu(A^{\text{ren}} + \sqrt{\frac{Z_Q}{Z_A}} Q^{\text{ren}}) c_{\text{ren}}] \\
&+ Z_g {Z_Q}^{1/2} Z_{gh} L^{\text{ren}} \left(\frac{1}{2} u c^{\text{ren}} \times c^{\text{ren}}\right) \\
&+ {Z_A}^{1/2} {Z_Q}^{-1/2} [{K_{a,\text{ren}}}^\mu - D^\mu(A^{\text{ren}}) b^{\text{ren}}] {M_\mu}^{\text{ren}} \\
&+ u {f^a}_{bc} {b_a}^{\text{ren}} {Q_{\text{ren}}}^{\mu b} {M_\mu}^{c,\text{ren}} \text{ with } Z_g {Z_A}^{1/2} = 1
\end{aligned} \tag{7.3.25}$$

The interested reader may check a few terms; we suggest first to look at the terms of the form bMM in $Q\hat{\Gamma} = 0$ and the counter terms proportional to $K_a^\mu M_\mu^a$. The last term in $S^{\text{ren}} + \Delta S^{\text{ren}}$ does not contain any Z factors, hence "it does not renormalize", i.e., there are no divergences of this structure. (It is also not allowed by the ghost field equation which only allows terms with $(K_a^\mu - D^\mu b_a)$). It is easy to check that the term $ubQM$ does not renormalize at the one-loop level: only triangle graphs could reproduce this structure, but they are not divergent.



(7.3.26)

There are thus an equal number of possible divergences and Z factors, and these divergences can be removed by rescaling the fields and parameters. This concludes the proof that background gauge field theory is renormalizable.

We now come back to the problem we signalled at the beginning of this section: the danger of an extra divergence proportional to $S_{YM}(A)$. The coefficient of this term is Z_A, which is determined in terms of Z_g, and not an independent coefficient. Having extended the classical symmetry $\delta A_\mu{}^a = -M_\mu{}^a$ and $\delta Q_\mu{}^a = M^a$ to part of the BRST symmetry, $\delta_B{}' A_\mu{}^a = -M_\mu{}^a\Lambda$ and $\delta_B{}' Q_\mu{}^a = M_\mu{}^a\Lambda$, the requirement of BRST symmetry at the quantum level, namely $Q\hat{\Gamma}^{\rm ren}_{\rm div} = 0$, has ruled out separate divergences proportional to $S_{YM}(A)$ and $S_{YM}(A+Q)$. Only divergences obtained by expanding $S_{YM}(\sqrt{Z_A}A + \sqrt{Z_Q}Q)$ can occur. This is easily checked by using the explicit form of Q in (7.3.19). For graphs with only external A-fields, the divergences are given by $Z_A S_{YM}(A_{\rm ren})$ because $gA = g_{\rm ren}A_{\rm ren}$.

Having gone through the whole proof of renormalizability of the background field theory, the weary reader may wonder whether one could not just have started with ordinary field theory in the peculiar gauge we mentioned at the beginning of this section. After all, both formalisms are equivalent. However, if one does not use the knowledge that the theory has a background gauge invariance, one would expect too many divergences for the theory to be renormalizable. Of course, the background gauge invariance is still present in the equivalent ordinary background field method; it is only hidden. This raises another question. Suppose we had chosen another peculiar gauge fixing function, one that this time destroys the background gauge invariance (for example, the gauge-fixing term obtained by multiplying A_μ by a factor). The theory should still be renormalizable, but the proof could not be given as simply as before.

4 Gauge parameter independence of the beta function

The background field formalism allows a relatively simple proof that the β function is ξ independent [8]. We shall begin by assuming that the theory is multiplicatively renormalizable but at the end of this section we shall briefly disucss how to proceed if only additive renormalization is possible. (For example in the case of the Standard Model where chiral fermions prevent multiplicate renormalizability or supersymmetric theories due to the occurrence of γ_5 matrices.) The basic idea is to prove that the wave function renormalization constant $Z(A)$ is ξ independent. Because β can be expressed in terms of $Z(A)$ only, as we prove in the next section, this then proves that β is ξ independent. Since $Z(A)$ appears as the coefficient in front of $-\frac{1}{4}(F_{\mu\nu}{}^a(A^{\text{ren}}))^2$ in $\Gamma(\text{div})$, as we proved in the previous section, one tries to find an equation for $\frac{\partial}{\partial\xi}\Gamma(\text{div})$ with $Q^{\text{ren}} = 0$. If this is a Ward identity, it should follow from an invariance of the quantum action. Because $\frac{\partial}{\partial\xi}$ maps S_{fix} into $-\frac{1}{\xi}$ times itself, a natural strategy is to consider instead $\xi\frac{\partial}{\partial\xi}$ which maps S_{fix} into minus itself. Since $\xi\frac{\partial}{\partial\xi}$ is the operator form of $\delta\xi \sim \xi$, we try to find a symmetry of the action containing the transformation law $\delta\xi \sim \xi$. Since $\delta\xi \sim \xi$ maps $S(\text{fix})$ into minus itself, we must produce further variations which cancel $-S(\text{fix})$. We may start again from the observation that $D_\mu(A)Q^\mu$ in $\mathcal{L}(\text{fix})$ is proportional to the BRST transformation of the antighost b, and try to view $\delta\xi \sim \xi$ as part of a further extension of the already extended BRST transformation rules. Then we are led to consider

$$\delta_B\xi = 2\xi N\Lambda \tag{7.4.1}$$

where N is a real constant anticommuting parameter whose quantum numbers are opposite to those of Λ. If we define

$$\delta_B N = 0 \tag{7.4.2}$$

the pair (ξ, N) forms a contractible pair.

In the ordinary BRST formalism, one can follow the same ideas [12], and finds then a new term of the form bNd in the action

$$\delta_B\left[b_a\left(F^a + \frac{1}{2}\xi d^a\right)\right] = \left[d_a\left(F^a + \frac{1}{2}\xi d^a\right) + \mathcal{L}(\text{ghost}) + \xi b_a N d^a\right]\Lambda \tag{7.4.3}$$

Eliminating d^a, one finds $\mathcal{L}(\text{fix}) = -\frac{1}{2\xi}(F^a + \xi b^a N)^2$ and hence a new term $\mathcal{L}(N) = N b_a F^a$ in the action . In our case, we therefore add the background gauge invariant term

$$\mathcal{L}(N) = N b_a D^\mu(A) Q_\mu{}^a \tag{7.4.4}$$

The BRST variation of b_a in $\mathcal{L}(N)$ cancels then the BRST variation of ξ in $\mathcal{L}(\text{fix})$. Furthermore, the BRST variation $\delta_B Q_\mu{}^a \sim D_\mu(A+Q)c\Lambda$ in $\mathcal{L}(N)$ produces a variation $Nb[D(A)D(A+Q)c]\Lambda$ which clearly can be cancelled by an extra law $\delta_B b = Nb\Lambda$. This motivates to study the following action and transformation laws in more detail

$$\begin{aligned}
\mathcal{L} &= \mathcal{L}_{YM}(A+Q) - \frac{1}{2\xi}(D(A)Q)^2 - (D(A)b)D(A+Q)c \\
&+ KD(A+Q)c + L(\frac{1}{2}u\, c\times c) + M(D(A+Q)b - K) \\
&+ N[b\cdot D(A)Q + KQ + aLc]
\end{aligned} \tag{7.4.5}$$

$$\begin{aligned}
\delta\xi &= 2\xi N\Lambda, \delta A_\mu = -M_\mu\Lambda, \delta Q_\mu = D_\mu(A+Q)c\Lambda + M_\mu\Lambda \\
\delta c &= \frac{1}{2}u\, c\times c\Lambda, \delta b = -\frac{1}{\xi}D(A)Q\Lambda - N\Lambda b; \delta M_\mu = \delta N = 0 \\
\delta K &= -N\Lambda K, \delta L = -aN\Lambda L
\end{aligned} \tag{7.4.6}$$

The reason for the last two terms in the action and transformation laws and the value of the constant a will be given shortly.

Since the action without N-terms was already BRST invariant, and since N itself does not vary under BRST transformations, we only need to check that the variations proportional to N cancel. They are given by

$$\begin{aligned}
&\frac{N}{\xi}(D(A)Q)^2 - N\mathcal{L}(\text{ghost}) - NKD(A+Q)c - NaL\frac{1}{2}u\, c\times c - NMD(A+Q)b \\
&+NMK + N\left[-\frac{1}{\xi}(D(A)Q)^2 + \mathcal{L}(\text{ghost}) + bD(A)M + bQ\times M + K_\mu D^\mu(A+Q)c\right. \\
&\left.+KM + aL\frac{1}{2}u\, c\times c\right]
\end{aligned} \tag{7.4.7}$$

and their sum vanishes.

The reason for adding the last two terms in the action is that they are allowed by dimensions and ghost number and so might be (and indeed are) produced as divergences in loops. The variations of these N dependent terms are proportional to $NKD(A+Q)c + NKM + NaL\frac{1}{2}c \times c$, and these variations are cancelled by letting K and L transform into themselves times the scale factor $N\Lambda$. We fixed the normalization of the NKQ term such that in the N sector one finds again the combination $K_\mu - D_\mu(A)b$, but the constant a in the term $aNLc$ is not fixed by BRST invariance.

Having established that the action is invariant under this extended BRST symmetry, we write down the Ward identity for $\tilde{W}[A, J, \beta, \gamma, K, L, M, N, \xi]$.

$$
\begin{aligned}
&J\frac{\partial}{\partial K}\tilde{W} + \beta\frac{\partial}{\partial L}\tilde{W} + \left(\frac{1}{\xi}D_\mu(A)Q^\mu + bN\right)\gamma + M\frac{\partial}{\partial A}\tilde{W} \\
&+N\left[-2\xi\frac{\partial}{\partial \xi} + K\frac{\partial}{\partial K} + aL\frac{\partial}{\partial L} + J\frac{\partial}{\partial J} + a\beta\frac{\partial}{\partial \beta}\right]\tilde{W} = 0
\end{aligned}
\tag{7.4.8}
$$

The term $NJ\frac{\delta}{\delta J}\tilde{W}$ is needed because $\frac{\partial}{\partial K^\mu}S = \delta_B Q_\mu/\Lambda - NQ_\mu$. Similarly, the term $Na\beta\partial/\partial\beta\tilde{W}$ is needed to compensate the last term in $\frac{\partial}{\partial L}S = \delta_B c/\Lambda + aNc$. After the Legendre transform, we find[8]

$$\begin{aligned}&\left(\frac{\partial}{\partial Q}\tilde{\Gamma}\right)\,\frac{\partial}{\partial K}\tilde{\Gamma}-\left(\frac{\partial}{\partial c}\tilde{\Gamma}\right)\,\frac{\partial}{\partial L}\tilde{\Gamma}+\frac{1}{\xi}\left(D_\mu(A)Q^\mu\right)\frac{\partial}{\partial b}\tilde{\Gamma}-M\frac{\partial}{\partial A}\tilde{\Gamma}\\ +\quad& N\left[2\xi\frac{\partial}{\partial\xi}-K\frac{\partial}{\partial K}-aL\frac{\partial}{\partial L}+Q\frac{\partial}{\partial Q}+ac\frac{\partial}{\partial c}-b\frac{\partial}{\partial b}\right]\tilde{\Gamma}=0\end{aligned} \tag{7.4.9}$$

The ghost field equation yields

$$< D(A)D(A+Q)c + D(A+Q)M - ND(A)Q + \gamma >= 0 \tag{7.4.10}$$

Factoring out the derivative $D(A)$ this becomes

$$< D(A)[D(A+Q)c + M - NQ] + Q\times M + \gamma >= 0 \tag{7.4.11}$$

From here it is easy to obtain

$$D(A)\frac{\partial}{\partial K}\tilde{\Gamma} + Q\times M - \frac{\partial}{\partial b}\tilde{\Gamma} = 0 \tag{7.4.12}$$

In terms of $\hat{\Gamma} = \tilde{\Gamma} - S(\text{fix})$ one then finds the following two Ward identities for the unrenormalized effective action (which contains both terms without N and terms

[8]Introducing the BRST auxiliary field d one may simplify these results somewhat. The action can now be written as

$$\mathcal{L} = \mathcal{L}_{YM}(A+Q) + \delta_{BRST}\left[b\left(D_\mu(A)Q^\mu + \frac{1}{2}\xi d\right) + K_\mu(Q+A)^\mu + Lc\right]$$

Coupling K_μ to $Q^\mu + A^\mu$ instead of only to Q^μ leads to a small further simplification: it removes the terms $-M_\mu K^\mu$ from the action. The transformation rules are $\delta Q_\mu = D_\mu(A+Q)c\Lambda + M_\mu\Lambda, \delta A_\mu = -M_\mu\Lambda, \delta c = \frac{1}{2}(c\times c)\Lambda$ but now $\delta b = d\Lambda$ and $\delta d = 0$. If one then defines $\delta\xi = N'\Lambda$ (hence redefines $2\xi N = N'$) then the action is invariant without having to vary K, L, M or N'.

$$\delta K^a_\mu = \delta L^a = \delta M^a_\mu = \delta N' = 0$$

Working out the action, one finds that it differs in the following way from (7.4.5): there is no $-M_\mu K^\mu$ term, the N' term is simply $\frac{1}{2}N'bd$ and the gauge fixing term appears in the familiar form $\frac{1}{2}\xi d^2 + dD^\mu(A)Q_\mu$. The BRST Ward identity simplifies to

$$\left(\frac{\partial}{\partial Q}\tilde{\Gamma}\right)\tilde{\Gamma} - \left(\frac{\partial}{\partial c}\tilde{\Gamma}\right)\frac{\partial}{\partial L}\tilde{\Gamma} - d\frac{\partial}{\partial b}\tilde{\Gamma} - M\frac{\partial}{\partial A}\tilde{\Gamma} + N'\frac{\partial}{\partial\xi}\tilde{\Gamma} = 0$$

proportional to N)

$$
\begin{aligned}
&\left(D^\mu(A)\frac{\partial}{\partial K^\mu}-\frac{\partial}{\partial b}\right)\hat{\Gamma}+Q^\mu\times M_\mu=0\\
&\left(\frac{\partial}{\partial Q_\mu}\hat{\Gamma}\right)\left(\frac{\partial}{\partial K^\mu}\hat{\Gamma}\right)-\left(\frac{\partial}{\partial c}\hat{\Gamma}\right)\left(\frac{\partial}{\partial L}\hat{\Gamma}\right)-M_\mu\frac{\partial}{\partial A_\mu}\hat{\Gamma}+\frac{1}{\xi}(D_\mu(A)Q^\mu)\cdot(Q_\nu\times M^\nu)\\
+\quad & N\left[2\xi\frac{\partial}{\partial\xi}+Q_\mu\frac{\partial}{\partial Q_\mu}-K_\mu\frac{\partial}{\partial K_\mu}-b\frac{\partial}{\partial b}+a\left(c\frac{\partial}{\partial c}-L\frac{\partial}{\partial L}\right)\right]\hat{\Gamma}=0
\end{aligned}
\tag{7.4.13}
$$

As usual we have used the ghost equation to remove the gauge-fixing terms from the $\Gamma\Gamma$ Ward identity, but we already saw in the previous section that then the last term in the second line is left. As we have shown, this extra term in the $\Gamma\Gamma$ identity, and also the term $Q\times M$ in the local Ward identity, both are finite after removing the overall Z-factor from the Ward identities, and we shall therefore drop them from now on.

The terms in the ghost field equation which are independent of N were already discussed in the previous section. The terms proportional to N yield

$$\left(D^\mu(A)\frac{\partial}{\partial K^\mu}-\frac{\partial}{\partial b}\right)\frac{\partial}{\partial N}\hat{\Gamma}=0 \tag{7.4.14}$$

Note that we have not yet renormalized $\hat{\Gamma}$. Taking the divergent parts, one finds

$$\left(D^\mu(A)\frac{\partial}{\partial K^\mu}-\frac{\partial}{\partial b}\right)\frac{\partial}{\partial N}\hat{\Gamma}(\text{div})=0. \tag{7.4.15}$$

We can also decompose the $\Gamma\Gamma$ Ward identity into terms without N and terms linear in N. The latter tell us more about $\frac{\partial}{\partial N}\hat{\Gamma}$. Differentiating (7.4.13) w.r.t. N one finds for $\hat{\Gamma}_N\equiv\frac{\partial}{\partial N}\hat{\Gamma}\mid_{N=0}$

$$
\begin{aligned}
&\left[2\xi\frac{\partial}{\partial\xi}+\ldots\right]\hat{\Gamma}-\left(\frac{\partial}{\partial Q_\mu}\hat{\Gamma}\right)\left(\frac{\partial}{\partial K}\hat{\Gamma}_N\right)-\left(\frac{\partial}{\partial K}\hat{\Gamma}\right)\left(\frac{\partial}{\partial Q_\mu}\hat{\Gamma}_N\right)\\
&+\left(\frac{\partial}{\partial c}\hat{\Gamma}\right)\left(\frac{\partial}{\partial L}\hat{\Gamma}_N\right)+\left(\frac{\partial}{\partial c}\hat{\Gamma}_N\right)\left(\frac{\partial}{\partial L}\hat{\Gamma}\right)+M\frac{\partial}{\partial A}\hat{\Gamma}_N=0\ \text{ at }\ N=0
\end{aligned}
\tag{7.4.16}
$$

The divergences at the n-loop level can either be in the order $\hbar^n$ terms in $\hat{\Gamma}$ or in the order $\hbar^n$ terms in $\hat{\Gamma}_N$. We find the following equation for these divergences

$$\begin{aligned}&\left(2\xi\frac{\partial}{\partial\xi}+\ldots\right)\hat{\Gamma}(\text{div})=\left[\left(\frac{\partial}{\partial Q}\hat{S}\right)\frac{\partial}{\partial K^\mu}+\left(\frac{\partial}{\partial K^\mu}\hat{S}\right)\frac{\partial}{\partial Q^\mu}-\left(\frac{\partial}{\partial c}\hat{S}\right)\frac{\partial}{\partial L}\right.\\ &\left.-\left(\frac{\partial}{\partial L}\hat{S}\right)\frac{\partial}{\partial c}-M_\mu\frac{\partial}{\partial A_\mu}\right]\hat{\Gamma}_N(\text{div})+\left[-\left(\frac{\partial}{\partial Q_\mu}\hat{S}_N\right)\frac{\partial}{\partial K^\mu}+\left(\frac{\partial}{\partial K^\mu}\hat{S}_N\right)\frac{\partial}{\partial Q_\mu}\right.\\ &\left.-\left(\frac{\partial}{\partial c}\hat{S}_N\right)\frac{\partial}{\partial L}+\left(\frac{\partial}{\partial L}\hat{S}_N\right)\frac{\partial}{\partial c}\right]\hat{\Gamma}(\text{div})=0 \text{ at } N=0\end{aligned}\tag{7.4.17}$$

(Some extra signs are due to the fact that $\hat{\Gamma}_N$ is anticommuting). Substituting

$$\hat{S}_N=\int(b\cdot D^\mu(A)Q_\mu+K^\mu\cdot Q_\mu+aL\cdot c)d^4x \tag{7.4.18}$$

and using $bD^\mu(A)\frac{\partial}{\partial K^\mu}\hat{\Gamma}(\text{div})=b\frac{\partial}{\partial b}\hat{\Gamma}(\text{div})$, all terms involving $\hat{\Gamma}(\text{div})$ cancel except the term with $\frac{\partial}{\partial\xi}$

$$\begin{aligned}2\xi\frac{\partial}{\partial\xi}\hat{\Gamma}(\text{div}) &= \left[\left(\frac{\partial}{\partial Q_\mu}\hat{S}\right)\frac{\partial}{\partial K^\mu}+\left(\frac{\partial}{\partial K^\mu}\hat{S}\right)\frac{\partial}{\partial Q_\mu}-\left(\frac{\partial}{\partial c}\hat{S}\right)\frac{\partial}{\partial L}-\left(\frac{\partial}{\partial L}\hat{S}\right)\frac{\partial}{\partial c}\right.\\ &- \left.M_\mu\frac{\partial}{\partial A_\mu}\right]\hat{\Gamma}_N(\text{div}) \text{ at } N=0\end{aligned}\tag{7.4.19}$$

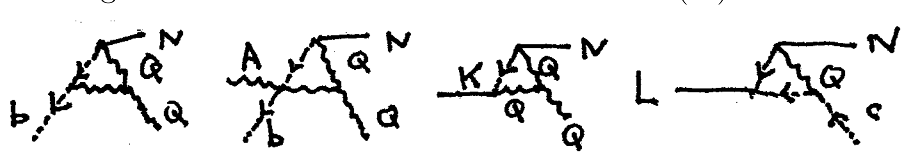
We must now come to grips with renormalization. It is at this point that a surprise occurs. **If** multiplicative renormalization were to hold, the $\hat{\Gamma}\hat{\Gamma}$ equation in (7.4.13) would lead to a series of identities among the Z factors, including a Z factor for N (defined by $N={Z_N}^{1/2}N_{\text{ren}}$)

$$Z_QZ_KZ_N=Z_{gh}Z_LZ_N=Z_AZ_N{Z_M}^{-1}=1 \tag{7.4.20}$$

This would imply that none of the terms in $\mathcal{L}(N)$ would renormalize where we recall

$$\mathcal{L}(N)=N(b\cdot D(A)Q+K\cdot Q+aLc) \tag{7.4.21}$$

It is clear that the vertices NKQ and NLc cannot contribute to proper graphs. However, it is easy to write down triangle graphs with one vertex $NbD(A)Q$ which produce divergences of the same form as the terms in $\mathcal{L}(N)$.



(7.4.22)

These diagrams are (logarithmically) divergent by power counting. Hence, the terms with N are not multiplicatively renormalizable! Rather the most general N-dependent divergences at the n-loop level which satisfy the antighost equation (7.4.14) and which are background gauge invariant are given by

$$\hat{\Gamma}(\mathrm{div}, N) = N\left[z_{11}(K^\mu - D^\mu(A)b)Q_\mu + z_{12}Lc\right] \tag{7.4.23}$$

where all fields and coupling constants except N have been $(n-1)$-loop renormalized. This is the way composite operators renormalize when they mix under renormalization. In addition to the Z factors which are produced when one renormalizes the basis fields and constants which make up the composite operator, there is also an overall Z factor, here corresponding to z_{11} and z_{12}. Hence, we have here a case where certain terms in the action which seem on the same footing as the other terms in the action, nevertheless should be viewed as composite operators.[9]

Renormalizing all fields and constants in $\hat{\Gamma}_{\mathrm{div}}$ at $N = 0$ with n-loop Z factors, see (7.3.25), we find that the n-loop divergences in the $(n-1)$-loop renormalized effective action with one insertion of $\mathcal{L}(N)$ satisfy the following equation (this is (7.4.19) with (7.3.25) substituted)

$$\begin{aligned}
&-\int 2\xi\frac{\partial}{\partial\xi}\Bigg[Z_A\mathcal{L}_{YM}(A+\zeta Q, u)\\
&+Z_{gh}(K^\mu - D^\mu(A)b)\left(D_\mu(A+\zeta Q)c + \frac{1}{\zeta Z_{gh}}M_\mu\right) + Z_{gh}\zeta L\frac{1}{2}u\, c\times c\Bigg]d^4x\\
&=\int\Bigg[\left(\frac{\partial}{\partial Q_\mu}S_{cl}(A+Q) + uc\times D^\mu(A)b - uc\times K^\mu\right)Q_\mu z_{11}\\
&+\left(D_\mu(A+Q)c + M_\mu\right)\ (K^\mu - D^\mu(A)b)\, z_{11} - (D_\mu(A+Q)D^\mu(A)b)\, c\, z_{12}\\
&-\left(\frac{1}{2}u\, c\times c\right)Lz_{12}\Bigg]\, d^4x
\end{aligned} \tag{7.4.24}$$

Here $\zeta = Z_Q^{1/2}/{Z_A}^{1/2}$, and all fields, sources and ξ are $(n-1)$-loop renormalized. We used that $\hat{\Gamma}(\mathrm{div})$ is equal to $-\Delta S$ at the n-loop level, and $2\xi\frac{\partial}{\partial\xi}\Delta S = 2\xi\frac{\partial}{\partial\xi}(\hat{S} + \Delta S)$.

[9] In [9], it is proposed to renormalize N such that $N(K - D(A)b)Q = z_{11}N_{\mathrm{ren}}(K_{\mathrm{ren}} - D(A)b)Q_{\mathrm{ren}})$. So, $z_{11} = (Z_N Z_Q Z_{gh})^{1/2}$. Then the freedom in the value of a is used to also make the term $aNLc$ multiplicatively renormalizable: $aNLc = z_{12}N_{\mathrm{ren}}L_{\mathrm{ren}}c_{\mathrm{ren}}$. So $a = Z_a a_{\mathrm{ren}}$ with $Z_a z_{11} = z_{12}$.

This is the Ward identity which will give information about the ξ dependence of Z factors. Putting all fields and sources except A_μ equal to zero, we find

$$\frac{\partial}{\partial\xi} Z_A S_{YM}(A) = 0 \tag{7.4.25}$$

Since $S_{YM}(A)$ is evidently ξ independent, we arrive at

$$\frac{\partial}{\partial\xi} Z_A = 0 \tag{7.4.26}$$

This is the main result of this section, and implies that the β function is gauge-parameter independent.

Further information is obtained by equating coefficients in different sectors. Here several consistency conditions are satisfied: there are 5 relations for the 2 parameters z_{11} and z_{12}. One finds that the Z factors for the composite operators are ξ-derivatives of the Z factors of the elementary fields

$$2\xi\frac{\partial}{\partial\xi}(Z_{gh}\zeta) = -z_{12}; \qquad 2\xi\frac{\partial}{\partial\xi}\zeta = -z_{11} \tag{7.4.27}$$

Together with $\partial/\partial\xi\ Z_A = 0$ this is all the extra information one obtains from the extension of the BRST transformations to include the gauge parameter ξ.

Let us now discuss how to proceed if there is no regularization scheme available that maintains BRST symmetries [6, 12] (or if one uses a scheme that breaks BRST symmetry). Suppose that up to and including the $(n-1)$-loop level the extended $\{\Gamma,\Gamma\} = 0$ equation holds. Then at the n-loop level one finds possible anomalies Δ, namely

$$\{\Gamma,\Gamma\} = 2Q'\Gamma^{(n)} = \hbar^n\Delta \ \ ; \Delta = \Delta^{(1)} + N\Delta^{(2)} \tag{7.4.28}$$

As we shall show, $\Delta^{(1)}$ is BRST trivial (if there are no gauge anomalies), while $\Delta^{(2)}$ is either BRST trivial, or it contains cohomological classes. The latter are, however, of the form $\Delta^{(2)}(coh) = \frac{\partial}{\partial g}S_{cl}$ where $S_{cl} = \frac{1}{g^2}\int F^2 d^4x$. (We thus use here $A' = gA$ instead of A, and then g only appears in front of S_{cl}).

The nilpotent BRST operator

$$Q = \left(\frac{\partial}{\partial Q} \hat{\Gamma} \frac{\partial}{\partial K} \cdots - \frac{\partial}{\partial L} \hat{\Gamma} \frac{\partial}{\partial c} \right) - M \frac{\partial}{\partial A} \tag{7.4.29}$$

is extended to another nilpotent operator

$$Q' = Q + N' \frac{\partial}{\partial \xi} \ \ ; \left\{ Q, N' \frac{\partial}{\partial \xi} \right\} = 0 \tag{7.4.30}$$

(Q is ξ independent because the terms with K, Q, L and c in S_{qu} are ξ independent, and $\hat{\Gamma}$ is independent of the gauge fixing term). The consistency condition on Δ reads

$$Q' \Delta = 0 \tag{7.4.31}$$

Decomposing $(Q + N' \frac{\partial}{\partial \xi})(\Delta^{(1)} + N' \Delta^{(2)}) = 0$ we find two relations

$$Q \Delta^{(1)} = 0 \ \ ; \frac{\partial}{\partial \xi} \Delta^{(1)} - Q \Delta^{(2)} = 0 \tag{7.4.32}$$

We know that the solution of $Q\Delta^{(1)} = 0$ is $\Delta^{(1)} = QP + A$ where A denotes genuine gauge anomalies (if present). Substituting this result for $\Delta^{(1)}$ leads to $\frac{\partial}{\partial \xi}(QP + A) = Q\Delta^{(2)}$. Since $\{\frac{\partial}{\partial \xi}, Q\} = 0$ this becomes

$$\frac{\partial}{\partial \xi} A = Q \left(\Delta^{(2)} - \frac{\partial}{\partial \xi} P \right) \tag{7.4.33}$$

The gauge anomalies do not depend on the gauge parameter ξ, hence

$$Q \left(\Delta^{(2)} - \frac{\partial}{\partial \xi} P \right) = 0 \tag{7.4.34}$$

Since the only nontrivial cohomology in the sector with ghost number zero are the gauge invariant terms in the action if one is dealing with simple groups[10], we find

$$\Delta^{(2)} - \frac{\partial}{\partial \xi} P = X \text{ (gauge inv) } + QY \tag{7.4.35}$$

[10]If there are $U(1)$ groups, one can write down further BRST but not gauge invariant terms which are not BRST trivial.

We have thus obtained that

$$\begin{aligned}\Delta &= QP + A + N'\left(X + QY + \frac{\partial}{\partial\xi}P\right)\\ &= A + \left(Q + N'\frac{\partial}{\partial\xi}\right)P + \left(Q + N'\frac{\partial}{\partial\xi}\right)(-N'Y) + N'X\\ &= A + Q'(P - N'Y) + N'X \end{aligned} \tag{7.4.36}$$

The Q' terms can be removed by adding $-P + N'Y$ as counter terms to the action (at this point we do additive renormalization). The anomaly terms A are there and stay; for a consistent theory they should be absent. The main point of this discussion is that the gauge invariant terms X can all be written as $\frac{\partial}{\partial g}S_{cl}$, or $\frac{\partial}{\partial m^2}S_{cl}$ if there are fermions with masses. (The Dirac action is BRST trivial, as one might guess from the fact that it vanishes on-shell. Namely $\bar\psi \not{D}(A+Q)\psi = Q'\int(\bar\psi^*\psi + \bar\psi\psi^*)d^4x - \bar\psi^*\frac{\partial}{\partial\bar\psi^*}S_{qu} - \partial S_{qu}/\partial\psi^*\psi^*$ where $\bar\psi^*$ and ψ^* are the BRST sources for the fermions). Rescaling g, m^2 (and $\bar\psi^*$ and ψ^*), all these terms can be removed. This procedure eliminates the potential anomalies Δ loop-by-loop from the $\{\Gamma,\Gamma\}$ equations if there are no genuine gauge anomalies.

5 Calculation of the β function at two loops

The beta function is defined by $\beta = \mu\frac{\partial}{\partial\mu}u|_{g,\xi,M}$, where μ is the renormalization mass which appears in dimensional regularization in the relation between the unrenormalized and renormalized coupling constant, $g = Z_g u \mu^{\frac{1}{2}\epsilon}$ with $\epsilon = 4 - n$. One should differentiate w.r.t. μ while keeping all bare (unrenormalized) parameters constant (g, ξ and M, where M denotes possible masses). For fixed ϵ-independent g, and ϵ-dependent Z_g and $\mu^{\frac{1}{2}\epsilon}$, also u is ϵ-dependent. Hence this function β is a function of u and $\epsilon, \beta = \beta(u,\epsilon)$. By applying the chain rule to $\mu\frac{\partial}{\partial\mu}g = 0$ one finds

$$\frac{1}{2}\epsilon u Z_g + \beta(u,\epsilon)\frac{d}{du}(uZ_g) = 0 \tag{7.5.1}$$

from which one obtains

$$\beta(u,\epsilon) = \frac{-\frac{1}{2}\epsilon}{\frac{d}{du}\ln uZ_g} \tag{7.5.2}$$

We used that $\beta(u,\epsilon)$ is independent of ξ_{ren}, a result which we proved in the previous section. By its definition, $\beta(u,\epsilon)$ is nonsingular in ϵ. We shall use minimal subtraction so that Z_g consists of a power series in $1/\epsilon$. Substituting the expansions

$$uZ_g = u + \sum_{k=1}^{\infty} a_k(u)\epsilon^{-k}; \beta(u,\epsilon) = \sum_{k=0}^{\infty} b_k(u)\epsilon^k \tag{7.5.3}$$

into (7.5.1) one discovers that $\beta(u,\epsilon)$ has only a term linear in ϵ and a term which is independent of ϵ

$$\beta(u,\epsilon) = -\frac{1}{2}\epsilon u + \beta(u);\ \beta(u) = \beta_0\frac{u^3}{16\pi^2} + \cdots \tag{7.5.4}$$

Often one means the function $\beta(u)$ when one speaks of "the β function". From (7.5.1) one finds the recursion relations

$$u^2\frac{d}{du}(a_{k+1}(u)/u) = 2\beta(u)\frac{d}{du}a_k(u) \tag{7.5.5}$$

which shows that the coefficients of the higher order poles in $Z_g = \sum_{j=0}^{\infty} Z_g^{(j)}\epsilon^{-j}$ are determined by the coefficients of the first order poles. These first order poles are, of course, the most difficult to compute, and yield the beta function

$$\beta(u) = \frac{1}{2}u^2\frac{d}{du}{Z_g}^{(1)} \tag{7.5.6}$$

In ordinary quantum gauge field theory, the anomalous dimension γ_Q of a gauge field ${Q_\mu}^a$ is defined by $\gamma_Q = \mu\frac{\partial}{\partial\mu}\ln Z_Q^{1/2}$. It is a function of u and $\xi_{\text{ren}}, \gamma_Q = \gamma_Q(u,\xi_{\text{ren}})$. In the background field formalism, the anomalous dimension of the background field, γ_A, is related to the β function as a result of the relation $Z_g{Z_A}^{1/2} = 1$

$$\begin{aligned}\gamma_A &= \frac{1}{2}\mu\frac{\partial}{\partial\mu}\ln Z_A = -\beta(u,\epsilon)\frac{\partial}{\partial u}\ln Z_g = \frac{1}{2}u\frac{d}{du}{Z_g}^{(1)} = \frac{1}{u}\beta(u)\\ \beta(u) &= u\gamma_A(u)\end{aligned} \tag{7.5.7}$$

Using $Z_g^{(1)} = -\frac{1}{2}Z_A^{(1)}$ yields then **an expression for the beta function in terms of the first order poles of Z_A**

$$\beta(u) = -\frac{1}{4}u^2\frac{d}{du}Z_A{}^{(1)} \tag{7.5.8}$$

One can also obtain these results directly from (7.5.2). If $Z_g{}^2$ is given by

$$Z_g{}^2 = 1 + \frac{a}{\epsilon}\frac{u^2}{16\pi^2} + \left(\frac{b}{\epsilon^2} + \frac{c}{\epsilon}\right)\left(\frac{u^2}{16\pi^2}\right)^2 + \mathcal{O}(u^6) \tag{7.5.9}$$

then (7.5.2) yields

$$\beta(u,\epsilon) = -\frac{1}{2}\epsilon u + \frac{1}{2}\frac{au^3}{16\pi^2} + \frac{cu^5}{(16\pi^2)^2} - \frac{1}{\epsilon}(a^2-b)\frac{u^5}{(16\pi^2)^2} + \mathcal{O}(u^7) \tag{7.5.10}$$

and $a^2 = b$ for consistency. Using $Z_g{}^2 = Z_A{}^{-1}$, we find that **in the background field formalism Z_A has no ϵ^2 pole at the two loop level**

$$Z_A = 1 - \frac{a}{\epsilon}\frac{u^2}{16\pi^2} - \frac{c}{\epsilon}\left(\frac{u^2}{16\pi^2}\right)^2 + \mathcal{O}(u^6) \tag{7.5.11}$$

It is customary to parametrize $\beta(u)$ as

$$\beta(u) = u\left[\beta_0\left(\frac{u^2}{16\pi^2}\right) + \beta_1\left(\frac{u^2}{16\pi^2}\right)^2 + \mathcal{O}(u^6)\right] \tag{7.5.12}$$

Then

$$\beta_0 = \frac{1}{2}a, \beta_1 = c \tag{7.5.13}$$

In ordinary field theory $Z_g = Z_1/Z_3{}^{3/2}$, and β_0 and β_1 have been computed using the 2 loop results for Z_1 and Z_3 (the vertex corrections and wave function renormalization constants), with the result that for pure gauge theory (gauge theory without matter couplings)

$$\beta_0 = \frac{-11}{3}C_2(G)\ ,\ \beta_1 = -\frac{34}{3}C_2(G)^2 \tag{7.5.14}$$

Here $C_2(G)$ is the quadratic Casimir operator, defined by

$$f^a{}_{bp}f^b{}_{aq} = -C_2(G)\delta_{pq} \tag{7.5.15}$$

if in the defining represention of $SU(N)$ one normalizes the generators to $Tr\; T_aT_b = -\frac{1}{2}\delta_{ab}$. (So, for $SU(2)$, T_a are $-\frac{i}{2}\tau_a$ with τ_a the Pauli matrices. Then $f^a{}_{bc} = \epsilon_{abc}$, and $C_2(G) = 2$. For $SU(n)$ one has $C_2(G) = n$).

We shall now compute the $\frac{1}{\epsilon}$ terms in Z_A to two loop order. This will give the β function according to (7.5.8). The result should be the same as (7.5.14) because $\beta(u)$ is gauge-choice independent. Since the computation of Z_1 in ordinary quantum field theory is much more complicated than the calculation of Z_A in the background field method, the latter method greatly simplifies the calculation of β.

The easiest way to determine Z_A is to calculate the $1/\epsilon$ divergences in selfenergy graphs with two external A fields. The Q-propagators in these graphs are obtained from the Q^2 terms in the action. Of course one could also (in principle at least) compute the full Q propagators in the background of A fields, but propagators can also be written as an infinite series of Q propagators with AQQ vertices, and the more AQQ vertices, the less divergent the graphs are. Power counting shows that only graphs with 2,3 and 4 external A fields are divergent. It does not matter whether one uses the set of graphs with two or more external A fields because the third or fourth A field only enter in the combination gA which does not renormalize. Of course the set of graphs with only two external A fields is easiest to calculate, and that is what we shall do.

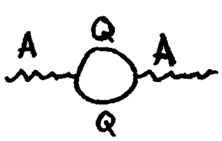
To compute graphs with only external $A_\mu{}^a$ fields, but no external $Q_\mu{}^a, b_a$ and c^a fields, it is useful not to replace $Q_\mu{}^a, b_a$ and c^a by $Z^{1/2}Q_\mu{}^{a,\mathrm{ren}}, Z_{gh}{}^{1/2}b_a{}^{\mathrm{ren}}$ and $Z_{gh}{}^{1/2}c_{\mathrm{ren}}{}^a$ in the path integral since this avoids having to determine Z_Q and Z_{gh}. (For selfenergy graphs with external *quantum* lines these would have to be calculated.) One can also understand from a diagrammatic approach why the Z_Q and Z_{gh} cancel when Q and b, c fields only appear inside loops. If one does not decompose the action into $\mathcal{L}(\mathrm{ren}) + \Delta\mathcal{L}(\mathrm{ren})$, but directly writes all unrenormalized quantum fields $\phi = \{Q, b, c\}$ as $\sqrt{Z}\phi_{\mathrm{ren}}$, then at each vertex each quantum field acquires a factor $\sqrt{Z}$, but each (quantum) propagator acquires a factor $\frac{1}{Z}$, and since all quantum fields at the vertices are contracted to yield propagators, all Z factors of quantum fields cancel.

The renormalization of ξ requires the computation of Z_Q. If one would calculate at arbitrary ξ_{ren}, see Capper and McLean [27], one could in the end take the limit $\xi_{\text{ren}} \to 0$ (the Landau gauge) and in that case one would not even have to compute Z_Q (because if $\xi \to 0$, also $Z_\xi \xi_{\text{ren}} \to 0$). However, it is far easier to work with $\xi_{\text{ren}} = 1$ at the expense of having to calculate Z_Q. The Z factors Z_g and Z_A would enter at the two-loop level from renormalizing the $gbAc$ and $gAQQ$ vertices in one-loop graphs, but they always occur in the combination $Z_g Z_A^{1/2}$, and because $Z_g Z_A^{1/2} = 1$ there are no counter terms with Z_g and Z_A. The action from which one computes Z_A at the two-loop level is thus given by

$$\begin{aligned} S &= -\frac{1}{4}\left[F_{\mu\nu}\left(A_\mu{}^{\text{ren},a} + Q_\mu{}^a, u\right)\right]^2 \\ &- \frac{1}{2Z_Q\xi_{\text{ren}}}\left[D^\mu\left(A^{\text{ren}}\right)Q_\mu{}^a\right]^2 - (D^\mu(A^{\text{ren}})b_a)(D_\mu(A^{\text{ren}}+Q)c^a) \end{aligned} \tag{7.5.16}$$

Terms with only one Q field do not contribute since Z_A is computed from proper graphs.

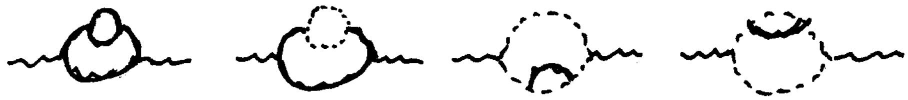
At the one-loop level, the diagrams needed to compute Z_A are given by



$$Z_A = 1 + 2\left(\frac{10}{3} + \frac{1}{3}\right)\frac{u^2 C_2(G)}{16\pi^2(4-n)} \tag{7.5.17}$$

No tadpole graphs contribute, since massless tadpoles vanish in dimensional regularization. Each graph is separately transversal (since S(ghost) and S(fix) are separately gauge invariant) and separately ξ independent, and the usual $\beta_0 = -\frac{11}{3}C_2(G)$ gets a factor $-\frac{10}{3}$ from the gauge loop and a factor $-\frac{1}{3}$ from the ghost loop. In ordinary field theory, only the sum of the Q-loop and the bc loop is transversal, and $\beta_0 = -\frac{11}{3}C_2(G)$ is obtained from $Z_g = Z_1/(Z_3)^{3/2}$ where Z_1 gives a factor $\frac{4}{3}C_2(G)$ and Z_3 gives a factor $\frac{10}{3}C_2(G)$.

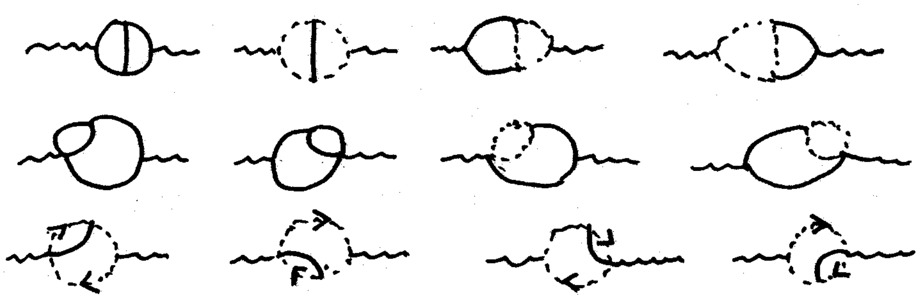
At the two-loop level there are graphs with a selfenergy correction inserted into the one-loop graph



(7.5.18)

Wiggly lines denote A fields, solid lines Q fields, and dotted lines ghosts.

The graphs with a vertex correction inserted into the one-loop graph are given by



(7.5.19)

There are also graphs with a counter term obtained by expanding Z_Q^{-1} in

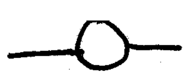
$$\mathcal{L}_{\text{fix}} = -\frac{1}{2}(Z_Q \xi_{\text{ren}})^{-1}(D_\mu(A)Q)^2 \tag{7.5.20}$$

These graphs are

(7.5.21)

There are no contributions from graphs with a counter term of the form  because such contributions are not one-particle irreducible.

The value of Z_Q is computed from the QQ selfenergy graphs

 $Z_Q = 1 + \dfrac{10}{3}\dfrac{u^2 C_2(G)}{16\pi^2\ (4-n)}$  (7.5.22)

The vertex counter term follows from $\Delta\mathcal{L} = -\frac{1}{2}(Z_Q^{-1} - 1)\xi_{\text{ren}}^{-1}(D_\mu(A)Q^\mu)^2$. Since the AQQ vertices are different from the QQQ vertices, Z_Q differs from Z_A. (Note that for the computation of Z_Q is the same as in ordinary field theory. There the factor $\frac{10}{3}$ for Z_3 combines with a factor $\frac{4}{3}$ for Z_1 to yield the familiar $-\frac{11}{3}$ of β_0 as we already mentioned: $\frac{4}{3} - \frac{3}{2} \cdot \frac{10}{3} = -\frac{11}{3}$).

There is subtlety concerning the counter terms. Only the transversal part of the propagator in gauge theories renormalizes, but not the longitudinal part. We claimed that one can drop the factors Z_Q from the action because in a one-loop graph for every factor Z_Q at a vertex there is a factor Z_Q^{-1} in the propagator. Let us study this argument further. If one renormalizes both Q^a_μ and as always ξ, the counter terms are given by

$$\Delta\mathcal{L}(Q^2 - \text{terms}) = -\frac{1}{4}(Z_Q - 1)G^2_{\mu\nu}(Q, A)^2 \tag{7.5.23}$$

There are the usual counter terms for the QQ selfenergy and the AQQ vertex, and the counter term for the Q-selfenergy is transversal. In this approach one treats the counter terms as in ordinary field theory; in particular, one does not consider possible cancellations of factors Z_Q and Z_Q^{-1} at vertices and in propagators. On the other hand, one can start from the two terms $-\frac{1}{4}Z_Q G^2_{\mu\nu}(Q, A) - \frac{1}{2\xi}(D_\mu(A)Q^\mu)^2$ and rewrite the last term such that one obtains

$$Z_Q\left[-\frac{1}{4}G^2_{\mu\nu}(Q, A) - \frac{1}{2\xi}(D_\mu(A)Q^\mu)^2\right] + (Z_Q - 1)\frac{1}{2\xi}(D_\mu(A)Q^\mu)^2 \tag{7.5.24}$$

The first term leads to a propagator proportional to Z_Q^{-1} and can be entirely omitted for reasons explained, while the second term yields the counter terms. Note that now the counter term for the selfenergy is longitudinal. So one should obtain the same result for closed Q loops whether one uses $\Delta\mathcal{L} = -\frac{1}{4}(Z_Q - 1)G^2_{\mu\nu}(Q, A)$ or $\Delta\mathcal{L} = (Z_Q - 1)\frac{-1}{2\xi}(D_\mu(A)Q^\mu)^2$. In both cases the propagators are ξ-dependent, and it has been checked at the two-loop level that the sum of the result for the sum of both graphs in (7.5.21) (but not the results for each graph separately) are the same [24].

In fact, the sum of the three graphs in (7.5.21) with counter term insertions (but not each graph separately) is proportional to ξ. (The $\frac{1}{\xi}$ singularities in the vertices are cancelled by the ξ in the propagator $[(\eta_{\mu\nu} - k_\mu k_\nu/k^2) + \xi k_\mu k_\nu/k^2](k^2 - i\epsilon)^{-1}$.) Hence one could have calculated all graphs for arbitrary ξ, and by dropping the graphs with Z_Q and taking the ξ-independent terms in the remainder, one would have gotten the correct result. Another way of doing the same is to take the limit $\xi \to 0$ in all graphs; again, since the graphs with Z_Q contain only terms of order ξ, ξ^2 and ξ^3, one can omit the graphs with Z_Q counter terms. The limit $\xi \to 0$ leads to the Landau propagator with $\eta_{\mu\nu} - k_\mu k_\nu/k^2$ is the numerator, and is sometimes used in theoretical arguments, but from a practical point of view calculations with $\xi_{\text{ren}} \neq 1$ are a nightmare. It is much easier to work with $\xi_{\text{ren}} = 1$, at the expense in our case of the two extra diagrams with a Z_Q counter term.

Finally, there are the two loop graphs which are not obtained from the one-loop graphs.

 (7.5.25)

We shall now discuss the evaluation of these one- and two-loop graphs using ordinary dimensional regularization. The basic n-dimensional Minkovski integral

$$I_M = \int d^n k/(k^2 + M^2 - i\epsilon)^\alpha \tag{7.5.26}$$

is Wick rotated to a Euclidean integral with $d^n k \to i d^n k_E$. (Recall that $d^n k = d\vec{k}\; dk_0$ and $t \to -i\tau$ while $k_0 \to i k_0^E$ such that $\vec{k} \cdot \vec{x} + k_0 x^0 \to \vec{k} \cdot \vec{x} + k_0^E \tau$.) The Euclidean integral itself is evaluated by first writing it in terms of polar coordinates $\int_0^\infty dk k^{n-1} \Omega_n/(k^2 + M^2)^\alpha$. The angular factor Ω_n follows from

$$\prod_{j=1}^n \int_{-\infty}^\infty e^{-x_j{}^2} dx_j = \pi^{n/2} = \int_0^\infty dr r^{n-1} \Omega_n e^{-r^2} = \frac{1}{2}\int_0^\infty dr^2 (r^2)^{\frac{n}{2}-1} e^{-r^2} \Omega_n = \frac{1}{2}\Gamma\left(\frac{n}{2}\right)\Omega_n$$

$$\Omega_n = 2\pi^{n/2}/\Gamma\left(\frac{n}{2}\right); \Gamma(x) = \int_0^\infty d\tau\, \tau^{x-1} e^{-\tau} \tag{7.5.27}$$

It is easy to check this result for $n = 2$ using $\Gamma(1) = 1$. The integral over k^2 can be evaluated by changing variables

$$k^2 = \frac{z}{1-z}, k^2 + 1 = \frac{1}{1-z}, dk^2 = dz/(1-z)^2 \tag{7.5.28}$$

and yields a beta function

$$B(\alpha, \beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx = \Gamma(\alpha)\Gamma(\beta)/\Gamma(\alpha+\beta) \tag{7.5.29}$$

In this way one obtains

$$I_M = i\frac{\pi^{n/2}\Gamma(\alpha - \frac{n}{2})}{(M^2)^{\alpha-\frac{n}{2}}\Gamma(\alpha)} \tag{7.5.30}$$

The quadratic divergence for $\alpha = 1$ shows up as a pole (= logarithmic divergence) at $n = 2$, while the logarithmic divergence for $\alpha = 2$ shows up as a pole at $n = 4$. This explains the factor $\Gamma(\alpha - \frac{n}{2})$. The factor $(M^2)^{n/2-\alpha}$ follows from the dimensions of the integral, and the factor $\Gamma(\alpha)^{-1}$ is needed because for $M = 1$ and fixed $\alpha - \frac{n}{2}$, the integral decreases for increasing α. Finally the factor i is due to the Wick rotation of dk_0.

Let us say a few words more about this Wick rotation. In Feynman diagrams, one has propagators of the form $[-(\sum \alpha_i \ell_{io} + \sum \beta_j p_{jo})^2 + (\sum \alpha_i \vec{\ell}_i + \sum \beta_j \vec{p}_j)^2 - i\epsilon]^{-1}$, where $\ell_{i\mu}$ are the loop momenta and $p_{j\mu}$ the external momenta. After rotating all the energies **simultaneously** over an angle $e^{i\theta}$ and extracting an overall factor $e^{-2i\theta}$, the resulting expression for the denominator

$$-\left(\sum \alpha_i \ell_{i0} + \sum \beta_j p_{j0}\right)^2 + \left(\sum \alpha_i\ \vec{\ell}_i + \sum \beta_j \vec{p}_j\right)^2 e^{-2i\theta} - i\epsilon e^{-2i\theta} \tag{7.5.31}$$

never vanishes for $0 \leq \theta \leq \pi/2$ since its imaginary part is never zero. Thus, for each θ the integral is well-defined and this yields one possible analytic continuation from Minkovskian to Euclidean Green's functions. For convergent integrals (small n) one should be able to show that the results for the integrals corresponding to θ and $\theta + d\theta$ can be obtained from each other by only rotating the energies of the external particles over an angle $e^{id\theta}$. Suppose one begins with the integral at θ and rotates in this expression all p_{j0} to $p_{j0}e^{id\theta}$ without encountering any singularities; then one must show that the ℓ_{jo} integrals over the hyperplane with θ are equal to those over the hyperplane with $\theta + d\theta$. The difference would be proportional to integrals of all the l_{j0} from θ to $\theta + d\theta$ at fixed large radius. This contribution should vanish, but it is not easy to prove this in general because it is a problem in the theory of several complex variables. However, one may check it for special cases separately.

For $\theta = \pi/2$, one finds Euclidean Green's functions which one evaluates with, for example, dimensional regularization, and to obtain the Minkovski Green's functions, one must rotate the p_{j0} back. Then one ends up with the same four-momenta p_j^μ as one started with since a phase $+i$ for the Wick rotation is compensated by a phase $-i$ for the rotation back to Minkovski spacetime.

For loop calculations, one needs to combine propagators using Feynman's trick

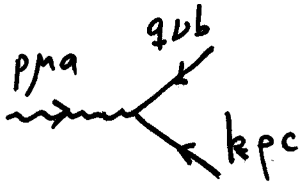
$$\frac{1}{A^\alpha B^\beta} = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 dx \frac{x^{\alpha-1}(1-x)^{\beta-1}}{[Ax + B(1-x)]^{\alpha+\beta}} \tag{7.5.32}$$

(There are various proofs of this formula. One may substitute $x = \frac{t}{1-t}$ and then $\frac{At}{B} = \frac{u}{1-u}$ to obtain the beta function $B(\alpha, \beta)$. Or one may use that the operator $A\frac{\partial}{\partial A} + \alpha$ yields zero when it acts on left-hand side, while on the right-hand side one obtains a total x-derivative whose integral vanishes. Finally one may prove the result first for integer α and β and then continue analytically).

Let us now, as a warming up for the two-loop calculation, first compute the one-loop contribution to Z_A. The vertices one needs are the AQQ vertex and the Abc vertex. The former gets contributions from the gauge action but also from the gauge fixing term. The AQQ vertex from the gauge action is obtained from the QQQ vertex of ordinary gauge theory by identifying one of the Q fields as the A field in all 3 possible ways

$$\mathcal{L}_{int} = -\partial_\mu A_\nu{}^a u f_{abc} Q_\mu{}^b Q_\nu{}^c - \left(\partial_\mu Q_\nu{}^a - \partial_\nu Q_\mu{}^a\right) u f_{abc} A_\mu{}^b Q_\nu{}^c \tag{7.5.33}$$

Further, $\mathcal{L}(\text{fix})$ yields the vertex $-\frac{1}{\xi_{\rm ren}} u f_{abc} A_\mu{}^a Q^{\mu b} \partial^\nu Q_\nu{}^c$. Together they yield

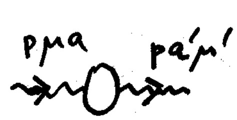


$$= iuf_{abc}[\eta_{\mu\nu}(p-q-\xi_{\rm ren}^{-1}k)_\rho + \eta_{\nu\rho}(q-k)_\mu + \eta_{\rho\mu}(k-p+\xi_{\rm ren}^{-1}q)_\nu] \tag{7.5.34}$$

We shall use $\xi_{\rm ren}=1$, in which case the vertex simplifies upon substituting $k=-p-q$

$$V(p\mu a; q\nu b, k\rho c) = iuf_{abc}[\eta_{\mu\nu}2p_\rho + \eta_{\nu\rho}(p+2q)_\mu - \eta_{\rho\mu}2p_\nu] \tag{7.5.35}$$

The AA selfenergy graph due to a Q loop then yields



$$= \int \frac{1}{2} i^2 V \begin{pmatrix} p\mu a \\ q\nu b \\ k\rho c \end{pmatrix} \left(\frac{-i\eta_{\nu\nu'}\delta_{bb'}}{q^2-i\epsilon}\right)\left(\frac{-i\eta_{\rho\rho'}\delta_{cc'}}{k^2-i\epsilon}\right) V \begin{pmatrix} -pa'\mu' \\ -q\nu' b' \\ -k\rho' c' \end{pmatrix} \frac{d^n k}{(2\pi)^n} \tag{7.5.36}$$

The factor $\frac{1}{2}$ is a combinatorial factor which accounts for the fact that there are only 2 ways to contract the Q's, whereas in each V we have 2 contractions of Q's with a given external line. One obtains for the numerator $N^{aa'}_{\mu\mu'}$ of the integrand in (7.5.36)

$$\begin{aligned} N^{aa'}_{\mu\mu'} &= \frac{1}{2} u^2 \delta^{aa'} C_2(G)\,[\eta_{\mu\nu}2p_\rho + \eta_{\nu\rho}(p+2q)_\mu - \eta_{\rho\mu}2p_\nu] \times \\ &\quad [\eta_{\mu'\nu}2p_\rho + \eta_{\nu\rho}(p+2q)_{\mu'} - \eta_{\rho\mu'}2p_\nu] \\ &= \frac{1}{2} u^2 \delta_{aa'} G_2(G)\left[8\left(\eta_{\mu\mu'}p^2 - p_\mu p_{\mu'}\right) + n(p+2q)_\mu (p+2q)_{\mu'}\right] \end{aligned} \tag{7.5.37}$$

(The two factors i in each V have canceled the minus signs in the second V).

Since background gauge invariance tells us that the result will be transversal, this result should after integration over k be of the form $N_{\mu\mu'} = (\eta_{\mu\mu'} - p_\mu p_{\mu'}/p^2)\Pi(p^2)$. We may then simplify the calculation by contracting $N_{\mu\mu'}$ with the n-dimensional metric $\eta^{\mu\mu'}$. This yields

$$\Pi(p^2) = \left(\frac{1}{n-1}\right)\frac{1}{2} u^2 \delta_{aa'} C_2(G)\left[8(n-1)p^2 + n(p^2+4p\cdot q+4q^2)\right] \tag{7.5.38}$$

Next we use the information that tadpole integrals such as $\int d^n q/(p+q)^2$ and $\int d^n q/q^2$ vanish in dimensional regularization. Hence we drop the term with q^2 in (7.5.38) and replace $p\cdot q$ by $-\frac{1}{2}p^2$ (using $p\cdot q = \frac{1}{2}(p+q)^2 - \frac{1}{2}p^2 - \frac{1}{2}q^2$). Then the selfenergy becomes

$$\begin{aligned} &= \left(p^2\eta_{\mu\mu'} - p_\mu p_{\mu'}\right)\left(\frac{1}{n-1}\right)\frac{1}{2}u^2\delta_{aa'}C_2(G)[7n-8]\left\{\int \frac{1}{(2\pi)^n}\frac{d^n q}{(p+q)^2 q^2}\right\} \\ &= \left(p^2\eta_{\mu\mu'} - p_\mu p_{\mu'}\right)u^2\delta_{aa'}C_2(G)\left[\frac{20}{6}\right]\left\{\frac{2i}{16\pi^2\epsilon}\right\} + \mathcal{O}(1) \end{aligned} \tag{7.5.39}$$

where $\epsilon = 4 - n$. The vertex $-\frac{1}{4}Z_A(\partial_\mu A_\nu{}^a - \partial_\nu A_\mu{}^a)^2$ contributes $-Z_A i(p^2\eta_{\mu\mu'} - p_\mu p_{\mu'})\delta_{aa'}$ to the selfenergy, and requiring that the sum be finite we find

$$Z_A(Q-\text{loop}) = \frac{20}{3}\frac{1}{4-n}\frac{u^2}{16\pi^2}C_2(G) \tag{7.5.40}$$

The AA selfenergy due to a ghost loop is easier to evaluate. The vertex $\mathcal{L}_{int} = uf_{abc}A_\mu{}^a(\partial^\mu b^b c^c - b^b\partial^\mu c^c)$ yields a result proportional to $(k-q)_\mu(k-q)_{\mu'} = (p+2q)_\mu(p+2q)_{\mu'}$, and using that $p\cdot(p+2q) = 0$ inside the integral (since $p\cdot q \sim -\frac{1}{2}p^2$ as we discussed before), the result is transverse, as expected. In ordinary gauge field theory, the ghost loop contribution to the gauge field selfenergy is, of course, not transversal. Tracing with $\eta^{\mu\mu'}$ one finds

$$Z_A(\text{ghost loop}) = \frac{2}{3}\frac{1}{4-n}\frac{u^2}{16\pi^2}C_2(G) \tag{7.5.41}$$

Adding the results of these two loops, and using $Z_A{}^{-1/2} = Z_g$, we find the familiar result for the one-loop β function[11]

$$\beta_0 = -\frac{11}{3}C_2(G) \tag{7.5.42}$$

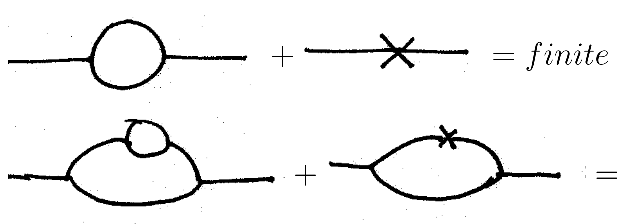
The reader may show that, and explain why, also the AA selfenergy due to a Q loop is separately ξ independent and transversal.

[11] S. Weinberg [16] discusses an alternative way to compute β, namely by using **constant** background fields. Writing the terms quadratic in Q as $Q^a_\mu D^{\mu\nu}_{ab} Q^b_\nu$ where $D^{\mu\nu} = \eta^{\mu\nu}\delta_{ab}D(A)^2 - gf_{ab}{}^c G^{\mu\nu}{}_c(A)$, and idem for the ghosts with $D^a{}_b$, he evaluates $(\det D^{\mu\nu}_{ab})^{-1/2}(\det D^a{}_b)$ for the one-loop effective action. Writing $\det D = \exp tr \ln D$ and expanding D into term with none, one and two A fields, $D = D^{(0)} + D^{(1)} + D^{(2)}$, the factors $(D^{(0)})^{-1}$ become propagators, and one obtains again a kind of Feynman graphs. The calculation is not significantly simpler.

The contribution to the one-loop beta function from a fermion loop is, of course, the same as in ordinary field theory because the Dirac action depends only on $A^a_\mu + Q^a_\mu$. One finds now a result proportional to $TrT^{(R)}_a T^{(R)}_b \equiv -\delta_{ab}T(R)$ where R denotes the representation of the fermions. The trace over the fermion indices is proportional to $k_\mu(k+p)_{\mu'} + k_{\mu'}(k+p)_\mu - \eta_{\mu\mu'}(k^2+m^2+k\cdot p)$, which is to be divided by $[k^2+m^2-i\epsilon][(k+p)^2+m^2-i\epsilon]$. One may again take the trace over μ and μ', and finds then that in the divergence the m^2 terms cancel, as they should because the divergences should be proportional to $F_{\mu\nu}(A)^2 \sim p^2$. Note, however, that because $\int \frac{1}{k^2+m^2}$ is nonvanishing, one can no longer simply drop terms proportional to k^2+m^2 or $(k+p)^2+m^2$. The result for one complex fermion is $\beta_0 = \frac{4}{3}T(A)$.

Consider next the two loop contributions to Z_A. We make a series of comments and technical statements which together should enable to reader to complete the actual two-loop calculation.

The QQ selfenergy insertion in a QQ loop gives, of course, terms with ϵ^{-2} and ϵ^{-1}. In ordinary field theory one would also add a graph with a counter term which makes the selfenergy finite. It is a misconception to expect that the sum of these two graphs therefore will only have a single ϵ^{-1} pole.



$$:= \frac{A}{\epsilon} + \frac{B}{\epsilon^2} \qquad (7.5.43)$$

Figure caption: Even though the one-loop subgraphs are finite, the sum of the two two-loop graphs contains double poles in $\frac{1}{\epsilon}$ in addition to single poles.

The total QQ selfenergy, though finite, is proportional to $\ln(k^2+m^2)$, and in dimensional regularization the integral

$$\int d^n k \ln(k^2+m^2)(k^2-i\epsilon)^{-1}[(p+k)^2-i\epsilon]^{-1} \qquad (7.5.44)$$

has a double pole. To show this use the identity

$$\int d^n q \left(\frac{\partial}{\partial q^\mu} q^\mu\right) f(q) = n \int d^n q f(q) = -\int d^n q\, q^\mu \frac{\partial}{\partial q^\mu} f(q) \tag{7.5.45}$$

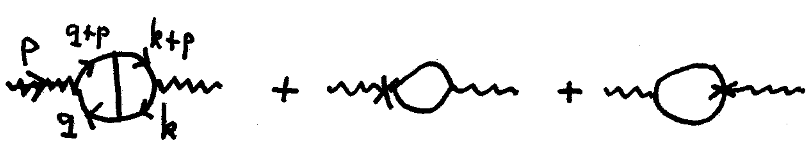
which may be proven by partial integration, using that boundary terms may be dropped in dimensional regularization. Applied to $f(q) = \ln(q^2+m^2)/(q^2+m^2)^\alpha$, one obtains

$$(n-2\alpha)\int \frac{\ln(q^2+m^2)}{(q^2+m^2)^\alpha} d^n q = -2\int \frac{q^2}{(q^2+m^2)^{\alpha+1}} d^n q - 2\alpha m^2 \int \frac{\ln(q^2+m^2)}{(q^2+m^2)^{\alpha+1}} d^n q \tag{7.5.46}$$

Dividing by $n-2\alpha$, and using (7.5.30), it is clear that the first term on the right-hand side has a double pole at $n-2\alpha=0$. In this way one shows that m-loop graphs have poles in $\epsilon = 4-n$ of order $m, m-1, \ldots, 1$, even after all $(m-1)$-loops have been made finite by renormalization.

In practice it is easiest to keep the loop insertions in n-dimensions as factors $(k^2)^{\frac{n}{2}-2}$ and evaluate the integrals for the loop insertions and the counter term insertions separately.

A similar misconception may arise as far as the vertex corrections are concerned. The sum of the contributions from the following graphs



(7.5.47)

contains $\frac{1}{\epsilon^2}$ and $\frac{1}{\epsilon}$ terms. It is incorrect to argue that the counter terms remove the $\frac{1}{\epsilon^2}$ terms for two reasons: one counter term would already suffice for this, and furthermore one-loop integrals of integrands with logarithms yield double poles. Again it is best to evaluate all these graphs separately.

Let us now consider the two-loop graph with overlapping divergences depicted in (7.5.47).

$$I_{\mu\nu} = \int \frac{N_{\mu\nu}[p,q,k]\, d^n k\, d^n q}{(k+p)^2 k^2 (q+p)^2 q^2 (k-q)^2} \tag{7.5.48}$$

where $N_{\mu\nu}$ is a polynomial of degree four in p, q and k. Tracing over μ and ν, and using identities such as $p \cdot k = \frac{1}{2}(k+p)^2 - \frac{1}{2}k^2 - \frac{1}{2}p^2$ yields sums of scalar diagrams, some still with 5 propagators, others with less propagators. Graphs with less than 5 propagators are easy to evaluate. For example

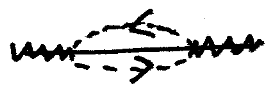
$$\begin{aligned}\int \frac{d^n k d^n q}{k^2 q^2 (k+p)^2 (k-q)^2} &= \int \frac{d^n k}{k^2 (k+p)^2} \left(\int \frac{d^n q}{q^2 (q-k)^2} \right) \\ &= i\pi^{n/2} \Gamma(2 - \frac{n}{2}) B(\frac{n}{2} - 1, \frac{n}{2} - 1) \int \frac{d^n k}{(k+p)^2 (k^2)^{3-\frac{n}{2}}} \end{aligned} \tag{7.5.49}$$

Combining the two denominators one finds an integral over Feynman auxiliary variables $\int dx [x(1-x)]^{\frac{n}{2}-2}$ which yields the beta function. The final k integral can be performed after Feynman combining the denominators, using (7.5.32).

After using the identities for $k \cdot p$ etc., diagrams which still have 5 propagators have $N_{\mu\nu}$ equal to $p^4 \eta_{\mu\nu}$, $k_\mu q_\nu p^2$ or $k_\mu k_\nu p^2$. Using the background gauge invariance, we may trace over (μ, ν) to obtain the coefficient of $(\eta_{\mu\nu} - p_\mu p_\nu / p^2)$. Again graphs with 4 or less propagators are easily evaluated. The only hard integral has a numerator equal to unity.

$$I = \int \frac{d^n k d^n q}{k^2 (k+p)^2 q^2 (q+p)^2 (k-q)^2} \tag{7.5.50}$$

Expanding factors like $(p^2)^{\frac{n}{2}-2} \frac{1}{\epsilon^2}$, one may obtain divergences of the form $\frac{1}{\epsilon} \ln p^2$. Such nonlocal terms should cancel, and this provides a check on the results. Furthermore, single poles with $\ln \pi$ and the Euler constant γ (from expanding $\Gamma(\epsilon)$) cancel in the Z factors. In some diagrams the $\frac{1}{\epsilon^2}$ contributions cancel already by themselves and not only in the sum. For example, in the setting sun diagrams with $3Q$ propagators or one Q and 2 ghost propagators, one only finds a single pole.

 $= \dfrac{A}{\epsilon} + \cdots$;  $= \dfrac{A'}{\epsilon} + \cdots$ (7.5.51)

The reason for this cancellation is known from massless $\lambda\varphi^4$ theory: in a theory with only AQ^3 vertices where massless tadpoles vanish, locality of counter terms requires that $\frac{1}{\epsilon^2}$ poles in this graph vanish.[12] Similar remarks apply to a theory with only massless $AQbc$ vertices.

6 Further applications of the background field method

We now discuss some areas where the background field method has simplified proofs. We begin with an application to renormalization in Yang-Mills theory. It is known that when $U(1)$ groups are present (as in the case of the Standard Model), one can write down exceptional counter terms in ordinary (i.e. non-background) quantum field theory which are BRST nontrivial and yet not gauge invariant [16]. Explicit computations in particular models show that these exceptional divergences do not show up. Using the background field method, one can prove that these counter terms are absent [5].[13]

Another area of applications is supersymmetry. Here one must use additive (algebraic) renormalization, both in x-space and in superspace [11], because there are γ_5 matrices present. Again the background field method simplifies the analysis considerably. In superspace Yang-Mills theory the background field formalism is very useful for the proof of renormalizability. There exists in superspace a dimensionless superfield $V(x,\theta)$ which allows local counterterms with any number of V fields, but requiring background gauge invariance one only needs the constant rescaling of V by

[12] In massive $\lambda\varphi^4$ theory the two graphs   cancel each others nonlocal divergences, but in massless $\lambda\varphi^4$ theory the latter graph vanishes [27].

[13] The very fact that these exceptional divergences are manifestly absent in the background field formalism suggests that also in ordinary field theory they are absent. In fact, this was already conjectured by Kluberg-Stern and Zuber [9]. One can indeed prove that in actual calculations such non-gauge invariant cohomologies will never appear by analyzing the $\{\Gamma,\Gamma\} = 0$ equation one step further then is usually done [10]. In the equation $\{\Gamma^{(0)},\Gamma^{(1)}\} = 0$ terms with are BRST nontrivial but not gauge invariant can appear in $\Gamma^{(1)}$. However, the next equation $2\{\Gamma^{(0)},\Gamma^{(2)}\} + \{\Gamma^{(1)},\Gamma^{(1)}\} = 0$ can be used to rule them out. In the proof one adds higher-dimensional nonrenormalizable local terms to $\Gamma^{(0)}$ with a coefficient α. The exceptional terms are then independent of α.

a Z factor, just as in x-space [13].

In gravity, the background field formalism plays a crucial role in functional methods. Since one is anyhow dealing in gravity with various curved backgrounds, it is natural to consider completely arbitrary gravitational backgrounds, and this is how the background field method originated [1]. Quantum gravity cannot be renormalized in the same manner as Yang-Mills theories because the gravitational coupling constant (the Newton constant) is dimensionful, as a result of which possible counterterms have a different functional form (typically $R^2_{\mu\nu}$ and R^2 at the one-loop level, when one starts with the Hilbert-Einstein action R). At each higher loop the number of counterterms increases, and thus the theory loses its power to predict[14]. To study the renormalizability (or, rather, the non-renormalizability) of quantum gravity, the background field method has turned out to be almost indispensable because it allows only general coordinate-invariant counterterms. In one-loop calculations of pure gravity and gravity coupled to matter, it was found that in pure gravity the one-loop divergences could be removed by a local but nonlinear field redefinition, but as soon as matter was coupled, nonremovable divergences remained: quantum gravity is nonrenormalizable [12]. In supergravity the same situation occurs: at the one- and two-loop level in pure $N = 1$ supergravity the divergences can be removed, but as soon as matter is coupled it was shown by explicit calculation that already at the one-loop level renormalizability is lost. [28] Thus for a consistent quantum theory of gravity ordinary field theory is inadequate, and one must turn to other approaches such as string theory. Supergravity then reappears as the effective action for "low" energies (energies accessible to accelerators).

[14]One expands in perturbative gravity in terms of the dimensionless constant $k\sqrt{G_N}$, where k is a loop momentum and G_N is Newton's constant. Clearly, perturbation theory is not a good approximation for ultraviolet momenta.

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A The Slavnov identity with background fields.

The Slavnov identity [18] states that the measure is gauge-invariant

Lemma: in ordinary field theory the measure

$$\Pi_x dQ_\mu{}^a(x)\Delta_F(x) \tag{7.A.1}$$

with Δ_F the Faddeev-Popov determinant, is invariant under the nonlocal gauge transformation $\delta Q_\mu{}^a = D_\mu(Q)(M^{-1}\zeta)^a$ where M is the ghost matrix.

We shall consider the following extension to background gauge field theories [11]

Lemma: in the background field formalism the measure $dQ_\mu{}^a\Delta_F(A,Q)$ transforms into the variation of the ghost action w.r.t. $A_\mu{}^a$.

$$\det \frac{\partial(Q+\delta_g Q)}{\partial Q}\Delta_F(Q+\delta_g Q) = \Delta_F(Q)\exp\frac{\partial}{\partial A_\mu{}^a}S(\text{fix})\delta A_\mu{}^a \tag{7.A.2}$$

where

$$\delta_g Q_\nu^{ab} = D_\nu(A+Q)M^{-1}\zeta^b \text{ with } \zeta^b = D^\sigma(A+Q)\delta A_\sigma^b \tag{7.A.3}$$

The matrix M is the ghost matrix, $\mathcal{L}$ (ghost) $= b_a M^a{}_b c^b$, hence $\det M = \Delta_F$.

To keep the notation relatively simple, we shall use the condensed notation of DeWitt. In this notation gauge fields are denoted by ϕ_i where i denotes both the spacetime coordinates x, the Lorentz index μ and the group index a. We identify

ϕ_i with $A_\mu{}^a(x)+Q_\mu{}^a(x)$ in the background field formalism. An infinitesimal gauge transformation can then be written as

$$\delta\phi_i = \partial_i{}^\alpha\lambda_\alpha + \Gamma_i^{\alpha j}\phi_j\lambda_\alpha \equiv D_i{}^\alpha[\phi]\lambda_\alpha \tag{7.A.4}$$

where $\partial_i{}^\alpha = \partial/\partial x^\mu \delta^a{}_b\delta(x-y)$ and $\lambda_\alpha = \lambda^b(y)$. Contractions like $\Lambda_i^\alpha\lambda_\alpha$ and $\Gamma_i^{\alpha j}\phi_j$ include integration over the spacetime coordinates y. Hence

$$\begin{aligned}
\partial_i{}^\alpha\lambda_\alpha &= \int \partial/partial x^\mu \delta^a{}_b\delta(x-y)\lambda^b(y)dy = \partial_\mu\lambda^a(x) \\
\Gamma_i^{\alpha j}\phi_j\lambda_\alpha &= \int\int \left(f^a{}_{bc}\delta_\mu^\nu\delta(x-y)\delta(x-z)\right)\left(A_\nu^b(y)+Q_\nu^b(y)\right)\lambda^c(z)d^4yd^4z \\
&= f^a{}_{bc}\left(A_\mu^b(x)+Q_\mu^b(x)\right)\lambda^c(x)
\end{aligned} \tag{7.A.5}$$

Hence $\delta\phi_i$ is indeed the usual gauge transformation $\delta(A_\mu{}^a(x)+Q_\mu{}^a(x)) = D_\mu(A+Q)\lambda^a$.

The nonlocal gauge transformation becomes in this notation

$$\begin{aligned}
\delta\phi_i &= D_i{}^\alpha[\phi]g_\alpha \\
g_\alpha &= (M^{-1})_\alpha{}^\beta\lambda_\beta \ , \ \ \lambda_\beta = (D^\sigma(A+Q)\delta A_\sigma)^b(z)
\end{aligned} \tag{7.A.6}$$

For ordinary field theories (with only $Q_\mu{}^a$ but not $A_\mu{}^a$), the Slavnov transformation has been used with λ_β a field-independent parameter, but we need the field-dependent λ_β given in (7.A.6).

Consider now first the Jacobian

$$\begin{aligned}
&J = 1 + Tr\partial\delta\phi_i/\partial\phi_j = 1 - (D_i{}^\alpha[\phi])(M^{-1})_\alpha{}^\beta\left(\delta M_\beta{}^\gamma/\delta\phi_i\right)g_\gamma \\
&+(D_i{}^\alpha[\phi])M^{-1}{}_\alpha{}^\beta\partial\lambda_\beta/\partial\phi_i
\end{aligned} \tag{7.A.7}$$

We used that differentiation of the factor in $D_i{}^\alpha[\phi]$ in $\delta\phi_i$ w.r.t. ϕ_i yields a vanishing result because $\Gamma_i{}^{\alpha i}=0$ (the structure constants are traceless. This is the counterpart of $f_\alpha{}^{\beta\alpha}=0$). We claim that the last term in J is equal to minus the change in S(ghost) under the variation $A_\mu{}^a \to A_\mu{}^a + \delta A_\mu{}^a$ and $Q_\mu{}^a \to Q_\mu{}^a - \delta A_\mu{}^a$. Indeed, the ghost action reads $b_b(D^\mu(A)D_\mu(A+Q))^b_a c^a$ and $< b_b(y)c^a(x) >= (_\alpha^{-1\beta}$. Furthermore, the variation of A in $D^\mu(A)$ yields $[\delta A^\mu, D_\mu(A+Q)]$, which is equal to $\partial\lambda^\beta/\partial\phi^i D_i{}^\gamma[\phi] = [D_\mu(A+Q), \delta A^\mu]$. Hence, our claim is that

$$Tr_i D_i{}^\alpha[\phi] M^{-1}{}_\alpha{}^\beta \partial\lambda_\beta/\partial\phi_j = Tr_\alpha M^{-1}{}_\alpha{}^\beta \partial\lambda_\beta/\partial\phi_i (D_i{}^\gamma[\phi]) \tag{7.A.8}$$

This equation holds due to the cyclicity of the trace.

[Another way to derive that the second term in (7.A.8) is equal to the change in the ghost action under $A \to A + \delta A$ and $Q \to Q\underline{\delta}A$ is to rewrite this term plus unity as a determinant.

$$\begin{aligned} &1 + Tr_\alpha M^{-1}{}_\alpha{}^\beta \partial\lambda_\beta/\partial\phi_i (D_i{}^\alpha[\phi]) \\ &= \det\left[1 + M^{-1}\partial\lambda/\partial\phi(D[\phi])\right] \end{aligned} \tag{7.A.9}$$

and combining with $\Delta_F = \det M$, one finds, after exponentiation, the ghost action together with a second term

$$\begin{aligned} &\det\left[M + \partial\lambda/\partial\phi(D[\phi]\right] = \\ &\int DbDc \exp\left\{b^\alpha M_\alpha{}^\beta c_\beta + b^\alpha \partial\lambda_\alpha/\partial\phi_i (D_i{}^\beta[\phi])c_\beta\right\} \end{aligned} \tag{7.A.10}$$

Since $\lambda_\alpha = D^\sigma(A+Q)\delta A^a_\sigma$, the last term is equal to

$$b_a g f^a{}_{bc} \left(D^\sigma(A+Q)c\right)^b \delta A^c_\sigma \tag{7.A.11}$$

This is indeed equal to minus the change in the ghost action].

We must now prove Slavnov's theorem, namely that the remaining term in J cancels the variation of Δ_F. The variation of Δ_F under the nonlocal gauge transformation is given by

$$\Delta_F + \delta\Delta_F = \det(M + \delta M) = \Delta_F\left(1 + TrM^{-1}\partial M/\partial\phi_i\delta\phi_i\right) \tag{7.A.12}$$

The sum of the remaining contribution from the Jacobian and the Faddeev-Popov determinant is thus

$$\begin{aligned} &-(D_i{}^\alpha[\phi])M^{-1}{}_\alpha{}^\beta(\partial M_\beta{}^\gamma/\partial\phi_i)g_\gamma \\ &+M^{-1}{}_\alpha{}^\beta(\partial M_\beta{}^\alpha/\partial\phi_i)(D_i{}^\gamma[\phi])g_\gamma \end{aligned} \tag{7.A.13}$$

Since both terms are traces, we can use cyclicity to move the matrix M^{-1} to the far right

$$\{-\partial M_\beta{}^\gamma/\partial\phi_i(D_i{}^\alpha[\phi]) + \partial M_\beta{}^\alpha/\partial\phi_i(D_i{}^\gamma[\phi])\}M^{-1}{}_\alpha{}^\beta g_\gamma \tag{7.A.14}$$

Thanks to the condensed notation, it is still clear which operators act on which terms. All derivatives end on delta functions $\delta(x-y)$ inside the same symbol (for example $D_i{}^\alpha = \frac{\partial}{\partial x^\mu}\delta_b{}^a\delta(x-y)$), and the index contractions contain spacetime integrals which effectively carry the derivatives to other fields or parameters. In ordinary notation (without all these extra $\delta(x-y)$) it becomes very hard to move $M^{-1}{}_\alpha{}^\beta g_\gamma$ to the far right.

We must show that the expression between curly brackets vanishes. To this purpose we evaluate $\partial M_\beta{}^\gamma/\partial\phi_i$. For a gauge fixing term $F_\beta(\phi)$ we have

$$\begin{aligned} M_\beta{}^\gamma &= \partial F_\beta/\partial\phi_j\,(D_j{}^\gamma[\phi]) \\ \partial M_\beta{}^\gamma/\partial\phi_i &= \partial^2 F_\beta/\partial\phi_i\partial\phi_j\,(D_j{}^\gamma[\phi]) + \partial F_\beta/\partial\phi_j\Gamma_j{}^{\gamma i} \end{aligned} \tag{7.A.15}$$

The terms with $\partial^2 F_\beta/\partial\phi_i\partial\phi_j$ cancel (they are multiplied by $(D_j{}^\gamma[\phi])(D_i{}^\alpha[\phi]) - i \leftrightarrow j$ and thus cancel by symmetry).

Hence we arrive at the following expression

$$\partial F_\beta/\partial\phi_j\left\{-\left(\Gamma_j{}^{\gamma i}\right)(D_i{}^\alpha[\phi]) + \left(\Gamma_j{}^{\alpha i}\right)(D_i{}^\gamma[\phi])\right\}(M^{-1})_\alpha{}^\beta g_\gamma \tag{7.A.16}$$

The two terms inside the curly brackets are a commutator of two gauge transformations with field-independent parameters. Indeed the latter reads

$$\left[\delta_g(g^1_\beta), \delta_g(g^2_\alpha)\right]\phi_i = \Gamma_i{}^{\alpha j}\left(D_j{}^\beta[\phi]\right)g^1_\beta g^2_\alpha - g^1 \leftrightarrow g^2 = D_i{}^\gamma[\phi]\tilde{g}_\gamma \tag{7.A.17}$$

with

$$\tilde{g}_\gamma = g f_\gamma{}^{\beta\alpha} g^1_\beta g^2_\alpha \tag{7.A.18}$$

Hence the terms within curly brackets in (7.A.16) are equal to

$$\partial F_\beta/\partial\phi_j \left\{ f_\delta{}^{\gamma\alpha} \left(D_j{}^\delta[\phi] \right) \right\} M^{-1}{}_\alpha{}^\beta g_\gamma \tag{7.A.19}$$

However $\partial F_\beta/\partial\phi_j(D_j{}^\delta[\phi])$ is equal to $M_\beta{}^\delta$. Hence we can simplify the expression to

$$M_\beta{}^\delta f_\delta{}^{\gamma\alpha} M^{-1}{}_\alpha{}^\beta g_\gamma = f_\delta{}^{\gamma\delta} g_\gamma = 0 \tag{7.A.20}$$

This expression vanishes because the trace of the structure constants is zero.

Chapter 8

Instantons*

In the last decades enormous progress has been made in understanding nonperturbative effects, both in supersymmetric field theories and in superstring theories. By non-perturbative effects we mean effects due to solitons and instantons, whose masses and actions, respectively, are inversely proportional to the square of the coupling constant. Typical examples of solitons are the kink, the vortex, and the magnetic monopole in field theory, and some D-branes in supergravity or superstring theories. In supersymmetric field theories these solutions preserve half of the supersymmetry and saturate BPS bounds. As for instantons, we have the Yang-Mills (YM) instantons in four dimensions [1–3], or tunnelling phenomena in quantum mechanics with a double-well potential as described by the kink, see e.g. [4], and there are various kinds of instantons in string theory, for example the D-instantons [5]. Also instantons preserve half the number of supersymmetries in supersymmetric field theories. Instantons can also be defined in field theories in dimensions higher than four [6], but we discuss in this chapter mainly the case of four dimensions.

Instantons in ordinary (i.e., nongravitational) quantum field theories are by definition solutions of the classical field equations in Euclidean space

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with finite action.[1] Only for a finite classical action $S_{\rm cl}$ is the factor $\exp[-\frac{1}{\hbar}S_{\rm cl}]$ in the path integral nonvanishing. We shall consider instantons in nonabelian gauge theories in flat spacetime (there are no instantons in abelian gauge theories in flat space), both regular instantons (the one-instanton solutions which actually have a singularity at $|x|^2 = \infty$) and singular instantons (the one-instanton solutions which have a singularity at a point $x = x_0$ but not at $|x|^2 = \infty$). A singular gauge transformation maps the first into the second, and vice-versa[2]. Around a given instanton solution, there are the quantum fluctuations. The action contains terms with 2, 3, 4 ... quantum fields, and one can perform perturbation theory around the instanton. The terms quadratic in quantum fields yield the propagators, which are complicated background-dependent expressions, and the terms cubic and higher in quantum fields yield the vertices. However, there is a subtlety with an instanton background: there are zero modes. A zero mode is by definition a solution of the linearized field equations for the fluctuations which is **normalizable**. (It is an eigenfunction of the quantum field operator with eigenvalue zero). In a trivial vacuum there are no zero modes: there are, of course, solutions of the linearized field equations, but they are not normalizable. We must treat the zero modes in instanton physics separately from the nonzero modes; for example, they have their own measure in the path integral. The

[1] In gravity there are various definitions of instantons: Einstein spaces with selfdual Weyl tensors, selfdual Riemann tensors, solutions of the Einstein equations with/without finite action etc. Since in gravity spacetime is part of the solution, one usually considers spacetime topologies which are different from that of $\mathbf{R^4}$. A selfdual Riemann tensor leads to an Einstein space ($R_{\mu\nu} = \Lambda g_{\mu\nu}$) whose Einstein-Hilbert action is either infinite (if the cosmological constant Λ is nonvanishing), or it only gets contributions from the Gibbons-Hawking boundary term [7]. In general, the semiclassical approximation of the Einstein-Hilbert action is not well defined due to the unboundedness of the action inside the path integral. To cure this, one probably has to discuss gravitational instantons inside a full theory for quantum gravity. For instanton solutions in flat space but using curvilinear coordinates (for example S^4, or cylindrical coordinates) see [8].

[2] For the "regular solution", A^a_μ is finite on $\mathbf{R^4} \cup \infty = \mathbf{S^4}$ everywhere, but this does not mean that it is regular. It is finite only because one can use two different patches to cover S^4, and A^a_μ is regular in each patch. If one maps infinity to the origin by a space-inversion transformation ($x^\mu = y^\mu/y^2$), then one finds a singularity at the origin. In this sense the "regular solution" is singular. We further clarify this issue in the next section.

nonzero modes live in the space orthogonal to the zero modes and in this space one can invert the linearized field equations for the fluctuations and construct propagators, and build Feynman diagrams and do perturbation theory.

Instantons describe tunnelling processes in Minkowski space-time from one vacuum at time t_1 to another vacuum at time t_2. The simplest model which exhibits this phenomenon is a quantum mechanical point particle with a double-well potential having two vacua, or a periodic potential with infinitely many vacua. Classically there is no trajectory for a particle to travel from one vacuum to the other, but quantum mechanically tunnelling occurs. The tunnelling amplitude can be computed in the WKB approximation, and is typically exponentially suppressed. In the Euclidean picture, after performing a Wick rotation, the potential is turned upside down, and it is possible for a particle to propagate between the two vacua, as described by the classical solution to the Euclidean equations of motion. The claim is then that the contributions from instantons in Euclidean space yield a good approximation of the path integral in Minkowski space. We shall prove this for the case of quantum mechanics.

Also in Yang-Mills theories, instantons are known to describe tunnelling processes between different vacua of the Minkowski theory, labeled by an integer winding number, and lead to the introduction of the CP-violating θ-term in the action for the Minkowski theory. [9, 10] It was hoped that instantons could shed some light on the mechanism of quark confinement. Although this was successfully shown in three-dimensional gauge theories (based on the Georgi-Glashow model) [11], the role of instantons in relation to confinement in four dimensions is less clear. Together with the non-perturbative chiral $U(1)$ anomaly in an instanton background, which leads to baryon number violation and a solution of the $U(1)$ problem [2, 3], instantons are used in phenomenological applications to QCD and the Standard Model. To avoid confusion, note that the triangle chiral anomalies in perturbative field theories in Minkowski space-time are canceled by choosing suitable multiplets of fermions.

There remain, however, chiral anomalies at the non-perturbative level. It is hard to compute the non-perturbative terms in the effective action which lead to a breakdown of the chiral symmetry by using methods in Minkowski space-time. However, by using instantons in Euclidean space, one can relatively easily determine these terms. The nonperturbative chiral anomalies are due to fermionic zero modes which appear in the path integral measure (in addition to bosonic zero modes). One must saturate the Grassmann integrals over these zero modes, and this leads to correlation functions of composite operators with fermionic fields which do violate the chiral $U(1)$ symmetry. The new non-perturbative terms are first computed in Euclidean space, but then continued to Minkowski space where they give rise to new physical effects [3]. They have the following generic form in the effective action (we suppress here possible flavor or adjoint indices that the fermions can carry)

$$S_{\text{eff}} \propto \mathrm{e}^{\left\{-\frac{8\pi^2}{g^2}\left(1+\mathcal{O}(g^2)\right)+i\theta\right\}}(\bar{\lambda}\bar{\lambda})^n \ , \tag{8.0.1}$$

where $2n$ is the number of fermionic zero modes (n depends on the representation of the fermions and the gauge group). The prefactor is due to the classical instanton action and is clearly non-perturbative. The terms indicated by $\mathcal{O}(g^2)$ are due to standard radiative corrections computed by using Feynman graphs in an instanton background. The term $\left(\bar{\lambda}\bar{\lambda}\right)^n$ involving antichiral spinors $\bar{\lambda}$ is produced if one saturates the integration in the path integral over the fermionic collective coordinates, and it violates in general the chiral symmetry. On top of (8.0.1) we have to add the contributions from anti-instantons, generating $(\lambda\lambda)^n$ terms in the effective action, where λ denotes chiral spinors. The sum of the $(\bar{k}\bar{k})^n$ terms and the $(\lambda\lambda)^n$ terms preserves chiral parity (the finite transformation $\lambda \leftrightarrow \bar{k}$), but it violates continuous chiral symmetry ($\delta\lambda = \dot{\alpha}\lambda$ and $\delta\bar{k} = -\alpha\bar{k}$). As we shall discuss, for Majorana spinors in Euclidean space the chiral and anti-chiral spinors are independent, but in Minkowski space-time they are related by complex conjugation, and one needs the sum of instanton and anti-instanton contributions to obtain a hermitean effective action.

We shall also apply the results of the general formalism to supersymmetric gauge theories, especially to the $\mathcal{N} = 4$ $SU(N)$ supersymmetric Yang-Mills (SYM) theory. Here $\mathcal{N}$ stands for the number of supersymmetries. Because we need Euclidean SYM theories to describe instantons, a first issue we shall discuss is how to define Majorana spinors in Euclidean space. Instantons in $\mathcal{N} = 1, 2$ models have been extensively studied in the past, see e.g. [12] for an early reference, and still are a topic of current research. For the $\mathcal{N} = 1$ models, one is mainly interested in the calculation of the superpotential and the gluino condensate [13,14]. In some specific models, instantons also provide a mechanism for supersymmetry breaking [14], see [15] for a review on these issues. In the case of $\mathcal{N} = 2$ models, there are exact results for the prepotential [16] based only on general symmetry principles and electric-magnetic duality; the prepotential acquires contributions from all multi-instanton sectors. These predictions were successfully tested against direct field theoretical calculations in the one-instanton sector in [17], and for a two-instanton background in [18]. More recently, new techniques were developed to perform multi-instantons calculations in [19]. Finally, the nonperturbative structure of $\mathcal{N} = 4$ SYM has been studied thoroughly in the context of the AdS/CFT correspondence [20]. SYM instantons in the limit of large number of colors were succesfully shown to reproduce the D-instanton contributions to certain correlation functions, both for single instantons [21, 22] and for multi-instantons [23]. Other correlation functions were studied in [24, 25]. For a recent review of instantons in supersymmetric gauge theories, see [26].

This chapter is organized as follows. In section 1, we discuss the winding number of gauge fields, and we present the standard one-instanton solution in $SU(2)$ and in $SU(N)$. This already raises the question how to embed $SU(2)$ into $SU(N)$, and we discuss the various embeddings. In section 2 we discuss instanton solutions in general: we solve the duality condition and find multi-instanton solutions which depend on their position and their scale. We concentrate on the one-instanton solutions, and first determine the singular solutions, but then we make a (singular) gauge transformation and obtain the regular solutions. In section 3 we start the study of "collective coordinates", the parameters on which the most general instanton solutions depend. We show that the number of collective coordinates is given by an index theorem for the Dirac operator in an instanton background. We then give a derivation of this index theorem, and conclude that a k-instanton solution in $SU(N)$ has $4Nk$ bosonic collective coordinates, $2Nk$ fermionic collective coordinates for fermions in the adjoint representation, and k fermionic collective coordinates for fermions in the defining (vector) representation. In section 4 we explicitly construct the zero modes for gauge group $SU(N)$ in a one-instanton background. First we construct the bosonic zero modes; these are associated to the collective coordinates for translations, dilatations and gauge orientations. Next we derive the explicit formula for the general solution of the fermionic zero modes of the Dirac equation in a one-instanton background, first for $SU(2)$ and then for $SU(N)$.

In section 5 we construct the one-instanton measure for the bosonic and fermionic collective coordinates. We explain in detail the normalization of the zero modes since it is crucial for the construction of the measure. We convert the integration over the coefficients of the bosonic zero modes to an integration over the corresponding bosonic collective coordinates by the Faddeev-Popov trick, but for fermionic zero modes we do not need this procedure because in this case the coefficients of fermionic zero modes are already the fermionic collective coordinates. In section 6 we discuss the one-loop determinants in the background of an instanton, arising from integrating out the quantum fluctuations. We then apply this to supersymmetric theories, and we use an index theorem to prove that the determinants for all supersymmetric Yang-Mills theories cancel each other, at least formally. Furthermore, we compute the complete nonperturbative β function for supersymmetric Yang-Mills theories by assuming that the measure for the zero modes does not depend on the renormalization scale μ. However, since it is not known to which regularization scheme this procedure corresponds, this result cannot be checked by standard perturbative calculations. In section 7 we discuss the general problem how to define Majorana fermions in Euclidean space, and apply the result to $\mathcal{N} = 4$ supersymmetric Yang-Mills theory in Euclidean space and its instantons.

The remaining sections contain applications. Section 8 discusses the problem of large instantons and its solution in terms of Higgs fields and spontaneous symmetry breaking. Section 9 gives a detailed discussion how instantons can describe tunnelling. In section 10 we use a quantum mechanical model with a double-well potential to discuss the phase transition from a false vacuum to the true vacuum by bubble formation. Section 11 contains the strong CP problem, the mystery that the θ angle is so small. Section 12 discusses that instantons solve the $U(1)$ problem, the problem that there is no Goldstone boson for the spontaneously broken arial $U(1)$ symmetry. In section 13 we discuss how instantons lead to baryon decay. We end this chapter with a construction of instanton solutions in finite-temperature field theory!

In a few appendices we set up our conventions and give a detailed derivation of some technical results in order to make this chapter self-contained. In appendix A we provide details of the calculation of the winding number. In appendix B we discuss the 't Hooft tensors and the spinor formalism in Euclidean space. In appendix C we calculate the volume of the moduli space of gauge orientations. Finally, in appendix D we show that conformal boosts and Lorentz rotations do not lead to additional zero modes.

1 Winding number and embeddings

We start with some elementary facts about instantons in $SU(N)$ Yang-Mills theories. The action, continued to Euclidean space, is

$$S = -\frac{1}{2g^2}\int \mathrm{d}^4x \, \mathrm{tr}\, F_{\mu\nu}F_{\mu\nu} \; ; \qquad F_{\mu\nu} = F^a_{\mu\nu}T_a \; . \tag{8.1.2}$$

The generators T_a are traceless anti-hermitean N by N matrices satisfying $[T_a, T_b] = f_{ab}{}^c T_c$ with real structure constants and $\mathrm{tr}(T_a T_b) = -\frac{1}{2}\delta_{ab}$. For instance, for $SU(2)$ one has $T_a = -\frac{i}{2}\tau_a$, where τ_a are the Pauli matrices and $f^a{}_{bc} = \epsilon^a{}_{bc}$. Notice that with these conventions the action is positive. Further conventions are $D_\mu Y = \partial_\mu Y + [A_\mu, Y]$ for any Lie algebra valued field Y, and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$, so that $F_{\mu\nu} = [D_\mu, D_\nu]$. The Euclidean metric is $\delta_{\mu\nu} = \mathrm{diag}(+,+,+,+)$. In (8.1.2), the only appearance of the coupling constant is in front of the action. In general one defines the group metric g_{ab} by $trT_aT_b = -\frac{1}{2}g_{ab}$, so in our case $g_{ab} = \delta_{ab}$. The group metric is an invariant tensor, and indices are raised and lowered by g^{ab} and g_{ab}, respectively. Since with out normalization $g_{ab} = \delta_{ab}$,[3], so we may raise and lower indices with δ^{ab} and δ_{ab}. Thus we may also write $[T_a, T_b] = f_{abc}T_c$, and from now on we shall write group and Lorentz indices either as covariant indices or as contravariant indices, depending on which way is most convenient.

By definition, a Yang-Mills instanton is a solution of the classical Euclidean equations of motion with finite action. The classical equations of motion read

$$D_\mu F_{\mu\nu} = 0 \ . \tag{8.1.3}$$

To find solutions with finite action, we require that the field strength tends to zero at infinity faster than $|x|^{-2} \equiv r^{-2}$, hence the gauge fields asymptotically approach a

[3]From $tr[T_c, T_aT_b] = tr([T_c, T_a]T_b + trT_a[T_c, T_b])$ it follows that g_{ab} is an invariant tensor, by which we mean that transforming its indices by an adjoint transformation with parameter λ^c yields zero. Indeed, $\delta g_{ab} = \lambda^c f_{ca}{}^d g_{db} + \lambda^c f_{cb}{}^d g_{ad} = 0$.

pure gauge[4]

$$A_\mu \overset{|x|^2\to\infty}{=} U\partial_\mu U^{-1} \ , \tag{8.1.4}$$

for some $U\epsilon SU(N)$.

To prove that gauge fields are pure gauge if the curvature $F_{\mu\nu}$ vanishes, is easy. Note, however, that if two gauge field configurations, say A_μ^I and A_μ^{II}, yield the same curvature, $F_{\mu\nu}(A^I) = F_{\mu\nu}(A^{II})$, they need not be gauge equivalent. A simple example proves this. Consider

$$A_\mu^I = \left\{-\tfrac{1}{2}ByT_3, \tfrac{1}{2}BxT_3, 0, 0\right\}; \qquad A_\mu^{II} = \left\{A_1^{II} = \sqrt{B}T_1, A_2^{II} = \sqrt{B}T_2, 0, 0\right\} \tag{8.1.5}$$

where B is a constant and T_a are the generators of $SU(2)$ with structure constants ${f_{ab}}^c = \epsilon_{abc}$. Clearly $F_{12}(A^I) = BT_3$ and also $F_{12}(A^{II}) = BT_3$ while all other components of $F_{\mu\nu}$ vanish. To prove that A_μ^I cannot be written as $U^{-1}(\partial_\mu + A_\mu^{II})U$ we note that if there was such a group element U, it should satisfy $U^{-1}F_{\mu\nu}U = F_{\mu\nu}$, hence U should commute with T_3. This implies that U would be given by $\exp(f(x)T_3)$ for some real function $f(x)$. Then $-\frac{1}{2}ByT_3 = \partial_x f\, T_3 + \mathrm{e}^{-fT_3}\sqrt{B}\, T_1 \mathrm{e}^{fT_3}$ which has no solution.[5]

There is actually a way of classifying fields which satisfy the boundary condition in (8.1.4). It is known from homotopy theory that all gauge fields with vanishing field

[4]Another way of satisfying the finite action requirement is to first formulate the theory on a compactified $\mathbf{R}^4$, by adding and identifying points at infinity. Then the topology is that of the four-sphere, since $\mathbf{R}^4 \cup \infty \simeq S^4$. The stereographic map from $\mathbf{R}^4 \cup \infty$ to S^4 preserves the angles, and is therefore conformal. Also the YM action is conformally invariant, implying that the action and the field equations on $\mathbf{R}^4 \cup \infty$ are the same as on S^4. (The action on the sphere is $\sqrt{g}g^{\mu\rho}g^{\nu\sigma}F_{\mu\nu}F_{\rho\sigma}$. Using stereographic coordinates, the metric on the sphere is $g_{\mu\nu} = \delta_{\mu\nu}(1+x^2)^{-2}$. The expression $\sqrt{g}g^{\mu\rho}g^{\nu\sigma}$ is then equal to $\delta^{\mu\rho}\delta^{\nu\sigma}$). The finiteness requirement is satisfied when the gauge potentials can be smoothly extended from $\mathbf{R}^4$ to S^4. The action is then finite because S^4 is compact and A_μ is well-defined on the whole of the four-sphere.

[5]One can also calculate a Wilson loop $W = \mathrm{tr} P \exp \oint A dl$. This expression is gauge invariant, and if one chooses as loop a square in the $x-y$ plane with sides L_1 and L_2, one finds

$$W^I = BL_1L_2T_3; \qquad W^{II} = 2\sqrt{B}(L_1T_1 + L_2T_2)$$

If A_μ^I and A_μ^{II} were gauge equivalent, W^I should have been equal to W^{II}).

strength at infinity can be classified into sectors characterized by an integer number called the Pontryagin class, or the winding number, or the instanton number, or the topological charge

$$k = -\frac{1}{16\pi^2}\int \mathrm{d}^4x \,\mathrm{tr}\, F_{\mu\nu}{}^*F_{\mu\nu} \ , \tag{8.1.6}$$

where ${}^*F_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F_{\rho\sigma}$ is the dual field strength, and $\epsilon_{1234} = 1$. Note that it is not necessary that these gauge fields satisfy the field equations, only that their field strength vanishes sufficiently fast at $r = \infty$. The derivation of this result can be found in Appendix A. As part of the proof, one shows that the integrand in (8.1.6) is the divergence of a current

$$K_\mu = -\frac{1}{8\pi^2}\epsilon_{\mu\nu\rho\sigma}\mathrm{tr}\, A_\nu\left(\partial_\rho A_\sigma + \tfrac{2}{3}A_\rho A_\sigma\right) \ . \tag{8.1.7}$$

The four-dimensional integral in (8.1.6) then reduces to an integral over a three-sphere at spatial infinity (large radius in space), and one can use (8.1.4) to show that the integer k counts how many times this spatial three-sphere covers the gauge group three-sphere $S^3 \approx SU(2) \subset SU(N)$. In more mathematical terms, the integer k corresponds to the third homotopy group $\pi_3(SU(2)) = \mathbf{Z}$. So k as defined in (8.1.6) does not depend on the values of the fields in the interior, but only on the fields at large $|x|^2$. This can also directly be seen: under a small variation $A_\mu \to A_\mu + \delta A_\mu$ one has $F_{\mu\nu} \to F_{\mu\nu} + D_\mu\delta A_\nu - D_\nu\delta A_\mu$, and partial integration (allowed when δA_μ is only nonzero in a region in the interior) yields $\delta A_\nu D_\mu{}^*F_{\mu\nu}$ which vanishes due to the Bianchi identity $D_{[\mu}F_{\nu\rho]} = 0$. (To prove this Bianchi identity one may use $F_{\nu\rho} = [D_\nu, D_\rho]$. In $[D_\mu,[D_\nu,D_\rho]] + [D_\nu,[D_\rho,D_\mu]] + [D_\rho,[D_\mu,D_\nu]]$ there are then 12 terms which cancel pairwise.)

Since we require instantons to have finite action, they satisfy the above boundary conditions at infinity, and hence they are classified by k, which we call the instanton number. Gauge potentials leading to field strengths with different instanton number can not be related by continuous gauge transformations. This follows from the fact that the instanton number is a gauge invariant quantity. In a given topological sector, the field configuration which minimizes the action is a solution of the field equations. (It is a priori not obvious that there exist field configurations that minimize the action, but we shall construct such solutions, thereby explicitly proving that they exist). We now show that, in a given topological sector, the solution to the field equations that minimizes the action has either a selfdual or anti-selfdual field strength

$$F_{\mu\nu} = \pm {}^*F_{\mu\nu} = \pm\tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}F_{\rho\sigma} \ . \tag{8.1.8}$$

This equation is understood in Euclidean space, where $({}^*)^2 = 1$. In Minkowski space there are no real solutions to the selfduality equations since $({}^*)^2 = -1$. As seen from (8.1.6), **instantons (with selfdual field strength) have $\boldsymbol{k > 0}$ whereas anti-instantons (with anti-selfdual field strength) have $\boldsymbol{k < 0}$**. (Recall that trT_aT_b is negative). To see that minimum action solutions are indeed selfdual or anti-selfdual, we perform a trick similar to the one used for deriving the BPS bound for solitons: we write the action as the square of a sum plus a total derivative term

$$\begin{aligned} S &= -\frac{1}{2g^2}\int \mathrm{d}^4x\,\mathrm{tr}\,F^2 = -\frac{1}{4g^2}\int \mathrm{d}^4x\,\mathrm{tr}\,(F \mp {}^*F)^2 \mp \frac{1}{2g^2}\int \mathrm{d}^4x\,\mathrm{tr}\,F\ {}^*F \\ &\geq \mp\frac{1}{2g^2}\int \mathrm{d}^4x\,\mathrm{tr}\,F\ {}^*F = \frac{8\pi^2}{g^2}(\pm k) \ . \end{aligned} \tag{8.1.9}$$

We used that $\mathrm{tr}{}^*F{}^*F = \mathrm{tr}FF$ and omitted Lorentz indices to simplify the notation. The equality is satisfied if and only if the field strength is (anti-) selfdual. The value of the action is then $S_{\mathrm{cl}} = (8\pi^2/g^2)|k|$, and has the same value for the instanton as for the anti-instanton. However, we can also add a theta-angle term to the action, which reads

$$S_\theta = -i\frac{\theta}{16\pi^2}\int \mathrm{d}^4x\,\mathrm{tr}\,F_{\mu\nu}{}^*F^{\mu\nu} = i\theta k = \pm i\theta|k| \ . \tag{8.1.10}$$

The plus or minus sign corresponds to the instanton and anti-instanton respectively, so the theta-angle distinguishes between them. The presence of a factor i in the Euclidean action may seem puzzling, but it can be explained by considering the θ-term in Minkowski spacetime. In Minkowski spacetime the θ-term appears in the path integral as $\exp \frac{i}{\hbar}\frac{\theta}{16\pi^2}\int d^4 \times \mathrm{tr}\, F_{\mu\nu}{}^*F^{\mu\nu}$ but the factor i stays if one goes to Euclidean space because both d^4x and $F_{\mu\nu}{}^*F^{\mu\nu}$ produce a factor i under a Wick rotation. We give a more detailed treatment of the theta-angle term and its applications in Section 11.

It is interesting to note that the energy-momentum tensor for a selfdual (or anti-selfdual) field strength always vanishes[6]

$$T_{\mu\nu} = -\frac{2}{g^2}\mathrm{tr}\left\{F_{\mu\rho}F_{\nu\rho} - \tfrac{1}{4}\delta_{\mu\nu}F_{\rho\sigma}F_{\rho\sigma}\right\} = 0 \ . \tag{8.1.11}$$

(Because in Euclidean space $T_{44} = -\frac{2}{g^2}\mathrm{tr}\,(\vec{E}^2 - \vec{B}^2)$, the Euclidean "energy" T_{44} need not be positive definite). This agrees with the observation that the instanton action $\int \mathrm{d}^4x\, \mathrm{tr}\, F^2 = \int \mathrm{d}^4x\, \mathrm{tr}\, {}^*FF$ is metric independent in curved space. The vanishing of the energy-momentum tensor is consistent with the fact that instantons are topological in nature. It implies that instantons do not curve Euclidean space, as follows from the Einstein equations.

An explicit construction of finite action solutions of the Euclidean classical equations of motion was given by Belavin et al. [1]. We shall derive this solution, and others, in section 2, but to get oriented we present it here, and discuss some of its properties. The gauge configuration for one instanton ($k = 1$) in $SU(2)$ contains the matrices $\sigma_{\mu\nu}$ or $\bar{\sigma}_{\mu\nu}$. One often writes it in terms of the 't Hooft η tensors, related to $\bar{\sigma}_{\mu\nu}$ by $\bar{\sigma}_{\mu\nu} = i\eta^a{}_{\mu\nu}\tau_a$ where τ_a are the generators of $SU(2)$. We discuss these tensors in Appendix B. The regular one-instanton solution reads then

$$\begin{aligned} A^a_\mu(x; x_0, \rho) &= 2\frac{\eta^a{}_{\mu\nu}(x - x_0)^\nu}{(x - x_0)^2 + \rho^2} \ , \\ A_\mu \equiv A^a_\mu\left(\frac{\tau_a}{2i}\right) &= -\frac{\bar{\sigma}_{\mu\nu}(x - x_0)^\nu}{(x - x_0)^2 + \rho^2} \ , \end{aligned} \tag{8.1.12}$$

where x_0 and ρ are arbitrary parameters called collective coordinates. They correspond to the position and the size of the instanton. The above expression solves

[6]Note that T_{12} is proportional to $\mathrm{tr}(F_{13}F_{23} + F_{14}F_{24})$, which is equal to minus itself due to the selfduality relations $F_{12} = F_{34}, F_{13} = -F_{24}$ and $F_{14} = F_{23}$. Similarly T_{11} vanishes because it is proportional to the trace of $(F_{12}^2 + F_{13}^2 + F_{14}^2) - (F_{23}^2 + F_{24}^2 + F_{34}^2)$.

the selfduality equations for any value of the collective coordinates. Notice that it is regular at $x = x_0$, as long as $\rho \neq 0$. The real antisymmetric eta-symbols are defined as follows

$$\begin{aligned} \eta^a{}_{\mu\nu} &= \epsilon^a{}_{\mu\nu} \qquad \mu,\nu = 1,2,3\,, \qquad \eta^a{}_{\mu 4} = -\eta^a{}_{4\mu} = \delta^a_\mu\ , \\ \bar\eta^a{}_{\mu\nu} &= \epsilon^a{}_{\mu\nu} \qquad \mu,\nu = 1,2,3\,, \qquad \bar\eta^a{}_{\mu 4} = -\bar\eta^a{}_{4\mu} = -\delta^a_\mu\ . \end{aligned} \tag{8.1.13}$$

The η and $\bar\eta$-tensors are selfdual and anti-selfdual respectively, for fixed index a. They form a basis for antisymmetric four by four matrices, and we have listed their properties in Appendix B. They are linear combinations of the Euclidean Lorentz generators $L_{\mu\nu}$, namely $\eta^a{}_{\mu\nu} = (J^a + K^a)_{\mu\nu}$ and $\bar\eta^a{}_{\mu\nu} = (J^a - K^a)_{\mu\nu}$, where $J^a = \epsilon^{abc} L_{bc}$ and $K^a = L_{a4}$, and $(L_{mn})_{\mu\nu} = \delta_{m\mu}\delta_{n\nu} - \delta_{m\nu}\delta_{n\mu}$ with $m, n = 1, 4$. In this subsection we use η tensors, but in later sections we shall use the matrices $\sigma_{\mu\nu}$ and $\bar\sigma_{\mu\nu}$.

The field strength corresponding to this gauge potential is (use 8.B.348)

$$F^a_{\mu\nu} = -4\eta^a{}_{\mu\nu}\frac{\rho^2}{[(x-x_0)^2+\rho^2]^2}\ , \tag{8.1.14}$$

and it is selfdual. Thus (8.1.12) is a solution of the classical field equations. Far away, A^a_μ becomes proportional to the inverse radius $\frac{1}{r}$ so that it contributes a finite amount to the integral for the winding number which is of the form $\int A^3(r^3 \mathrm{d}\Omega)$, while $F_{\mu\nu}$ becomes proportional to $\frac{1}{r^4}$, yielding a finite action. However, A^a_μ itself vanishes at $r \to \infty$, hence we have a smooth configuration on S^4. Notice that the special point $\rho = 0$, corresponding to zero size instantons, leads to zero field strength and corresponds to pure gauge. Strictly speaking, this point must therefore be excluded from the instanton moduli space of collective coordinates. Finally one can compute the value of the action by integrating the density

$$\mathrm{tr}\, F_{\mu\nu}F^{\mu\nu} = -96\,\frac{\rho^4}{[(x-x_0)^2+\rho^2]^4}\ . \tag{8.1.15}$$

Using the integral given at the end of Appendix B, one finds that this solution corresponds to $k = 1$.

One may show by direct calculation that the regular one-anti-instanton solution is also given by (8.1.12) but with $\bar{\eta}^a{}_{\mu\nu}$. (In the proof one uses that the first formula in (B.5) also holds for $\bar{\eta}^a{}_{\mu\nu}$).

We shall also derive the one-instanton solution in the singular gauge. In terms of η symbols it reads

$$A_\mu^a = 2\,\frac{\rho^2 \bar{\eta}^a_{\mu\nu}(x-x_0)_\nu}{(x-x_0)^2[(x-x_0)^2+\rho^2]} = -\bar{\eta}^a_{\mu\nu}\partial_\nu \ln\left\{1+\frac{\rho^2}{(x-x_0)^2}\right\} \ . \tag{8.1.16}$$

This gauge potential is singular for $x = x_0$, where it approaches a pure gauge configuration as we shall show in the next section, $A_\mu \overset{x\to x_0}{=} U\partial_\mu U^{-1}$. The gauge transformation U is singular and relates the regular gauge instanton (8.1.12) to the singular one (8.1.16) at all points. The field strength in singular gauge is then (taking the instanton at the origin, $x_0 = 0$, otherwise replace $x \to x - x_0$)

$$F^a_{\mu\nu} = -\frac{4\rho^2}{(x^2+\rho^2)^2}\left\{\bar{\eta}^a_{\mu\nu} - 2\bar{\eta}^a_{\mu\rho}\frac{x_\rho x_\nu}{x^2} + 2\bar{\eta}^a_{\nu\rho}\frac{x_\rho x_\mu}{x^2}\right\} \ . \tag{8.1.17}$$

Notice that despite the presence of the anti-selfdual eta-tensors $\bar{\eta}$, this field strength is still selfdual, as can be seen by using the properties of the eta-tensors given in (8.B.348). The singular gauge is frequently used, because, as we will see later, zero modes fall off more rapidly at large x in the singular gauge. One can compute the winding number again in singular gauge. Then one finds that there is no contribution coming from infinity. Instead, all the winding is coming from the singularity at the origin. The singular solution is singular at x_0, so one would expect that the regular solution is singular at infinity. This may seem puzzling since we saw that the regular solution was smooth on S^4. However, to decide whether a configuration is smooth at $r \to \infty$, one should first transform the point at infinity to the origin and then study how the transformed configuration behaves near the origin. Making the coordinate transformation $x^\mu = y^\mu/y^2$ or $x^\mu = -y^\mu/y^2$, not forgetting that a vector field transforms as $A'_\mu(y) = (\partial x^\nu/\partial y^\mu)A_\nu(x)$, one finds that the transformed regular

$k = 1$ solution is indeed singular at the origin[7]. In fact, it is equal to the singular $k = -1$ solution with ρ replaced by $\frac{1}{\rho}$.

At first sight it seems that there are five collective coordinates for the $k = 1$ solution. There are however extra collective coordinates corresponding to the gauge orientation. One can act with an $SU(2)$ matrix on the solution (8.1.12) to obtain another solution,

$$A_\mu(x; x_0, \rho, \vec{\theta}) = U^{-1}(\vec{\theta})\, A_\mu(x; x_0, \rho)\, U(\vec{\theta}) \ , \qquad U \epsilon SU(2) \ . \tag{8.1.18}$$

with constant $\vec{\theta}$. One might think that these configurations should not be considered as a new solution since they are gauge equivalent to the expression given above. This is not true, however, the reason being that, after we fix the gauge, we still have left a rigid $SU(2)$ symmetry which acts as in (8.1.18). So in total there are eight collective coordinates, also called moduli. In principle, one could also act with the (space-time) rotation matrices $SO(4)$ on the instanton solution, and construct new solutions. However, these rotations can be undone by suitably chosen gauge transformations [27]. Actually, the Yang-Mills action is not only invariant under the Poincaré algebra (and the gauge algebra), but it is also invariant under the conformal algebra which contains the Poincaré algebra and further the generators for dilatations (D) and conformal boosts (K_μ). As shown in Appendix D, for the Euclidean conformal group $SO(5,1)$, the subgroup $SO(5)$ consisting of $SO(4)$ rotations and a combination of conformal boosts and translations ($R^\mu \equiv K^\mu + \rho^2 P^\mu$), leaves the instanton invariant up to gauge transformations. This leads to a 5 parameter instanton moduli space $SO(5,1)/SO(5)$, which is the Euclidean version of the five-dimensional anti-de Sitter space AdS_5. The coordinates on this manifold correspond to the four positions and

[7]This coordinate transformation in $\mathbf{R}^4$ can be viewed as a product of two conformal projections, one from the plane to the coordinate patch on the sphere S^4 containing the south pole, and the other from the other coordinate patch on S^4 with the north pole back to the plane. The transformed metric is $g'_{\mu\nu}(y) = \delta_{\mu\nu}/y^4$, so conformally flat. Then the action for the $A'_\mu(y)$ in y-coordinates is again the usual flat space action in 8.1.2, and the transformed instanton solution is an anti-instanton solution.

the size ρ of the instanton. On top of that, there are still three gauge orientation collective coordinates, yielding a total of eight moduli for the $k = 1$ instanton in $SU(2)$.

Instantons in $SU(N)$ can be obtained by embedding $SU(2)$ instantons into $SU(N)$. For instance, a particular embedding is given by the following N by N matrix

$$A_\mu^{SU(N)} = \begin{pmatrix} 0 & 0 \\ 0 & A_\mu^{SU(2)} \end{pmatrix} . \tag{8.1.19}$$

where the instanton resides in the 2×2 matrix on the lower right. Of course this is not the most general solution, as one can choose different embeddings, see below.

One can act with a general $SU(N)$ element on the solution (8.1.19) and obtain a new one. Not all elements of $SU(N)$ generate a new solution. There is a stability group that leaves (8.1.19) invariant, acting only on the zeros, or commuting trivially with the $SU(2)$ embedding. Such group elements should be divided out, so we consider, for $N > 2$,

$$A_\mu^{SU(N)} = U \begin{pmatrix} 0 & 0 \\ 0 & A_\mu^{SU(2)} \end{pmatrix} U^\dagger, \qquad U \,\epsilon\, \frac{SU(N)}{SU(N-2) \times U(1)} . \tag{8.1.20}$$

One can now count the number of collective coordinates. From counting the dimension of the coset space in (8.1.20), one finds there are $4N - 5$ parameters. Together with the position and the scale of the $SU(2)$ solution, we find in total $4N$ collective coordinates for a one-instanton solution in $SU(N)$. It is instructive to work out the example of $SU(3)$. Here we use the eight Gell-Mann matrices $\{\lambda_\alpha\}, \alpha = 1, \ldots, 8$. The first three λ_a, $a = 1, 2, 3$, form an $SU(2)$ algebra and are used to define the $k = 1$ instanton by contracting (8.1.12) or (8.1.16) with λ_a. The generators $\lambda_4, \ldots, \lambda_7$ form two doublets under this $SU(2)$, so they act on the instanton and can be used to generate new solutions. This yields four more collective coordinates. Then there is λ_8, corresponding to the $U(1)$ factor in (8.1.20). It commutes with the $SU(2)$ subgroup spanned by λ_a, and so it belongs to the stability group leaving the instanton invariant.

So for $SU(3)$ and $k = 1$, there are seven gauge orientation zero modes, which agrees with $4N - 5$ for $N = 3$.

The embedding of instanton solutions as a 2×2 block inside the $N \times N$ matrix representation of $SU(N)$ is not the only embedding possible. For example, one can also use the 3×3 matrix representation T_a of $SU(2)$, and put the instanton inside a 3×3 block of the **N** of $SU(N)$. This **3** of $SU(2)$ is sometimes called "the other $SU(2)$ in $SU(3)$", but it is simply the adjoint representation of $SU(2)$, which is also the defining representation of $SO(3)$, and is given by $(T_a)_{ij} = \epsilon_{iaj}$,

$$T_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}; \quad T_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}; \quad T_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (8.1.21)$$

This representation has the same structure constants $f_{abc} = \epsilon_{abc}$ as the representation $T_a = \tau_a/(2i)$, but now $\text{tr}\{T_a T_b\} = -2\delta_{ab}$, four times larger.

In fact, going back to the construction of the instanton, we note that **any** representation T_a of $SU(2)$ yields an instanton solution for $SU(N)$ as long as it fits inside the $N \times N$ matrices of $SU(N)$ [28]

$$A_\mu = 2\eta^a_{\mu\nu} T_a \frac{x^\nu}{x^2 + \rho^2} \, . \quad (8.1.22)$$

The **2** of $SU(2)$ with $T_a = \frac{\tau_a}{2i}$ yields (8.1.19), but any other representation yields another embedding.

For $SU(3)$ there are only two possibilities. We can embed the instanton using the **2** of $SU(2)$; this yields (8.1.19). But we can also use the matrices T_a given in (8.1.21) as the first 3 generators of $SU(3)$. For $SU(N)$ we can use any spin j representation of $SU(2)$ provided it fits inside the $N \times N$ matrices. Since the action and winding number are proportional to the trace $\delta^{ab}\text{tr}\, T_a T_b$, which is proportional to the quadratic Casimir operator $j(j+1)$ times the dimension $2j+1$ of the spin j representation[8], we

[8] Use $\delta^{ab} tr T_a T_b = -tr C_2(R) = -(2j+1) C_2(R)$ where the quadratic Casimir operator for the representation R with spin j is given by $C_2(R) = -\delta^{ab} T(R)_a T(R)_b = j(j+1)$.

see that we get instanton solutions with winding number $k = \pm\frac{2}{3}j(j+1)(2j+1)$. For $j = 1/2$ this reduces to $k = \pm 1$. For the first few $SU(N)$ the results are as follows

$$\begin{aligned}
SU(3): &\quad k = \pm 1; k = \pm 4 \ (j = 1/2 \text{ and } j = 1) \\
SU(4): &\quad k = \pm 1; k = \pm 4; k = \pm 10 \ (j = 1/2, 1, 3/2) \\
&\quad k = \pm 2 \ (\text{two } j = 1/2 \text{ in block form}) \\
SU(5): &\quad k = \pm 1, \pm 4, \pm 10, \pm 20 \quad (j = \tfrac{1}{2}, 1, \tfrac{3}{2}, 2) \\
&\quad k = \pm 2, \pm 5 \ (j = \tfrac{1}{2} \oplus \tfrac{1}{2} \text{ and } j = \tfrac{1}{2} \oplus 1) \ .
\end{aligned} \tag{8.1.23}$$

All these instanton solutions with winding number $|k| > 1$ still are (anti-) selfdual, so they still have minimal action, determined by the winding number, so the same as k instantons embedded as 2×2 matrices but far apart. Two instantons far apart and each of the form (8.1.12) repel each other (as opposed to an instanton and anti-instanton) with an interaction energy proportional to $1/r$. Bringing k instantons together such that they sit all at the same point, gives solutions of the kind above. So, far apart there is a small positive interaction, but when they are brought together the interaction energy vanishes. Hence, there must be domains of attraction in between. This already shows that the interaction of instantons is a complicated problem [28]. In fact, one can deform these single-instanton solutions such that a multi-instanton solution is obtained in which the single-instantons do not attract or repel each other at all. In other words, in such a multi-instanton solution the positions, sizes and gauge orientations of the single instantons are collective coordinates.

For the general multi-instanton solution, the dependence on all collective coordinates is in implicit form given by the ADHM construction [29]. For a recent review, see [30]. In the next section we will obtain explicit formulas for the dependence on $5k$ collective coordinates. Explicit formulas for the dependence on all collective coordinates only exist for the $k = 2$ instanton solution [29–32] and the $k = 3$ instanton solution [33].

We end this section with some remarks on embeddings into other gauge groups [34]. For $k = 1$ and gauge group $SO(N)$, it is known that there are $4N - 8$ collective coordinates. This can be understood as follows. The one-instanton solution is constructed by choosing an embedding of $SO(4) = SU(2) \times SU(2)$ generated by $\eta^a_{\mu\nu}$ and $\bar{\eta}^a_{\mu\nu}$, and putting the instanton in one of the $SU(2)$ groups. The stability group of this instanton is $SO(N-4) \times SU(2)$, so we obtain (for $N > 4$)

$$A_\mu^{SO(N)} = U \begin{pmatrix} 0 & 0 \\ 0 & A_\mu^{SU(2)} \end{pmatrix} U^\dagger, \qquad U \infty \frac{SO(N)}{SO(N-4) \times SU(2)} \,. \tag{8.1.24}$$

The number of collective coordinates of such solutions follows from the dimension of the coset (which is $4N - 13$). Including the positions and size of the $SU(2)$ instanton, we arrive at $4N - 8$ for the total number of collective coordinates. Notice that for $N = 6$, we can use the isomorphism between $SO(6)$ and $SU(4)$. For both countings, we arrive at 16 moduli.

Similarly, we can analyze the symplectic gauge groups $USp(2N)$. Here we can simply choose the lower diagonal $SU(2) = USp(2)$ embedding inside $USp(2N)$ for a $k = 1$ instanton. The stability group of this embedding is now $USp(2N-2)$, so for we have the following instanton solution:

$$A_\mu^{Sp(N)} = U \begin{pmatrix} 0 & 0 \\ 0 & A_\mu^{SU(2)} \end{pmatrix} U^\dagger, \qquad U \infty \frac{USp(2N)}{USp(2N-2)} \,. \tag{8.1.25}$$

The dimension of $USp(2N)$ is $N(2N + 1)$,[9] and so the total number of collective coordinates that follows from this construction is $5 + (4N - 1) = 4(N + 1)$, which is the correct number [34]. For $N = 2$, we have the isomorphism $USp(2) = SO(5)$, which in both countings leads to 12 collective coordinates.

[9]The dimension of $U(2N)$ is $4N^2$ and the generators have the form $\begin{pmatrix} a_1 + is_1 & b \\ -b^\dagger & a_2 + is_2 \end{pmatrix}$ where a_i is antisymmetric and s_i is symmetric. Complex symplectic matrices $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ satisfy $M^T\Omega + \Omega\, M = 0$ where $\Omega = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$. The restriction that the unitary generators be also symplectric leads to $N^2 + N(N-1)$ constraints ($D + A^T = 0$ and $C - C^T = B - B^T = 0$).

For higher instanton number, not all instantons can be constructed from a properly chosen embedding. There the ADHM formalism must be used. We just mention here that the total number of collective coordinates is $4kN, 4k(N-2)$ and $4k(N+1)$ for the gauge groups $SU(N), SO(N)$ and $USp(2N)$ respectively. The geometric relation between instanton moduli spaces and quaternionic manifolds (whose dimension is always a multiple of four) can e.g. be found in [35].

1.1 Some remarks on nonselfdual instanton solutions

Note that we have not shown that all solutions of (8.1.3) with finite action are given by selfdual (or anti-selfdual) field strengths. In principle there could be configurations which are extrema of the action, but are neither selfdual nor anti-selfdual[10]. For the gauge group $SU(2)$ this has been a long standing question. The first result was established in [36–38] where it was shown that for gauge groups $SU(2)$ and $SU(3)$, nonselfdual solutions cannot be local minima, hence if they exist, they should correspond to saddle points. The existence of nonselfdual solutions with finite action and gauge group $SU(2)$ was first established in [39], for $k = 0$, and later for $k \neq 0$ in [40]. For gauge group $SU(3)$ some results have been obtained in [41, 42]. The situation seems to be quite complicated, and no elegant and simple framework to address these issues has been found so far. For bigger gauge groups, it is easier to construct non-selfdual (or anti-selfdual) solutions. This becomes clear in the example of $SO(4) = SU(2)\times SU(2)$. If we associate a selfdual instanton to the first factor, and an anti-selfdual instanton to the second factor, the total field strength satisfies the equations of motion (8.1.3) but is neither selfdual nor anti-selfdual. Even simpler is the example of $SU(4)$. By choosing two commuting $SU(2)$ subgroups, we can embed

[10] It is possible to construct solutions for $SU(2)$ that are not selfdual, but not with finite action. An example is $A_\mu = -\frac{1}{2}\sigma_{\mu\nu}\frac{x_\nu}{r^2}$. Its field strength is $F_{\mu\nu} = \frac{1}{2}\sigma_{\mu\nu}/r^2 + \frac{1}{2}(x^\mu\sigma_{\mu\rho} - x^\nu\sigma_{\mu\rho})x^\rho/r^4$. One can check that it satisfies the second order equation of motion (8.1.3) (both $\partial_\mu F_{\mu\nu}$ and $[A_\mu, F_{\mu\nu}]$ vanish), but this configuration is not selfdual since $F_{\mu\nu} - {}^*F_{\mu\nu} = \frac{1}{2}\sigma_{\mu\nu}\frac{1}{r^2}$. Because this field strength does not tend to zero fast enough at infinity, the action evaluated on this solution diverges logarithmically.

both an $SU(2)$ instanton and an anti-instanton inside $SU(4)$,

$$A_\mu^{SU(4)} = \begin{pmatrix} A_\mu^+ & 0 \\ 0 & A_\mu^- \end{pmatrix} , \tag{8.1.26}$$

where $A_\mu^\pm$ denotes the (anti-) selfdual $SU(2)$ gauge potentials with topological charges $k^\pm$. Clearly the total field strength is neither selfdual nor anti-selfdual, but satisfies the second order equations of motion. The instanton action is finite and the total topological charge is $k^+ - k^-$.

From the embedding 8.1.26 one can generate more solutions by acting on the gauge potential with a global gauge transformation $U \infty SU(4)$. In this way, one generates new exact and nonselfdual solutions which are not of the form 8.1.26.

For $SU(N)$ gauge groups, one has even more possibilities. One can embed k_+ instantons and k_- anti-instantons on the (block)-diagonal of $SU(N)$, as long as $2(k_+ + k_-) \leq N$. If we take both $k_+ > 0$ and $k_- > 0$, the solution is clearly not selfdual or anti-selfdual and the instanton action, including the theta-angle, is given by

$$S = \frac{8\pi^2}{g^2}(k_+ + k_-) + i\theta(k_+ - k_-) \ . \tag{8.1.27}$$

In a supersymmetric theory, these solutions will not preserve any supersymmetry. This is interesting in the context of the AdS/CFT correspondence that relates $\mathcal{N} = 4$ SYM theory to type IIB superstrings. In [43], it is shown that these non-selfdual Yang-Mills instantons are related to non-extremal (non BPS) D-instantons in IIB supergravity.

2 Regular and singular instanton solutions

To find explicit instanton solutions, we solve the selfduality (or anti-selfdualty) equations $F_{\mu\nu} = {}^*F_{\mu\nu}$ where ${}^*F_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F_{\rho\sigma}$ with $\mu,\nu = 1,4$ and $\epsilon_{1234} = \epsilon^{1234} = 1$. Since $D_\mu {}^*F_{\mu\nu}$ vanishes identically due to the Bianchi identity, we then have a solution of the field equations, $D_\mu F_{\mu\nu} = 0$. The main idea is to make a suitable ansatz, and then to check that it yields solutions. The ansatz is (we restrict ourselves for the moment to the gauge group $SU(2)$)

$$A_\mu(x) = \alpha\ \sigma_{\mu\nu}\partial_\nu \ln\phi(x^2)\ , \tag{8.2.28}$$

where α is a real constant to be fixed and $\sigma_{\mu\nu}$ is the 2×2 matrix representation of the Lorentz generators in Euclidean space. Since we shall be using these matrices $\sigma_{\mu\nu}$ a lot, we first discuss their properties in some detail, and then we shall come back below (8.2.50) to the construction of instanton solutions.

2.1 Lorentz and spinor algebra

In Euclidean space, a suitable 4×4 matrix representation of the Dirac matrices is given by

$$\gamma^\mu = \begin{pmatrix} 0 & -i(\sigma^\mu)^{\alpha\beta'} \\ i(\bar\sigma^\mu)_{\alpha'\beta} & 0 \end{pmatrix}\ , \qquad \begin{array}{l} \sigma^\mu = (\vec\tau, iI) \\ \bar\sigma^\mu = (\vec\tau, -iI) \end{array}\ , \tag{8.2.29}$$

where $\vec\tau$ are the Pauli matrices. We use slashes instead of dots on the spinor indices to indicate that we are in Euclidean space. All four Dirac matrices are hermitian, and satisfy $\{\gamma^\mu,\gamma^\nu\} = 2\delta^{\mu\nu}$. The matrix γ^5 is diagonal

$$\gamma^5 \equiv \gamma^1\gamma^2\gamma^3\gamma^4 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}\ , \tag{8.2.30}$$

and chiral spinors correspond to projections with $\frac{1}{2}(1\pm\gamma_5)$ which yield the upper or lower two components of a nonchiral four-component spinor.

$$\psi = \begin{pmatrix} \lambda^\alpha \\ \bar\chi_{\alpha'} \end{pmatrix} \tag{8.2.31}$$

Since we are in Euclidean space, it does not matter whether we write the index μ as a contravariant or covariant index. In Minkowski space this representation (with γ^4 replaced by γ^0 where $\gamma^4 = i\gamma^0$, so that $(\gamma^k)^2 = +1$ but $(\gamma^0)^2 = -1$) is used for two-component spinor formalism. Four-component spinors are then decomposed into two-component spinors as $\psi = \binom{\lambda^\alpha}{\bar\chi_{\dot\alpha}}$, and this explains the position of the spinor indices on

σ^μ and $\bar{\sigma}^\mu$ in (8.2.29). The Euclidean Lorentz generators ($SO(4)$ generators) acting on 4-component spinors are $M_{\mu\nu} = \frac{1}{4}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)$ and satisfy the Euclidean Lorentz algebra

$$[M_{\mu\nu}, M_{\rho\sigma}] = \delta_{\nu\rho}M_{\mu\sigma} - \delta_{\nu\sigma}M_{\mu\rho} - \delta_{\mu\rho}M_{\nu\sigma} + \delta_{\mu\sigma}M_{\nu\rho} \ . \tag{8.2.32}$$

However, this representation is reducible: the upper and lower components of ψ form separate representations

$$M_{\mu\nu} = \frac{1}{2}\begin{pmatrix} (\sigma^{\mu\nu})^\alpha{}_\beta & 0 \\ 0 & (\bar{\sigma}^{\mu\nu})_{\alpha'}{}^{\beta'} \end{pmatrix} \ . \tag{8.2.33}$$

In terms of σ^μ and $\bar{\sigma}^\mu$ we then find the following two inequivalent spinor representations of $SO(4) : M_{\mu\nu} = \frac{1}{2}\sigma_{\mu\nu}$ and $M_{\mu\nu} = \frac{1}{2}\bar{\sigma}_{\mu\nu}$, where

$$\sigma^{\mu\nu} = \tfrac{1}{2}(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu) \ ; \qquad \bar{\sigma}^{\mu\nu} = \tfrac{1}{2}(\bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu) \ . \tag{8.2.34}$$

(It is customary not to include the factor $\frac{1}{2}$ in $M_{\mu\nu} = \frac{1}{2}\sigma_{\mu\nu}$ into the definition of $\sigma^{\mu\nu}$).

The matrices $\sigma_{\mu\nu}$ and $\bar{\sigma}_{\mu\nu}$ satisfy some properties which we shall need repeatedly. First of all, they are anti-selfdual and selfdual, respectively

$$\sigma_{\mu\nu} = -\tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}\sigma_{\rho\sigma} \ ; \qquad \bar{\sigma}_{\mu\nu} = \tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}\bar{\sigma}_{\rho\sigma} \ . \tag{8.2.35}$$

This follows most easily by noting that the matrices γ_μ satisfy $\gamma_{[\mu}\gamma_{\nu]} = -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\gamma_\rho\gamma_\sigma\gamma_5$ where $\gamma_{\mu\nu} \equiv \gamma_{[\mu}\gamma_{\nu]} = \frac{1}{2}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)$. For example, $\gamma_1\gamma_2 = -\gamma_3\gamma_4\gamma_5$ because $\epsilon_{1234} = +1$. From this (anti)-selfduality one derives another useful property

$$\epsilon_{\mu\nu\rho\sigma}\sigma_{\sigma\tau} = \delta_{\mu\tau}\sigma_{\nu\rho} - \delta_{\nu\tau}\sigma_{\mu\rho} + \delta_{\rho\tau}\sigma_{\mu\nu} \ . \tag{8.2.36}$$

For example, $\epsilon_{123\sigma}\sigma_{\sigma 1} = \sigma_{23}$. It is easiest to prove (8.2.36) by substituting (8.2.35) into the left-hand side, and decomposing the product of two ϵ-tensors into a sum of products of Kronecker tensors. Another proof is based on the "Schouten identity"[11]

[11] This obvious statement is not due to the Dutch mathematician D.J. Schouten, but was stumbled on in the construction of supergravity by D.Z. Freedman and the author. For a while they studied the textbook by Schouten to find identities which proved that a complicated expression involving a Riemann tensor vanishes. In the end, they realized that this expression vanished due to antisymmetrization. The expressions "Schouten identity" and "to Schoutenize" became household expressions in the supergravity community.

which is the observation that a totally antisymmetric tensor with 5 indices vanishes in 4 dimensions (because there are always at least two indices equal). Writing the left-hand side of (8.2.36) as $\epsilon_{\mu\nu\rho\alpha}\delta_\beta{}^\tau\sigma_{\alpha\beta}$ and using the Schouten identity

$$\epsilon_{\mu\nu\rho\alpha}\delta_\beta{}^\tau = \epsilon_{\beta\nu\rho\alpha}\delta_\mu{}^\tau + \epsilon_{\mu\beta\rho\alpha}\delta_\nu{}^\tau + \epsilon_{\mu\nu\beta\alpha}\delta_\rho{}^\tau + \epsilon_{\mu\nu\rho\beta}\delta_\alpha{}^\tau \ , \tag{8.2.37}$$

the identity (8.2.35) can be used to prove the property (8.2.36) (the last term in (8.2.37) yields minus the contribution of the term on the left-hand side). In a similar way one may prove

$$\epsilon_{\mu\nu\rho\sigma}\bar{\sigma}_{\sigma\tau} = -\delta_{\mu\tau}\bar{\sigma}_{\nu\rho} + \delta_{\nu\tau}\bar{\sigma}_{\mu\rho} - \delta_{\rho\tau}\bar{\sigma}_{\mu\nu}. \tag{8.2.38}$$

The extra overall minus sign is due to the extra minus sign in the selfduality relation in (8.2.35).

Further identities are the commutator of two Lorentz generators, and the anti-commutator which is proportional to the unit matrix in spinor space

$$\begin{aligned}[]
[\sigma_{\mu\nu}, \sigma_{\rho\sigma}] &= 2\delta_{\nu\rho}\sigma_{\mu\sigma} + \text{three more terms} \ , \\
\{\sigma_{\mu\nu}, \sigma_{\rho\sigma}\} &= 2(\delta_{\mu\sigma}\delta_{\nu\rho} - \delta_{\mu\rho}\delta_{\nu\sigma}) + 2\epsilon_{\mu\nu\rho\sigma} \ .
\end{aligned} \tag{8.2.39}$$

One easy way to prove or check these identities is to use 4×4 Dirac matrices; for example $\{\gamma_1\gamma_2, \gamma_3\gamma_4\} = 2\gamma_5$ and $\{\gamma_{12}, \gamma_{13}\} = 0$ but $\{\gamma_{12}, \gamma_{12}\} = -2$ and $[\gamma_{12}, \gamma_{13}] = -2\gamma_{23}$. Because $\gamma_5 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$, it is clear that the $\bar{\sigma}_{\mu\nu}$ satisfy the same commutation and anticommutation relations, but with a different sign for the term with the ϵ symbol. In particular,

$$\{\bar{\sigma}_{\mu\nu}, \bar{\sigma}_{\rho\sigma}\} = 2(\delta_{\mu\sigma}\delta_{\nu\rho} - \delta_{\mu\rho}\delta_{\nu\sigma}) - 2\epsilon_{\mu\nu\rho\sigma} \ . \tag{8.2.40}$$

All these identities can also be derived using two-component spinor formalism for vectors. For example, a vector v^μ is written as $v^{\alpha\alpha'} \equiv v^\mu \sigma_\mu{}^{\alpha\alpha'}$, and then one may use such identites as

$$\delta_\mu{}^\nu \sim \delta_{\alpha\alpha'}{}^{\beta\beta'} \sim \delta_\alpha{}^\beta\delta_{\alpha'}{}^{\beta'}; \qquad \delta_{\mu\nu} \sim \epsilon_{\alpha\beta}\epsilon_{\alpha'\beta'} \ . \tag{8.2.41}$$

If one never introduces any vector indices at all but only uses spinor indices, this spinor formalism turns about all identities into trivialities, but we prefer to also keep vector indices around. The other extreme is to expand $\sigma_{\mu\nu}$ and $\bar{\sigma}_{\mu\nu}$ into Pauli matrices τ_a as $\sigma_{\mu\nu} = i\bar{\eta}^a_{\mu\nu}\tau_a$ and $\bar{\sigma}_{\mu\nu} = i\eta^a_{\mu\nu}\tau_a$ where $\eta^a_{\mu\nu}$ and $\bar{\eta}^a_{\mu\nu}$ are constructed from ϵ_{aij} and δ_{ai} tensors, as in (8.1.13). A whole calculus of these "'t Hooft-tensors" can be set-up, and is often used. We discuss it in appendix B. We shall not limit ourselves to one of these extremes; proofs are given either by using 2-component spinors or 4×4 Dirac matrices, depending on which approach is simplest for a given problem.

The index structure of the ansatz for A_μ in (8.2.28) merits a short discussion. A Lie-algebra valued gauge field A_μ has indices i, j for a representation R of an $SU(N)$ group. For $SU(2)$ the generators in the defining representation are the Pauli matrices τ^a divided by $2i$, hence $A_\mu = (A_\mu)^i{}_j = A^a_\mu \left(\frac{\tau_a}{2i}\right)^i{}_j$. The ansatz for the instanton can then be written as

$$(A_\mu)^i{}_j = (\sigma_{\mu\nu})^i{}_j\, x^\nu f(x^2) \; . \tag{8.2.42}$$

The indices μ, ν are Lorentz indices, but the indices i, j are $SU(2)$ indices. Hence the matrix $(\sigma_{\mu\nu})^i{}_j$ carries simultaneously spacetime indices and internal $SU(2)$ indices. The matrices $\sigma_{\mu\nu}$ are indeed proportional to $\tau_a, \sigma_{\mu\nu} = i\bar{\eta}^a_{\mu\nu}\tau_a$, as one may check for specific values of μ and ν, using

$$\begin{aligned}(\sigma_{\mu\nu})^i{}_j &= \tfrac{1}{2}\left\{(\sigma_\mu)^{i\beta'}(\bar{\sigma}_\nu)_{\beta' j} - (\sigma_\nu)^{i\beta'}(\bar{\sigma}_\mu)_{\beta' j}\right\} \\ (\sigma_\mu)^{i\beta'} &= \{\vec{\tau}, i\} \quad , (\bar{\sigma}_\mu)_{\beta' j} = \{\vec{\tau}, -i\} \; .\end{aligned} \tag{8.2.43}$$

The matrices $\eta^a_{\mu\nu}$ and $\bar{\eta}^a_{\mu\nu}$ are actually invariant tensors of a particular $SU(2)$ group. There are three groups $SU(2)$: the gauge group $SU(2)_g$ and the rotation group $SO(4) = SU(2)_L \times SU(2)_R$ generated by $\eta^a_{\mu\nu}$ and $\bar{\eta}^a_{\mu\nu}$. The tensor $\eta^a_{\mu\nu}$ is invariant under the combined $SU(2)_g$ gauge transformations acting on the index a generated by ϵ_{abc}, and the $SU(2)_L$ Lorentz transformations generated by $\eta^b_{\rho\sigma}$. Indeed, under infinitesimal variations with parameter λ^a we find, using (8.B.350),

$$\delta\eta^a = \epsilon^{abc}\eta^b\lambda^c_g + \tfrac{1}{2}\lambda^c_L[\eta^c, \eta^a] = 0 \qquad \text{if } \lambda^a_g = \lambda^a_L \; . \tag{8.2.44}$$

Furthermore, $\eta^a_{\mu\nu}$ is separately invariant under the $SU(2)_R$ subgroup of the Lorentz group generated by $\bar{\eta}^b_{\rho\sigma}$; this follows from $[\eta^a, \bar{\eta}^b] = 0$. In fact, $\eta^a = L^a{}_4 + \frac{1}{2}\epsilon^{abc}L_{bc}$ and $\bar{\eta}^a = -L^a{}_4 + \frac{1}{2}\epsilon^{abc}L_{bc}$ from which $[\eta^a, \bar{\eta}^b] = 0$ easily follows.[12]

Spinor indices are raised and lowered by ϵ-tensors following the northwest-southeast convention: $v^{\alpha'} = \epsilon^{\alpha'\beta'}v_{\beta'}$ and $v^{\alpha} = \epsilon^{\alpha\beta}v_{\beta}$. So $(\bar{\sigma}^\mu)^{\beta'\alpha} = \epsilon^{\beta'\delta'}\epsilon^{\alpha\gamma}(\bar{\sigma}^\mu)_{\delta'\gamma}$. There are various definitions of these ϵ tensors in the literature; we define

$$\epsilon^{\alpha\beta} = -\epsilon^{\alpha'\beta'} . \tag{8.2.45}$$

Note that numerically $\epsilon^{\alpha\beta} = \epsilon_{\alpha\beta}$ but also $\epsilon^{\alpha'\beta'} = \epsilon_{\alpha'\beta'}$ because one needs two ϵ tensors to raise or lower both indices of an ϵ tensor. We fix the overall sign by $\epsilon^{\alpha\beta} = \epsilon^{ij}$ where $\epsilon^{12} = 1$. A crucial relation in the spinor formalism which we shall frequently use is

$$\bar{\sigma}_{\mu,\alpha' i} = \sigma_{\mu,i\,\alpha'} \ , \tag{8.2.46}$$

where we recall that $\sigma_{\mu,i\,\alpha'} = \sigma_\mu^{j\,\beta'}\epsilon_{ji}\epsilon_{\beta'\alpha'}$.

Using 2-component spinor indices for vectors,

$$(\bar{\sigma}^\mu)_{\alpha'\alpha}A_\mu \equiv A_{\alpha'\alpha} \text{ and } (\bar{\sigma}_\nu)_{\beta' j}x^\nu \equiv x_{\beta' j} \ , \tag{8.2.47}$$

the ansatz for the instanton solution in (8.2.28) with spinor indices for A_μ becomes

$$\begin{aligned} &(\bar{\sigma}^\mu)_{\alpha'\alpha}(A_\mu)^i{}_j \equiv A_{\alpha'\alpha}{}^i{}_j = (\bar{\sigma}^\mu)_{\alpha'\alpha}(\sigma_{\mu\nu})^i{}_j x^\nu f(x^2) \\ &= \left\{\delta^{\beta'}_{\alpha'}\delta^i_\alpha x_{\beta' j} - \epsilon_{\alpha'\beta'}\epsilon_{\alpha j}x^{i\beta'}\right\} f(x^2) = \left\{\delta^i_\alpha x_{\alpha' j} + \epsilon_{\alpha j}x^i{}_{\alpha'}\right\} f(x^2) \ . \end{aligned} \tag{8.2.48}$$

The trace over (ij) clearly vanishes, and this fixes the relative sign. We worked out the matrix $(\bar{\sigma}^\mu)_{\alpha'\alpha}(\sigma_{\mu\nu})^i{}_j$ using

$$\bar{\sigma}^\mu_{\alpha'\alpha}\sigma^{i\beta'}_\mu = 2\delta^{\beta'}_{\alpha'}\delta^i_\alpha \ , \tag{8.2.49}$$

and

$$\bar{\sigma}^\mu_{\alpha'\alpha}(\bar{\sigma}_\mu)_{\beta' j} = \bar{\sigma}^\mu_{\alpha'\alpha}(\sigma_\mu)_{j\beta'} = \bar{\sigma}^\mu_{\alpha'\alpha}\sigma^{k\gamma'}_\mu\epsilon_{kj}\epsilon_{\gamma'\beta'} = 2\epsilon_{\alpha'\beta'}\epsilon_{\alpha j} \ . \tag{8.2.50}$$

[12] The 4×4 matrices $L^a{}_4$ and L_{bc} have entries $(L^a{}_4)_{\mu\nu} = \delta^a_\mu\delta_{4\nu} - \delta^a_\nu\delta_{4\mu}$ and $(L_{bc})_{\mu\nu} = \delta_{b\mu}\delta_{c\nu} - \delta_{c\mu}\delta_{b\nu}$. They form the defining representation of the Euclidean Lorentz algebra.

2.2 Solving the selfduality equations

Let us now come back to the construction of instanton solutions. Substituting the ansatz for A_μ in (8.2.28) into the definition of $F_{\mu\nu}$ yields with (8.2.39)

$$\begin{aligned} F_{\mu\nu} &= \alpha\sigma_{\nu\rho}\partial_\mu\partial_\rho \ln\phi - \alpha\sigma_{\mu\rho}\partial_\nu\partial_\rho \ln\phi + \alpha^2[\sigma_{\mu\rho},\sigma_{\nu\sigma}](\partial_\rho \ln\phi)(\partial_\sigma \ln\phi) \\ &= (\alpha\sigma_{\nu\rho}\partial_\mu\partial_\rho \ln\phi - \mu\leftrightarrow\nu) + 2\alpha^2(\sigma_{\mu\sigma}\partial_\nu \ln\phi\ \partial_\sigma \ln\phi - \mu\leftrightarrow\nu) \\ &- 2\alpha^2\sigma_{\mu\nu}(\partial\ln\phi)^2\ . \end{aligned} \tag{8.2.51}$$

We want to solve the equation $F_{\mu\nu} = {}^*F_{\mu\nu}$. The dual of $F_{\mu\nu}$ can be written as an expression without any ϵ tensor by using the identities for $\epsilon_{\mu\nu\rho\sigma}\sigma_{\sigma\tau}$ and $\epsilon_{\mu\nu\rho\sigma}\sigma_{\rho\sigma}$ in (8.2.35) and (8.2.36). One finds

$$\begin{aligned} {}^*F_{\mu\nu} &= \tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}F_{\rho\sigma} = \alpha\epsilon_{\mu\nu\rho\sigma}\sigma_{\sigma\alpha}\partial_\rho\partial_\alpha \ln\phi \\ &+ 2\alpha^2\epsilon_{\mu\nu\rho\sigma}\sigma_{\rho\beta}\partial_\sigma \ln\phi\ \partial_\beta \ln\phi - \alpha^2\epsilon_{\mu\nu\rho\sigma}\sigma_{\rho\sigma}(\partial\ln\phi)^2 \\ &= \sigma_{\nu\rho}(\alpha\partial_\rho\partial_\mu \ln\phi - 2\alpha^2\partial_\rho \ln\phi\ \partial_\mu \ln\phi) - \mu\leftrightarrow\nu \\ &+ \sigma_{\mu\nu}(\alpha\partial^2\ln\phi)\ . \end{aligned} \tag{8.2.52}$$

Equating $F_{\mu\nu}$ to ${}^*F_{\mu\nu}$ yields two equations for ϕ, namely one for the terms with $\sigma_{\nu\rho}$ and the other for the terms with $\sigma_{\mu\nu}$

$$\begin{aligned} \alpha\partial_\mu\partial_\rho \ln\phi - 2\alpha^2\partial_\mu \ln\phi\ \partial_\rho \ln\phi &= \alpha\partial_\mu\partial_\rho \ln\phi - 2\alpha^2\partial_\mu \ln\phi\ \partial_\rho \ln\phi\ , \\ -2\alpha^2(\partial\ln\phi)^2 &= \alpha\partial^2\ln\phi\ . \end{aligned} \tag{8.2.53}$$

The first equation is identically satisfied (for that reason we equated $F_{\mu\nu}$ to $+{}^*F_{\mu\nu}$), while the second equation can be rewritten as $\partial^2\ln\phi + 2\alpha(\partial\ln\phi)^2 = 0$. For $\alpha = \frac{1}{2}$ it simplifies to $\partial^2\phi/\phi = 0$. (Setting $\alpha = \frac{1}{2}$ is not a restriction because rescaling $\phi \to \phi^{1/2\alpha}$ achieves the same result). Thus the problem of finding an instanton solution has been reduced to solving

$$\frac{1}{\phi}\Box\phi = 0\ . \tag{8.2.54}$$

Setting $\phi = \frac{1}{x^2}$ yields for $x \neq 0$ a solution: $\partial^2\phi/\phi = x^2\partial_\mu(-2x^\mu/x^4) = 0$. However, this is also a solution at $x = 0$ because $\partial^2 x^{-2}$ is proportional to a delta function (note that the dimensions match) and $x^2\delta^4(x) = 0$

$$\partial^2 \frac{1}{x^2} = -4\pi^2\delta^4(x) \ . \tag{8.2.55}$$

(To check the coefficient, we integrate over a small ball, which includes the point $x = 0$; we obtain then $\int \partial^2 \frac{1}{x^2} \mathrm{d}^4x = \int r^3 \mathrm{d}r \mathrm{d}\Omega_\mu \partial_\mu \frac{1}{x^2} = \int r^3 \mathrm{d}\Omega_\mu(-2x^\mu/x^4) = -4\pi^2$. The surface of a sphere in 4 dimensions is $2\pi^2$).

We have thus found a selfdual solution

$$A_\mu(x) = \tfrac{1}{2}\sigma_{\mu\nu}\partial_\nu \ln\left[1 + \frac{\rho^2}{(x-a)^2}\right] \ . \tag{8.2.56}$$

We have added unity to ϕ in order that $A_\mu(x)$ vanishes for large $|x|$.[13] A more general solution is given by $A_\mu(x) = \frac{1}{2}\sigma_{\mu\nu}\partial_\nu \ln\phi$ with

$$\phi = 1 + \sum_{i=1}^{k} \frac{\rho_i^2}{(x-a_i)^2} \ , \tag{8.2.57}$$

which also solves $\partial^2\phi/\phi = 0$. These are a class of k-instanton solutions, parameterized by $5k$ collective coordinates. In particular for $k = 1$ we find the one-instanton solution

$$\begin{aligned} A_\mu^{\text{sing}}(x) &= \tfrac{1}{2}\sigma_{\mu\nu}\partial_\nu \ln\left[1 + \frac{\rho^2}{(x-a)^2}\right] \ , \\ &= -\sigma_{\mu\nu}\frac{\rho^2(x-a)_\nu}{(x-a)^2((x-a)^2+\rho^2)} \qquad (k=1, \text{singular}) \ . \end{aligned} \tag{8.2.58}$$

This solution is clearly singular at $x = a$, but one can remove the singularity at $x = a$ by a singular gauge transformation (which maps the singularity to $x^2 = \infty$). To determine this gauge transformation we first study the structure of the singularity. Near $x = a$ the singular solution becomes

$$A_\mu^{\text{sing}}(x) \approx -\sigma_{\mu\nu}\frac{(x-a)_\nu}{(x-a)^2} \ , \tag{8.2.59}$$

[13] Actually, one must add unity because without adding unity the solution is pure gauge.

which is a pure gauge field with $U(x-a)$ in (A.9)

$$U^{-1}\partial_\mu U = -\sigma_{\mu\nu}\frac{(x-a)_\nu}{(x-a)^2}\ ; \qquad U(x) = \frac{x_4 + ix_k\sigma_k}{\sqrt{x^2}} = i\bar{\sigma}_\mu x_\mu/\sqrt{x^2}. \tag{8.2.60}$$

Note that U is unitary, and U^{-1} equals $-i\sigma_\mu x_\mu/\sqrt{x^2}$, which follows from the property $\sigma_\rho\bar{\sigma}_\mu + \sigma_\mu\bar{\sigma}_\rho = 2\delta_{\rho\mu}$.

From (8.2.58) and (8.2.59) it follows that we can write A_μ^{sing} as

$$A_\mu^{\text{sing}}(x) = \frac{\rho^2}{(x-a)^2+\rho^2}U^{-1}\partial_\mu U\ . \tag{8.2.61}$$

It is now clear that an opposite gauge tranformation removes the singularity at $x=0$

$$\begin{aligned} A_\mu^{\text{reg}}(x) &= U(\partial_\mu + A_\mu^{\text{sing}})U^{-1} = \partial_\mu U U^{-1}\left(-1 + \frac{\rho^2}{(x-a)^2+\rho^2}\right) \\ &= (U\partial_\mu U^{-1})\frac{(x-a)^2}{(x-a)^2+\rho^2}\ . \end{aligned} \tag{8.2.62}$$

The expressions $U^{-1}\partial_\mu U$ and $U\partial_\mu U^{-1}$ are closely related; in fact, one finds by direct evaluation

$$U\partial_\mu U^{-1} = -\bar{\sigma}_{\mu\nu}\frac{(x-a)_\nu}{(x-a)^2}\ . \tag{8.2.63}$$

Thus the regular one-instanton solution is given by

$$A_\mu^{\text{reg}} = -\bar{\sigma}_{\mu\nu}\frac{(x-a)_\nu}{(x-a)^2+\rho^2} \qquad (k=1, \text{regular})\ . \tag{8.2.64}$$

Of course, the singular and the regular solution are both selfdual, because selfduality is a gauge-invariant property, but the field strengths differ by a gauge transformation. Setting $a=0$ for simplicity, one finds for the field strengths in the regular and singular gauge

$$\begin{aligned} F_{\mu\nu}^{\text{reg}} &= 2\bar{\sigma}_{\mu\nu}\frac{\rho^2}{[x^2+\rho^2]^2} && (k=1, \text{regular})\ , \\ F_{\mu\nu}^{\text{sing}} &= U^{-1}F_{\mu\nu}^{\text{reg}}U = -\frac{ix^\rho\sigma_\rho}{\sqrt{x^2}}\,\frac{2\bar{\sigma}_{\mu\nu}\rho^2}{(x^2+\rho^2)^2}\,\frac{ix^\sigma\bar{\sigma}_\sigma}{\sqrt{x^2}} && (k=1, \text{singular})\ . \end{aligned} \tag{8.2.65}$$

It is clear that $F^{\rm reg}_{\mu\nu}$ is selfdual because $\bar\sigma_{\mu\nu}$ is selfdual, but also $F^{\rm sing}_{\mu\nu}$ is selfdual[14] as is clear from acting with $\epsilon^{\mu\nu\rho\sigma}$ on $\bar\sigma_{\mu\nu}$.

The action for the one-instanton solution is, of course, proportional to the winding number

$$S = -\frac{1}{2g^2}\int \mathrm{tr}\, F_{\mu\nu}F_{\mu\nu}\,\mathrm{d}^4x = -\frac{1}{2g^2}\int \mathrm{tr}\, F_{\mu\nu}{}^*F_{\mu\nu}\,\mathrm{d}^4x = \frac{8\pi^2}{g^2}\ . \tag{8.2.66}$$

The same result is obtained by direct evaluation of this integral.

The anti-instanton (the solution with $k=-1$) is closely related to the instanton solution. Recall that we derived the instanton solution by making the ansatz $A_\mu = \alpha\sigma_{\mu\nu}\partial_\nu \ln\phi$, evaluating $F_{\mu\nu}$ and ${}^*F_{\mu\nu}$ in terms of $\sigma_{\mu\nu}$ matrices, and then setting $F_{\mu\nu} = {}^*F_{\mu\nu}$. For the anti-instanton solution we make the ansatz $A_\mu = \beta\bar\sigma_{\mu\nu}\partial_\nu\ln\phi$. The expression for $F_{\mu\nu}$ is unchanged (except that A_μ contains $\bar\sigma_{\mu\nu}$ instead of $\sigma_{\mu\nu}$), but the $\bar\sigma_{\mu\nu}$ are selfdual instead of anti-selfdual, hence the expression for $\epsilon_{\mu\nu\rho\sigma}\bar\sigma_{\sigma\tau}$ has opposite signs from $\epsilon_{\mu\nu\rho\sigma}\sigma_{\sigma\tau}$. The equation with $\partial_\mu\partial_\rho\ln\phi$ again cancels if $F_{\mu\nu} = -{}^*F_{\mu\nu}$, which leads to opposite winding number ($k=-1$). The other equation is again $\partial^2\ln\phi + 2\beta(\partial\ln\phi)^2 = 0$, hence $\beta=\frac{1}{2}$ and again $\phi = 1+\sum_{i=1}^N \frac{\rho_i^2}{(x-a_i)^2}$. This yields for the singular-gauge anti-instanton solution

$$A^{\rm sing}_\mu = -\bar\sigma_{\mu\nu}\frac{\rho^2(x-a)_\nu}{(x-a)^2[(x-a)^2+\rho^2]}\ , \qquad (k=-1,\ \text{singular}) \tag{8.2.67}$$

Setting again temporarily $a=0$, we find near $x=0$

$$A^{\rm sing}_\mu \approx -\bar\sigma_{\mu\nu}x_\nu/x^2 = U\partial_\mu U^{-1}\ , \tag{8.2.68}$$

[14] Using some further identities which follow from the results for $[\gamma_{\mu\nu},\gamma_\rho]$ and $\{\gamma_{\mu\nu},\gamma_\rho\}$

$$\bar\sigma_\mu\sigma_{\nu\rho} = \delta_{\mu\nu}\bar\sigma_\rho - \delta_{\mu\rho}\bar\sigma_\nu - \epsilon_{\mu\nu\rho\sigma}\bar\sigma_\sigma\ ; \qquad \bar\sigma_{\mu\nu}\bar\sigma_\rho = \delta_{\nu\rho}\bar\sigma_\mu - \delta_{\mu\rho}\bar\sigma_\nu - \epsilon_{\mu\nu\rho\sigma}\bar\sigma_\sigma\ ,$$
$$\sigma_{\mu\nu}\sigma_\rho = \delta_{\nu\rho}\sigma_\mu - \delta_{\mu\rho}\sigma_\nu + \epsilon_{\mu\nu\rho\sigma}\sigma_\sigma\ ; \qquad \sigma_\rho\bar\sigma_{\mu\nu} = \delta_{\rho\mu}\sigma_\nu - \delta_{\rho\nu}\sigma_\mu + \epsilon_{\rho\mu\nu\sigma}\sigma_\sigma\ ,$$

one finds for the $k=1$ singular solution

$$F^{\rm sing}_{\mu\nu} = \frac{2\rho^2}{(x^2+\rho^2)^2}\left(-2\frac{x_\mu x_\rho}{x^2}\sigma_{\rho\nu} + 2\frac{x_\nu x_\rho}{x^2}\sigma_{\rho\mu} + \sigma_{\mu\nu}\right)$$

In this form the selfduality is no longer manifest.

with the same $U = i\bar{\sigma}^\mu x_\mu/\sqrt{x^2}$ as before. Similarly as for the instanton, we have

$$\begin{aligned} A_\mu^{\rm sing} &= U\partial_\mu U^{-1}\frac{\rho^2}{x^2+\rho^2} \\ A_\mu^{\rm reg} &= U^{-1}(\partial_\mu + A_\mu^{\rm sing})U \\ &= \partial_\mu U^{-1}U\left(\frac{\rho^2}{x^2+\rho^2}-1\right) \\ &= U^{-1}\partial_\mu U\left(\frac{x^2}{x^2+\rho^2}\right) . \end{aligned} \tag{8.2.69}$$

Using the expression for $U^{-1}\partial_\mu U$ in (8.2.60) one finds

$$A_\mu^{\rm reg} = -\sigma_{\mu\nu}\frac{(x-a)_\nu}{(x-a)^2+\rho^2} \qquad (k=-1, \text{regular}) \tag{8.2.70}$$

The curvatures for the anti-instanton solution are obtained by interchanging $\sigma_{\mu\nu}$ and $\bar{\sigma}_{\mu\nu}$ in the instanton solution

$$F_{\mu\nu}^{\rm reg} = 2\sigma_{\mu\nu}\frac{\rho^2}{[(x-a)^2+\rho^2]^2} \qquad (k=-1, \text{regular}) . \tag{8.2.71}$$

So, the only difference between the instanton and anti-instanton solutions is the exchange between $\sigma_{\mu\nu}$ and $\bar{\sigma}_{\mu\nu}$ in $F_{\mu\nu}$ and A_μ. For the instanton solution, $F_{\mu\nu}^{\rm reg}$ and $A_\mu^{\rm reg}$ depend on $\bar{\sigma}_{\mu\nu}$, but $A_\mu^{\rm sing}$ depends on $\sigma_{\mu\nu}$, and $F_{\mu\nu}^{\rm sing}$ also depends on $\sigma_{\mu\nu}$ (setting $a = 0$ again for notational simplicity),

$$F_{\mu\nu}^{\rm sing} = U\,F_{\mu\nu}^{\rm reg}\,U^{-1} = \frac{ix^\rho\bar{\sigma}_\rho}{\sqrt{x^2}}\,\frac{2\sigma_{\mu\nu}\rho^2}{(x^2+\rho^2)^2}\,\frac{-ix^\sigma\sigma_\sigma}{\sqrt{x^2}} \quad (k=-1, \text{singular}) . \tag{8.2.72}$$

If one evaluates the product of the σ matrices as in footnote 14, one finds an expression for $F_{\mu\nu}^{\rm sing}$ in which the anti-selfduality is no longer manifest.

3 Collective coordinates, the index theorem and fermionic zero modes

We found in section 2 one-instanton solutions ($k = 1$) in $SU(N)$ with $4N$ parameters. The question arises whether these are all the solutions. To find this out, one can consider small deformations of the solution, $A_\mu + \delta A_\mu$, and study when they preserve selfduality. Expanding to first order in the deformation, and using that the variation of a curvature is the covariant derivative of the variation of the gauge field, this leads to the condition

$$D_\mu \delta A_\nu - D_\nu \delta A_\mu = {}^*(D_\mu \delta A_\nu - D_\nu \delta A_\mu) \ , \tag{8.3.73}$$

where the covariant derivative depends only on the classical solution but not on δA_μ. In addition we require that the new solution is not related to the old one by a gauge transformation. This can be achieved by requiring that the small deformations are orthogonal to any small gauge transformation $D_\mu \Lambda$, for any function Λ, i.e.

$$\int \mathrm{d}^4 x \, \mathrm{tr} \, \{(D_\mu \Lambda) \delta A_\mu\} = 0 \ . \tag{8.3.74}$$

This certainly rules out deformations of the form $\delta A_\mu = D_\mu \Lambda$. After partial integration the orthogonality requirement leads to the usual gauge condition in the background field formalism

$$D_\mu \delta A^\mu = 0 \ . \tag{8.3.75}$$

At this point the reader may start feeling uneasy because the conditions (8.3.73) and (8.3.74) may seem too strong. First of all, the deformation should be a solution but need not be (anti-) selfdual. Furthermore, the field equation for the fluctuations consists of the sum of a classical piece and a piece from the gauge fixing term, so that, requiring each part to vanish separately may seem too restrictive. However, one can prove the following general result [44]. Arbitrary solutions of the fluctuations around an (anti-) instanton which are square-integrable so that they do not change the winding number, are themselves also (anti-) selfdual and transversal. To prove this property, note that the field equations for the fluctuations read $D_\mu(A + \delta A) F^{\mu\nu}(A + \delta A) + D_\nu(A + \delta A)(D^\mu \delta A_\mu) = 0$. The second term comes from the gauge-fixing term. Taking the $D^\nu(A + \delta A)$ derivative, the first term vanishes while the second term yields $D^2(D^\mu \delta A_\mu) = 0$ to first order in δA_μ, hence $D^\mu \delta A_\mu = 0$ on-shell. So, on-shell the gauge-fixing term vanishes. The terms in the classical action which are quadractic in the fluctuations can be written as $-\frac{1}{8}(f_{\mu\nu} - {}^*f_{\mu\nu})^2$ where $f_{\mu\nu} = D_\mu \delta A_\nu - D_\nu \delta A_\mu$. To prove this one has to partially integrate $f_{\mu\nu}{}^* f_{\mu\nu}$ and here one needs the square integrability of δA_μ to exclude extra boundary terms. The minimum of the action yields a solution, hence $f_{\mu\nu} = {}^*f_{\mu\nu}$ on-shell. Thus imposing (8.3.73) and (8.3.75) is not too restrictive.

The requirement that δA_μ be square integrable is due to the fact that the inner product of zero modes δA_μ will later give us the metric or moduli space, which in turn will give us the integration measure of the moduli space. Also, for the index theorem which will be used to determine the number of zero modes, one needs the L^2 norm for fluctuations. It is remarkable that the zero modes which satisfy the differential equations in (8.3.73) and (8.3.75) are all square integrable.

In references [34, 44] the solutions of (8.3.73) subject to the condition (8.3.75) were studied using the Atiyah-Singer index theorem. Index theory turns out to be a useful tool when counting the number of solutions to a certain linear differential equation of the form $\hat{D}T = 0$, where $\hat{D}$ is some differential operator and T is a tensor. We will elaborate on this in the next subsection and also when studying fermionic collective coordinates. The ultimate result of [34] is that there are $4Nk$ solutions, leading indeed to $4N$ collective coordinates for $k = 1$ [44]. An assumption required to apply index theorems is that the space has to be compact. One must therefore compactify Euclidean space to a four-sphere S^4, as was already discussed in footnote 4.

3.1 Bosonic collective coordinates and the Dirac operator

In this section we will make more precise statements about the number of solutions to the selfduality equations by relating it to the index of the Dirac operator. The problem is to determine the number of solutions to the (anti-)selfduality equations with topological charge k. For definiteness we consider anti-instantons, so we look for deformations which satisfy an anti-selfduality equation.

As explained in the last subsection, we study deformations of a given classical solution $A_\mu^{\rm cl} + \delta A_\mu$. Let us define $\phi_\mu \equiv \delta A_\mu$ and $f_{\mu\nu} \equiv D_\mu \phi_\nu - D_\nu \phi_\mu$. The covariant derivative here contains only $A_\mu^{\rm cl}$. The constraints can then be written as

$$\bar{\sigma}_{\mu\nu} D_\mu \phi_\nu = 0 \ ; \qquad D_\mu \phi_\mu = 0 \ , \tag{8.3.76}$$

which are $3+1$ relations. Indeed, more explicitly, $(\bar{\sigma}_{\mu\nu})_\alpha{}^{\alpha'} D_\mu \phi_\nu$ are 3 Lie-algebra valued expressions because $\alpha, \alpha' = 1, 2$ and $\mathrm{tr}\bar{\sigma}_{\mu\nu} = 0$. To prove the first relation, multiply by $\bar{\sigma}_{\rho\sigma}$ and take the trace. Since the trace of $[\bar{\sigma}_{\rho\sigma}, \bar{\sigma}_{\mu\nu}]$ vanishes, while $\{\bar{\sigma}_{\rho\sigma}, \bar{\sigma}_{\mu\nu}\} = 2(\delta_{\mu\sigma}\delta_{\rho\nu} - \delta_{\nu\sigma}\delta_{\rho\mu}) - 2\epsilon_{\rho\sigma\mu\nu}$, one finds $D_\sigma \phi_\rho - D_\rho \phi_\sigma - \epsilon_{\mu\nu\rho\sigma} D_\mu \phi_\nu = 0$, which is the anti-selfduality condition (3 relations). Both equations can be written as one simple equation as follows

$$\bar{\sigma}_\mu \sigma_\nu D_\mu \phi_\nu = 0 \ , \tag{8.3.77}$$

because $\bar{\sigma}_\mu \sigma_\nu = \delta_{\mu\nu} + \bar{\sigma}_{\mu\nu}$, and the spinor structures of $\delta_{\mu\nu}$ and $\bar{\sigma}_{\mu\nu}$ are independent.

Introducing two-component spinor notation with

$$\bar{\not{D}} = \bar{\sigma}_{\mu,\alpha'\beta} D_\mu = \bar{D}_{\alpha'\beta} \ ; \qquad \sigma_\nu^{\alpha\beta'} \phi_\nu = \Phi^{\alpha\beta'} \ , \tag{8.3.78}$$

the deformations of an anti-instanton can be written as follows

$$\bar{\not{D}} \Phi = \bar{D}_{\alpha'\beta} \Phi^{\beta\gamma'} = 0 \ . \tag{8.3.79}$$

Note that $\Phi^{\beta\gamma'}$ is in the adjoint representation, so 8.3.79 stands for $\partial_{\alpha'\beta}\Phi^{\beta\gamma'} + [A_{\alpha'\beta}, \Phi^{\beta\gamma'}] = 0$. Using the explicit representation of the matrices σ_μ in 8.B.352, we can represent the quaternion Φ by

$$\Phi = \begin{pmatrix} a & b^* \\ b & -a^* \end{pmatrix} \ , \tag{8.3.80}$$

with a and b complex adjoint-valued functions. Then (8.3.79) reduces to two spinor equations, one for

$$\lambda = \binom{a}{b} \ ; \qquad \bar{\not{D}}\lambda = 0 \ , \tag{8.3.81}$$

and one for $i\sigma^2\lambda^* = \begin{pmatrix} b^* \\ -a^* \end{pmatrix}$. Conversely, for each spinor solution λ to the Dirac equation, one may show that also $i\sigma^2\lambda^*$ is a solution. (Use $(\bar{\sigma}^\mu)^* = -\sigma^2\bar{\sigma}^\mu\sigma^2$). Indeed, if λ yields a deformation $(\delta A_1, \delta A_2, \delta A_3, \delta A_4)$, then $i\sigma^2\lambda^*$ corresponds to the deformation $(\delta A_1', \delta A_2', \delta A_3', \delta A_4')$ with $\delta A_1' = -\delta A_3, \delta A_3' = \delta A_1, \delta A_2' = \delta A_4$ and $\delta A_4' = -\delta A_2$. They are not related by a Lorentz transformation because the coordinates x^μ are not transformed. Thus given λ, we obtain two linearly independent deformations of the (anti-) instanton. As we already stressed, the spinors λ are in the adjoint representation. We shall discuss other representations later.

Given a solution λ of the spinor equation, one can still construct **two** other solutions of the deformation of the anti-instanton, which differ by a factor i

$$\Phi^{(1)} = \begin{pmatrix} a & b^* \\ b & -a^* \end{pmatrix} , \ \Phi^{(2)} = \begin{pmatrix} ia & -ib^* \\ ib & ia^* \end{pmatrix} . \tag{8.3.82}$$

The reason we do not count $i\lambda$ as a different solution for the spinors but treat $\Phi^{(1)}$ and $\Phi^{(2)}$ as independent has to do with reality properties: δA_μ^a should be real, and $\Phi^{(1)}$ and $\Phi^{(2)}$ yield different variations δA_μ. Namely, $a = \phi_3 + i\phi_4$ and $b = \phi_1 + i\phi_2$, so

$$\begin{aligned} \Phi^{(1)} &: \ \delta A_4 = \phi_4 \ , \delta A_3 = \phi_3 \ , \delta A_1 = \phi_1 \ , \delta A_2 = \phi_2 \ , \\ \Phi^{(2)} &: \ \delta A_4 = \phi_3 \ , \delta A_3 = -\phi_4 \ , \delta A_1 = -\phi_2 \ , \delta A_2 = \phi_1 \ . \end{aligned} \tag{8.3.83}$$

It may seem miraculous that we find a second solution without any hard work, but closer inspection reveals that no miracle is at work: under the substitutions $\delta A_1 \to \delta A_2, \delta A_2 \to -\delta A_1, \delta A_3 \to \delta A_4, \delta A_4 \to -\delta A_3$, one of the anti-selfduality equations is exchanged with the gauge condition, and the other two duality equations get interchanged. Also for solitons this way of counting zero modes is encountered: for example for vortices one complex fermion zero mode corresponds to two real bosonic zero modes [45].

In fact, because $\Phi^{(2)} = \Phi^{(1)} i\sigma_3$, one might wonder whether $\Phi^{(3)} = \Phi^{(1)}(-i\sigma_1)$ and $\Phi^{(4)} = \Phi^{(1)}(-i\sigma_2)$ yield further solutions. One obtains

$$\Phi^{(3)} = \begin{pmatrix} -ib^* & -ia \\ ia^* & -ib \end{pmatrix} \; ; \; \Phi^{(4)} = \begin{pmatrix} b^* & -a \\ -a^* & -b \end{pmatrix} \tag{8.3.84}$$

which are just the $\Phi^{(1)}$ constructed from $\sigma_2\lambda^*$ and $i\sigma_2\lambda^*$. So there are no further independent solutions [44]. Therefore, the number of solutions for Φ is twice the number of solutions for a single two-component adjoint spinor. So, the problem of counting the number of bosonic collective coordinates is now translated to the computation of the Dirac index, which we discuss next.

3.2 Fermionic moduli and the index theorem

Both motivated by the counting of bosonic collective coordinates, as discussed in the last subsection, and by the interest of coupling Yang-Mills theory to fermions, we study the Dirac equation in the background of an anti-instanton. We start with a massless four-component complex (Dirac) fermion ψ, in an arbitrary representation (adjoint, fundamental, etc) of an arbitrary gauge group

$$\gamma_\mu D_\mu \psi \; = \; \not{D}\psi = 0 \; . \tag{8.3.85}$$

We recall that a Dirac spinor can be decomposed into its chiral and anti-chiral components

$$\psi = \begin{pmatrix} \lambda^\alpha \\ \bar{\chi}_{\alpha'} \end{pmatrix} \; ; \qquad \lambda \equiv \tfrac{1}{2}\left(1+\gamma^5\right)\psi \; , \qquad \bar{\chi} \equiv \tfrac{1}{2}\left(1-\gamma^5\right)\psi \; . \tag{8.3.86}$$

We use the Euclidean representation for the Clifford algebra discussed before

$$\gamma^\mu = \begin{pmatrix} 0 & -i\sigma^{\mu\,\alpha\beta'} \\ i\bar{\sigma}^\mu_{\alpha'\beta} & 0 \end{pmatrix} \; , \qquad \gamma^5 = \gamma^1\gamma^2\gamma^3\gamma^4 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \; . \tag{8.3.87}$$

In Euclidean space the Lorentz group decomposes according to $SO(4) = SU(2) \times SU(2)$. The spinor indices α and α' correspond to the doublet representations of these two $SU(2)$ factors. As opposed to the case of Minkowski space, λ^α and $\bar{\chi}_\alpha{}'$ are not in complex-conjugate representations. The Dirac equation then becomes

$$\bar{\not{D}}\lambda = 0 \; , \qquad \not{D}\bar{\chi} = 0 \; , \tag{8.3.88}$$

where $\not{D}$ and $\bar{\not{D}}$ are two-by-two matrixes, see 8.3.78, and λ and $\bar{\chi}$ are independent complex two-component spinors. We now show that in the presence of an anti-instanton, (8.3.88) has zero modes for λ, but not for $\bar{\chi}$. Conversely, in the background of an instanton, $\not{D}$ has zero modes, but $\bar{\not{D}}$ has not. A zero mode is by definition a solution of the linearized field equations for the quantum fluctuations **which is normalizable**. The fermionic fields are treated as quantum fields (there are no background fermionic fields), so normalizable solutions of (8.3.88) are zero modes.

The argument goes as follows. Given a zero mode $\bar{\chi}$ for $\not{D}$, it also satisfies $\bar{\not{D}}\,\not{D}\bar{\chi} = 0$. In other words, $\ker \not{D} \subset \ker\left\{\bar{\not{D}}\,\not{D}\right\}$ where ker denotes the kernel. Next we evaluate

$$\bar{\not{D}}\,\not{D} = \bar{\sigma}_\mu \sigma_\nu D_\mu D_\nu = D^2 + \frac{1}{2}\bar{\sigma}_{\mu\nu}F_{\mu\nu} \; , \tag{8.3.89}$$

where we have used $\bar{\sigma}_\mu\sigma_\nu + \bar{\sigma}_\nu\sigma_\mu = 2\delta_{\mu\nu}$, and $\bar{\sigma}_{\mu\nu}$ was defined in (8.2.34). But notice that the anti-instanton field strength is anti-selfdual whereas the tensor $\bar{\sigma}_{\mu\nu}$ is selfdual, so the second term vanishes. From this it follows that $\bar{\chi}$ satisfies $D^2\bar{\chi} = 0$. Now we can multiply $D^2\bar{\chi}$ with its conjugate $\bar{\chi}^*$ and integrate to get, after partial integration and assuming that the fields go to zero at infinity[15], $\int \mathrm{d}^4x\, |D_\mu\bar{\chi}|^2 = 0$. From this it follows that $\bar{\chi}$ is covariantly constant, $D_\mu\bar{\chi} = 0$, and so $F_{\mu\nu}\,\bar{\chi} = 0$. Since $F_{\mu\nu}\,\bar{\chi} = F^a_{\mu\nu}\,T_a\,\bar{\chi}$, with T_a the generators of the gauge group $SU(2)$ in a representation R, we conclude that $F^a_{\mu\nu}(x)\,T_a\,\bar{\chi}(x)$ must vanish at all points x. Since $F^a_{\mu\nu}$ is proportional to $\eta_{a\mu\nu}$ (or $\bar{\eta}_{a\mu\nu}$), and $\eta_{a\mu\nu}\eta_{b\mu\nu}$ is proportional to δ_{ab}, we find that $T_a\,\bar{\chi}(x)$ vanishes for all a and

[15] Normalizability of zero modes requires that $\bar{\chi}$ tends to zero faster than $1/r^2$ (usually like $1/r^3$ or sometimes $1/r^4$). Then the boundary term with $\bar{\chi}^* D_\mu \bar{\chi}$ indeed vanishes.

all x. Then $D_\mu \bar{\chi} = 0$ reduces to $\partial_\mu \bar{\chi} = 0$, and this implies that $\bar{\chi} = 0$. We conclude that $\not{D}\bar{\chi}$ has no square-integrable solutions. Stated differently, $-D^2$ is a positive definite operator and has no zero modes. Note that this result is independent of the representation of the fermion.

For the λ-equation, we have $\not{D}\,\bar{\not{D}}\lambda = 0$, i.e. $\ker \bar{\not{D}} \subset \ker \left\{\not{D}\,\bar{\not{D}}\right\}$, and we obtain

$$\not{D}\,\bar{\not{D}} = D^2 + \frac{1}{2}\sigma_{\mu\nu}F_{\mu\nu} \ . \tag{8.3.90}$$

This time the second term does not vanish in the presence of an anti-instanton, so zero modes cannot be ruled out. In fact, there do exist fermionic zero modes, because we shall construct them. Knowing that $\not{D}$ has no zero modes, one easily concludes that $\ker \bar{\not{D}} = \ker\left\{\not{D}\,\bar{\not{D}}\right\}$ and $\ker \not{D} = \ker\left\{\bar{\not{D}}\,\not{D}\right\} = 0$.

For massive spinors no zero modes are possible. To prove this one may repeat the same steps as for massless spinors, but now one finds that $\bar{\not{D}}\lambda = im\bar{\chi}$ and $\not{D}\bar{\chi} = -im\lambda$, and iteration yields $\bar{\not{D}}\ \not{D}\bar{\chi} = m^2\bar{\chi}$. The crucial observation is that m^2 is positive, while $\bar{\not{D}}\ \not{D}$ is negative definite. Hence, no zero modes exist for massive spinors.

Now we can count the number of solutions using index theorems. The index of the Dirac operator is defined as

$$\text{Ind}\ \bar{\not{D}} = \dim\ker \not{D}\,\bar{\not{D}} - \dim\ker \bar{\not{D}}\,\not{D} \ . \tag{8.3.91}$$

This index will give us the number of zero modes, since the second term is zero and since any renormalizable solution of $\not{D}\ \bar{\not{D}}\lambda = 0$ satisfies $\bar{\not{D}}\lambda = 0$ as we have shown. There are several ways to compute its value. We begin by writing the index as follows

$$\text{Ind}\ \bar{\not{D}} = \lim_{M^2 \to 0} Tr\left\{\frac{M^2}{-\not{D}\,\bar{\not{D}} + M^2} - \frac{M^2}{-\bar{\not{D}}\,\not{D} + M^2}\right\} \ , \tag{8.3.92}$$

where M is an arbitrary parameter. The trace Tr stands for a sum over group indices and spinor indices, and includes an integration over space-time. We shall discuss that this expression (before taking the limit) is independent of M. This implies that the operators $\not{D}\,\bar{\not{D}}$ and $\bar{\not{D}}\,\not{D}$ not only have the same spectrum but also the same density of states for non-zero eigenvalues[16]. That they have the same non-zero eigenvalues is clear: if ψ is an eigenfunction of $\bar{\not{D}}\,\not{D}$, then $\not{D}\psi$ is an eigenfunction of $\not{D}\,\bar{\not{D}}$ with the same nonvanishing eigenvalue and $\not{D}\psi$ does not vanish. Conversely, if ψ is an eigenfunction of $\not{D}\,\bar{\not{D}}$ with nonzero eigenvalue, then $\bar{\not{D}}\psi$ does not vanish and is an eigenfunction of $\bar{\not{D}}\,\not{D}$ with the same nonvanishing eigenvalue.

To show that 8.3.92 is independent of M^2, we rewrite the index in terms of four-dimensional Dirac matrices,

$$I(M^2) \equiv \text{Ind}\,\bar{\not{D}} = Tr\left\{\frac{M^2}{-\not{D}^2 + M^2}\gamma_5\right\}\ , \tag{8.3.93}$$

where now $\not{D} = D_\mu \gamma^\mu$.

$$\not{D}_{4\times 4} = \begin{pmatrix} 0 & \not{D}_{2\times 2} \\ \bar{\not{D}}_{2\times 2} & 0 \end{pmatrix} \tag{8.3.94}$$

We rewrote the trace of the two terms in (8.3.92) over a two-dimensional spinor space as the trace of one term over a four-dimensional spinor space. It has been argued that independence of M^2 follows by taking the M^2-derivative (see [44]),

$$\frac{\partial}{\partial M^2} I(M^2) = -Tr\left\{\frac{\not{D}^2}{(-\not{D}^2 + M^2)^2}\gamma_5\right\}\ . \tag{8.3.95}$$

Using that γ_5 anticommutes with $\not{D}$ and that the trace is cyclic, we find

$$-Tr\,\frac{\not{D}\,\not{D}\gamma_5}{A} = Tr\,\frac{\not{D}\gamma_5\,\not{D}}{A} = Tr\frac{\not{D}\,\not{D}\gamma_5}{A}\ , \tag{8.3.96}$$

where $A = (-\not{D}^2 + M^2)^2$. Hence $Tr(\not{D}^2\gamma_5/A)$ would seem to vanish and this would prove that $I(M^2)$ is independent of M^2. The problem with this proof is that one can

[16] One can also (as is customary in the literature) place the system in a large box to discretize the spectrum, and let the boundary conditions for the eigenfunctions of $\not{D}\,\bar{\not{D}}$ determine the boundary conditions for the eigenfunctions of $\bar{\not{D}}\,\not{D}$, and vice-versa, such that the non-zero eigenvalues are the same. Such a treatment for the kink has been worked out in detail in [45]. However, in the limit of infinite volume, the densities of states can become different, as we shall discuss.

give a counter example: one can repeat all the steps for the supersymmetric kink, and this would then imply that the densities for chiral and anti-chiral fermion modes are equal. However, one can directly calculate these densities for the supersymmetric kink, and one then finds that they are different [46]

$$\Delta\rho(k^2) = -\frac{2m}{k^2+m^2} \ , \tag{8.3.97}$$

where m is the mass of the fluctuating fields far away from the kink. Applied to the case of instantons, the situation was considered in [44]. In [47–49] it was noted that the proof of [44] was incomplete. Cyclicity of the trace (on which the proof in [44] that $\Delta\rho(k^2)$ vanishes is based), breaks down due to the presence of massless fluctuating fields[17]. One can directly compute $I(M^2)$, using a more detailed index theorem [48, 49], and then finds that the densities of chiral and antichiral fermionic modes in an instanton background are equal,

$$\Delta\rho(k^2) = 0 \qquad \text{for instantons} \ . \tag{8.3.98}$$

Given that the density of states of the operator $\not{D}\,\bar{\not{D}}$ in 8.3.92 is the same as the density of states of the operator $\bar{\not{D}}\,\not{D}$, there is a pairwise cancellation in (8.3.92) coming from the sum over eigenstates with non-zero eigenvalues, both for the discrete and continuous spectrum. So the only contribution is coming from the zero modes, for which the first term in 8.3.92 simply gives one for each zero mode, and the second term vanishes because there are no zero modes. The result is then clearly an integer, namely $\dim\left\{\ker\bar{\not{D}}\right\}$. Since $I(M^2)$ is independent of M^2, one can evaluate it in the large M^2 limit instead of the small M^2 limit. The calculation is then identical to the calculation of the chiral anomaly, which we now review.

[17]One would expect that at the regularized level the trace is cyclic but one may expect that one should also regularize infrared aspects of the problem. Consider for example quantum mechanics for a harmonic oscillator with mass term $\frac{1}{2}m^2q^2$. Define $a = \sqrt{\frac{m}{2}}q + ip/\sqrt{2m}$ and $a^\dagger = \sqrt{\frac{m}{2}}q - ip/\sqrt{2m}$. For m tending to zero, the vacuum is annihilated by $p + \mathcal{O}(m)$ but the vacuum becomes non-normalizable when m vanishes. Still, at finite m, $tr[p,q] = 0$.

The chiral anomaly is equal to the regulated trace of the matrix γ_5. It can be written as

$$\text{Ind}\,\bar{\not{D}} = \text{tr}\int \mathrm{d}x < x \mid \frac{M^2}{-\not{D}\bar{\not{D}} + M^2} - \frac{M^2}{-\bar{\not{D}}\not{D} + M^2} \mid x > , \tag{8.3.99}$$

where tr denotes the trace over group indices and spinor indices. Because $\gamma_5 = \text{diag}(+1, +1, -1, -1)$ one finds in (8.3.99) a relative minus sign between the first and the second term. We have chosen a quantum mechanical representation for the trace in a Hilbert space spanned by the eigenfunctions $|x>$ of the position operator. The operators D_μ depend on the operators $\hat{x}^\mu$ and the operators $\hat{p}_\mu$. When $\hat{x}$ reaches $|x>$, it becomes a c-number x. Similarly $\hat{p}_\mu|x> = -\frac{\hbar}{i}\frac{\partial}{\partial x^\mu}|x>$. The latter statement follows by contracting with a complete set of momentum eigenstates, using that $< k|\hat{p}_\mu = \hbar k_\mu < k|$ because $\hat{p}_\mu$ is hermitian

$$\begin{aligned} &< k \mid \hat{p}_\mu \mid x > = \hbar k_\mu < k \mid x > = \hbar k_\mu \frac{\mathrm{e}^{-ikx}}{(2\pi)^2} \\ &= -\frac{\hbar}{i}\frac{\partial}{\partial x^\mu}\frac{\mathrm{e}^{-ikx}}{(2\pi)^2} = -\frac{\hbar}{i}\frac{\partial}{\partial x^\mu} < k \mid x > = < k \mid \frac{-\hbar}{i}\frac{\partial}{\partial x^\mu} \mid x > \ . \end{aligned} \tag{8.3.100}$$

So, from now on we will replace the operators $D_\mu(\hat{x}, \hat{p}_x)$ by $D_\mu(x, -\frac{\hbar}{i}\frac{\partial}{\partial x})$. These $\frac{\partial}{\partial x}$ act on the x in $|x>$ and do not act on $|k>$.

Let us now insert a complete set of eigenstates of $\bar{\not{D}}\not{D}$ and $\not{D}\bar{\not{D}}$, respectively. The index becomes then

$$\text{Ind}\,\bar{\not{D}} = \text{tr}\sum_{m,n}\int \mathrm{d}x < x \mid n_L >< n_L \mid \mathcal{O}_L \mid m_L >< m_L \mid x > - \text{ same with } L \leftrightarrow R\ , \tag{8.3.101}$$

where $\mathcal{O}_L$ is the first operator in (8.3.99) and $\mathcal{O}_R$ the second. As we already discussed the eigenfunctions $< x|n_L > = \varphi_n^{(L)}(x)$ and $< x|n_R > = \varphi_n^{(R)}(x)$ have the same nonvanishing eigenvalues λ_n and the same densities.

So the eigenfunctions with nonzero eigenvalues do not contribute to the index. (Note that it does not make sense to look for eigenfunctions of $\not{D}$ or $\bar{\not{D}}$ because these operators change the helicity of the spinors). There are in general a finite number of zero modes in the L sector but none in the R sector. Hence

$$\begin{aligned}\mathrm{Ind}\ \bar{\not{D}} &= \int \mathrm{d}^4x \left(\sum_n \varphi_n^{(L)}(x)\varphi_m^{(L)}(x)^* - \sum_m \varphi_n^{(R)}(x)\varphi_m^{(R)}(x)^*\right)\frac{M^2\delta_{mn}}{\lambda_n^2+M^2} \\ &+ \sum_\alpha \int \mathrm{d}^4x \varphi_\alpha^{(L)}(x)\varphi_\alpha^{(L)}(x)^* = n^{(L)}\ ,\end{aligned} \tag{8.3.102}$$

where $\varphi_\alpha^{(L)}(x)$ are the (square-integrable) zero modes, and $n^{(L)}$ is the number of these. A sum over spinor indices is taken in (8.3.102).

To actually compute the index (namely, to compute the integer $n^{(L)}$), we use momentum eigenstates instead of eigenfunctions of $\not{D}\bar{\not{D}}$ and $\bar{\not{D}}\not{D}$

$$\mathrm{Ind}\ \bar{\not{D}} = \int \mathrm{d}^4x \int \mathrm{d}^4k \int \mathrm{d}^4k'\ \mathrm{tr}\ < x \mid k' >< k' \mid \frac{M^2}{-\not{D}^2+M^2}\gamma_5 \mid k >< k \mid x >\ , \tag{8.3.103}$$

where we recall that $\gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\not{D} = \begin{pmatrix} 0 & -i\not{D} \\ i\bar{\not{D}} & 0 \end{pmatrix}$.

As we have discussed, the operator $D_\mu = \frac{\partial}{\partial x^\mu} + [A_\mu(x), \cdot]$ acts on the coordinates x in $|x>$ but not on the k in $|k><k|$, and the trace tr sums over the group indices and the spinor indices of γ^μ in $\not{D} = \gamma^\mu D_\mu$ and γ_5. Using $<k|x> = \mathrm{e}^{-ikx}/(2\pi)^2$ and pulling these plane waves to the left, the derivatives ∂_μ act on the c-numbers x in e^{-ikx} and are replaced by $\partial_\mu - ik_\mu$. The matrix element $< k'|M^2(-\not{D}^2+M^2)^{-1}\gamma_5 \mid k >$ is equal to $< k' \mid k >$ times the operator $[M^2/(-\not{D}^2+M^2)]\gamma_5$ and $<k'|k> = \delta^4(k-k')$. When the plane wave e^{-ikx} has been pulled all the way to the left, the plane waves $\mathrm{e}^{ik'x}$ and e^{-ikx} in $<x|k'> = \mathrm{e}^{ik'x}/(2\pi)^2$ and $<k|x> = \mathrm{e}^{-ikx}/(2\pi)^2$ cancel each other, and one is left with

$$\mathrm{Ind}\ \bar{\not{D}} = \int \mathrm{d}^4x \int \frac{\mathrm{d}^4k}{(2\pi)^4}\mathrm{tr}\left\{\frac{M^2}{-(-i\not{k}+\not{D})^2+M^2}\gamma_5\right\}\ . \tag{8.3.104}$$

The denominator can be written as

$$\frac{1}{(k^2+M^2)-(-2ik\cdot D + D_\mu D_\mu + \frac{1}{2}\gamma_\mu\gamma_\nu F_{\mu\nu})}\ , \tag{8.3.105}$$

and we can exhibit the M^2 dependence by rescaling $k_\mu = M\kappa_\mu$, yielding

$$\mathrm{Ind}\,\bar{\not{D}} = \int \mathrm{d}^4x\, M^4 \int \frac{d^4\kappa}{(2\pi)^4}\,\mathrm{tr}\left\{ \frac{1}{(\kappa^2+1) - \left(-\frac{2i\kappa_\mu D_\mu}{M} + \frac{D_\mu D_\mu}{M^2} + \frac{1}{2}\frac{\gamma_\mu\gamma_\nu F_{\mu\nu}}{M^2}\right)}\gamma_5 \right\} . \tag{8.3.106}$$

Expanding the denominator, only terms due to expanding two, three or four times can contribute in the limit $M \to \infty$, but only the terms with at least four Dirac matrices can contribute to the trace due to the matrix γ_5. Thus we only need retain the square of $\frac{1}{2}\gamma_\mu\gamma_\nu F_{\mu\nu}$, and the index becomes

$$\begin{aligned} \mathrm{Ind}\,\bar{\not{D}} &= \int \mathrm{d}^4x \int \frac{\mathrm{d}^4\kappa}{(2\pi)^4}\frac{1}{(\kappa^2+1)^3}\mathrm{tr}\,\left(\tfrac{1}{2}F_{\mu\nu}\gamma_{\mu\nu}\tfrac{1}{2}F_{\rho\sigma}\gamma_{\rho\sigma}\gamma_5\right) \\ &= \int \mathrm{d}^4x \frac{2\pi^2}{(2\pi)^4}\int_0^\infty \frac{r^3\mathrm{d}r}{(r^2+1)^3}\tfrac{1}{4}(\mathrm{tr}\,T_aT_b)(\mathrm{tr}\gamma_{\mu\nu}\gamma_{\rho\sigma}\gamma_5)F^a_{\mu\nu}F^b_{\rho\sigma} \\ &= \int \mathrm{d}^4x \frac{1}{32\pi^2}(\mathrm{tr}\,T_aT_b)(\epsilon_{\mu\nu\rho\sigma}F^a_{\mu\nu}F^b_{\rho\sigma}) , \end{aligned} \tag{8.3.107}$$

where we used that $\int \mathrm{d}\Omega_\mu = 2\pi^2$ and $\int_0^\infty \frac{r^3\mathrm{d}r}{(r^2+1)^3} = \frac{1}{4}$. Note that both a trace over group indices and a trace over spinor indices has been taken. The result for the index is twice the product of the winding number in (8.1.6) and a group theory factor

$$\mathrm{Ind}\,\bar{\not{D}} = 2\left(\frac{1}{32\pi^2}\int \mathrm{d}^4x F^a_{\mu\nu}\,{}^*F^b_{\mu\nu}\right)\mathrm{tr}\,T_aT_b\ . \tag{8.3.108}$$

For a representation R of $SU(N)$ for the fermions, we define $\mathrm{tr}\,T^R_aT^R_b = -\delta_{ab}T(R)$. By definition one has $T(R) = \frac{1}{2}$ for the fundamental representation, and then $T(R) = N$ for the adjoint representation[18]. Hence, finally,

$$\mathrm{Ind}\,\bar{\not{D}} \;=\; |k| \qquad \text{for the fundamental representation,}$$

[18]To compute $T(R)$ for the adjoint representation, write the carrier space for the adjoint representation of $SU(N)$ as $u^i\bar{v}_j - \frac{1}{N}\delta^i{}_j(u^k\bar{v}_k)$. Then, for $i \neq j$, $T^{\mathrm{adj}}_a u^i\bar{v}_j = (T^{(f)}_a)^i{}_{i'}u^{i'}\bar{v}_j + (T^{f^*}_a)_j{}^{j'}u^i\bar{v}_{j'}$. For a diagonal generator A of the fundamental representation of $SU(N)$ with entries $(i\alpha_1,\ldots,i\alpha_N)$ with real α_j one has $Au^i = i\alpha^i u^i$ and $Au^i\bar{v}_j = (i\alpha^i - i\alpha^j)u^i\bar{v}_j$, so $A(u^i\bar{v}_j - \frac{1}{N}\delta^i_j u^k\bar{v}_k) = (i\alpha^i - i\alpha^j)(u^i\bar{v}_j - \frac{1}{N}\delta^i_j u^k\bar{v}_k)$ and $\sum\alpha_i = 0$. Hence $\mathrm{tr}\,A^2 = -\sum_{i=1}^N(\alpha^i)^2$ for the fundamental representation, but $\mathrm{tr}A^2 = -\sum_{i,j}(\alpha^i-\alpha^j)^2$ for the adjoint representation. The latter sum can also be written as

$$\sum_{i,j=1}^N(\alpha_i-\alpha_j)^2 = \left(\sum\alpha_i^2\right)N - 2\left(\sum\alpha_i\right)\left(\sum\alpha_j\right) + \left(\sum\alpha_j^2\right)N = 2\left(\sum\alpha_i^2\right)N\ .$$

So $T(R^{\mathrm{adj}}) = 2NT(R^f)$.

$$= \quad 2N|k| \qquad \text{for the adjoint representation .} \tag{8.3.109}$$

(For an anti-instanton, k is negative. The factor 2 corresponds to our earlier observation that $i\sigma_2\lambda^*$ is also a zero mode if λ is a zero mode.) Furthermore, as shown in the last subsection, an (anti-) instanton in $SU(N)$ has twice as many bosonic collective coordinates as there are fermionic zero modes in the adjoint representation. This proves that there are $4Nk$ bosonic collective coordinates for an instanton with winding number k and gauge group $SU(N)$.

4 Construction of zero modes

In two later sections we will show how to set up and do (one-loop) perturbation theory around an (anti-) instanton. This will require the reduction of the path integral measure over instanton field configurations to an integral over the moduli space of collective coordinates. In order to achieve this we need to know the explicit form of the bosonic and fermionic zero modes. This is the content of this section. We follow closely [50].

4.1 Bosonic zero modes and their normalization

In order to construct the bosonic zero modes and discuss perturbation theory, we first decompose the fields into a background part and quantum fields

$$A_\mu = A_\mu^{\mathrm{cl}}(\gamma) + A_\mu^{\mathrm{qu}} \ . \tag{8.4.110}$$

Here γ_i denote a set of collective coordinates, and, for gauge group $SU(N)$, $i = 1, \ldots, 4Nk$. Before we make the expansion of the action, we should first fix the gauge and introduce ghosts, c, and anti-ghosts, b. We choose the background gauge condition

$$D_\mu^{\mathrm{cl}} A_\mu^{\mathrm{qu}} = 0 \ . \tag{8.4.111}$$

The gauge-fixing term is then $\mathcal{L}_{\text{fix}} = -\frac{1}{g^2}\text{tr}(D_\mu A_\mu^{\text{qu}})^2$ and the ghost action is $\mathcal{L}_{\text{ghost}} = -b^a(D_\mu(A_\mu^{\text{cl}})D_\mu(A_\mu^{\text{cl}} + A_\mu^{\text{qu}})c)^a$. The action, expanded through quadratic order in the quantum fields, is of the form

$$S = \frac{8\pi^2}{g^2} \mid k \mid + \frac{1}{g^2}\text{tr} \int \text{d}^4x \left\{ A_\mu^{\text{qu}} \, M_{\mu\nu} \, A_\nu^{\text{qu}} + 2b \, M^{\text{gh}} \, c \right\} \ , \tag{8.4.112}$$

with $M^{\text{gh}} = D^2$ and

$$\begin{aligned} M_{\mu\nu} &= \left(D^2\delta_{\mu\nu} - D_\nu D_\mu + F_{\mu\nu}\right) + D_\mu D_\nu \equiv M_{\mu\nu}^{(1)} + M_{\mu\nu}^{(2)} \ , \\ &= D^2\delta_{\mu\nu} + 2F_{\mu\nu} \ , \end{aligned} \tag{8.4.113}$$

where we have dropped the subscript cl. Here, $M^{(1)}$ stands for the quadratic operator coming from the classical action, and $M^{(2)}$ is due to the gauge fixing term[19]. (Recall that $F_{\mu\nu}$ acts on A_ν^{qu} as $[F_{\mu\nu}, A_\nu^{\text{qu}}]$). In an expansion as in (8.4.112), one encounters zero modes (i.e. normalizable eigenfunctions of the operator $M_{\mu\nu}$ with zero eigenvalues). They are of the form

$$Z_\mu^{(i)} \equiv \frac{\partial A_\mu^{\text{cl}}}{\partial \gamma_i} + D_\mu^{\text{cl}}\Lambda^i \ , \tag{8.4.114}$$

where the gauge parameter Λ^i is chosen to keep Z_μ in the background gauge, so that

$$D_\mu^{\text{cl}} Z_\mu^{(i)} = 0 \ . \tag{8.4.115}$$

The first term in (8.4.114) is a solution of $M^{(1)}$ (i.e. an eigenfunction with zero eigenvalue), as follows from taking the derivative with respect to γ_i of the field equation. Namely, $\delta S^{\text{cl}}/\delta A_\mu^{\text{cl}} = 0$ for all γ_i, so

$$0 = \frac{\partial}{\partial \gamma_i} \frac{\delta S^{\text{cl}}}{\delta A_\mu^{\text{cl}}(x)} \quad = \quad \int \frac{\delta^2 S^{\text{cl}}}{\delta A_\nu^{\text{cl}}(y)\delta A_\mu^{\text{cl}}(x)} \partial_{\gamma_i} A_\nu^{\text{cl}}(y) \text{d}^4 y \ . \tag{8.4.116}$$

[19]To arrive at this expression for $M_{\mu\nu}^{(1)}$, use that $F_{\mu\nu} = F_{\mu\nu}^{\text{cl}} + (D_\mu^{\text{cl}} A_\nu^{\text{qu}} - D_\nu^{\text{cl}} A_\mu^{\text{qu}}) + [A_\mu^{\text{qu}}, A_\nu^{\text{qu}}]$ and note that $-\frac{1}{2g^2}\text{tr}2F_{\mu\nu}^{\text{cl}}[A_\mu^{\text{qu}}, A_\nu^{\text{qu}}] = \frac{1}{g^2}\text{tr}A_\mu^{\text{qu}}[F_{\mu\nu}^{\text{cl}}, A_\nu^{\text{qu}}]$.

The term $D_\mu\Lambda$ is also a solution of $M^{(1)}$, since it is a pure gauge transformation.[20] The sum of the two terms is also a solution of $M^{(2)}$, because Λ is chosen such that Z_μ is in the background gauge. As we shall show, the solutions in 8.4.114 are normalizable, hence they are zero modes. Due to these zero modes, we cannot integrate over all quantum fluctuations, since the corresponding determinants would vanish and yield divergences in the path integral. They must therefore be extracted from the quantum fluctuations, in a way we will describe in a more general setting in the next subsection. It will turn out to be important to compute the matrix of inner products

$$U^{ij} \equiv \langle Z^{(i)}|Z^{(j)}\rangle \equiv -\frac{2}{g^2}\int \mathrm{d}^4x \,\mathrm{tr}\,\left\{Z_\mu^{(i)}Z^{\mu(j)}\right\} = \frac{1}{g^2}\int Z_\mu^{(i)a}Z_\mu^{(j)a}\mathrm{d}^4x \;. \qquad (8.4.117)$$

We put a factor $\frac{1}{g^2}$ in front of the usual L^2 inner product because the metric U^{ij} will be used to construct a measure $(\det U^{ij})^{1/2}$ for the zero modes, and this measure is also needed if one considers the quantum mechanics of zero modes $\gamma_i(t)$. The action for these time-dependent $\gamma_i(t)$ is $U^{ij}\dot\gamma_i\dot\gamma_j$ with the same prefactor $\frac{1}{g^2}$ as in the Yang-Mills gauge action.

We now evaluate this matrix for the anti-instanton. For the four translational zero modes, one can easily keep the zero mode in the background gauge by choosing $\Lambda^i = A_\nu^{\mathrm{cl}}$. Indeed,

$$Z_\mu^{(\nu)} = \frac{\partial A_\mu^{\mathrm{cl}}}{\partial x_0^\nu} + D_\mu A_\nu^{\mathrm{cl}} = -\partial_\nu A_\mu^{\mathrm{cl}} + D_\mu A_\nu^{\mathrm{cl}} = F_{\mu\nu}^{\mathrm{cl}} \;, \qquad (8.4.118)$$

which satisfies the background gauge condition. The norms of these zero modes are

$$U^{\mu\nu} = \frac{8\pi^2|k|}{g^2}\delta^{\mu\nu} = S_{\mathrm{cl}}\,\delta^{\mu\nu} \;. \qquad (8.4.119)$$

As indicated, this result actually holds for any k, and arbitrary gauge group.

[20]This is also easy to prove by direct calculation: $(D^2\delta_{\mu\nu} - D_\nu D_\mu + F_{\mu\nu})D_\nu\Lambda$ is equal to $D_\nu[D_\nu, D_\mu]\Lambda + F_{\mu\nu}D_\nu\Lambda$, and this vanishes since $[D_\nu, D_\mu] = F_{\nu\mu}$ and $D_\nu F_{\nu\mu} = 0$. More generally, $\delta S^{\mathrm{cl}}/\delta A_\nu^{\mathrm{cl}} \sim D_\mu^{\mathrm{cl}}F_{\mu\nu}^{\mathrm{cl}}$ is gauge-covariant, hence $D_\mu^{\mathrm{cl}}F_{\mu\nu}^{\mathrm{cl}}(A_\rho + D_\rho\Lambda) - D_\mu^{\mathrm{cl}}F_{\mu\nu}^{\mathrm{cl}}(A_\rho) = [D_\mu^{\mathrm{cl}}F_{\mu\nu}^{\mathrm{cl}}, \Lambda]$ which vanishes on-shell (field equations transform into field equations). Hence $\frac{\delta^2 S^{\mathrm{cl}}}{\delta A_\mu^{\mathrm{cl}}\delta A_\rho^{\mathrm{cl}}}(D_\rho\Lambda) = M_{\mu\nu}^{(1)}D_\nu\Lambda$ vanishes.

Next we consider the dilatational zero mode corresponding to ρ and limit ourselves to $k = -1$. Taking the derivative with respect to ρ leaves the zero mode in the background gauge, so we can set $\Lambda^\rho = 0$. In the singular gauge of (8.2.67) we have

$$Z_\mu^{(\rho)} = -2\,\frac{\rho\,\bar\sigma_{\mu\nu}\,x_\nu}{(x^2+\rho^2)^2}\ . \tag{8.4.120}$$

(To show that (8.4.115) is satisfied, note that $(\partial/\partial x^\mu)Z_\mu^{(\rho)} = 0$ since $\bar\sigma_{\mu\nu}$ is antisymmetric, while $[A_\mu^{\rm cl}, Z_\mu^{(\rho)}] = 0$ since both involve $\bar\sigma_{\mu\nu}x^\nu$). Using (8.B.361) and (8.B.363), one easily computes that

$$U^{\rho\rho} = \frac{16\pi^2}{g^2} = 2S_{\rm cl}\ . \tag{8.4.121}$$

In regular gauge one finds $Z_\mu^{(\rho)} = \frac{2\rho\sigma_{\mu\nu}x^\nu}{(x^2+\rho^2)^2}$ which has clearly the same norm. This result can also be derived from $\frac{\partial}{\partial\rho}A_\mu^{\rm reg}(k=-1) = \frac{\partial}{\partial\rho}U^{-1}(\partial_\mu + A_\mu^{\rm sing}(k=-1))U$ and the identity $U^{-1}\bar\sigma_{\mu\nu}x^\nu U = -\sigma_{\mu\nu}x^\nu$.

The gauge-orientation zero modes can be obtained from (8.1.18). By expanding $U(\theta) = \exp(\theta^a T_a)$ infinitesimally in (8.1.18) we get to lowest order in θ (the case of general θ will be discussed shortly)

$$\frac{\partial A_\mu}{\partial\theta^a} = [A_\mu, T_a]\ , \tag{8.4.122}$$

which is not in the background gauge (the matrices T_a are in the fundamental representation). To satisfy (8.4.115) we have to add appropriate gauge transformations, which differ for different generators of $SU(N)$. First, for the $SU(2)$ subgroup corresponding to the instanton embedding, we add, for the singular gauge,

$$\Lambda_a = -\frac{\rho^2}{x^2+\rho^2}\,T_a\ , \tag{8.4.123}$$

and find that

$$Z_{\mu\,(a)} = D_\mu\left[\frac{x^2}{x^2+\rho^2}T_a\right]\ . \tag{8.4.124}$$

(using $\partial_\mu\frac{x^2}{x^2+\rho^2} = -\partial_\mu\frac{\rho^2}{x^2+\rho^2}$). One can now show, using (8.B.348), that the zero mode

(8.4.124) is in the background gauge, and its norm reads[21]

$$U_{ab} = \frac{4\pi^2}{g^2}\rho^2\delta_{ab} = \tfrac{1}{2}\delta_{ab}\rho^2 S_{\rm cl}\ . \tag{8.4.125}$$

We need the gauge-orientation zero modes for arbitrary values of θ because this is needed for the group (Haar) measure. They are obtained as follows. By differentiating $U(\theta)$ and using that $U^{-1}\frac{\partial}{\partial\theta^\alpha}U$ is equal to $e_\alpha{}^a(\theta)T_a$, where the function $e_\alpha{}^a(\theta)$ is called the group vielbein (with α a curved and a a flat index according to the usual

[21] A few details may be helpful. One finds for this zero mode in the singular gauge, using 8.2.67 and 8.B.358,

$$Z_\mu^{(a)} = 2x_\mu\rho^2(x^2+\rho^2)^{-2}T_a + 2\eta_{b\mu\nu}\epsilon_{bac}T_c x_\nu\rho^2/(x^2+\rho^2)^2.$$

It is covariantly transversal: ∂_μ acting on the first term plus the commutator of $A_\mu^{\rm sing}$ with the second term vanishes upon using (8.B.348). (The commutator of the first term with $A_\mu^{\rm sing}$ is proportional to $(\bar\sigma_{\mu\nu}x^\nu)x^\mu$ and vanishes). The norm is due to integrating the sum of the square of the first term and the second term, using (8.B.363) with $n=1$ and $m=4$. All terms which contribute to $Z_\mu^{(a)}$, namely $\frac{\partial}{\partial\theta^a}A_\mu^{\rm sing}$ and $\partial_\mu\Lambda^{a,{\rm sing}}$ and $[A_\mu^{\rm sing},\Lambda^{a,{\rm sing}}]$ fall off as $1/r^3$ for large $|x|$, and $Z_\mu^{(a)}$ itself is nonsingular at $x=0$.

In regular gauge one finds from (8.2.69)

$$\partial_\gamma A_\mu^{\rm reg} = \partial_\gamma U^{-1}(\partial_\mu + A_\mu^{\rm sing})U = U^{-1}\partial_\gamma A_\mu^{\rm sing}U\ ,$$

and the transversality condition becomes

$$\begin{aligned}&U^{-1}D_\mu(A^{\rm sing})U[U^{-1}\partial_\gamma A_\mu^{\rm sing}U + U^{-1}D_\mu(A^{\rm sing})UU^{-1}\Lambda^{a,{\rm sing}}U]\\ &= D_\mu(A^{\rm reg})[\partial_\gamma A_\mu^{\rm reg} + D_\mu(A_\mu^{\rm reg})U^{-1}\Lambda^{a,{\rm sing}}U] = 0\ .\end{aligned}$$

Hence, $\Lambda^{a,{\rm reg}} = U^{-1}\Lambda^{a,{\rm sing}}U$, and now all contributions to $Z_\mu^{(a)}$ in the regular gauge fall only off as $1/r$. Only their sum $Z_\mu^{(a),{\rm reg}}$ falls off as $1/r^3$, just as $Z_\mu^{(a),{\rm sing}}$. It is clearly simpler to work in the singular gauge, because then all integrals separately converge.

terminology[22]), one obtains

$$\frac{\partial}{\partial\theta^\alpha} A_\mu(\theta) = [A_\mu(\theta), e_\alpha{}^a(\theta)T_a] \tag{8.4.126}$$

For $\Lambda_{(\alpha)}$ we take now $\Lambda_{(\alpha)}(\theta) = -\frac{\rho^2}{x^2+\rho^2} e_\alpha{}^a(\theta)T_a$, and then we obtain for the gauge zero modes at arbitrary θ

$$\begin{aligned} Z_{\mu(\alpha)}(\theta) &= D_\mu(A(\theta))\left(\frac{x^2}{x^2+\rho^2} e_\alpha{}^a(\theta)T_a\right) \\ &= U^{-1}\left[D_\mu(A(\theta=0))\left(\frac{x^2}{x^2+\rho^2}\partial_\alpha U U^{-1}\right)\right] U \ . \end{aligned} \tag{8.4.127}$$

We define[23] $\partial_\alpha U U^{-1} = f_\alpha{}^a(\theta)T_a$. Note that $\mathrm{tr}\,\partial_\alpha U U^{-1}\partial_\beta U U^{-1} = \mathrm{tr}\,(U^{-1}\partial_\alpha U\ U^{-1}\partial_\beta U) = e_\alpha{}^a e_\beta{}^b \mathrm{tr}\,T_a T_b = f_\alpha{}^a f_\beta{}^b \mathrm{tr}\,T_a T_b$. Hence the left-invariant metric $e_\alpha{}^a e_\beta{}^b \delta_{ab}$ is equal to the right-invariant metric. There is a geometrical interpretation of these results [56, 57].

There are only two differences with the $\theta = 0$ case
(i) the factors $U(\theta)$ and $U^{-1}(\theta)$ in front and at the back; these drop out in the trace
(ii) the factors of $f_\alpha{}^a$ multiplying T_a. Taking the trace one obtains the group metric

$$U^{\alpha\beta}(\theta) = \langle Z_\mu^{(\alpha)} \mid Z_\mu^{(\beta)}\rangle = e_\alpha{}^a(\theta)e_\beta{}^b(\theta)U_{ab}(\theta=0) = e_\alpha{}^a(\theta)e_\beta{}^a(\theta)\ \left(\tfrac{1}{2}\rho^2 S_{\mathrm{cl}}\right)\ . \tag{8.4.128}$$

Hence, in the square root of the determinant of U one finds a factor $\det e_\alpha{}^a$ (because $\det(e_\alpha{}^a\delta_{ab}e_\beta{}^b) = (\det e_\alpha{}^a)^2$), and this yields the Haar measure

$$\mu(\theta) = \det e_\alpha{}^a(\theta)\mathrm{d}^3\theta \ . \tag{8.4.129}$$

[22] The group vielbein is given by

$$e_\alpha{}^a(\theta)T_a = T_\alpha + \frac{1}{2!}[T_\alpha, \theta\cdot T] + \frac{1}{3!}[(T_\alpha, \theta\cdot T], \theta\cdot T] + \cdots\ ,$$

whereas the adjoint matrix representation $M^{\mathrm{adj}}(\theta)$ is given by

$$\mathrm{e}^{-\theta\cdot T}T_a\mathrm{e}^{\theta\cdot T} = M^{\mathrm{adj}}(\theta)_a{}^b T_b = T_a + [T_a, \theta\cdot T] + \cdots\ .$$

One has $M^{\mathrm{adj}}(\theta)_a{}^b = (\exp\theta^c f_{\cdot\, c}{}^{\cdot})_a{}^b$. There is a relation between the group vielbein and the adjoint matrix: $\left(\theta^\beta\frac{\partial}{\partial\theta^\beta}+1\right)e_\alpha{}^b(\theta) = (M^{\mathrm{adj}}(\theta))_\alpha{}^b$

[23] The functions $e_\alpha{}^a(\theta)$ are sometimes called the left-invariant one-forms, while $f_\alpha{}^a(\theta)$ are the right-invariant one-forms.

Using this measure one can calculate the group volume V of $SU(2)$, $V = \int(\det e_\alpha{}^a)\mathrm{d}^3\theta$, which is independent of the choice of coordinates θ. (We chose the parametrization $U(\theta) = \exp\theta^a T_a$, but any other parametrization yields the same result.)

We have now calculated all norms. It is fairly easy to prove that there is no mixing between the different modes, for example $U^{\mu(\rho)} = U^\mu{}_a = U^{(\rho)}{}_a = 0$. Thus the matrix U^{ij} for $SU(2)$ is eight by eight, with non-vanishing entries along the block-diagonal

$$U^{ij} = \begin{pmatrix} \delta^{\mu\nu}S_{\rm cl} & & \\ & 2S_{\rm cl} & \\ & & \frac{1}{2}g_{\alpha\beta}(\theta)\rho^2 S_{\rm cl} \end{pmatrix}_{8\times 8} , \tag{8.4.130}$$

The square root of the determinant is

$$\sqrt{U} = \tfrac{1}{2}S_{\rm cl}^4\rho^3\sqrt{\det g_{\alpha\beta}(\theta)} = \frac{2^{11}\pi^8\rho^3}{g^8}\sqrt{\det g_{\alpha\beta}(\theta)} \quad (\text{for } SU(2)) \ . \tag{8.4.131}$$

Let us now consider the remaining generators of $SU(N)$ by first analyzing the example of $SU(3)$. For simplicity, we restrict ourselves again to lowest order in θ^a. There are seven gauge orientation zero modes, three of which are given by (8.4.124) by taking for T_a the first three Gell-Mann matrices $\lambda_1, \lambda_2, \lambda_3$ multiplied by $-\frac{i}{2}$. For the other four zero modes, corresponding to $\lambda_4, \ldots, \lambda_7$, the formula (8.4.122) still holds, but we have to change the gauge transformation in order to keep the zero mode in background gauge,

$$\Lambda_k = \left[\sqrt{\frac{x^2}{x^2+\rho^2}} - 1\right] T_k \ , \qquad k = 4,5,6,7 \ , \tag{8.4.132}$$

with $T_k = (-i/2)\lambda_k$. The difference in x-dependence of the gauge transformations (8.4.123) and (8.4.132) is due to the change in commutation relations. Namely, $\sum_{a=1}^3[\lambda_a,[\lambda_a,\lambda_\beta]] = -(3/4)\lambda_\beta$ for $\beta = 4,5,6,7$, whereas it is $-2\lambda_\beta$ for $\beta = 1,2,3$. (These are the values of the Casimir operator of $SU(2)$ on doublets and triplets, respectively). As argued before, there is no gauge orientation zero mode associated with λ_8, since it commutes with the $SU(2)$ embedding. The zero modes are then

$$Z_{\mu\,(k)} = D_\mu\left[\sqrt{\frac{x^2}{x^2+\rho^2}}T_k\right] \ , \qquad k = 4,5,6,7 \ , \tag{8.4.133}$$

with norms[24]

$$U_{kl} = \tfrac{1}{4}\delta_{kl}\rho^2 S_{\rm cl} \ , \tag{8.4.134}$$

and are orthogonal to (8.4.124), such that $U_{ka} = 0$. This construction easily generalizes to $SU(N)$. One first chooses an $SU(2)$ embedding, and this singles out 3 generators. The other generators can then be split into $2(N-2)$ doublets under this $SU(2)$ and the rest are singlets. There are no zero modes associated with the singlets, since they commute with the $SU(2)$ chosen. For the doublets, each associated zero mode has the form as in (8.4.133), with the same norm $\frac{1}{4}\rho^2 S_{\rm cl}$. This counting indeed leads to $4N-5$ gauge orientation zero modes. Straightforward calculation for the square-root of the complete determinant then yields an extra factor $(\frac{1}{4}\rho^2 S_{\rm cl})^{2(N-2)}$, and so

$$\sqrt{U} = \frac{2^{2N+7}}{\rho^5}\left(\frac{\pi\rho}{g}\right)^{4N} \quad (\text{for } SU(N)) \ . \tag{8.4.135}$$

This result is a factor 2^{4N-5} smaller than [**?**, 50], since we chose $U(\theta) = \exp\theta^a T_a$ instead of $\exp(2\theta^a T_a)$. This ends the discussion about the (bosonic) zero mode normalization.

4.2 Construction of the fermionic zero modes

In this subsection we will explicitly construct the fermionic zero modes (normalizable solutions of the Dirac equation) in the background of a single anti-instanton. For an $SU(2)$ adjoint fermion, there are 4 zero modes according to 8.3.109, and these can be written as follows [61]

$$\lambda^\alpha = -\tfrac{1}{2}\sigma_{\rho\sigma}{}^\alpha{}_\beta\left(\xi^\beta - \sigma_\nu^{\beta\gamma'}\bar\eta_{\gamma'}(x-x_0)^\nu\right)F_{\rho\sigma} \ . \tag{8.4.136}$$

The $SU(2)$ indices u and v are carried by $(\lambda^\alpha)^u{}_v$ and $(F_{\rho\sigma})^u{}_v$.

[24]These zero modes are given by $Z_{\mu\,(k)} = \rho^2 x^\nu/(\sqrt{x^2}(x^2+\rho^2)^{3/2})(\delta_{\mu\nu}T_k + 2\eta_{a\mu\nu}[T_a,T_k])$ in the singular gauge, see 8.2.67. For the first three zero modes we found instead $Z_{\mu\,(a)} = \rho^2 x^\nu/(x^2+\rho^2)^2(2\delta_{\mu\nu}T_a + 2\eta_{b\mu\nu}[T_b,T_a])$ with $[T_b,T_a] = \epsilon_{bac}T_c$. The norm of 8.4.133 is proportional to ${\rm tr}\,T_kT_l + 4{\rm tr}\,[T_a,T_k][T_a,T_l] = 4{\rm tr}\,T_kT_l$, where we used 8.B.348.

To prove that these spinors are solution of the Dirac equation, use $\bar{\sigma}_\mu \sigma_{\rho\sigma} = \delta_{\mu\rho}\bar{\sigma}_\sigma - \delta_{\mu\sigma}\bar{\sigma}_\rho - \epsilon_{\mu\rho\sigma\tau}\bar{\sigma}_\tau$. Then $\bar{\not{D}}\lambda$ vanishes since $D_\mu F_{\rho\sigma}$ vanishes when contracted with $\eta_{\mu\rho}, \eta_{\mu\sigma}$ or $\epsilon_{\mu\rho\sigma\tau}$. Actually, this expression also solves the Dirac equation for higher order k, but there are then additional solutions, $4|k|$ in total for $SU(2)$, see (8.3.109). The four fermionic collective coordinates are denoted by ξ^α and $\bar{\eta}_{\gamma'}$, where $\alpha, \gamma' = 1, 2$ are spinor indices in Euclidean space[25]. They are the fermionic partners of the translational and dilatational collective coordinates in the bosonic sector. These solutions take the same form in any gauge, one just takes the corresponding gauge for the field strength. The canonical dimension of ξ and $\bar{\eta}$ is $-1/2$ and $1/2$, respectively.

For $SU(N)$ (and always $k = -1$) there are a further set of $2 \times (N-2)$ zero modes in the adjoint representation, and their explicit form depends on the gauge chosen. In regular gauge, with color indices $u, v = 1, \ldots, N$ explicitly written, the gauge field is given by (8.2.70) (setting $x_0 = 0$, otherwise replace $x \to x - x_0$)

$$A_\mu{}^u{}_v = A_\mu^a \, (T_a)^u{}_v = -\frac{\sigma_{\mu\nu}{}^u{}_v x_\nu}{x^2 + \rho^2} \,, \qquad \sigma_{\mu\nu}{}^u{}_v = \begin{pmatrix} 0 & 0 \\ 0 & \sigma_{\mu\nu}{}^\alpha{}_\beta \end{pmatrix} \,. \tag{8.4.137}$$

Then the corresponding fermionic instanton in the adjoint representation reads

$$\lambda^{\alpha\, u}{}_v = \frac{\rho}{\sqrt{(x^2+\rho^2)^3}} \left(\mu^u \delta^\alpha{}_v + \epsilon^{\alpha u} \bar{\mu}_v \right) \,. \tag{8.4.138}$$

Here we have introduced Grassmann collective coordinates

$$\mu^u = (\mu^1, \ldots, \mu^{N-2}, 0, 0) \,; \epsilon^{\alpha u} = \begin{pmatrix} 0, \ldots, 0, & \\ & \epsilon^{\alpha\beta'} \\ 0, \ldots, 0, & \end{pmatrix} \text{ with } N - 2 + \beta' = u \tag{8.4.139}$$

and similarly for $\bar{\mu}_v$ and $\delta^\alpha{}_v$. Thus the $SU(N)$ structure for the fermionic instanton is as follows

$$\lambda \propto \begin{pmatrix} 0 & \mu \\ \bar{\mu} & \xi, \bar{\eta} \end{pmatrix} \,. \tag{8.4.140}$$

[25]To check that the expression with $\bar{\eta}$ is a solution, one may use that $\bar{\sigma}_\rho \sigma_{\mu\nu} \sigma_\rho = 0$. Note that one may change the value of x_0 in (8.4.136) while keeping $F_{\mu\nu}$ fixed, because the difference is a solution with ξ^β.

The canonical dimension of μ and $\bar{\mu}$ is $-1/2$. To prove that $(\lambda^\alpha)^u{}_v$ in (8.4.138) satisfies the Dirac equation $\bar{\sigma}^\mu(\partial_\mu \lambda + [A_\mu, \lambda]) = 0$, note that the terms $(A_\mu)^u{}_w \mu^w$ and $\bar{\mu}_w (A_\mu)^w{}_v$ vanish due to the index structure of A_μ and $\mu, \bar{\mu}$. Because A_μ has only nonzero entries in the lower right block, there cannot be fermionic instantons in the upper left block.

In singular gauge, the gauge field is given by (8.2.67)

$$A_{\mu\, u}{}^v = -\frac{\rho^2}{x^2(x^2+\rho^2)} \bar{\sigma}_{\mu\nu\, u}{}^v x_\nu \ . \tag{8.4.141}$$

Notice that the position of the color indices is different from that in regular gauge. This is due to the natural position of indices on the sigma matrices[26]. The fermionic anti-instanton in singular gauge reads [58]

$$\lambda^\alpha{}_u{}^v = \frac{\rho}{\sqrt{x^2(x^2+\rho^2)^3}} \left(\mu_u x^{\alpha v} + x^\alpha{}_u \bar{\mu}^v\right) \ , \tag{8.4.142}$$

where for fixed α, the N-component vectors μ_u and $x^{\alpha v}$ are given by

$$\mu_u = (\mu_1, \ldots, \mu_{N-2}, 0, 0) \ , \qquad x^{\alpha v} = \left(0, \ldots, 0, x^\mu \sigma_\mu^{\alpha\beta'}\right) \quad \text{with} \quad N-2+\beta' = v \ . \tag{8.4.143}$$

Further, $x^\alpha{}_u = x^{\alpha v}\epsilon_{vu}$ and $\bar{\mu}^v$ also has $N-2$ nonvanishing components. The particular choice of zeros in the last two entries corresponds to the choice of embedding the $SU(2)$ instanton in the lower-right block of $SU(N)$. Notice that the adjoint field λ is indeed traceless in its color indices. This follows from the observation that μ and $\bar{\mu}$ only appear at the off-diagonal blocks inside $SU(N)$. In general μ and $\bar{\mu}$ are independent, but if there is a reality condition on λ in Euclidean space, the μ and $\bar{\mu}$ are related by complex conjugation. We will discuss this in a concrete example when we discuss instantons in $\mathcal{N} = 4$ super Yang-Mills theory. We should also mention

[26] To be very precise, we could have used different Pauli matrices $(\tau^a)^u{}_v$ for the internal $SU(2)$ generators. Then we could have defined a matrix $(\bar{\sigma}_{\mu\nu})^u{}_v$ by $\bar{\sigma}_{\mu\nu} = i\eta_{a\mu\nu}\tau^a$, and the $SU(2)$ indices in (8.4.141) and (8.4.142) would have appeared in the same position as in (8.4.138). It is simpler to work with only one kind of Pauli matrices.

that while the bosonic collective coordinates are related to the rigid symmetries of the theory, this is not obviously true for the fermionic collective coordinates, although, as we will see later, the ξ and $\bar{\eta}$ collective coordinates can be obtained from ordinary supersymmetry and conformal supersymmetry in super Yang-Mills theories.

A similar construction holds for a fermion in the fundamental representation. Now there is only one fermionic collective coordinate, see (8.3.109), which we denote by $\mathcal{K}$. The explicit expression for $k=-1$ in singular gauge is[27]

$$(\lambda^\alpha)_u = \frac{\rho}{\sqrt{x^2(x^2+\rho^2)^3}}\, x^\alpha{}_u\, \mathcal{K}\ . \tag{8.4.144}$$

In regular gauge it is given by

$$(\lambda^\alpha)^u = \frac{\epsilon^{\alpha u}}{(x^2+\rho^2)^{3/2}}\mathcal{K}\ . \tag{8.4.145}$$

The Dirac equation for $(\lambda^\alpha)^u$ is proportional to

$$-3x^\mu \bar{\sigma}_{\mu,\alpha'\beta}\epsilon^{\beta u} - \bar{\sigma}_{\mu,\alpha'\beta}\epsilon^{\beta v}(\sigma_{\mu\nu})^u{}_v x^\nu \tag{8.4.146}$$

and to show that this vanishes one may use $(\sigma_{\mu\nu})^u{}_v\epsilon^{\beta v} = (\sigma_{\mu\nu})^{u\beta}$ and the symmetry of the Lorentz generators $(\sigma_{\mu\nu})^{u\beta} = (\sigma_{\mu\nu})^{\beta u}$ and $\bar{\sigma}_\mu\sigma_{\mu\nu} = 3\bar{\sigma}_\nu$.

5 The measure for zero modes

Having determined the bosonic and fermionic zero modes for $k=\pm 1$ instantons with $SU(N)$ gauge group, we now discuss the measure for the zero mode sector of path integrals. The one-loop corrections due to the nonzero modes, will be discussed in the next section.

[27]The color index should again be written as $(\lambda^\alpha)_{u'}$ because $\lambda^{\mathrm{reg},u} = (U)^{uv'}\lambda^{\mathrm{sing}}_{v'}$ with $U^{uv'} = \sigma^{uv'}_\mu x_\mu/\sqrt{x^2}$. However, we drop these primes. The proof that (8.4.144) satisfies the Dirac equation uses $\bar{\sigma}_{\mu\alpha'\beta}\sigma^\beta_{\rho u}x^\mu x^\rho = \epsilon_{\alpha' u}x^2$ and $(\bar{\sigma}_{\mu\rho})_{\alpha' v}(\bar{\sigma}_{\mu\nu})_u{}^v = -(\bar{\sigma}_{\mu\rho})_{\alpha'}{}^v(\bar{\sigma}_{\mu\nu})_{vu} = 3\delta_{\rho\nu}\epsilon_{\alpha' u}$.

5.1 The measure for the bosonic collective coordinates

We now construct the measure on the moduli space of bosonic collective coordinates, and show that the matrix U plays the role of a Jacobian. We first illustrate the idea for a generic system without gauge invariance, with fields ϕ^A, and action $S[\phi]$ (for example, the kink in one dimension). We expand around the instanton solution

$$\phi^A(x) = \phi^A_{\rm cl}(x,\gamma) + \phi^A_{\rm qu}(x,\gamma) \ . \tag{8.5.147}$$

The collective coordinates are denoted by γ and, for notational simplicity, we assume there is only one. At this point the fields $\phi^A_{\rm qu}$ can still depend on the collective coordinate, as they can include zero modes. The action, up to terms quadratic in the quantum fields, is

$$S = S_{\rm cl} + \tfrac{1}{2}\phi^A_{\rm qu} M_{AB}(\phi_{\rm cl})\,\phi^B_{\rm qu} \ . \tag{8.5.148}$$

The operator M has zero modes given by

$$Z^A = \frac{\partial \phi^A_{\rm cl}}{\partial\gamma} \ , \tag{8.5.149}$$

since, as we explained in 8.4.116, $M_{AB}Z^B$ is just the derivative of the field equation $\partial S_{\rm cl}/\partial\phi^A_{\rm cl}$ with respect to the collective coordinate. More generally, if the operator M is hermitian (or rather self-adjoint[28]), it has a complete set of eigenfunctions F_α with eigenvalues ϵ_α,

$$M_{AB}F^B_\alpha = \epsilon_\alpha F^A_\alpha \ . \tag{8.5.150}$$

One of the solutions is of course the zero mode $Z = F_0$ with $\epsilon_0 = 0$. Any function can be expanded into a basis of eigenfunctions, in particular the quantum fields,

$$\phi^A_{\rm qu} = \sum_\alpha \xi_\alpha F^A_\alpha \ , \tag{8.5.151}$$

[28] More precisely, if there is an inner product $(\phi_1,\phi_2) = \int \phi_1^A H_{AB}\phi_2^B {\rm d}^4x$ with real ϕ_1,ϕ_2 and with metric H_{AB}, and $H_{AB}H^{BC} = \delta_A{}^C$, then one may define $\phi^A H_{AB} = \phi_B$ so that $(\phi_1,\phi_2) = \int \phi_{1\,A}\phi_2^A{\rm d}^4x$. If one further defines $H^{BC}M_{CD} = M^B{}_D$, then $M^A{}_B$ is hermitian if $(\phi_1, M\phi_2) = (M\phi_1,\phi_2)$. The need for a matrix to define an inner product is familiar from spinors, but for bosons the metric is in general trivial ($H_{AB} = \delta_{AB}$).

with coefficients ξ_α. The eigenfunctions have norms, determined by their inner product

$$\langle F_\alpha | F_\beta \rangle = \int \mathrm{d}^4x\, F^A_\alpha(x) F^A_\beta(x) \ . \tag{8.5.152}$$

The eigenfunctions can always be chosen orthogonal, such that $\langle F_\alpha | F_\beta \rangle = \delta_{\alpha\beta} u_\alpha$. The action then becomes

$$S = S_{\rm cl} + \tfrac{1}{2} \sum_\alpha \xi_\alpha \xi_\alpha \epsilon_\alpha u_\alpha \ . \tag{8.5.153}$$

If there is a coupling constant in front of the action (8.5.148), we rescale the inner product with the coupling, such that (8.5.153) still holds. This was done in (8.4.117). The path-integral measure is now *defined* as

$$[\mathrm{d}\phi] \equiv \prod_{\alpha=0}^{\infty} \sqrt{\frac{u_\alpha}{2\pi}}\, \mathrm{d}\xi_\alpha \ . \tag{8.5.154}$$

We perform the Gaussian integration over the ξ_α and get

$$\int [\mathrm{d}\phi]\ \mathrm{e}^{-S[\phi]} = \int \sqrt{\frac{u_0}{2\pi}} \mathrm{d}\xi_0\, \mathrm{e}^{-S_{\rm cl}} (\det{}' M)^{-1/2} \ . \tag{8.5.155}$$

One sees that if there were no zero modes, the measure in (8.5.154) produces the correct result with the determinant of M. In the case of zero modes, the determinant of M is zero, and the path integral would be ill-defined. Instead, we must leave out the zero mode in M, take the amputated determinant (denoted by $\det{}'$), and integrate over the mode ξ_0. By slightly changing some parameters in the action (for example by adding a small mass term) the zero mode turns into a non-zero mode, and then one needs $\sqrt{\frac{u_0}{2\pi}}\mathrm{d}\xi_0$ as measure. So, continuity fixes the measure for the zero modes as in (8.5.154).

The next step is to convert the ξ_0 integral to an integral over the collective coordinate γ [53]. This can be done by inserting unity into the path integral. Consider the identity

$$1 = \int \mathrm{d}\gamma\, \delta\left(f(\gamma)\right) \frac{\partial f}{\partial \gamma} \ , \tag{8.5.156}$$

which holds for any (invertible) function $f(\gamma)$. Taking $f(\gamma) = -\langle \phi - \phi_{\rm cl}(\gamma)|Z\rangle$, and recalling that the original field ϕ is independent of γ, we get

$$1 = \int \mathrm{d}\gamma \left(u_0 - \left\langle \phi_{\rm qu} \left| \frac{\partial Z}{\partial \gamma} \right.\right\rangle \right) \delta\Big(\langle \phi_{\rm qu}|Z\rangle \Big) = \int \mathrm{d}\gamma \left(u_0 - \left\langle \phi_{\rm qu} \left| \frac{\partial Z}{\partial \gamma} \right.\right\rangle \right) \delta\Big(\xi_0 u_0 \Big) \ . \tag{8.5.157}$$

This trick is similar to the Faddeev-Popov trick for gauge fixing. In the semiclassical approximation, the term $\langle \phi_{\rm qu} \left| \frac{\partial Z}{\partial \gamma} \right.\rangle$ is subleading and we will neglect it[29]. The integration over ξ_0 is now trivial and one obtains

$$\int [\mathrm{d}\phi]\ \mathrm{e}^{-S} = \int \mathrm{d}\gamma \sqrt{\frac{u_0}{2\pi}} \mathrm{e}^{-S_{\rm cl}} \left(\det{}' M\right)^{-1/2} \ . \tag{8.5.158}$$

For a system with more zero modes Z^i with norms-squared U^{ij}, the result is[30]

$$\int [\mathrm{d}\phi]\ \mathrm{e}^{-S} = \int \prod_{i=1} \frac{\mathrm{d}\gamma_i}{\sqrt{2\pi}} \left(\det\, U\right)^{1/2} \mathrm{e}^{-S_{\rm cl}} \left(\det{}' M\right)^{-1/2} \ . \tag{8.5.159}$$

Notice that this result is invariant under rescalings of Z, which can be seen as rescalings of the collective coordinates. More generally, the matrix U_{ij} can be interpreted as a metric on the moduli space of collective coordinates. The measure is then invariant under general coordinate transformations on the moduli space.

One can repeat the analysis for gauge theories to show that (8.5.159) also holds for Yang-Mills instantons in singular gauge. For regular gauges, there are some complications due to the fact that neither of the two terms in (8.4.114) does fall off fast at infinity, but only their sum is convergent. In singular gauge, each term separately falls off fast at infinity. For this reason, it is more convenient to work in singular

[29] It will contribute however to a two-loop contribution. To see this, one first writes this term in the exponential, where it enters without $\hbar$, so it is at least a one-loop effect. Then $\phi_{\rm qu}$ has a part proportional to the zero mode, which drops out by means of the delta function insertion. The other part of $\phi_{\rm qu}$ is genuinely quantum and contains a power of $\hbar$ (which we have suppressed). Therefore, it contributes at two loops [55] (see also [54] for related matters).

[30] One obtains from (8.5.157) $\det\langle \partial_{\gamma_i} A^{\rm cl}_\mu | Z^{(j)}\rangle$ times $(\det U^{ab})^{-1/2}$. The matrix elements $\langle \partial_{\gamma_i} A^{\rm cl}_\mu | Z^{(j)}\rangle$ are equal to $\langle Z^{(i)}|Z^{(j)}\rangle = U^{ij}$ minus $\langle D_\mu \Lambda^{(a)} | Z^{(b)}_\mu \rangle$. The latter term can be partially integrated, and vanishes since there are no boundary contributions, neither in the singular nor in the regular gauge. (For the regular gauge one needs an explicit calculation to check this statement.)

gauge. The measure for the bosonic collective coordinates for $k = 1$ $SU(N)$ YM theories, without the determinant from integrating out the quantum fluctuations which will be analyzed in the next section, becomes

$$\frac{2^{4N+2}\pi^{4N-2}}{(N-1)!(N-2)!}\frac{1}{g^{4N}}\int \mathrm{d}^4x_0\,\frac{\mathrm{d}\rho}{\rho^5}\rho^{4N}\ . \tag{8.5.160}$$

This formula contains the square-root of the determinant of U in 8.4.135, $4N$ factors of $1/\sqrt{2\pi}$, and we have also integrated out the gauge orientation zero modes. This may be done only if we are evaluating gauge invariant correlation functions. The result of this integration follows from the volume of the coset space

$$\mathrm{Vol}\left\{\frac{SU(N)}{SU(N-2)\times U(1)}\right\} = \frac{2^{4N-5}\pi^{2N-2}}{(N-1)!(N-2)!}\ , \tag{8.5.161}$$

which is a factor 2^{4N-5} larger than in [?, 50], because we have used the normalization $\mathrm{tr}(T_aT_b) = -\frac{1}{2}\delta_{ab}$, while in [?, 50] $\mathrm{tr}(T_aT_b) = -2\delta_{ab}$ was used. We found in 8.4.135 another factor $2^{-(4N-5)}$, and indeed the result for the total measure in 8.5.160 is the same as in [?, 50]. The derivation of this formula can be found in Appendix C, which is a detailed version of [50].

5.2 The measure for the fermionic collective coordinates

We must also construct the measure on the moduli space of fermionic collective coordinates. Consider (8.4.136). The fermionic zero modes are linear in the Grassmann parameters ξ^α and $\bar\eta_{\alpha'}$. Thus these ξ^α and $\bar\eta_{\alpha'}$ correspond to the coefficients ξ^α in (8.5.151). One obtains the zero modes by differentiating λ^α in (8.4.136) w.r.t. ξ^α and $\bar\eta_{\alpha'}$, and for this reason one often calls these ξ^α and $\bar\eta_{\alpha'}$ the fermionic collective coordinates. This is not quite correct, because collective coordinates appear in the classical solution (the instanton) but we shall use this terminology nevertheless because it is common practice. We use again the measure in (8.5.154). There are in this case no factors $\frac{1}{\sqrt{2\pi}}$ because of the Grassmann integration, and instead of $(\det M')^{-1/2}$ we now obtain $(\det M')^{1/2}$ in 8.5.155. Because the parameters $\xi^\alpha, \bar\eta_\alpha$, etc. appear linearly in the zero modes, we do not need the Faddeev-Popov trick to convert the integration over zero modes into an integration over collective coordinates. So *for fermions the Grassmannian coefficients of the zero modes are at the same time the collective coordinates.*

We shall discuss these issues in more detail when we come to supersymmetric gauge theories, but now we turn to computing the norms of the fermionic zero modes.

For the zero modes with ξ in (8.4.136), one finds

$$Z^{\alpha}_{(\beta)} = \frac{\partial \lambda^{\alpha}}{\partial \xi^{\beta}} = -\tfrac{1}{2}\sigma_{\mu\nu}{}^{\alpha}{}_{\beta} F_{\mu\nu} \ . \tag{8.5.162}$$

The norms of these two zero modes are given by

$$(U_{\xi})_{\beta}{}^{\gamma} = -\frac{2}{g^2}\int \mathrm{d}^4x \,\mathrm{tr}\left\{ Z_{\alpha(\beta)} Z^{\alpha(\gamma)} \right\} = 4S_{\mathrm{cl}}\delta_{\beta}{}^{\gamma} \ , \tag{8.5.163}$$

where we have used the definition in (8.4.117) and contracted the spinor indices with the usual metric for spinors. This produces a term in the measure[31]

$$\int \mathrm{d}\xi^1 \mathrm{d}\xi^2 \ (4S_{\mathrm{cl}})^{-1} \ . \tag{8.5.164}$$

The result (8.5.164) actually holds for any k. We get the square root of the determinant in the denominator for fermions. One really gets the square root of the super determinant of the matrix of inner product, but because there is no mixing between bosonic and fermionic moduli, the superdeterminant factorizes into the bosonic determinant divided by the fermionic determinant.

For the $\bar{\eta}$ zero modes, we obtain, using some algebra for the σ-matrices,

$$Z^{\alpha\beta'} = \partial\lambda^{\alpha}/\partial\bar{\eta}_{\beta'} = \tfrac{1}{2}(\sigma_{\mu\nu}\sigma_{\rho})^{\alpha\beta'} F_{\mu\nu} x_{\rho} \ , \tag{8.5.165}$$

and

$$(U_{\bar{\eta}})_{\alpha'}{}^{\beta'} = 8S_{\mathrm{cl}}\delta_{\alpha'}{}^{\beta'}\rho^2 \ , \tag{8.5.166}$$

[31] Sometimes one finds in the literature that $U_{\xi} = 2S_{\mathrm{cl}}$. This is true when one uses the conventions for Grassmann integration $\int \mathrm{d}^2\xi\, \xi^{\alpha}\xi^{\beta} = \frac{1}{2}\epsilon^{\alpha\beta}$. In our conventions $\mathrm{d}^2\xi \equiv \mathrm{d}\xi^1\mathrm{d}\xi^2$.

so that the corresponding measure is

$$\int \mathrm{d}\bar{\eta}_1 \mathrm{d}\bar{\eta}_2 \, (8\rho^2 S_{\rm cl})^{-1} \ , \tag{8.5.167}$$

which only holds for $k=1$.

Finally we compute the Jacobian for the fermionic "gauge orientation" zero modes. For convenience, we take the solutions in regular gauge (the Jacobian is gauge invariant anyway), and find from (8.4.138)

$$\left(Z^{\alpha}_{(\mu^w)}\right)^u{}_v = \frac{\rho}{\sqrt{(x^2+\rho^2)^3}}\,\delta^\alpha{}_v\,\Delta^u{}_w \ , \qquad \left(Z^{\alpha}_{(\bar{\mu}_w)}\right)^u{}_v = \frac{\rho}{\sqrt{(x^2+\rho^2)^3}}\,\epsilon^{\alpha u}\Delta^w{}_v \ , \tag{8.5.168}$$

where the N by N matrix Δ is the unity matrix in the $(N-2)$ by $(N-2)$ upper diagonal block, and zero elsewhere. So Δ restricts the values of u, w and v to up to $N-2$ while in $\delta^\alpha{}_v$ and $\epsilon^{\alpha v}$ the index runs over the next two values. Consequently, the norms of Z_μ and $Z_{\bar{\mu}}$ are easily seen to be zero, but the nonvanishing inner product is

$$(U_{\mu\bar{\mu}})^u{}_v = -\frac{2}{g^2}\int \mathrm{d}^4x \, \mathrm{tr}\, Z^{\alpha}_{(\bar{\mu}_u)} Z_{\alpha\,(\mu^v)} = \frac{2\pi^2}{g^2}\Delta^u{}_v \ , \tag{8.5.169}$$

where we have used the integral (8.B.363). It also follows from the index structure that the ξ and $\bar{\eta}$ zero modes are orthogonal to the μ zero modes, so there is no mixing in the Jacobian.

Putting everything together, the fermionic part of the measure for $\mathcal{N}$ adjoint fermions coupled to $SU(N)$ YM theory, with $k=1$, is given by

$$\int \left(\prod_{A=1}^{\mathcal{N}} \mathrm{d}^2\,\xi^A\right)\left(\frac{g^2}{32\pi^2}\right)^{\mathcal{N}} \left(\prod_{A=1}^{\mathcal{N}} \mathrm{d}^2\,\bar{\eta}^A\right)\left(\frac{g^2}{64\pi^2\rho^2}\right)^{\mathcal{N}} \prod_{A=1}^{\mathcal{N}}\left(\prod_{u=1}^{N-2}\mathrm{d}\mu^{A,u}\,\mathrm{d}\bar{\mu}^A_u\right)\left(\frac{g^2}{2\pi^2}\right)^{\mathcal{N}(N-2)} . \tag{8.5.170}$$

Similarly, one can include fermions in the fundamental representation, for which the Jacobian factor is

$$U_{\mathcal{K}} \equiv \int \mathrm{d}^4x \, Z^\alpha{}_u Z_\alpha{}^u = \pi^2 \ , \tag{8.5.171}$$

for each species. Here $\mathcal{K}$ is the Grassmann collective coordinate of (8.4.145). Hence in this case the fermionic part of the measure is

$$\int \left(\prod_{A=1}^{N_f} \mathrm{d}\mathcal{K}^A\right)\left(\sqrt{\frac{1}{\pi^2}}\right)^{N_f} \tag{8.5.172}$$

for N_f fundamental Weyl spinors coupled to $SU(N)$ YM theory with $k = 1$.

Note that we did not put a factor $\frac{1}{g^2}$ in front of the integral in (8.5.171), whereas we used such a factor for fermions in the adjoint representation. The reason we do not use such a factor for fermions in the fundamental representation has to do with the action. One finds a factor $\frac{1}{g^2}$ in front of the Yang-Mills action, and therefore also, by susy, in front of the Dirac action for gluinos. However, in the matter action the g-dependence has been absorbed by the gluons, so there is no factor $\frac{1}{g^2}$ in front of the matter fermions. The measure of the zero modes uses the metric of the collective coordinates. In soliton physics (and instantons can be considered as solitons in one higher dimension) one obtains this metric if one lets the collective coordinates become time dependent and integrates over d^4x in the action one ends up with a quantum mechanical action of the

$$\mathcal{L} = (U^{ij}\dot{\gamma}_i\dot{\gamma}_j + U_{\alpha\beta}\dot{\xi}^{\alpha}\dot{\xi}^{\beta} + U^{\alpha'\beta'}\dot{\bar{y}}_{\alpha'}\dot{\bar{y}}_{\beta'} + U_{AB}\dot{K}^A\dot{K}^B \tag{8.5.173}$$

Since $U^{ij}, U_{\alpha\beta}$ and $U^{\alpha'\beta'}$ are produced by the Yang-Mills action and its susy partner, while U_{AB} is due to the matter action, there is no g-dependence in (8.5.172).

6 One loop determinants

Having determined the measure on the moduli space of collective coordinates, we now compute the determinants that arise by Gaussian integration over the quantum fluctuations. Before doing so, we extend the model by adding real scalar fields and Majorana fermions in the adjoint representation. The action is

$$S = -\frac{1}{g^2}\int \mathrm{d}^4x\, \mathrm{tr}\left\{\tfrac{1}{2}F_{\mu\nu}F_{\mu\nu} + (D_\mu\phi)(D_\mu\phi) - i\bar{\lambda}\,\bar{\not{D}}\lambda - i\lambda\,\not{D}\bar{\lambda}\right\}\ . \tag{8.6.174}$$

Here, λ is a two-component Weyl spinor which we take in the adjoint representation[32]. In Minkowski space there is a reality condition between the two complex 2-component spinors λ and $\bar{\lambda}$, and as a result $\bar{\lambda}_{\dot{\alpha}}$ transforms in the complex conjugate of the representation of λ_α, but in Euclidean space this reality condition is dropped. So λ^α and $\bar{\lambda}_{\alpha'}$ are independent complex variabes. For the Grassmann integration this makes no difference. Written with indices the Euclidean Dirac action in 8.6.174 reads $\{-i\lambda_\alpha(\sigma^\mu)^{\alpha\beta'}D_\mu\bar{\lambda}_{\beta'} - i\bar{\lambda}^{\alpha'}(\bar{\sigma}^\mu)_{\alpha'\beta}D_\mu\lambda^\beta\}$ where $\lambda_\alpha = \lambda^\beta\epsilon_{\beta\alpha}$ and $\bar{\lambda}^{\alpha'} = \epsilon^{\alpha'\beta'}\bar{\lambda}_{\beta'}$. Generalization to fundamental fermions is straightforward. The anti-instanton solution around which we will expand is

$$A_\mu^{\rm cl}\ , \qquad \phi_{\rm cl} = 0\ , \qquad \lambda_{\rm cl} = 0\ , \qquad \bar{\lambda}_{\rm cl} = 0\ , \tag{8.6.175}$$

where $A_\mu^{\rm cl}$ is the anti-instanton. This background represents an exact solution to the field equations. The bosonic and fermionic zero modes are taken care of by the measure for the collective coordinates, while in the orthogonal space of nonzero modes, one can define propagators and vertices, and perform perturbation theory around the (anti-) instanton.

After expanding $A_\mu = A_\mu^{\rm cl} + A_\mu^{\rm qu}$, and similarly for the other fields, we add gauge fixing and ghost terms

$$S_{\rm gf} = -\frac{1}{g^2}\int {\rm d}^4x\, {\rm tr}\left\{\left(D_\mu^{\rm cl}\,A_\mu^{\rm qu}\right)^2 - 2\,b\,D_{\rm cl}^2\,c\right\}\ , \tag{8.6.176}$$

such that the total gauge field action is given by (8.4.112). The integration over A_μ gives

$$[\det{}'\Delta_{\mu\nu}]^{-1/2}\ , \qquad \Delta_{\mu\nu} = -D^2\delta_{\mu\nu} - 2F_{\mu\nu}\ , \tag{8.6.177}$$

[32]As before $\lambda = \begin{pmatrix}\lambda^\alpha \\ \bar{\lambda}_{\dot{\alpha}}\end{pmatrix}$, but the 4-component Majorana spinor $\bar{\lambda}$ is defined by $\lambda^T C$ both in Minkowski and in Euclidean space, where C is the charge conjugation matrix, $C = \begin{pmatrix}\epsilon_{\alpha\beta} & 0 \\ 0 & \epsilon^{\dot{\alpha}\dot{\beta}}\end{pmatrix}$. Then $\bar{\lambda} = \left(\lambda_\alpha\ , -\bar{\lambda}^{\dot{\alpha}}\right)$ and Lorentz (or rather $SO(4)$) invariance is preserved in Euclidean space because the relation $C\gamma^\mu = -\gamma^{\mu,T}C$ holds in both spaces. In Euclidean space we denote the indices of $\bar{\lambda}$ by α' instead of $\dot{\alpha}$.

where the prime stands for the amputated determinant, with zero eigenvalues left out. We have suppressed the subscript 'cl' and Lie algebra indices. Integration over the scalar fields results in

$$[\det \Delta_\phi]^{-1/2} \ , \qquad \Delta_\phi = -D^2 \ , \tag{8.6.178}$$

and the ghost system yields similarly

$$[\det \Delta_{\rm gh}] \ , \qquad \Delta_{\rm gh} = -D^2 \ . \tag{8.6.179}$$

For the fermions λ and $\bar{\lambda}$, we need a bit more explanation. Since neither $\not{D}$ nor $\bar{\not{D}}$ is hermitean(even worse, $\not{D}$ maps antichiral spinors into chiral spinors), we cannot evaluate the determinants in terms of their eigenvalues. But both products

$$\Delta_- = -\not{D}\,\bar{\not{D}} = -D^2 - \tfrac{1}{2}\sigma_{\mu\nu}F_{\mu\nu} \ , \qquad \Delta_+ = -\bar{\not{D}}\,\not{D} = -D^2 \ , \tag{8.6.180}$$

with spinor indices still suppressed, are hermitean. Let us label the nonzero modes by a subscript i. Then we can expand λ in terms of commuting eigenfunctions F_i of Δ_- with anticommuting coefficients ξ_i, and $\bar{\lambda}$ in terms of eigenfunctions $\bar{F}_i$ of Δ_+ with coefficients $\bar{\xi}_i$. We have seen before that both operators have the same spectrum of nonzero eigenvalues ϵ_i, and the relation between the eigenfunctions is $\bar{F}_i = \frac{1}{\sqrt{\epsilon_i}}\bar{\not{D}}F_i$ and $F_i = \frac{-1}{\sqrt{\epsilon_i}}\not{D}\bar{F}_i$. (The minus sign is needed in order that $\bar{F}_i = \frac{1}{\sqrt{\epsilon_i}}\bar{\not{D}}F_i = \frac{1}{\epsilon_i}(-\bar{\not{D}}\not{D})\bar{F}_i = \bar{F}_i$). Defining the path integral over λ and $\bar{\lambda}$ as the integration over ξ_i and $\bar{\xi}_i$, one gets the determinant over the nonzero eigenvalues[33]. The result for the integration

[33] Namely, the action becomes

$$-\frac{1}{g^2}{\rm tr}\int {\rm d}^4x\left[-i\left(\bar{\xi}_i\bar{F}_i\,\bar{\not{D}}\xi_j\frac{(-\not{D}\bar{F}_j)}{\sqrt{\epsilon_j}}\right) - i\xi_jF_j\,\not{D}\bar{\xi}_i\frac{\bar{\not{D}}F_i}{\sqrt{\epsilon_i}}\right] = -\frac{i}{2}\bar{\xi}_i\xi_j\sqrt{\epsilon_j}\left\langle\bar{F}_i^a|\bar{F}_j^a\right\rangle + \frac{i}{2}\xi_j\bar{\xi}_i\left\langle F_j^a|F_i^a\right\rangle\sqrt{\epsilon_i}\,. \tag{8.6.181}$$

Next we use that the norms of $\bar{F}_i$ and F_i are equal:

$$\begin{aligned}\langle\bar{F}_i|\bar{F}_j\rangle &= \frac{1}{\sqrt{\epsilon_i}}\left\langle\bar{\not{D}}F_i|\frac{1}{\sqrt{\epsilon_j}}\,\bar{\not{D}}F_j\right\rangle = \frac{1}{\sqrt{\epsilon_i\epsilon_j}}\langle F_i|-\not{D}\,\bar{\not{D}}F_j\rangle \\ &= \sqrt{\frac{\epsilon_j}{\epsilon_i}}\,\langle F_i|F_j\rangle = \frac{1}{\sqrt{\epsilon_i\epsilon_j}}\langle-\not{D}\,\bar{\not{D}}F_i|F_j\rangle = \sqrt{\frac{\epsilon_i}{\epsilon_j}}\,\langle F_i|F_j\rangle \ .\end{aligned}$$

over the fermions can be written in symmetrized form as

$$[\det{}'\Delta_-]^{1/4}\,[\det\Delta_+]^{1/4}\ . \tag{8.6.182}$$

As stated before, since all the eigenvalues of both Δ_- and Δ_+ are the same, the determinants are formally equal. This result can also be obtained by writing the spinors in terms of Dirac fermions; the determinant we have to compute is then

$$\left[\det{}'\,\Delta_D^2\right]^{1/2}\ , \qquad \Delta_D = \begin{pmatrix} 0 & \not{D} \\ \bar{\not{D}} & 0 \end{pmatrix}\ . \tag{8.6.183}$$

One would expect that in a supersymmetric model with vectors, spinors and scalars, the sum of all zero point energies cancel. These zero point energies correspond to the one-loop determinants in an external Yang-Mills field. So this suggests that all one-loop determinants are related, and since the one-loop determinants of fermions depend on Δ_+ and Δ_-, one would expect that the determinants for the bosons can be expressed in terms of the determinants of Δ_- and Δ_+. For the ghosts and adjoint scalars this is obvious,

$$\det\ \Delta_\phi = \det\ \Delta_{\text{gh}} = [\det\ \Delta_+]^{1/2}\ . \tag{8.6.184}$$

We get $\det\Delta_\phi = \det(-D^2) = \det\Delta_+^{1/2}$ and $\det\Delta_{\text{gh}} = \det(-D^2) = \det\Delta_+^{1/2}$ because the spinor space is two-dimensional.

For the vector fields, we rewrite the operator $\Delta_{\mu\nu}$ in (8.6.177) in terms of the fermion operator Δ_-. Using $\text{tr}(\bar\sigma_\mu\sigma_\nu) = 2\delta_{\mu\nu}$ and $\text{tr}(\bar\sigma_\mu\sigma_{\rho\sigma}\sigma_\nu) = 2(\delta_{\mu\rho}\delta_{\sigma\nu} - \delta_{\mu\sigma}\delta_{\rho\nu} - \epsilon_{\mu\rho\sigma\nu})$ we obtain the following identity for $\Delta_{\mu\nu} = -\delta_{\mu\nu}D^2 - 2F_{\mu\nu}$,

$$\begin{aligned} \Delta_{\mu\nu} &= \tfrac{1}{2}\text{tr}\left\{\bar\sigma_\mu\Delta_-\sigma_\nu\right\} = \tfrac{1}{2}\bar\sigma_{\mu\,\alpha'\beta}\left(\Delta_-{}^\beta{}_\gamma\right)\sigma_\nu^{\gamma\alpha'} \\ &= \tfrac{1}{2}(\bar\sigma_{\mu\alpha'\beta})(\Delta_-{}^\beta{}_\gamma\delta^{\alpha'}{}_{\delta'})(\bar\sigma_\nu{}^{\gamma\delta'}) \end{aligned} \tag{8.6.185}$$

Hence, as expected, the F_i and $\bar F_j$ for different eigenvalues are orthogonal to each other, and the norms of F_i and $\bar F_j$ are the same. Denoting $\frac{1}{g^2}\int \mathrm{d}^4x\,(F_i^a)^*F_i^a =< F_i^a|F_i^a >$ by u_i, one finds for the path integral

$$\int \mathrm{d}\bar\xi_i d\xi_i\, \mathrm{e}^{i\xi_i\bar\xi_i u_i\sqrt{\epsilon_i}} = iu_i\sqrt{\epsilon_i}\ .$$

Hence the measure is $\frac{\mathrm{d}\xi_i}{\sqrt{u_i}}\frac{\mathrm{d}\bar\xi_j}{\sqrt{u_j}}$, and the one-loop determinant is $\prod_i(\epsilon_i)^{1/2}$.

where $(\Delta_-)^\beta{}_\gamma \delta^{\alpha'}{}_{\delta'}$ is block-diagonal on the basis $\beta\alpha' = \gamma\delta' = (11), (21), (12), (22)$.

$$\Delta_-{}^\beta{}_\gamma \delta^{\alpha'}{}_{\delta'} = \begin{pmatrix} \Delta_-{}^1{}_1 & \Delta_-{}^1{}_2 & 0 & 0 \\ \Delta_-{}^2{}_1 & \Delta_-{}^2{}_2 & 0 & 0 \\ 0 & 0 & \Delta_-{}^1{}_1 & \Delta_-{}^1{}_2 \\ 0 & 0 & \Delta_-{}^2{}_1 & \Delta_-{}^2{}_2 \end{pmatrix} \tag{8.6.186}$$

This proves that[34]

$$\det{}'\Delta_{\mu\nu} = [\det{}'\Delta_-]^2 \ . \tag{8.6.187}$$

Now we can put everything together. The one-loop determinant for a Yang-Mills system, including the ghosts, coupled to n real adjoint scalars and $\mathcal{N}$ Weyl spinors (or Majorana spinors) also in the adjoint representation is

$$[\det{}'\Delta_-]^{-1+\mathcal{N}/4} [\det\Delta_+]^{\frac{1}{4}(2+\mathcal{N}-n)} \ . \tag{8.6.188}$$

This expression simplifies to the ratio of the determinants when $\mathcal{N} - \frac{n}{2} = 1$. Particular cases are

$$\begin{aligned} \mathcal{N} = 1 \quad n = 0 \quad &\rightarrow \quad \left[\frac{\det \Delta_+}{\det{}'\Delta_-}\right]^{3/4} \ , \\ \mathcal{N} = 2 \quad n = 2 \quad &\rightarrow \quad \left[\frac{\det \Delta_+}{\det{}'\Delta_-}\right]^{1/2} \ , \\ \mathcal{N} = 4 \quad n = 6 \quad &\rightarrow \quad \left[\frac{\det \Delta_+}{\det{}'\Delta_-}\right]^{0} \ . \end{aligned} \tag{8.6.189}$$

These cases correspond to supersymmetric Yang-Mills theories with $\mathcal{N}$-extended supersymmetry. Notice that for $\mathcal{N} = 4$, the determinants of Δ_+ and Δ_- separately cancel, so there is no one-loop contribution.

For $\mathcal{N} = 1, 2$ the determinants formally give unity since the non-zero eigenvalues are the same. However, one must first regularize the theory to define the determinants properly. After regularization, the renormalization procedure must be carried out and counterterms must be added. The counterterms are the same as in the theory

[34] Consider $\bar\sigma_{\mu,\alpha'\beta}$ and $\sigma_\nu^{\gamma\delta'}$ as 4×4 matrices. Then on the right-hand side of 8.6.185 one has the product of three 4×4 matrices. For fixed μ and ν one has $\bar\sigma^\mu_{\alpha'\beta}\sigma_\nu^{\beta\alpha'} = 2\delta_\mu{}^\nu$, hence $\det[\bar\sigma_{\mu,\alpha'\beta}] = 4$.

without instantons and their finite as well as infinite parts must be specified by physical renormalization conditions. The ratios of products of non-zero eigenvalues can be written as the exponent of the difference of two infinite sums

$$\frac{\det \Delta_+}{\det' \Delta_-} = \exp\left(\sum_n \omega_n^{(+)} - \sum_n \omega_n^{(-)}\right) \ , \tag{8.6.190}$$

with eigenvalues $\lambda_n = \exp \omega_n$. The frequencies $\omega_n^{(+)}$ and $\omega_n^{(-)}$ can be discretized by putting the system in a box of size R and imposing suitable boundary conditions on the quantum fields at R (for example, $\phi(R) = 0$, or $\frac{\mathrm{d}}{\mathrm{d}R}\phi(R) = 0$, or a combination thereof [2]). These boundary conditions may be different for different fields. The sums over $\omega_n^{(+)}$ and $\omega_n^{(-)}$ are divergent; their difference is still divergent (although less divergent than each sum separately) but after adding counterterms ΔS one obtains a finite answer. The problem is that one can combine the terms in both series in different ways, giving different answers. By combining $\omega_n^{(+)}$ with $\omega_n^{(-)}$ for each fixed n, one would find that the ratio $(\det \Delta_+ / \det{}' \Delta_-)$ equals unity. However, other values could result by using different ways to regulate these sums. We have discussed before that for susy instantons the densities of nonzero modes are equal, hence for susy instantons the contributions in 8.6.189 from the one-loop determinants cancel. This makes these models simpler to deal with than non-susy models. For ordinary (nonsusy) Yang-Mills theory, the results for the effective action due to different regularization schemes differ at most by a local finite counterterm. In the background field formalism we are using, this counterterm must be background gauge invariant, and since we consider only vacuum expectation values of the effective action, only one candidate is possible: it is proportional to the gauge action $\int \mathrm{d}^4x \, \mathrm{tr}\, F^2$ and multiplied by the one-loop beta-function for the various fields which can run in the loop,

$$\Delta S \propto \beta(g) \int \mathrm{d}^4x \, \mathrm{tr}\, F^2 \ \ln \frac{\mu^2}{\mu_0^2} \ . \tag{8.6.191}$$

The factor $\ln(\mu^2/\mu_0^2)$ parametrizes the freedom in choosing different renormalization schemes.

A particular regularization scheme used in [2] is Pauli-Villars regularization. In this case 't Hooft first used x-dependent regulator masses to compute the ratios of the one-loop determinants Δ in the instanton background and $\Delta^{(0)}$ in the trivial vacuum. Then he argued that the difference between using the x-dependent masses and using the more usual constant masses, was of the form ΔS given above. The final result for pure YM $SU(N)$ in the $|k=1|$ sector is [2, 50]

$$\left[\frac{\det'\Delta_-}{\det\Delta_-^{(0)}}\right]^{-1}\left[\frac{\det\Delta_+}{\det\Delta_+^{(0)}}\right]^{1/2} = \exp\left\{\tfrac{2}{3}N\,\ln(\mu\rho) - \alpha(1) - 2(N-2)\alpha\left(\tfrac{1}{2}\right)\right\}\ . \quad (8.6.192)$$

Here we have normalized the determinants against the vacuum, indicated by the superscript (0). From the unregularized zero mode sector one obtains a factor ρ^{4N}, see (8.5.160), and Pauli-Villars regularization of the $4N$ zero modes yields a factor M_{PV}^{4N}. All together one obtains $\frac{8\pi^2}{g_0^2}+\frac{22}{3}\ln(M_{PV}\rho)$ in the exponent for $SU(2)$, where g_0 is the unrenormalized coupling constant. Subtracting $\frac{22}{3}\ln(M_{PV}\rho_0)$ to renormalize at mass scale $1/\rho_0$, one is left for the effective action with $\frac{8\pi^2}{g_0^2}-\frac{11}{3}\ln(\rho/\rho_0)\equiv\frac{8\pi^2}{g^2(\rho)}$. Replacing $\ln(\rho/\rho_0)$ by $\ln(\mu/\mu_0)$, this is the correct one-loop renormalization equation for the running of the coupling constant. For supersymmetric theories, the nonzero mode corrections to the effective action cancel, and performing the same renormalization procedure as for the non-supersymmetric case, one now obtains only from the zero modes the correct β function. For $\mathcal{N}=4$ one finds a vanishing β function.

The fluctuations of the $SU(2)$ part of the gauge fields and the Faddeev-Popov ghosts yield the term $\alpha(1)$ in 8.6.192, while the fluctuations of the $2(N-2)$ doublets (corresponding to $\lambda_4,\cdots\lambda_7$ for $SU(3)$) yields the term with $\alpha\left(\frac{1}{2}\right)$. The numerical values of the function $\alpha(t)$ are related to the Riemann zeta function, and take the values $\alpha\left(\frac{1}{2}\right)=0.145873$ and $\alpha(1)=0.443307$. Notice that this expression for the determinant depends on ρ, and therefore changes the ρ-dependence of the integrand of the collective coordinate measure. Combined with (8.5.160) one correctly reproduces the β-function of $SU(N)$ YM theory. The calculation of the contribution of the nonzero modes can be simplified by using a so-called $O(5)$ formalism [51] which uses the conformal symmetries of instantons, in addition to the nonconformal symmetries. One still has to regulate the sums over zero-point energies, and both Pauli-Villars regularization [51] and zeta-function regularization [52] have been applied to the $O(5)$ formulation.

6.1 The exact β function for SYM theories

In supersymmetric gauge theories, the contributions to the one-loop partition function by the nonzero modes in the bosonic and fermionic loops cancel each other [59]. Although this has only been shown to occur in a gravitational background without winding, we assume here that still occurs in an instanton background. Actually, all contributions from the nonzero mode sector cancel: higher-loops as well as possible nonperturbative corrections. The zero mode sector can be regularized by Pauli-Villars fields, and since the partition function yields a physical observable, namely the cosmological constant (the sum over zero-point energies), the result for the partition function should not depend on the regularization parameter M_{PV} (the Pauli-Villars mass). From this observation one can derive a differential equation for the coupling constant $g(M_{PV})$, which yields the exact β function: it contains all perturbative contributions [60].

Before going on we should comment on the fact that from the 3-loop level on the result for the β function depends on the regularization scheme chosen. It is sometimes claimed that therefore higher-loop results for the β function have no meaning. This is incorrect: given a particular scheme, all orders in perturbation theory of β have meaning. In the derivation below of the β function we shall find an all-order result, but it is not (yet?) known which regularization scheme for Feynman graphs would reproduce these results. So the all-order expression for β has in principle meaning, but in practice one cannot do much with it. One can only say: there must exist a regularization scheme which, if used for the calculation of higher-loop Feynman graphs, will produce the all-order result for β obtained below.

We begin with pure supersymmetric gauge theory. We recall that the measure of the zero modes of a single instanton or anti-instanton ($k = \pm 1$) for $N = 1$ susy with gauge group $SU(N)$ and one Majorana or Weyl fermion in the adjoint representation is given by

$$\mathrm{d}\mathcal{M}_{k=\pm 1} = \mathrm{e}^{-\frac{8\pi^2}{g^2}} \left[\frac{\mathrm{d}^4x\mathrm{d}\rho}{\rho^5} 2^{4N+2} \left(\frac{\rho}{g}\right)^{4N} \left(\frac{M_{PV}}{\sqrt{2\pi}}\right)^{4N} \frac{\pi^{4N-2}}{(N-1)!(N-2)!} \right] \times \left[\frac{\mathrm{d}\xi_1\mathrm{d}\xi_2}{4S_{\mathrm{cl}}M_{PV}} \frac{\mathrm{d}\bar{\eta}_1\mathrm{d}\bar{\eta}_2}{8\rho^2 S_{\mathrm{cl}}M_{PV}} \frac{\prod_{u=1}^{N-2}\mathrm{d}\mu^u\mathrm{d}\bar{\mu}_u}{\left(\frac{1}{4}S_{\mathrm{cl}}M_{PV}\right)^{N-2}} \right] \mathrm{Vol}\left\{ \frac{SU(N)}{SU(N-2)\otimes U(1)} \right\} \quad (8.6.193)$$

where $S_{\mathrm{cl}} = 8\pi^2/g^2$. The volume of the gauge group was given in (8.5.161) but because it does not depend on g or M_{PV} it will play no role below. Note that this measure is dimensionless; $\mathrm{d}^4x\mathrm{d}\rho/\rho^5$ is dimensionless, and the remaining ρ and M_{PV} occur only in the combination ρM_{PV}. Also $d^2\xi/M_{PV}$ and $d^2\bar{\eta}/(\rho^2 M_{PV})$ are dimensionless. The prefactor $\mathrm{e}^{-8\pi^2/g^2}$ is of course the classical action for the one-instanton background, and we have left out the term with the term with the theta-angle. In the first square brackets we find the product of the measure for the bosonic zero modes in (8.5.160) and factors $\sqrt{\frac{M_{PV}^2}{2\pi}}$ for each bosonic zero mode from the corresponding Pauli-Villars modes.[35] The second expression in square brackets contains the contribution to the measure from the fermionic zero modes given by (8.5.170), with factors $\frac{1}{\sqrt{M_{PV}}}$ for each fermionic zero mode. Clearly, each bosonic zero mode contributes a factor M_{PV}/g and each fermionic zero mode contributes a factor $g/\sqrt{M}_{PV}$.

The dependence of $\mathrm{d}\mathcal{M}$ on M_{PV} and g is thus as follows

$$\mathrm{d}\mathcal{M} \propto \mathrm{e}^{-\frac{8\pi^2}{g^2}} (M_{PV})^{3N} \left(\frac{1}{g}\right)^{2N} , \quad (8.6.194)$$

[35] If the one-loop determinant for the bosonic fields is $(\det M_b)^{-1/2}$ and for the fermionic fields $\det M_f$, then the Pauli-Villars method yields further determinants $\det(M_b + M_{PV}^2)^{+1/2}$ and $\det(M_f + M_{PV})^{-1}$. The zero modes are eigenfunctions of M_b and M_f with eigenvalue zero, so their Pauli-Villars counterparts become nonzero modes with eigenvalues M_{PV}^2 and M_{PV}.

where g depends on M_{PV}, so $g = g(M_{PV})$. So g is the bare coupling constant in the regularized theory, and g and M_{PV} vary such that the renormalized coupling constant g_R is kept fixed. Usually one considers the renormalized coupling constant as a function of the renormalization mass μ, and then the bare coupling constant g satisfies $\mu\frac{\partial}{\partial\mu}g = 0$. Using dimensional regularization and $g = Z_g(g_{\text{ren}})g_{\text{ren}}\mu^{\epsilon/2}$ with $\epsilon = 4 - n$ yields then the β function. If one uses Pauli-Villars regularization there are two masses which play a role: the cut-off (regulator) mass M_{PV} and the physical renormalization mass μ. The bare coupling depends on one of them, the renormalized coupling on the other.

$$\begin{array}{ll} g = g(M_{PV}) & g_R = g_R(\mu) \\ M_{PV}\frac{\partial}{\partial M_{PV}}g(M_{PV}) = \beta(g) & \mu\partial/\partial\mu \;\; g_R(\mu) = \beta(g_R) \\ \mu\frac{\partial}{\partial\mu}g(M_{PV}) = 0 & M_{PV}\frac{\partial}{\partial M_{PV}}g_R(\mu) = 0 \end{array} \tag{8.6.195}$$

Physical quantities depend on μ but not on M_{PV}. If one wants to apply the renormalization group to the measure, one must use the approach based on $(M_{PV}\partial/\partial M_{PV})g_R(\mu)$ $= 0$ because the regularized measure depends on M_{PV}, not on μ. The results for the β function obtained from both schemes differ by a sign, because the logarithms in the theory depend on $\ln(M_{PV}/\mu)$.

Equating the derivative of the logarithm of the measure w.r.t. in (8.6.194) M_{PV} to zero yields then

$$M_{PV}\frac{\partial}{\partial M_{PV}}\left(-\frac{8\pi^2}{g^2} + 3N\ln M_{PV} - 2N\ln g\right) = 0\ . \tag{8.6.196}$$

Hence

$$M_{PV}\frac{\partial}{\partial M_{PV}}g \equiv \beta = \left(\frac{3N}{\frac{2N}{g} - \frac{16\pi^2}{g^3}}\right)\ , \tag{8.6.197}$$

or, written in terms of $\alpha = \frac{g^2}{4\pi}$

$$\frac{g}{2\pi}\beta = M_{PV}\frac{\partial}{\partial M_{PV}}\alpha = \frac{-3N\alpha^2}{2\pi}\frac{1}{1 - \frac{\alpha N}{2\pi}}\ . \tag{8.6.198}$$

This is the β-function for pure $\mathcal{N} = 1$ supersymmetric Yang-Mills theory. It is straightforward to extend this result to pure $\mathcal{N}$-extended supersymmetry with $\mathcal{N}$

Majorana or Weyl fermions in the adjoint representation. One finds for $SU(N)$

$$M_{PV}\frac{\partial}{\partial M_{PV}}\alpha = -\frac{\alpha^2}{2\pi}\frac{4N - \mathcal{N}N}{1 - \frac{\alpha}{2\pi}(2N - \mathcal{N}N)} \ . \tag{8.6.199}$$

It is clear that for $\mathcal{N} = 2$ there is only a one-loop contribution to β, and for $\mathcal{N} = 4$ the β function vanishes altogether. These are well-known properties of pure extended susy gauge theories. For $\mathcal{N} = 1$ one finds agreement for one- and two- loops. Beyond two loops the result for the beta function becomes scheme dependent, so it becomes then pointless to investigate whether agreement holds.

Let us now add matter. In susy QCD with N_f flavours the matter part consists of N_f pairs of chiral superfields Q^i and $\tilde{Q}_i$ with $i = 1, N_f$ in the $\underline{N}$ and $\underline{N}^*$ representations of $SU(N)$. Each fermion in Q and $\tilde{Q}$ has one zero mode, see 8.3.109 and (8.4.144), while the scalars do not have any zero modes. So the zero mode measure for the matter part is according to (8.5.172)

$$\mathrm{d}\mathcal{M}\ (\text{matter})\ = \left(\frac{1}{\pi^2}\right)^{2N_f}\frac{1}{(M_{PV})^{2N_f}}\prod_{u=1}^{N_f}\mathrm{d}K^u\mathrm{d}\tilde{K}_u \ . \tag{8.6.200}$$

Renormalization leads to a further term in the measure, and thus in the β function. In susy only the kinetic term $\bar{\phi}\,\mathrm{e}^V\phi$ of the matter fields gets a Z factor

$$\mathcal{L} = Z\bar{\phi}_{\text{ren}}e^{V\text{ren}}\phi_{\text{ren}} \ , \qquad \phi = \sqrt{Z}\phi_{\text{ren}} \ . \tag{8.6.201}$$

and rather than a factor $\frac{1}{\sqrt{M_{PV}}}$ for each fermion with one flavor, we now get in the measure a factor $(ZM_{PV})^{-1/2}$ for each zero mode. (The Pauli-Villars field operator becomes $ZM_f + M_{PV}$, so the zero modes continue to produce a factor $M_{PV}^{-1/2}$ in the Pauli-Villars sector, **not** $(ZM_{PV})^{-1/2}$. In the nonzero mode sector one can neglect the dependence on M_{PV}, and here the Z factors of bosons and fermions cancel due to susy).

For the gauge multiplet we factorized out a factor $1/g^2$ in front of the action of all fields of the gauge multiplet, so that the fields $gA_\mu = \tilde{A}_\mu$ do not renormalize. (We use here the background formalism in which $Z_g = Z_A^{-1/2}$, where Z_A is the wave function renormalization constant for the background fields.) Thus the renormalization of the gauge multiplet is taken care of by the renormalization of the factor $1/g^2$ in 8.6.196.

From here on we proceed as before. The measure for gauge group $SU(N)$ with N_f flavors is now given by

$$dM_{k=\pm 1} = e^{-8\pi^2/g^2}(M_{PV})^{3N}\left(\frac{1}{g}\right)^{2N}\left(\frac{1}{ZM_{PV}}\right)^{N_f} \tag{8.6.202}$$

We denote the anomalous dimension by γ_i where[36]

$$\begin{aligned} \gamma_i &= \mu\frac{\partial}{\partial\mu}\ln Z_i \\ &= -M_{PV}\frac{\partial}{\partial M_{PV}}\ln Z_i = \gamma \qquad \text{(the same for } i = 1, ..., N_f) \ , \end{aligned} \tag{8.6.203}$$

and obtain

$$\frac{g}{2\pi}\beta = M_{PV}\frac{\partial}{\partial M_{PV}}\alpha = -\frac{\alpha^2}{2\pi}\left(\frac{3N - N_f(1-\gamma)}{1-\frac{\alpha N}{2\pi}}\right) \ . \tag{8.6.204}$$

Expanding in terms of α, the result agrees with the results in the literature for the one-loop and two-loop β functions for $N = 1$ susy QCD with N_f pairs of chiral fields Q^i and $\tilde{Q}_i$. Namely, the one- and two-loop β function for an $N = 1$ vector multiplet coupled to a chiral multiplet in a representation R, including the effects of the Yukawa couplings whose coupling constant is also g (in fact, the renormalized coupling constant g_R), is given by [62]

$$\begin{aligned} &\frac{g}{2\pi}\mu\frac{\partial}{\partial\mu}g = \frac{\alpha^2}{2\pi}(-3C_2(G) + T(R)) \\ &+\frac{\alpha^3}{8\pi^2}(-6C_2^2(G) + 2C_2(G)T(R) + 4C_2(R)T(R)) \end{aligned} \tag{8.6.205}$$

For N_f pairs of chiral matter fields $\sum T(R) = N_f$, and $C_2(G) = N$ for $SU(N)$. Using also that the anomalous dimension $\gamma = \mu\frac{\partial}{\partial\mu}\ln Z$ for a complex fermion in

[36]Often one defines $\gamma = \mu\frac{\partial}{\partial\mu}\ln\sqrt{Z}$; here we follow [60]

the fundamental representation $\mathbf{N}$ of $SU(N)$ is equal to $-\alpha C_2(R)/\pi$, we indeed find agreement.[37]

The β function in 8.6.204 can be rewritten such that only the numbers of zero modes appear.

$$\beta(\alpha) = -\frac{\alpha^2}{2\pi}\left(n_g - \tfrac{1}{2}n_f - \tfrac{1}{2}\sum_g \gamma_g + \tfrac{1}{2}\sum_f \gamma_f\right) . \tag{8.6.206}$$

Here n_g is the number of bosonic zero modes $(4N), n_f$ the total (gluino and matter) number of fermionic zero modes $(2N+2N_f)$, and the sums $\sum_g$ and $\sum_f$ run over the gluon and fermion zero modes. For gluons and gluinos λ, the anomalous dimension is the same (due to susy) and proportional to the β function

$$\gamma_g = \gamma_\lambda = \beta/\alpha \, . \tag{8.6.207}$$

Substitution of this result yields back (8.6.204). This result does not yet agree with the results in the literature for the β-function of gauge fields minimally coupled to scalars and fermions, because in supersymmetry the Yukawa couplings between scalars and fermions have not an independent coupling constant λ but rather $\lambda = g^2$. At the two-loop level one therefore gets extra contributions which one must add to the results from the literature, and then one gets complete agreement.

7 $\mathcal{N}=4$ supersymmetric Yang-Mills theory

An interesting field theory with instantons is the $\mathcal{N}=4$ super Yang-Mills theory [63]. The action is of course well known in Minkowski space, but instantons require the formulation in $\mathcal{N}=4$ Euclidean space. Due to absence of a real representation of Dirac matrices in four-dimensional Euclidean space, one cannot straightforwardly define Majorana spinors in Euclidean space. This complicates the construction of Euclidean Lagrangians for supersymmetric models [64–66]. For $\mathcal{N}=2,4$ theories,

[37] With the usual normalization $\gamma = \mu\frac{\partial}{\partial\mu}\ln Z^{1/2}$ is equal to $\gamma = \frac{-\alpha}{2\pi}C_2(R)$ [62].

one can replace the Majorana condition by the so-called symplectic Majorana condition and then one can define (symplectic) Majorana spinors in Euclidean space. Equivalently, one can work with complex (Dirac) spinors [67, 68]. In the following subsection we write down the action in Minkowski space-time and discuss the reality conditions on the fields. Next we construct the hermitean $\mathcal{N}=4$ Euclidean model via the dimensional reduction of ten-dimensional $\mathcal{N}=1$ super Yang-Mills theory along the time direction. One can also define a continuous Wick rotation for the spinors directly in four dimensions [66].

7.1 Minkowskian $\mathcal{N}=4$ SYM

The $\mathcal{N}=4$ action in Minkowski space-time with the signature $\eta^{\mu\nu}=\mathrm{diag}(-,+,+,+)$ is given by

$$S=\frac{1}{g^2}\int \mathrm{d}^4x\,\mathrm{tr}\left\{\frac{1}{2}F_{\mu\nu}F^{\mu\nu}-i\bar{\lambda}^{\dot\alpha}_A\,\bar{\not{D}}_{\dot\alpha\beta}\lambda^{\beta,A}-i\lambda^A_\alpha\,\not{D}^{\alpha\dot\beta}\bar{\lambda}_{A\dot\beta}+\frac{1}{2}\left(D_\mu\bar{\phi}_{AB}\right)\left(D^\mu\phi^{AB}\right)\right.$$
$$\left.-\quad\sqrt{2}\bar{\phi}_{AB}\left\{\lambda^{\alpha,A},\lambda^B_\alpha\right\}-\sqrt{2}\phi^{AB}\left\{\bar{\lambda}^{\dot\alpha}_A,\bar{\lambda}_{\dot\alpha,B}\right\}+\frac{1}{8}\left[\phi^{AB},\phi^{CD}\right]\left[\bar{\phi}_{AB},\bar{\phi}_{CD}\right]\right\} \quad (8.7.208)$$

The on-shell $\mathcal{N}=4$ supermultiplet consists of a real gauge field A_μ, four complex Weyl spinors $\lambda^{\alpha,A}$ (equivalently, four Majorana spinors) and an antisymmetric complex scalar ϕ^{AB} with labels $A,B=1,\ldots,4$ of the internal R symmetry group $SU(4)$. The reality conditions on the components of this multiplet are[38] the Majorana conditions $\left(\lambda^{\alpha,A}\right)^*=-\bar{\lambda}^{\dot\alpha}_A$ and $(\lambda^A_\alpha)^*=\bar{\lambda}_{\dot\alpha,A}$ and

$$\bar{\phi}_{AB}\equiv\left(\phi^{AB}\right)^*=\tfrac{1}{2}\epsilon_{ABCD}\phi^{CD}\ . \quad (8.7.209)$$

These conditions are invariant under $SU(4)$ transformations. The sigma matrices are defined by $\sigma^{\mu\,\alpha\dot\beta}=(1,\tau^i)$, $\bar{\sigma}^\mu_{\dot\alpha\beta}=(-1,\tau^i)$ for $\mu=0,1,2,3$ and complex conjugation gives $\left(\sigma^{\alpha\dot\beta}_\mu\right)^*=\sigma^{\beta\dot\alpha}_\mu=\bar{\sigma}^{\dot\alpha\beta}_\mu=\epsilon^{\dot\alpha\dot\gamma}\epsilon^{\beta\delta}\bar{\sigma}_{\mu\,\dot\gamma\delta}$, with $\epsilon^{\dot\alpha\dot\beta}=\epsilon_{\dot\alpha\dot\beta}=-\epsilon^{\alpha\beta}=-\epsilon_{\alpha\beta}$. Since ϕ^{AB}

[38]Unless specified otherwise, equations which involve complex conjugation of fields will be understood as not Lie algebra valued, i.e. they hold for the components $\lambda^{a,\alpha,A}$, etc.

is antisymmetric, one can express it on a basis spanned by the real eta-matrices (see Appendix B)

$$\phi^{AB} = \frac{1}{\sqrt{2}}\left\{S^i\eta^{iAB} + iP^i\bar{\eta}^{iAB}\right\}, \qquad \bar{\phi}_{AB} = \frac{1}{\sqrt{2}}\left\{S^i\eta^i_{AB} - iP^i\bar{\eta}^i_{AB}\right\} \; , \tag{8.7.210}$$

in terms of real scalars S^i and real pseudoscalars P^i, $i = 1,2,3$. Because η^{iAB} is selfdual and $\bar{\eta}^{iAB}$ anti-selfdual, $\eta^{iAB} = \eta^i{}_{AB}$ and $\bar{\eta}^{iAB} = -\bar{\eta}^i{}_{AB}$. Then the reality conditions are fulfilled and the kinetic terms for the (S,P) fields take the standard form. The action in (8.7.208) is invariant under the following supersymmetry transformation laws with parameters ζ^A_α and $\bar{\zeta}_{\dot{\alpha}A}$

$$\begin{aligned}
\delta A_\mu &= -i\bar{\zeta}^{\dot{\alpha}}_A\bar{\sigma}_{\mu\,\dot{\alpha}\beta}\lambda^{\beta,A} + i\bar{\lambda}_{\dot{\beta},A}\sigma^{\alpha\dot{\beta}}_\mu\zeta^A_\alpha \; , \\
\delta\phi^{AB} &= \sqrt{2}\Big(\zeta^{\alpha,A}\lambda^B_\alpha - \zeta^{\alpha,B}\lambda^A_\alpha + \epsilon^{ABCD}\bar{\zeta}^{\dot{\alpha}}_C\bar{\lambda}_{\dot{\alpha},D}\Big) \; , \\
\delta\lambda^{\alpha,A} &= -\tfrac{1}{2}\sigma^{\mu\nu\,\alpha}{}_\beta F_{\mu\nu}\zeta^{\beta,A} - i\sqrt{2}\bar{\zeta}_{\dot{\alpha},B}\,\not{D}^{\alpha\dot{\alpha}}\phi^{AB} + \left[\phi^{AB},\bar{\phi}_{BC}\right]\zeta^{\alpha,C} \; ,
\end{aligned} \tag{8.7.211}$$

which are consistent with the reality conditions. Let us turn now to the discussion of the Euclidean version of this model and discuss the differences with the Minkowski theory.

7.2 Euclidean $\mathcal{N}=4$ SYM

To find out the $\mathcal{N}=4$ supersymmetric YM model in Euclidean $d=(4,0)$ space, we follow the same procedure as in [63]. We start with the $\mathcal{N}=1$ SYM model in $d=(9,1)$ Minkowski space-time, but contrary to the original papers we reduce it on a six-torus with one time and five space coordinates [67, 68]. As opposed to the action in (8.7.208) with the $SU(4)=SO(6)$ R-symmetry group, this reduction leads to a model with an internal non-compact $SO(5,1)$ R-symmetry group in Euclidean space. As we will see, the reality conditions on bosons and fermions will both use an internal metric for this non-compact internal symmetry group.

The $\mathcal{N}=1$ Lagrangian in $d=(9,1)$ dimensions reads

$$\mathcal{L}_{10} = \frac{1}{g^2_{10}}\mathrm{tr}\left\{\tfrac{1}{2}F_{MN}F^{MN} + \bar{\Psi}\Gamma^M D_M\Psi\right\} \; , \tag{8.7.212}$$

with the field strength $F_{MN} = \partial_M A_N - \partial_N A_M + [A_M, A_N]$ and the Majorana-Weyl spinor Ψ defined by the conditions

$$\Gamma^{11}\Psi = \Psi \ , \qquad \Psi^T C_{10}^- = \Psi^\dagger i\Gamma^0 \equiv \bar{\Psi} \ . \tag{8.7.213}$$

Here the hermitean matrix $\Gamma^{11} \equiv *\Gamma$ is a product of all Dirac matrices, $\Gamma^{11} = \Gamma^0 \ldots \Gamma^9$, normalized to $(*\Gamma)^2 = +1$. Furthermore, C_{10}^- is the charge conjugation matrix, satisfying $C_{10}^-\Gamma_M = -\Gamma_M^T C_{10}^-$. The Γ-matrices obey the Clifford algebra $\left\{\Gamma^M, \Gamma^N\right\} = 2\eta^{MN}$ with metric $\eta^{MN} = \mathrm{diag}(-,+,\ldots,+)$. The Lagrangian transforms into a total derivative under the standard transformation rules[39]

$$\delta A_M = \bar{\zeta}\Gamma_M\Psi \ , \qquad \delta\Psi = -\tfrac{1}{2}F_{MN}\Gamma^{MN}\zeta \ , \tag{8.7.214}$$

with $\Gamma^{MN} = \frac{1}{2}[\Gamma^M\Gamma^N - \Gamma^M\Gamma^N]$. The susy parameter is a Majorana-Weyl spinor, $\bar{\zeta} = \zeta^T C_{10}^- = \zeta^\dagger i\Gamma^0$ and $\star\Gamma\zeta = \zeta$. To proceed with the dimensional reduction we choose a particular representation of the gamma matrices in $d = (9,1)$, namely

$$\Gamma^M = \left\{\hat{\gamma}^a \otimes \gamma^5, \mathbb{1}_{[8]\times[8]} \otimes \gamma^\mu\right\}, \qquad \Gamma^{11} = \Gamma^0 \ldots \Gamma^9 = \hat{\gamma}^7 \otimes \gamma^5, \tag{8.7.215}$$

where the 8×8 Dirac matrices $\hat{\gamma}^a$ and $\hat{\gamma}^7$ of $d = (5,1)$ with $a = 1,\ldots,6$ can be conveniently defined by means of 't Hooft symbols as follows

$$\hat{\gamma}^a = \begin{pmatrix} 0 & \Sigma^{a,AB} \\ \bar{\Sigma}^a_{AB} & 0 \end{pmatrix}, \qquad \hat{\gamma}^7 = \hat{\gamma}^1 \ldots \hat{\gamma}^6 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ , \tag{8.7.216}$$

In Euclidean $d = (6,0)$ one defines

$$\begin{aligned} \Sigma^{a,AB} &= (\eta^{kAB}, i\bar{\eta}^{k,AB}) \\ \bar{\Sigma}^a_{AB} &= (-\eta^k{}_{AB}, i\bar{\eta}^k{}_{AB}) \end{aligned} \tag{8.7.217}$$

[39]After partial integrations, the Yang-Mills action transforms into $[\zeta\Gamma_N\psi^a)D_M F^{a,MN}$ and the Dirac action varies into $-\bar{\psi}^a\Gamma^M(-\frac{1}{2}\Gamma^{PQ}D_M F_{PQ}\zeta)$. The sum of these two variations cancels if one uses the Bianchi identity $D_{[M}F_{PQ]} = 0$. The variation of A_M in the covariant derivative in the Dirac action cancels separately due to the 3-spinor identity $(\bar{\psi}^a\Gamma^M\psi^b)(\bar{\zeta}\Gamma_M\psi^c)f_{abc} = 0$ which holds in 3,4,6 and 10 dimensions.

but in Minkowski space one puts a factor $-i$ in front of the first one. So explicitly $\Sigma^{a,AB} = \left\{-i\eta^{1,AB}, \eta^{2,AB}, \eta^{3,AB}, i\bar{\eta}^{k,AB}\right\}$, $\bar{\Sigma}^a_{AB} = \left\{i\eta^1_{AB}, -\eta^2_{AB}, -\eta^3_{AB}, i\bar{\eta}^k_{AB}\right\}$ so that $\frac{1}{2}\epsilon_{ABCD}\Sigma^{a\,CD} = -\bar{\Sigma}^a_{AB}$. The first three matrices $\hat{\gamma}^1, \hat{\gamma}^2, \hat{\gamma}^3$ are symmetric while the latter three matrices $\hat{\gamma}^4, \hat{\gamma}^5, \hat{\gamma}^6$ are antisymmetric. Meanwhile γ^μ and γ^5 are the usual Dirac matrices of $d = (4,0)$ introduced in (8.3.87). Note that in this construction we implicitly associated one of the Dirac matrices, namely $\hat{\gamma}^1$, in 6 dimensions with the time direction and thus it is anti-hermitean and has square -1; all others (as well as all Dirac matrices in $d = (4,0)$) are again hermitean with square $+1$. Let us briefly discuss the charge conjugation matrices in $d = (9,1)$, $d = (5,1)$ and $d = (4,0)$. One can prove by means of finite group theory [69] that all their properties are representation independent. In general there are two charge conjugation matrices C^+ and C^- in even dimensions, satisfying $C^\pm \Gamma^\mu = \pm\left(\Gamma^\mu\right)^T C^\pm$, and $C^+ = C^- {*\Gamma}$. These charge conjugation matrices do not depend on the signature of space-time and obey the relation $C^- *\Gamma = \pm\left(*\Gamma\right)^T C^-$ with $-$ sign in $d = 10, 6$ and $+$ sign in $d = 4$. The transposition depends on the dimension and leads to $\left(C^\pm\right)^T = \pm C^\pm$ in $d = 10$, $\left(C^\pm\right)^T = \mp C^\pm$ in $d = 6$, and finally $\left(C^\pm\right)^T = -C^\pm$ for $d = 4$. Explicitly, the charge conjugation matrix C^-_{10} is given by $C^-_6 \otimes C^-_4$ where

$$C^-_4 = \gamma^4\gamma^2 = \begin{pmatrix} \epsilon_{\alpha\beta} & 0 \\ 0 & \epsilon^{\alpha'\beta'} \end{pmatrix}, \qquad C^-_6 = i\hat{\gamma}^4\hat{\gamma}^5\hat{\gamma}^6 = \begin{pmatrix} 0 & \delta_{AB} \\ \delta^{AB} & 0 \end{pmatrix}. \tag{8.7.218}$$

Upon compactification to Euclidean $d = (4,0)$ space, the 10-dimensional Lorentz group $SO(9,1)$ reduces to $SO(4) \times SO(5,1)$ with compact space-time group $SO(4)$ and R-symmetry group $SO(5,1)$. In these conventions a Weyl spinor Ψ in ten dimensions with 16 (complex) nonvanishing components decomposes as follows into 8 and 4 component chiral-chiral and antichiral-antichiral spinors

$$\Psi = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} \lambda^{\alpha,A} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ \bar{\lambda}_{\alpha',A} \end{pmatrix}, \tag{8.7.219}$$

or more explicitly

$$\psi^T = [(\lambda^{\alpha 1}, 0); (\lambda^{\alpha 2}, 0); (\lambda^{\alpha 3}, 0); (\lambda^{\alpha 4}, 0); (0,0); (0,0); (0,0), (0,0)]$$

$$+ \quad [(0,0);(0,0);(0,0);(0,0);(0\bar{\lambda}_{\alpha'1});(0,\bar{\lambda}_{\alpha'2});(0,\bar{\lambda}_{\alpha'3});(0,\bar{\lambda}_{\alpha'4})] \tag{8.7.220}$$

Here $\lambda^{\alpha,A}$ ($\alpha = 1,2$) transforms only under the first $SU(2)$ in $SO(4) = SU(2)\times SU(2)$, while $\bar{\lambda}_{\alpha',A}$ changes only under the second $SU(2)$. Furthermore, $\bar{\lambda}_{\alpha',A}$ transforms in the complex conjugate of the $SO(5,1)$ representation of $\lambda^{\alpha,A}$. To understand this latter statement, note that the $SO(5,1)$ generators are $\hat{M}^{ab} = \frac{1}{2}(\hat{\gamma}^a\hat{\gamma}^b - \hat{\gamma}^b\hat{\gamma}^a)$, and $\hat{\gamma}^1$ is antihermitian. Furthermore, $\hat{\gamma}^1, \hat{\gamma}^4, \hat{\gamma}^5$ and $\hat{\gamma}^6$ are purely imaginary. Thus

$$(\hat{\gamma}^a)^* = -\hat{S}\hat{\gamma}^a\hat{S}^{-1}; \qquad \hat{S} = \hat{\gamma}^2\hat{\gamma}^3 \ . \tag{8.7.221}$$

This matrix $\hat{S}$ is not the charge conjugation matrix. The Lorentz generators $\hat{M}^{ab}$ and $\hat{S}$ are block diagonal

$$\hat{M}^{ab} = \begin{pmatrix} \Sigma^{ab} & 0 \\ 0 & \bar{\Sigma}^{ab} \end{pmatrix}; \qquad \hat{S} = \begin{pmatrix} S & 0 \\ 0 & S \end{pmatrix} \ , \tag{8.7.222}$$

where $\Sigma^{ab} = \frac{1}{2}(\Sigma^a\bar{\Sigma}^b - \Sigma^b\bar{\Sigma}^a)$ and $\bar{\Sigma}^{ab} = \frac{1}{2}(\bar{\Sigma}^a\Sigma^b - \bar{\Sigma}^b\Sigma^a)$ while $S = -\eta^2\eta^3 = \eta^1$. It follows that

$$\begin{aligned} S\Sigma^a S^{-1} &= -(\Sigma^a)^* \Rightarrow S\Sigma^{ab}S^{-1} = (\Sigma^{ab})^* \ , \\ S\bar{\Sigma}^b S^{-1} &= -(\Sigma^b)^* \Rightarrow S\bar{\Sigma}^{ab}S^{-1} = (\bar{\Sigma}^{ab})^* \ . \end{aligned} \tag{8.7.223}$$

Thus the two spinor representations of $SO(5,1)$ are each pseudoreal (they are not real since S is antisymmetric), but they are not equivalent to each other. For $SO(6) \simeq SU(4)$, the two spinor representations are of course complex and inequivalent to each other. For $SO(3,1)$ the opposite is the case: there the two spinor representations are complex, and equivalent to each other under complex conjugation, $(\sigma_{\mu\nu})^* = \sigma_2\bar{\sigma}^{\mu\nu}\sigma_2$ because $\hat{S} = \gamma_2$ is off-diagonal.

Substituting these results, the Lagrangian reduces to

$$\begin{aligned} \mathcal{L}_E^{\mathcal{N}=4} = \frac{1}{g^2}\mathrm{tr}\Big\{ &\tfrac{1}{2}F_{\mu\nu}F_{\mu\nu} - i\bar{\lambda}^{\alpha'}_A\, \bar{\not{D}}_{\alpha'\beta}\lambda^{\beta,A} - i\lambda^A_\alpha\, \not{D}^{\alpha\beta'}\bar{\lambda}_{\beta',A} + \tfrac{1}{2}\left(D_\mu\bar{\phi}_{AB}\right)\left(D_\mu\phi^{AB}\right) \\ &- \sqrt{2}\bar{\phi}_{AB}\left\{\lambda^{\alpha,A},\lambda^B_\alpha\right\} - \sqrt{2}\phi^{AB}\left\{\bar{\lambda}^{\alpha'}_A,\bar{\lambda}_{\alpha',B}\right\} + \tfrac{1}{8}\left[\phi^{AB},\phi^{CD}\right]\left[\bar{\phi}_{AB},\bar{\phi}_{CD}\right]\Big\} \ , \end{aligned} \tag{8.7.224}$$

where we still use the definition $\bar{\phi}_{AB} \equiv \frac{1}{2}\epsilon_{ABCD}\phi^{CD}$. These scalars come from the ten-dimensional gauge field, and can be grouped into $\phi^{AB} = \frac{1}{\sqrt{2}}\Sigma^{a\,AB}A_a$, where A_a are the first six real components of the ten dimensional gauge field A_M. Using $\eta^{ab} = \{\frac{1}{\sqrt{2}}\Sigma^{aAB}, \frac{1}{\sqrt{2}}\bar{\Sigma}^b_{AB}\}$ with $\eta^{ab} = (-1,+1,+1,+1,+1,+1)$ the vector indices are turned into $SU(4)$ indices. Writing the action in terms of the 6 scalars A_a, one of these fields, say A_0, has a different sign in the kinetic term, which reflects the $SO(5,1)$ symmetry of the theory. In the basis with the ϕ^{AB} fields, we obtain formally the same action for the Minkowski case by reducing on a torus with 6 space coordinates, but the difference hides in the reality conditions which we will discuss in the next subsection. The action is invariant under the dimensionally reduced supersymmetry transformation rules

$$\begin{aligned}
\delta A_\mu &= -i\bar{\zeta}^{\alpha'}_A\bar{\sigma}_{\mu\,\alpha'\beta}\lambda^{\beta,A} + i\bar{\lambda}_{\beta',A}\sigma^{\alpha\beta'}_\mu\zeta^A_\alpha \ , \\
\delta\phi^{AB} &= \sqrt{2}\Big(\zeta^{\alpha,A}\lambda^B_\alpha - \zeta^{\alpha,B}\lambda^A_\alpha + \epsilon^{ABCD}\bar{\zeta}^{\alpha'}_C\bar{\lambda}_{\alpha',D}\Big) \ , \\
\delta\lambda^{\alpha,A} &= -\tfrac{1}{2}\sigma^{\mu\nu\,\alpha}{}_\beta F_{\mu\nu}\zeta^{\beta,A} - i\sqrt{2}\bar{\zeta}_{\alpha',B}\,\not{D}^{\alpha\alpha'}\phi^{AB} + \Big[\phi^{AB},\bar{\phi}_{BC}\Big]\,\zeta^{\alpha,C} \ , \\
\delta\bar{\lambda}_{\alpha',A} &= -\tfrac{1}{2}\bar{\sigma}^{\mu\nu}{}_{\alpha'}{}^{\beta'}F_{\mu\nu}\bar{\zeta}_{\beta',A} + i\sqrt{2}\zeta^{\alpha,B}\,\bar{\not{D}}_{\alpha'\alpha}\bar{\phi}_{AB} + \Big[\bar{\phi}_{AB},\phi^{BC}\Big]\,\bar{\zeta}_{\alpha',C} \ .
\end{aligned} \tag{8.7.225}$$

Again, these rules are formally the same as in (8.7.211). Note that the indices A,B are lowered by complex conjugation, but the spinor indices α and α' are lowered by ϵ- symbols.

7.3 Involution in Euclidean space

The Majorana-Weyl condition (8.7.213) on Ψ leads in four-dimensional Euclidean space to reality conditions on λ^α which are independent of those on $\bar{\lambda}_{\alpha'}$, namely,

$$\Big(\lambda^{\alpha,A}\Big)^* = -\lambda^{\beta,B}\epsilon_{\beta\alpha}\eta^1_{BA} \ , \qquad \Big(\bar{\lambda}_{\alpha',A}\Big)^* = -\bar{\lambda}_{\beta',B}\epsilon^{\beta'\alpha'}\eta^{1,BA} \ . \tag{8.7.226}$$

These reality conditions are consistent and define a symplectic Majorana spinor in Euclidean space. The $SU(2)\times SU(2)$ covariance of (8.7.226) is obvious from the pseudoreality of the **2** of $SU(2)$, but covariance under $SO(5,1)$ can also be checked (use $[\eta^a, \bar{\eta}^b] = 0$). Since the first Σ matrix has an extra factor i in order that $(\Gamma^0)^2 = -1$, see (8.7.215), the reality condition on ϕ^{AB} involves η^1_{AB}

$$\left(\phi^{AB}\right)^* = \eta^1_{AC}\phi^{CD}\eta^1_{DB} \; . \tag{8.7.227}$$

The Euclidean action in (8.7.224) is hermitean under the reality conditions in (8.7.226) and (8.7.227). For the σ-matrices, we have under complex conjugation

$$\left(\sigma_\mu^{\alpha\beta'}\right)^* = \sigma_{\mu\,\alpha\beta'} \; , \qquad \left(\bar{\sigma}_{\mu\,\alpha'\beta}\right)^* = \bar{\sigma}_\mu^{\alpha'\beta} \; . \tag{8.7.228}$$

Due to the nature of the Lorentz group the involution cannot change one type of indices into another, as opposed to the Minkowskian case.

8 Large instantons and the Higgs effect

We have seen in previous sections that the instanton measure on the moduli space for pure $SU(2)$ gauge theory with one anti-instanton ($k = -1$) is given by (dropping overall multiplicative factors of two and π)

$$\mathrm{d}\mathcal{M} \propto \mathrm{d}^4x_0 \frac{\mathrm{d}\rho\rho^3}{g^8} M_{PV}^8 \mathrm{e}^{-\frac{8\pi^2}{g^2}} = \left(\frac{\mathrm{d}^4x_0\mathrm{d}\rho}{\rho^5 g^8}\right) \mathrm{e}^{-\frac{8\pi^2}{g^2}+4\ln(\rho M_{PV})^2} \; . \tag{8.8.229}$$

The one-loop corrections coming from the determinants further modify the factor 4 into $4 - \frac{1}{3} = \frac{11}{3}$, see (8.6.192), and in addition yield some constants in the exponent. The integral over ρ, the instanton size, is clearly nonsingular for small ρ as long as asymptotic freedom holds[40] , but for large ρ it diverges severely. However, in a Higgs model, the mass term for the gauge bosons ($\mathcal{L} = -\frac{1}{2}A_\mu^2 g^2 v^2$ if there are no instantons) yields further terms of the form

$$-\frac{1}{\hbar}\mathcal{L}_{\mathrm{cl}}\;(\mathrm{Higgs}) \;= -\frac{1}{\hbar} 2\pi^2 v^2 \rho^2 + \ldots \; . \tag{8.8.230}$$

[40]One integrates ρ up to the renormalization scale μ, and instantons with scale ρ yield the prefactor $\exp(-8\pi^2/g^2)$. The g^2 in this prefactor depends on ρ, not on μ. One finds then that $g^2(\rho) = 8\pi^2/(-\beta_1 \ln(\rho\Lambda))$ where $\beta_1 = -\frac{11}{3}C_2(G) + \cdots$ is negative if asymptotic freedom holds. So, if $-\beta_1 + 3 \geq 0$, there is no singularity at $\rho = 0$.

Thus for spontaneously broken gauge theories the ρ integral acquires a Gaussian cut-off, and yields a finite result. This solves the large-ρ problem for the electroweak interactions. For QCD the situation is more complicated; in fact, the large-ρ problem is presumably intimately related to confinement. We now give some details.

The Higgs action for an $SU(2)$ Higgs doublet is given by

$$\mathcal{L}_H = D_\mu\varphi^* D_\mu\varphi + \lambda(\varphi^*\varphi - v^2)^2$$
$$D_\mu\varphi = \partial_\mu\varphi + A_\mu\varphi; \qquad \varphi = \begin{pmatrix} \varphi^+ \\ \varphi^0 \end{pmatrix}; \qquad A_\mu = A_\mu^a \frac{\tau_a}{2i} \ . \tag{8.8.231}$$

With $<\varphi^0>= v$, the ordinary Higgs effect in Minkowski space gives a mass term $\mathcal{L} = -\frac{1}{4}A_\mu^2 v^2$ for the vector bosons[41]. We could also discuss other representations for the Higgs field but the analysis is very similar, and a doublet is of course the most interesting case. In Euclidean space we take for A_μ the regular selfdual instanton solution with $k = 1$

$$A_\mu = -\frac{\bar{\sigma}_{\mu\nu}x^\nu}{x^2+\rho^2} \ . \tag{8.8.232}$$

We next solve the φ field equation in this instanton background. For general λ, an exact solution to the coupled equations seems out of reach. We therefore drop the potential term and only require that $|\varphi| \to |v|$ at large $|x|$ [42]. So this is not an exact solution, but the first term in an approximate solution. We shall discuss the higher-order terms later. As we shall show, the solution to the equation $D^\mu D_\mu\varphi = 0, |\varphi(|x| \to \infty)| = v$ is of the form $\varphi = f(r^2)\begin{pmatrix} -ix_4 + x_3 \\ x^1 + ix^2 \end{pmatrix}$. This clearly looks awkward, and a more covariant way to construct the solution is to write φ as

$$\varphi = \begin{pmatrix} \varphi^+ & -(\varphi^0)^* \\ \varphi^0 & (\varphi^+)^* \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \ , \tag{8.8.233}$$

[41] One usually decomposes φ^0 into $\varphi^0 = \frac{1}{\sqrt{2}}(\sigma - i\chi_3)$, see below, with $<\sigma>= v_\sigma$. Then $v^2 = \frac{1}{2}v_\sigma^2$, and the mass of the vector boson is $m_A = \frac{1}{2}gv$.

[42] This is 't Hooft's approach [2]. Note that the field equation for A_μ is not restricted due to the backreaction of the Higgs field. Affleck [70] considered instead the case $v = 0$, λ arbitrary, in which case the usual instanton solution together with $\varphi = 0$ solves the coupled equations. Both approached yield equivalent results.

and to make the ansatz

$$\varphi = vf(x^2)\left(\bar\sigma_\mu x_\mu/\sqrt{x^2}\right)\begin{pmatrix}1\\0\end{pmatrix}\ , \tag{8.8.234}$$

with $f(x^2)\to 1$ as $x^2\to\infty$. (Recall that one can always write $\binom{\varphi^+}{\varphi^0}$ as $\frac{1}{\sqrt2}(\sigma+i\vec\chi\cdot\vec\sigma)\binom{1}{0}$ and this yields the form of φ given in 8.8.233 up to an inessential factor i).

The function $f(x^2)$ satisfies a second-order differential equation, but we do not analyze this equation, but present the result and check that it solves $D^\mu D_\mu\varphi = 0$:

$$\varphi = v\sqrt{\frac{x^2}{x^2+\rho^2}}\frac{\bar\sigma_\mu x_\mu}{\sqrt{x^2}}\begin{pmatrix}1\\0\end{pmatrix} = \frac{v}{\sqrt{x^2+\rho^2}}\bar\sigma_\mu x_\mu\begin{pmatrix}1\\0\end{pmatrix}\ . \tag{8.8.235}$$

The boundary condition is clearly satisfied because $\frac{\bar\sigma_\mu x_\mu}{\sqrt{x^2}}\binom{1}{0}$ has unit norm. It is straightforward to check that this expression for φ satisfies the field equation. Namely, omitting the overall factor v and the spinor $\binom{1}{0}$ one finds

$$\begin{aligned}
D_\mu\varphi &= \partial_\mu\varphi + A_\mu\varphi = \frac{\bar\sigma_\mu(x^2+\rho^2) - x_\mu x_\nu\bar\sigma_\nu}{(x+\rho^2)^{3/2}} - \frac{(\bar\sigma_{\mu\nu}x_\nu)(\bar\sigma_\rho x^\rho)}{(x^2+\rho^2)^{3/2}}\\
&= \bar\sigma_\mu\rho^2/(x^2+\rho^2)^{3/2}\\
D_\mu D_\mu\varphi &= \partial_\mu D_\mu\varphi + A_\mu D_\mu\varphi = \frac{-3\bar\sigma_\mu x_\mu\rho^2 - (\bar\sigma_{\mu\nu}x_\nu)\bar\sigma_\mu\rho^2}{(x^2+\rho^2)^{5/2}} = 0\ .
\end{aligned} \tag{8.8.236}$$

Having found the solution of the field equation of the Higgs scalar in the background of an instanton, we now substitute it into the action to find the corrections to the classical action. The kinetic term only yields a surface integral due to partial integration

$$\begin{aligned}
&\int D_\mu\varphi^\dagger D_\mu\varphi\ \mathrm{d}^4x = \int \mathrm{d}\Omega_\mu(\varphi^\dagger D_\mu\varphi)\\
&= \lim_{x^2\to\infty} 2\pi^2(x^2)^{3/2}v^2\begin{pmatrix}1\\0\end{pmatrix}^T\frac{\sigma_\nu x_\nu}{\sqrt{x^2+\rho^2}}\frac{1}{\sqrt{x^2}}\frac{x_\tau\bar\sigma_\tau\rho^2}{(x^2+\rho^2)^{3/2}}\begin{pmatrix}1\\0\end{pmatrix} = 2\pi^2v^2\rho^2\ .
\end{aligned} \tag{8.8.237}$$

This is the extra term mentioned in 8.8.230.

However, the contribution of the term with the potential is divergent

$$\lambda\int(\varphi^*\varphi - v^2)^2\mathrm{d}^4x = \lambda\int\left(\frac{v^2\rho^2}{x^2+\rho^2}\right)^2\mathrm{d}^4x = \infty\ . \tag{8.8.238}$$

The reason for this divergence is clear: we did not solve the full field equation, but rather took the instanton solution of pure Yang-Mills theory, and solved the field equation for the scalar in this background, omitting the potential term.

We enter here the difficult area of "constrained instantons" [70, 71]. There does not exist an exact and stable solution of the coupled field equations, as can be shown as follows. Suppose there was a solution with $\varphi \neq 0$, and a finite but nonvanishing action for the scalars. If one replaces $A_\mu(x)$ by $aA_\mu(ax)$ and $\varphi(x)$ by $\varphi(ax)$ (which preserves the boundary condition $|\varphi| \to v$) then the action becomes upon also setting $ax = y$

$$\begin{aligned} S_{\rm cl}(a) = \int {\rm d}^4 y \Big[& -\frac{1}{2g^2} {\rm tr} F^2_{\mu\nu}(y) + \frac{1}{a^2} \mid D_\mu \varphi(y) \mid^2 \\ & + \frac{1}{a^4} \lambda (\varphi^*(y)\varphi(y) - v^2)^2 \Big] \ . \end{aligned} \tag{8.8.239}$$

Note that all three terms in the action are positive. Replacing $A_\mu(x)$ by $aA_\mu(ax)$ for a near unity amounts to a particular small variation of A_μ, and similarly for φ. So one can make the value of the action slightly smaller by making a slightly larger then unity. This proves that no solution exists. In fact, if a tends to infinity, we approach the bound $S = 8\pi^2/g^2$, but this bound can never be reached. The expression for $aA_\mu(ax)$ is equal to the instanton solution with ρ replaced by ρ/a, and for $a \to \infty$ we get a zero-size instanton. That leaves open the possibility that a local minimum might stil exist, but detailed analysis shows that this is not the case. This scaling argument is called Derrick's theorem [72], and often yields valuable information without having to perform integrals.

One can still use an approximate solution to find a large part of the contributions to the path integral, and this approximate solution is obtained by first inserting a constraint into the path integral which yields an exact solution, and then to integrate over this constraint. The idea is as follows. There are one or at most a finite number of directions in field space along which the action decreases ("destabilizing directions", in our $SU(2)$ model the directions parametrized by a). Deformations in all other directions increase the action. The constraint prevents deformations in the destabilizing directions, and on first minimizes the action with the constraint present. The solution is called the constrained instanton. It looks like the instanton for pure Yang-Mills theory at short distances but decays exponentially at large distances. It has a particular value of ρ. Finally, one integrates with the measure for the zero modes over all values of ρ. The expectation is that this should capture most of the path integral, even though one is not expanding around a solution of the theory without constraint. For the $SU(2)$ instanton one may add a term $\sigma_1 \int d^4x[\mathrm{tr}F^3 - c_1\rho^{-2}]$ to the action to constrain deformations in the direction of the gauge zero mode $(\partial/\partial\rho)A_\mu^{\mathrm{cl}}$, and a term $\sigma_2[\int d^4x(\varphi^*\varphi - v^2)^3 - c_2\rho^{-2}]$ to freeze deformations in the directions of the matter zero mode $(\partial/\partial\rho)\varphi^{\mathrm{cl}}$, with φ^{cl} given by 8.8.235. One might fix the values of c_1 and c_2 such that the constraint is satisfied for the instanton solution and φ in 8.8.235. The Lagrange multipliers σ_1 and σ_2 are then fixed order by order in perturbation theory, by requiring suitable boundary conditions for the deformations.

The result is that one can make an expansion of the full approximate solution in terms of ρv and finds then the following results in the singular gauge [30, 70, 71]:
(i) inside a core of radius $\rho = \frac{1}{m_W}$ where $m_W = gv$, the approximate solution given in 8.8.235 is still valid
(ii) far away the solution decays exponentially, $A_\mu \sim \exp(-m_W|x|)$ and $|\varphi - v| \sim \exp(-m_H|x|)$ with $m_H = 2\sqrt{\lambda}v$.
(iii) the integral over $|D_\mu\varphi|^2$ has the same leading term $2\pi^2\rho^2v^2 + \mathcal{O}\Big(\lambda(v\rho)^4\ln(v\rho\sqrt{\lambda})\Big)$, but the potential term is now convergent and yields a result $\mathcal{O}\Big(\lambda(v\rho)^4\ln(v\rho\sqrt{\lambda})\Big)$.

Hence, the Higgs effect indeed solves the large ρ problem, and asymptotic freedom solves the small ρ problem. Constrained instantons are also relevant for $\mathcal{N} = 1, 2$ SYM theories. They can also be studied in the context of topological YM theories, as was discussed e.g. in [73].

9 Instantons as most probable tunnelling paths

Instantons of nonabelian gauge theories can be interpreted as amplitudes for tunnelling between vacua in Minkowski space with different winding numbers Q. We shall determine a path in Minkowski spacetime which yields the "most probable barrier tunnelling amplitude". We follow closely [74], but related work is found in [75,76].

We begin with one particular path $A_{I,\mu} = \{\vec{A}_I(\vec{x},t), A_{I,0}(\vec{x},t)\}$ from which we construct a class of paths which all differ by how fast one goes from one configuration at t_1 to the next at t_2. Namely, we make a coordinate transformation from t to $\lambda(t)$ **in Minkowski spacetime** and consider the following collection of paths

$$\vec{A}_I^{(\lambda)}(\vec{x},t) = \vec{A}_I(\vec{x},\lambda(t)) \; ; \qquad A_{I,0}^{(\lambda)}(\vec{x},t) = A_{I,0}(\vec{x},\lambda(t))\dot{\lambda}(t) \tag{8.9.240}$$

(Often one works in the temporal gauge $A_0^{(\lambda)} = 0$ because this makes the physical interpretation clearer. All our results are, however, gauge invariant). The case $\lambda(t) = t$ yields the original path, but different $\lambda(t)$ yield paths which all run through the same sequence of 3-geometries $\vec{A}_I(\vec{x},t_1), \vec{A}_I(x,t_2), \vec{A}_I(\vec{x},t_3)\ldots$ but at different speeds. The variable $\lambda(t)$ can be considered as a kind of collective coordinate which measures a kind of continuous winding number because we will start with one winding number and end up with another winding number. For t between t_1 and t_2 this continuous winding number is due to an integral $\int d^3x \int_{t_1}^t dt' \partial_\mu j^\mu$ over a surface where A_μ is not everywhere pure gauge. Only for $t = t_1$ and $t = t_2$ does A_μ everywhere on the surface become pure gauge and only at these times the winding number is an integer. These initial and final configurations describe vacua of the theory in Minkowski spacetime. We can also consider another particular path $A_{II,\mu} = \{\vec{A}_{II}(\vec{x},t), A_{II,0}(\vec{x},t)\}$, and then we can in the same way create a second class of paths, parametrized again by the function $\lambda(t)$. In this way we generate an infinite collection of classes of paths.

For a given class $A_\mu^{(\lambda)}(\vec{x},t)$, we can substitute $\vec{A}^{(\lambda)}$ and $A_0^{(\lambda)}$ into the action, and then we obtain, as we shall show, the Lagrangian for a point particle (one dynamical degree of freedom)

$$L = \frac{1}{2}m(\lambda)\dot{\lambda}^2 - V(\lambda) \tag{8.9.241}$$

where $m(\lambda)$ and $V(\lambda)$ depend on the choice for A_μ. We shall then determine for which $m(\lambda)$ and $V(\lambda)$ the tunnelling rate is maximal. The solution of this problem in Minkowski space involves instantons in Euclidean space. A crucial role is played by the notion of a winding number in Minkowski space, so we first discuss this subject.

One can define a winding number Q in Minkowski space in the same way as in Euclidean space because Q does not depend on the metric (in technical terms it is an affine quantity)

$$\begin{aligned} Q &= \frac{-1}{64\pi^2}\int_{\sigma_1}^{\sigma_2} F^a_{\mu\nu}F^a_{\rho\sigma}\epsilon^{\mu\nu\rho\sigma}\,\mathrm{d}^4x \\ &= \frac{1}{32\pi^2}\int(\mathrm{tr}\,F_{\mu\nu}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma})\,\mathrm{d}^4x \\ &= \frac{-1}{4\pi^2}\int \mathrm{tr}\,\vec{E}\cdot\vec{B}\,\mathrm{d}^4x \end{aligned} \tag{8.9.242}$$

where $\mathrm{d}^4x = \mathrm{d}^3x\mathrm{d}t$ and $\epsilon^{0123} = +1$, and we used that $A_\mu = A^a_\mu T_a$ with $T_a = -\frac{i}{2}\sigma_a$ so that $\mathrm{tr}(T_aT_b) = -\frac{1}{2}\delta_{ab}$ and the structure constants are given by $[T_a, T_b] = \epsilon_{ab}{}^cT_c$, so $f_{ab}{}^c = \epsilon_{ab}{}^c$. Furthermore, by definition $E_j = -F_{0j}$ and $B_j = \frac{1}{2}\epsilon_{jkl}F_{kl}$. Because we are (and stay all the time) in Minkowski space, $\epsilon^{0123} = -\epsilon_{0123} = +1$ and $-F_{\mu\nu}F^{\mu\nu} = 2F^2_{0i} - F^2_{ij}$. The integral is taken between two 3-dimensional hypersurfaces σ_1 and σ_2 at t_1 and t_2.

If at t_1 the configuration $A_\mu(\vec{x}, t)$ describes a vacuum, it has by definition vanishing energy. Since the energy density[43] is given by $\mathcal{H} = \frac{1}{2}(\vec{E}^a)^2 + \frac{1}{2}(\vec{B}^a)^2$, vanishing

[43] The gravitational stress tensor is $T_{\mu\nu} = F^a_{\mu\rho}F^{a\rho}_\nu - \frac{1}{4}\eta_{\mu\nu}F^a_{\rho\sigma}F^{a,\rho\sigma}$ and $T_{00} = \frac{1}{2}(E^a)^2 + \frac{1}{2}(B^a)^2$. One can also obtain $T_{\mu\nu}$ from canonical methods as follows. Evaluating $H = p\dot{q} - L$ with $q = A_j$ and $p = -E_j$ one finds upon using that $\dot{A}_j = F_{0j} + D_jA_0$ and partially integrating that $H = \int\left[\frac{1}{2}\left\{(E^a_j)^2 + (B^a_j)^2\right\} + A^a_0(D^jE^a_j)\right]\mathrm{d}^3x$ plus a boundary term. For solutions of the field equations such as the vacuum, $D^jE_j = 0$. For configurations with finite energy ($E = \mathcal{O}\frac{1}{r^2}$) the boundary term vanishes when A_0 falls off like $\mathcal{O}(\frac{1}{r})$. Moreover in the temporal gauge the last term vanishes.

energy means $F^a_{\mu\nu} = 0$, hence A_μ is pure gauge at $t = t_1$

$$A_\mu(\vec{x}, t_1) = \mathrm{e}^{-\alpha(\vec{x},t_1)}\partial_\mu \mathrm{e}^{\alpha(\vec{x},t_1)} \; . \tag{8.9.243}$$

Similarly, at t_2 we have $A_\mu(\vec{x}, t_2) = \mathrm{e}^{-\beta(\vec{x},t_2)}\partial_\mu \mathrm{e}^{\beta(\vec{x},t_2)}$. We now choose the temporal gauge

$$A_0(\vec{x}, t) = 0 \; . \tag{8.9.244}$$

Having fixed $A_0 = 0$, there are still residual space-dependent gauge transformations possible because they preserve the gauge $A_0 = 0$. To check this statement is easy:

$$A'_0(\vec{x}, t) = \mathrm{e}^{-g(\vec{x})}\partial_0 \mathrm{e}^{g(\vec{x})} = 0 \; . \tag{8.9.245}$$

We use these residual gauge transformations to set $\alpha(\vec{x}, t_1) = 0$.[44] Then $A_\mu(\vec{x}, t_1) = 0$ for all μ and all $\vec{r}$.

Note that even if there is winding in the vacuum at $t = t_1$ (such winding at one fixed time is discussed below (8.9.252)), one can still gauge it away by a time-independent gauge transformation, but then the winding at $t = t_2$ increases by just the same amount. This is as it should be, because the total winding is gauge-invariant.

We shall consider paths from σ_1 to σ_2 which at every time t have finite energy (finite integral $\int(E^2 + B^2)\mathrm{d}^3x$). This means that the energy density for fixed t must tend to zero for $|\vec{x}| \to\infty$ (to make the integral $\int(E^2 + B^2)\mathrm{d}^3x$ convergent), hence at large $|\vec{x}|$ the gauge fields become pure gauge

$$A_\mu(\vec{x}, t) \xrightarrow[|\vec{x}|\to\infty]{} \mathrm{e}^{-\alpha(\vec{x},t)}\partial_\mu \mathrm{e}^{\alpha(\vec{x},t)} \; . \tag{8.9.246}$$

But since $A_0(\vec{x}, t) = 0$, we see that $\alpha(\vec{x}, t)$ is independent of t. Because $\alpha(\vec{x}, t_1) = 0$ we obtain $\alpha(\vec{x}, t) = 0$ for all t and $|\vec{x}| \to\infty$. This means in particular that at t_2 for

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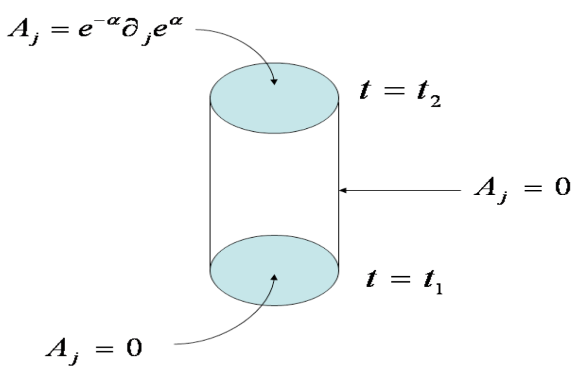
Actually, according to the Dirac formalism, the Gausz operator $D^j E_j$ is a first-class constraint, and should be omitted from the Hamiltonian. Thus, $H = \int \left[\frac{1}{2}(E^a_j)^2 + \frac{1}{2}(B^a_j)^2\right] \mathrm{d}^3x$ also according to canonical methods.

[44]With $A_j(\vec{x}, t_1) = \mathrm{e}^{-\alpha(\vec{r},t_1)}\partial_j \mathrm{e}^{\alpha(\vec{x},t_1)}$ we get $A'_j = \mathrm{e}^{-g(\vec{x})}\mathrm{e}^{-\alpha(\vec{x},t_1)}\partial_j(\mathrm{e}^{\alpha(\vec{x},t_1)}\mathrm{e}^{g(\vec{x})})$ and clearly $A'_j = 0$ if we take $\mathrm{e}^{g(\vec{x})}$ to be the inverse of $\mathrm{e}^{\alpha(\vec{x},t_1)}$.

large $|\vec{x}|$ the gauge fields tend to zero

$$A_j(\vec{x}, t_2) \xrightarrow[|\vec{x}|\to\infty]{} 0 \tag{8.9.247}$$

The fact that for large $|\vec{x}|$ all A_j vanish allows us to compactify the 3-dimensional spacelike hypersurfaces at fixed t into spheres S_3. The north pole of each sphere corresponds to all points with $|\vec{x}| = \infty$, and at this point on S_3 all A_j vanish. Thus, **all 3-spaces at fixed t compactify to a sphere** S_3. We summarize the results in a figure



(8.9.248)

Everywhere on the boundary of this cylinder the gauge fields vanish, except at the disk at $t = t_2$, but there $A_0 = 0$ and A_j are only pure gauge.

We now return to Q. First of all, Q can be written as a total derivative, using the same algebra as in Euclidean space

$$Q = \frac{1}{8\pi^2}\epsilon^{\mu\nu\rho\sigma}\int \partial_\mu \mathrm{tr}\,[A_\nu\partial_\rho A_\sigma + \frac{2}{3}A_\nu A_\rho A_\sigma]\mathrm{d}^4x\ . \tag{8.9.249}$$

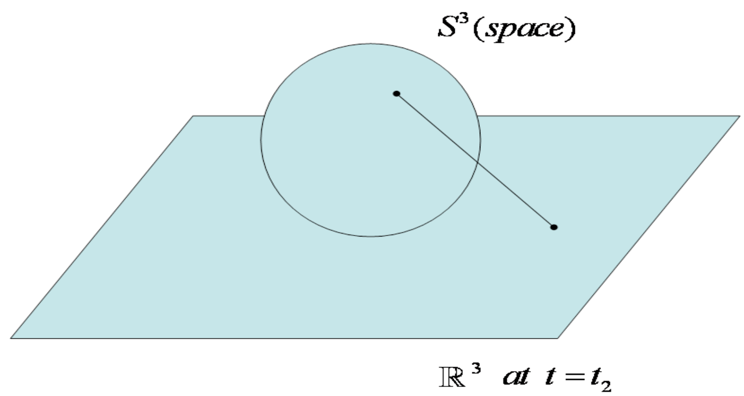
(we recall that $\mathrm{tr}\,[A_\mu A_\nu A_\rho A_\sigma \epsilon^{\mu\nu\rho\sigma}] = 0$). Furthermore, since on the boundary $F_{\mu\nu} = 0$, we can replace $\partial_\rho A_\sigma$ by $-A_\rho A_\sigma$ in (8.9.249). We then find

$$Q = \frac{-1}{24\pi^2}\epsilon^{\mu\nu\rho\sigma}\int \mathrm{d}\sigma_\mu \mathrm{tr}\,[A_\nu A_\rho A_\sigma] \qquad A_\nu = \mathrm{e}^{-\alpha}\partial_\nu \mathrm{e}^{\alpha}\ . \tag{8.9.250}$$

Since $A_0 = 0$, there is no contribution from the sides of the cylinder, and since $A_j = 0$ at the bottom, there is also no contribution from the bottom. Hence in the gauge we have chosen, all contributions to the winding come from the top of the cylinder:

$$Q = \frac{-1}{24\pi^2}\epsilon^{0ijk}\int Tr(\mathrm{e}^{-\alpha}\partial_i \mathrm{e}^{\alpha})(\mathrm{e}^{-\alpha}\partial_j \mathrm{e}^{\alpha})(\mathrm{e}^{-\alpha}\partial_k \mathrm{e}^{\alpha})\mathrm{d}^3 x \; . \qquad (8.9.251)$$

At the top of the cylinder the 3-space $t = t_2$ compactifies to a sphere S_3 (space). The map from this 3-sphere into the group $SU(2)$ is a map from one S_3 to another S_3[45] because (i) we can always compactify the $\mathbf{R}^3$ with coordinates $\vec{x}$ to an S_3 and (ii) the gauge fields at $|\vec{x}| = \infty$ are equal (and vanish)



(8.9.252)

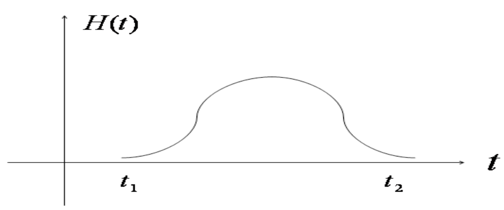
The maps S_3 (space) $\to$ S_3 (group) in Minkowski space fall into equivalence classes with a winding number $k\infty\mathbf{Z}$, just as the maps of instantons in Euclidean space give maps from S_3 (space) $\to$ S_3 (group). In the latter case S_3 (space) is the boundary of all of $\mathbf{R}^4$ while here it is the compactification of the whole $\mathbf{R}^3$ at $t = t_2$.

[45] The matrix elements of any 2×2 complex matrix can be written as $g = a_\mu \sigma^\mu$ with $\sigma^\mu = \{\vec{\sigma}, I\}$ and $\mu = 1, 2, 3, 0$. Unitarity requires that $g^\dagger = a^*_\mu \sigma^\mu$ equals g^{-1}, hence $g^\dagger g = \sum |a_\mu|^2 + (a^*_j a_k i\epsilon_{jkl} + a^*_0 a_l + a^*_l a_0)\sigma^l = 1$. Hence $|a_0|^2 + |a_k|^2 = 1$ and the coefficients of σ_l must vanish. The determinant yields $\det g = a_0^2 - a_k^2$, and since also $|a_0|^2 + |a_k|^2 = 1$, requiring $\det g = 1$ leads to $a_k = \pm i|a_k|$ and $a_0 = \pm|a_0|$. Then we are left with $g = a_0 I + i a_k \sigma_k$ with real a_0 and a_k satisfying $a_0^2 + a_k^2 = 1$ which defines S_3.

It follows that

$$Q = \pm k \qquad k \infty \mathbf{Z} \ . \tag{8.9.253}$$

We now can draw a picture of the energy $H = \int \mathcal{H} \mathrm{d}^3 x$ at times t as we move from $t = t_1$ to $t = t_2$. Initially and at the end one has $H = 0$, but in between we must have $H > 0$ (note that $\mathcal{H} \geq 0$) for the following reason.



(8.9.254)

There are no paths possible which connect the vacuum at t_1 to the vacuum at t_2 which are solutions of the field equations because if $F_{\mu\nu} = 0$ on σ_1 (or σ_2) and the field equations are satisfied, one has $F_{\mu\nu} = 0$ everywhere[46]. But, if $F_{\mu\nu}$ would vanish everywhere, $Q \sim \int E \cdot B \mathrm{d}^4 x$ would vanish, hence one could not change the winding number. The conclusion is that paths which go from one vacuum with winding number zero to another vacuum with nonvanishing winding number necessarily have positive energy at some intermediate times.

[46]For the proof, note that if at t_1 one has $F_{\mu\nu} = 0$ and at all t one has $D^\mu F_{\mu\nu} = 0$, then $\partial^\mu F_{\mu j} = \partial_0 F_{0j} = 0$ at t_1. Furthermore, the Bianchi identity $D_0 F_{ij} + D_i F_{j0} + D_j F_{0i} = 0$ yields $\partial_0 F_{ij} = 0$. Hence $\partial_0 F_{\mu\nu} = 0$ at t_1. Also $\partial_j F_{\mu\nu} = 0$ because $F_{\mu\nu} = 0$ at $t = t_1$ for all x. Hence $\partial_\rho F_{\mu\nu} = 0$ at $t = t_1$ for all ρ, μ, ν. We can rewrite this as $(D_\rho F_{\mu\nu}) = 0$ at $t = t_1$. Next we repeat this analysis by noting that also $D_\mu (D_\rho F^{\mu\nu}) = 0$ at $t = t_1$, because $D_\mu (D_\rho F^{\mu\nu}) = [D_\mu, D_\rho] F^{\mu\nu} + D_\rho (D_\mu F^{\mu\nu})$ and $D_\mu F^{\mu\nu} = 0$ everywhere. This shows that $\partial_0 (D_\rho F^{0j}) = 0$ at $t = t_1$. To also show that $\partial_0 (D_\rho F_{ij}) = 0$ at $t = t_1$ we rewrite $\partial_0 (D_\rho F_{ij}) = -\partial_0 D_i F_{j\rho} - \partial_0 D_j F_{\rho i}$ and then use $D_0 (D_i F_{\mu\nu}) = [D_0, D_i] F_{\mu\nu} + D_i (D_0 F_{\mu\nu}) = 0$. In this way we get $\partial_0^n F_{\mu\nu} = 0$ for any n. Hence $F_{\mu\nu} = 0$ at all t.

We are now ready to define a subset of paths which depend on one collective coordinate, and to which (we claim) we can restrict our attention. Consider first one given path corresponding to a fixed field configuration $A_j(\vec{x}, t)$. Instead of this single path, we consider the set of paths $A_j^{(\lambda)}(\vec{x}, t)$, as defined in (8.9.240). Each path is labeled by a different function $\lambda(t)$, and is defined by

$$A_j^{(\lambda)}(\vec{x}, t) = A_j(\vec{x}, \lambda(t)) \ . \tag{8.9.255}$$

As we already discussed, for $\lambda(t) = t$ we recover the original path, but for different $\lambda(t)$ we obtain paths which run through the same 3-dimensional configurations $\vec{A}(\vec{x}, t_1), \vec{A}(\vec{x}, t_2), \vec{A}(\vec{x}, t_3) \ldots$ at different speeds. For example if $\lambda(t)$ is constant for some time interval, the corresponding $\vec{A}(\vec{x}, t)$ do not change, but if $\lambda(t)$ changes rapidly, the sequence of $A(\vec{x}, t)$ is traversed rapidly.

Each path $A_j(\vec{x}, \lambda(t))$ should begin at $A_j(\vec{x}, t_1)$ and end at $A_j(\vec{x}, t_2)$, so we require $\lambda(t_1) = t$, and $\lambda(t_2) = t_2$, but between t_1 and t_2 the function $\lambda(t)$ is arbitrary. We shall later take $t_1 = -\infty$ and $t_2 = +\infty$, and then also require that $\lambda(t_1) = -\infty$ and $\lambda(t_2) = +\infty$. Given a path $A_j^{(\lambda)}(\vec{x}, t)$ we can compute the electric and magnetic fields

$$\begin{aligned} -E_j &= F_{0j} = \partial_0 A_j^{(\lambda)}(\vec{x}, t) = \frac{\partial A_j}{\partial \lambda}(\vec{x}, \lambda(t))\dot{\lambda} \text{ because } A_0(\vec{x}, t) = 0 \\ B_i &= \frac{1}{2}\epsilon_{ijk} F_{jk} = \frac{1}{2}\epsilon_{ijk}(\partial_j A_k(\vec{x}, \lambda(t)) + A_j(\vec{x}, \lambda(t)) A_k(\vec{x}, \lambda(t)) - j \leftrightarrow k) \ . \end{aligned} \tag{8.9.256}$$

The Lagrangian $L = \int \mathcal{L} \mathrm{d}^3 x$ with $\mathcal{L} = \frac{1}{2g^2} \mathrm{tr}\, F_{\mu\nu}^2 = \frac{-1}{g^2} \mathrm{tr}\, (\vec{E}^2 - \vec{B}^2)$ can then be written as

$$\begin{aligned} L &= \frac{1}{2} m(\lambda)\dot{\lambda}^2 - V(\lambda) \ , \\ m(\lambda) &= \frac{-2}{g^2} \int \mathrm{tr} \left(\frac{\partial \vec{A}}{\partial \lambda}\right)^2 \mathrm{d}^3 x \geq 0 \ , \\ V(\lambda) &= -\frac{1}{g^2} \int \mathrm{tr}\, \vec{B}^2 \, \mathrm{d}^3 x \geq 0 \end{aligned} \tag{8.9.257}$$

The momentum conjugate to $\lambda(t)$ is $p(\lambda) = \frac{\partial}{\partial\dot{\lambda}} L = m(\lambda)\dot{\lambda}$. Hence

$$H = \frac{(p(\lambda))^2}{2m(\lambda)} + V(\lambda) \ . \tag{8.9.258}$$

For a given path $A_j^{(\lambda)}(\vec{x},t)$ one can plot H as a function of t, and one finds then the profile in figure (8.9.254).

We have thus isolated a class of paths $A^{(\lambda)}(\vec{x},t)$ which depends on one collective coordinate $\lambda(t)$. For one given $A(\vec{x},t)$, this still yields an infinite set of paths, but all these paths run through the same set of 3-configurations $A_j(\vec{x},t_1), A_j(\vec{x},t_2), \ldots$. These are, of course, infinitely many other collective coordinates which describe a general path $A_j(x,t)$, but the idea is that $\lambda(t)$ is the relevant coordinate to describe tunnelling, while the other collective coordinates describe variations away from the paths $A_j^{(\lambda)}(\vec{x},t)$ which give only small corrections to the results obtained from $\lambda(t)$. It is, of course, difficult to prove this assertion; one could begin with two collective coordinates as a start, but even this would lead to a complicated analysis.

The action for $\lambda(t)$ in (8.9.257) can be viewed as the action for one point particle. This particle feels the potential barrier $V(\lambda)$, and to go from the vacuum at $t = t_1$ with $V(\lambda) = m(\lambda) = 0$ to the vacuum at t_2 with also $V(\lambda) = m(\lambda) = 0$, we need tunnelling. The tunnelling rate R in quantum mechanics is proportional to e^{-2R} where

$$R = \int_{\lambda_1}^{\lambda_2} \mathrm{d}\lambda \sqrt{2m(\lambda)(V(\lambda)-E)} \ , \tag{8.9.259}$$

with $\lambda(t_1) \equiv \lambda_1 = t_1, \lambda(t_2) \equiv \lambda_2 = t_2$ and $V(\lambda(t_1)) = V(\lambda(t_2)) = 0$ and $m(\lambda(t_1)) = m(\lambda(t_2)) = 0$. We also set $E = 0$ because we consider tunnelling from one vacuum (with $E = 0$) to another.

There is, of course, an important difference with ordinary quantum mechanics. The point particle $\lambda(t)$ feels a potential $V(\lambda)$, but both are derived from the same object, the fields $A_j(x,\lambda(t))$. In addition the mass is here "position"-dependent,

$m = m(\lambda)$. One can show that in quantum mechanics the formula for R also holds if the mass $m(\lambda)$ depends on the point particle $\lambda(t)$). The crucial step is now to pose the question: **for which set of paths $\vec{A}(\vec{x}, \lambda(t))$ is the tunnelling rate maximal?** The tunnelling rate for the quantum mechanical particle $\lambda(t)$ can be described by Minkowski path integrals, so we ask: for which $\vec{A}(\vec{x}, t)$ is there least destructive interference of the associated paths $A(\vec{x}, \lambda(t))$ in the path integral? Clearly, $V(\lambda)$ should be as small as possible, but it cannot be too small because it must produce winding.

The tunnelling rate is e^{-2R} where according to (8.9.259)

$$R = \int_{\lambda_1}^{\lambda_2} \mathrm{d}\lambda \, 2 \left[\left(\frac{1}{g^2} \int \mathrm{tr} \left(\frac{\partial \vec{A}}{\partial \lambda} \right)^2 \mathrm{d}^3 x \right) \left(\frac{1}{g^2} \int \mathrm{tr} \, \vec{B}^2 \mathrm{d}^3 x \right) \right]^{1/2}$$
$$= \frac{2}{g^2} \int_{t_1}^{t_2} \mathrm{d}t [(\mathrm{tr} \int \vec{E}^2 \mathrm{d}^3 x)(\mathrm{tr} \int \vec{B}^2 \mathrm{d}^3 x)]^{1/2} \; . \qquad (8.9.260)$$

We replaced $d\lambda$ by $dt\dot{\lambda}$ and brought $\dot{\lambda}$ inside the square root. The fields $\vec{E}$ and $\vec{B}$ still depend on $\lambda(t)$. Since tr $\int \vec{a}(\vec{x})\vec{b}(\vec{x})d^3x$ is an inner product, while $\int \mathrm{tr}\, \vec{E} \cdot \vec{B}$ is proportional to the winding number according to (8.9.242), we have the triangle inequality

$$R \geq \frac{2}{g^2} \Big| \int_{t_1}^{t_2} (\mathrm{tr}\, \vec{E} \cdot \vec{B}) \mathrm{d}^4 x \Big| = \frac{8\pi^2}{g^2} \mid Q \mid \; . \qquad (8.9.261)$$

Hence the tunnelling amplitude is bounded from above by

$$\mathrm{e}^{-R} \leq -\mathrm{e}^{-\frac{8\pi^2}{g^2}|Q|} \; . \qquad (8.9.262)$$

The inequality is saturated when $\vec{E}$ is parallel to $\vec{B}$ at each vector $\vec{x}$ and at each time $t : \vec{E}(\vec{x}, t) = \alpha(t)\vec{B}(\vec{x}, t)$. The claim is that among all paths with the same Q, the paths with the smallest R are the paths with $\vec{E}$ parallel to $\vec{B}$.

Let us discuss the meaning of this result. Paths which interpolate between vacua with different winding number must produce electric and magnetic fields $\vec{E}$ and $\vec{B}$ in between at finite $\vec{x}$ and t which cannot be too small, namely $|\int(E_j^a B_j^a)\mathrm{d}^4x|$ should be equal to $8\pi^2|Q|$. On the other hand, the tunnelling rate is proportional to the length of E^a times the length of B^a, so to make the tunnelling rate as large as possible, the product of these lengths should be as small as possible. One could set up a variational problem for R under the constraint that $\int \mathrm{tr}\,\vec{E}\cdot\vec{B}\,\mathrm{d}^4x$ be equal to $4\pi^2 Q$, but we shall not work this out.

The bound is reached, namely the tunnelling rate is maximal, when the set of paths $A_j(\vec{x},\lambda(t))$ produces parallel electric and magnetic fields

$$\vec{E}(\vec{x},\lambda(t)) = \alpha(t)\vec{B}(\vec{x},\lambda(t)) \ . \tag{8.9.263}$$

Of course, $\alpha(t)$ can also be viewed as a function of $\lambda(t)$ because $\lambda(t)$ is just another parametrization of the time interval. Note that this condition does not change if one changes the parametrization from $\lambda(t)$ to another function $\lambda'(t)$, because under such reparametrizations $\vec{E}$ scales by a constant factor $\partial\lambda'/\partial\lambda$, which cancels the Jacobian in (8.9.259) for this change of integration variables. We use this scaling property to select a particular $\lambda_0(t)$ such that $\vec{E}(\vec{x},\lambda_0(t)) = \pm\vec{B}(\vec{x},\lambda_0(t))$. The property of $\vec{E}$ and $\vec{B}$ being parallel is also a gauge-invariant property, and $\mathcal{L}$ and R are of course gauge-invariant. So, our characterization of paths with maximal tunnelling rate is gauge-invariant, as it should be. Thus the use of temporal gauge did not restrict the generality of the results.

We now can establish the connection between tunnelling and instantons. The fields for which $\vec{E}$ and $\vec{B}$ in Minkowski space are parallel are closely connected to instantons in Euclidean space. Namely, among the class of paths $\vec{A}(\vec{x},\lambda(t))$ parametrized by $\lambda(t)$, there is the path $\vec{E}(\vec{x},\lambda_0(t)) = \vec{B}(\vec{x},\lambda_0(t))$ (and another path with another $\lambda_0'(t)$ such that $\vec{E}(\vec{x},\lambda_0(t)) = -\vec{B}(\vec{x},\lambda_0(t))$). If we then define Euclidean gauge fields $A_\mu^E(x,t)$ by $A_j^E(\vec{x},t) = A_j(\vec{x},\lambda_0(t))$ and $A_4^E(\vec{x},t) = A_0(\vec{x},\lambda(t))\frac{d\lambda}{dt}$ then this $A_\mu^E(\vec{x},t)$ is self dual. The parameter t is Minkowski time, but in the expressions for $A_j^E(\vec{x},t)$ we should interpret t as the Euclidean time.

Summarizing: the most probable tunnelling paths are given by the set of paths $A_j(\vec{x},\lambda(t))$ with parallel $\vec{E}$ and $\vec{B}$ fields. A given class of paths with $\vec{E}$ parallel to $\vec{B}$ contains one path which, when viewed as a configuration in Euclidean space, is an instanton. Conversely, given an instanton $A^E_\mu(\vec{x},t)$ in Euclidean space, one can construct a corresponding set of paths $A^M_\mu(x,\lambda(t))$ in Minkowski space by setting

$$\begin{aligned} A_j^{M,(\lambda)}(x,t) &= A_j^E(\vec{x},\lambda(t)) \\ A_0^{(M,(\lambda)}(x,t) &= A_4^E(\vec{x},\lambda(t))\dot{\lambda}\ . \end{aligned} \tag{8.9.264}$$

As an example we take the $Q=-1$ anti-instanton solution in regular gauge, $A_\mu = -\sigma_{\mu\nu}x^\nu/(x^2+\rho^2)$, see (8.2.70), which yields the following set of paths in Minkowski space

$$\left.\begin{aligned} A_0^{(\lambda)}(\vec{x},t) &= \tfrac{-i\vec{x}\cdot\vec{\sigma}}{\vec{x}^2+\lambda(t)^2+\rho^2}\dot{\lambda}(t) \\ \vec{A}^{(\lambda)}(\vec{x},t) &= \tfrac{i\lambda(t)\vec{\sigma}-i\vec{x}\times\vec{\sigma}}{\vec{x}^2+\lambda(t)^2+\rho^2} \end{aligned}\right\} \begin{aligned} &\lambda(t\to-\infty)=-\infty \\ &\lambda(t\to+\infty)=+\infty\ . \end{aligned} \tag{8.9.265}$$

We are clearly not in the temporal gauge, but since our results are gauge-invariant, it does not matter which gauge we use. We still have $A_\mu \to 0$ at large $|\vec{x}|$, so that we still have the notion of winding as a map from S_3 (space) into S_3 (group) at each time.

Straightforward calculation yields for the curvatures in Minkowski space

$$\begin{aligned} F_{01} &= \partial_0 A_1 - \partial_1 A_0 + [A_0, A_1] = \frac{2i\rho^2\sigma_1}{(\vec{x}^2+\lambda^2+\rho^2)^2}\dot{\lambda}\ , \\ F_{23} &= \partial_2 A_3 - \partial_3 A_2 + [A_2, A_3] = \frac{2i\rho^2\sigma_1}{(\vec{x}^2+\lambda^2+\rho^2)^2}\ . \end{aligned} \tag{8.9.266}$$

Hence

$$\vec{E} = \frac{-2i\rho^2\vec{\sigma}}{(\vec{x}^2+\lambda^2+\rho^2)^2}\dot{\lambda}\ ; \qquad \vec{B} = \frac{2i\rho^2\vec{\sigma}}{(\vec{x}^2+\lambda^2+\rho^2)^2}\ , \tag{8.9.267}$$

which depend on $x^2 = \vec{x}^2 + \lambda(t)^2$ (**not** on $\vec{x}^2 - t^2$). Hence, $\vec{E}$ is indeed parallel to $\vec{B}$ (in fact, anti-parallel).

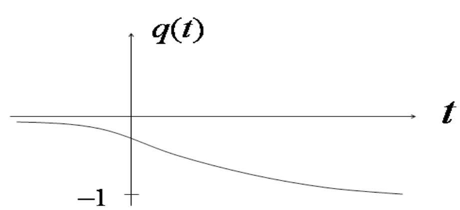
The winding number Q can be written in two ways

$$\begin{aligned} Q &= \frac{-1}{4\pi^2}\int\limits_{-\infty}^{\infty}[\mathrm{tr}\,\vec{E}\cdot\vec{B}\mathrm{d}^3x]\mathrm{d}t \\ &= \frac{-1}{24\pi^2}\epsilon^{\mu\nu\rho\sigma}\int\partial_\mu \mathrm{tr}\,[A_\nu A_\rho A_\sigma]\mathrm{d}^4x\ . \end{aligned} \tag{8.9.268}$$

In the latter expression Q receives only a contribution from the boundary,[47] but in the former expression we compute Q by integrating over all space and time. It is then natural to define a t-dependent function by integrating only up to a time t

$$\begin{aligned} q(t) &= \frac{-1}{4\pi^2}\int_{-\infty}^{t}\left[\int \mathrm{tr}\,\vec{E}\cdot\vec{B}\mathrm{d}^3x\right] \\ &= -\frac{1}{4\pi^2}\int_{-\infty}^{\lambda}\mathrm{d}\lambda\int \mathrm{d}^3x\frac{24\rho^4}{[\vec{x}^2+\lambda^2+\rho^2]^4} \\ &= -\frac{3}{4}\int_{-\infty}^{\lambda}\frac{\rho^4\mathrm{d}\lambda}{(\lambda^2+\rho^2)^{5/2}} \\ &= -\frac{3}{4}\int_{-\infty}^{\lambda/\rho}\frac{\mathrm{d}y}{(y^2+1)^{5/2}} \\ &= -\frac{3}{4}\left(t-\frac{1}{3}t^3\right)\Big|_{-1}^{x} \qquad \text{with } x=\frac{\lambda}{\sqrt{\lambda^2+\rho^2}}\ . \end{aligned} \tag{8.9.269}$$

Clearly, $q(t)$ is gauge-invariant and has the following form



(8.9.270)

[47]For example, the contribution to Q from the surface at $t = t_1$ is proportional to $\int \frac{t(\vec{x}^2+t^2)\mathrm{d}^3x}{(t^2+\vec{x}^2+\rho^2)^3}$ which is nonvanishing. On the other hand, the contribution to Q from the sides of the cylinder converges for large $|t|$.

It only receives contributions from regions where $\vec{E}$ and $\vec{B}$ are nonvanishing, hence where A^a_μ is not pure gauge.

To obtain the action for $\lambda(t)$ in this example we evaluate

$$\mathcal{L} = -\frac{1}{g^2}\mathrm{tr}\,(\vec{E}^2 - \vec{B}^2) = \frac{24}{g^2}\rho^4\left[\frac{\dot{\lambda}^2}{(x^2+\rho^2)^4} - \frac{1}{(x^2+\rho^2)^4}\right] . \tag{8.9.271}$$

Doing the space integral we obtain

$$L = \frac{1}{2}m(\lambda)\dot{\lambda}^2 - V(\lambda) = \frac{3\pi^2\rho^4}{g^2(\lambda^2+\rho^2)^{5/2}}(\dot{\lambda}^2 - 1) , \tag{8.9.272}$$

where we used

$$\int \frac{\mathrm{d}^3x}{(\vec{x}^2+\lambda^2+\rho^2)^4} = \frac{4\pi}{(\lambda^2+\rho^2)^{5/2}}\frac{1}{2}\int_{-\infty}^{\infty}\frac{y^2\mathrm{d}y}{(y^2+1)^4} = \frac{4\pi^2}{32}\frac{1}{(\lambda^2+\rho^2)^{5/2}} . \tag{8.9.273}$$

In this example, we were dealing with a gauge with $A_0 \neq 0$. We can map to a gauge in which $A_0 = 0$ by a suitable large gauge transformation

$$\begin{aligned} A'_\mu &= U^{-1}(\partial_\mu + A_\mu)U \\ U &= \exp\left[\frac{i\vec{x}\cdot\vec{\sigma}}{\sqrt{\vec{x}^2+\rho^2}}\,\mathrm{arctg}\,\frac{\lambda(t)}{\sqrt{\vec{x}^2+\rho^2}}\right] . \end{aligned} \tag{8.9.274}$$

Indeed, using the expression for A_0 in (8.9.265)

$$A_0 = \frac{-i\vec{x}\cdot\vec{\sigma}}{\vec{x}^2+\lambda(t)^2+\rho^2}\dot{\lambda}(t) , \tag{8.9.275}$$

one finds that $A'_0 = U^{-1}(\partial_t + A_0)U = U^{-1}\partial_t U + A_0$ vanishes

$$A'_0 = \frac{i\vec{x}\cdot\vec{\sigma}}{\sqrt{\vec{x}^2+\rho^2}}\frac{1}{1+\frac{\lambda(t)^2}{\vec{x}^2+\rho^2}}\frac{\dot{\lambda}(t)}{\sqrt{\vec{x}^2+\rho^2}} + A_0 = 0 , \tag{8.9.276}$$

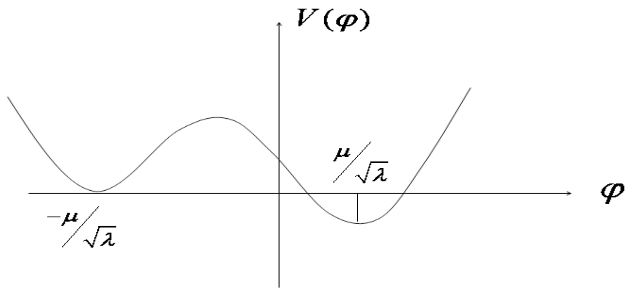
where we used that A_0 commutes with U. Of course, Q is gauge invariant because it can be written as a trace over $\vec{E}\cdot\vec{B}$ but it is instructive to see what happens if one writes Q as a surface integral and makes a gauge transformation with U. On the boundary of Minkowski space the $A_\mu = V^{-1}\partial_\mu V$ transform into $(VU)^{-1}\partial_\mu VU$ and the winding number of VU is the sum of the winding numbers of V and U. However,

U is connected to the identity element: $U \equiv \exp \alpha[\frac{i\vec{x}\cdot\vec{\sigma}}{\sqrt{\vec{x}^2+\rho^2}} \text{ arctg } \frac{\lambda(t)}{\sqrt{\vec{x}^2+\rho^2}}]$ traces an orbit as α runs from 0 to 1 which begins at the identity element and ends at U. Thus U does not produce any winding, and thus the answer for Q from the total derivative is the same, whether one uses a gauge in which A_0 vanishes or a gauge in which A_0 is nonvanishing. Note, however, that when $A_0 \neq 0$ one gets contributions from the timelike part of the boundary of the spacetime cylinder.

10 False vacua and phase transitions

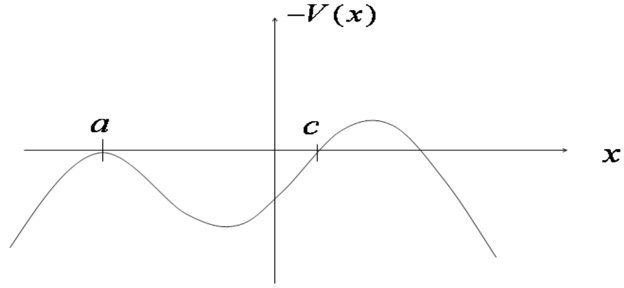
In spontaneously broken gauge theories, the potential has a local maximum and an absolute minimum. These extrema form a metastable and a stable vacuum, respectively. If a system is in the metastable vacuum at all points in spacetime, it could at some point and at some time, say $\vec{x} = 0$ and $t = 0$, make a quantum fluctuation to the stable vacuum. This transition costs energy, but if the region around x ("the bubble") is large enough, the energy needed for creation of a bubble (this energy is located in the boundaries of the bubble) is less than the energy gained by tunnelling to the lower vacuum (this energy is liberated in the volume of the bubble), and then the bubble will rapidly expand. In fact, since the rate of energy production increases the larger the bubble, the bubble will spread through space, with accelerating speed, converting the false vacuum to a true vacuum. As an application of this process one may consider the universe just after the Big Bang; at high temperature the universe is in the symmetric vacuum, but as cooling due to expansion sets in the potential develops a lower (true) vacuum, and if for some reason the universe remains stuck in the false vacuum, one can study the decay of the universe towards the true (asymmetric) vacuum. We shall consider another example: the perturbed double-well potential, with two classically stable minima, but one minimum (the true vacuum) below the other minimum (the false vacuum). We shall study the decay of the false vacuum in this model into the true vacuum [77, 78]. We follow [79].

As a preliminary to the calculation of the phase transition in field theory, we first revert to quantum mechanics and study the double-well. Let us pretend that we do not know that there are big differences between the double-well potential and the following potential.



$$(8.10.277)$$

We can then repeat the calculation of the nonperturbative corrections to the energy of the ground state. Already at this point it is clear that we should not blindly repeat all steps because previously we were dealing with two perturbatively degenerate vacua, and the kink-instantons provided the energy shift between both vacua. In the present case, the degeneracy is already broken at the classical level. Proceeding nevertheless we find a classical solution of the Euclidean equation $-\frac{\partial^2 x}{\partial t^2} + \frac{\partial V}{\partial x} = 0$ describing a point particle $x(t)$ in the inverted potential and use path integral methods.



(8.10.278)

The particle starts at $t = -\infty$ in the point $x = a$, rolls to the point $x = c$, "bounces" at time $t = X$, and ends up at $t = +\infty$ at the same point $x = a$. Clearly, X is the collective coordinate for this classical solution $x_{cl}(t)$. We then get for the "one-bounce solution"

$$\begin{aligned} T_{00} &\equiv < x = a \mid \mathrm{e}^{-\frac{1}{\hbar}H\tau_0} \mid x = a >= \mathrm{e}^{\frac{1}{\hbar}S_{cl}}\tau_0\sqrt{-S_{cl}}\, I_0 \ , \\ I_0 &= \mathcal{N}\int_{n.z.} \mathrm{d}q(\tau)\mathrm{e}^{\frac{1}{\hbar}S_E^{(2)}} \quad \text{with } q(\pm\tau_{0/2}) = 0 \ , \end{aligned} \tag{8.10.279}$$

where we used the Faddeev-Popov trick, and "n.z." indicates that the path integral is over the solutions of the field equation for the fluctuations about $x_{cl}(t)$ in the space orthogonal to the almost-zero mode. Assuming again that I_0 can be written as a factor K times the path integral of the harmonic oscillator we get

$$I_0 = K\sqrt{\frac{\omega}{\pi\hbar}}\mathrm{e}^{-\frac{1}{2}\omega\tau_0} \ ; \qquad K = \sqrt{\frac{\det(-\partial_t^2 + \omega^2)}{\det{}'(-\partial_t^2 + V''(x_{cl}))}} \tag{8.10.280}$$

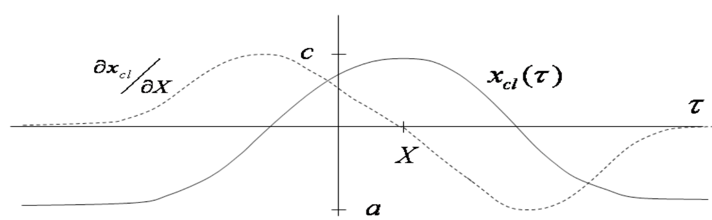
Continuing without further thought we would sum over multi-bounces and obtain

$$\begin{aligned} T_{00} &= \sqrt{\frac{\omega}{\pi\hbar}}\mathrm{e}^{-\frac{1}{2}\omega\tau_0}\sum_{n=0}^{\infty}\frac{(\sqrt{-S_{cl}}\tau_0 K\mathrm{e}^{\frac{1}{\hbar}S_{cl}})^n}{n!} \\ &= \frac{\omega}{\pi\hbar}\mathrm{e}^{-\frac{1}{2}\omega\tau_0}\exp(K\tau_0\mathrm{e}^{\frac{1}{\hbar}S_{cl}}) \ . \end{aligned} \tag{8.10.281}$$

Using the same arguments as used before for the unperturbed double-well potential, we would conclude that the ground state energy is given by

$$E_0 = \frac{1}{2}\hbar\omega - \hbar K \mathrm{e}^{\frac{1}{\hbar}S_{cl}} \ . \tag{8.10.282}$$

However, at this point we note that there are problems with this result
(i) first a small problem: the nonperturbative correction is exponentially suppressed, hence it should be neglected compared to the perturbative correction.
(ii) a more serious problem (actually a virture, as we shall see) is that K has a negative eigenvalue. This is easy to prove: $\frac{d}{dX}x_{cl}(t-X)$ is the zero mode fluctuation. It has a mode because x_{cl} bounces: unlike the kink, $x_{cl}(\tau)$ moves first forward and then backwards, yielding a kind of kink-antikink solution.



(8.10.283)

Hence there exists one mode for the fluctuations with lower eigenvalue and without a node, and since $\frac{\partial}{\partial X}x_{cl}(x-X)$ has zero eigenvalue, there exists an eigenfunction for the fluctuation with negative eigenvalue. Thus the nonperturbative correction is imaginary, reflecting the fact that the perturbative ground state near $x = 0$ is nonperturbatively unstable

$$ImE_0 = \hbar \mid K \mid \mathrm{e}^{\frac{1}{\hbar}S_{cl}} \equiv \Gamma/2 \ . \tag{8.10.284}$$

So, instantons (or rather bounces, still solutions of the classical field equations with finite action) yield in this case the width Γ of the unstable state.

Having seen that in quantum mechanics the path integral approach to nonperturbative corrections to the vacuum energy leads to the correct result that the ground state is unstable, we now return to the problem of phase transitions.

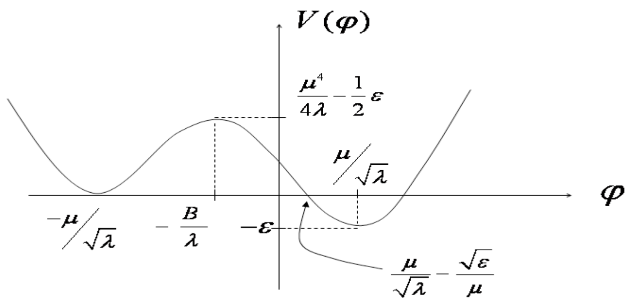
As a toy model for studying such decays we need a system with at least one space coordinate because bubbles have a finite extension in space. The simplest choice is a $1+1$ dimensional field theory. We choose the double-well potential with an extra term to destroy the degenacy between both minima. Since the double-well potential is symmetric under $\varphi \to -\varphi$, the extra term should be antisymmetric, and if it is to be a small perturbation compared to the leading $\lambda\varphi^4$ term, we need either a term linear in φ or cubic in φ, or both. It simplifies the mathematics if we keep the local minima of the perturbed potential at the same place as the minima of the unperturbated potential, namely at $\varphi = \pm\mu/\sqrt{\lambda}$. We are then led to the following model

$$\mathcal{L} = \frac{1}{2}\dot{\varphi}^2 - \frac{1}{2}(\varphi')^2 - \frac{\lambda}{4}\left(\varphi^2 - \frac{\mu^2}{\lambda}\right)^2 - B\left(\frac{1}{3}\varphi^3 - \frac{\mu^2}{\lambda}\varphi\right) + \frac{2}{3}B\left(\frac{\mu}{\sqrt{\lambda}}\right)^3 , \tag{8.10.285}$$

where we take B small and positive. For constant φ, the solutions of the classical field equations occur at

$$\frac{\partial V}{\partial\varphi} = \lambda\varphi\left(\varphi^2 - \frac{\mu^2}{\lambda}\right) + B\left(\varphi^2 - \frac{\mu^2}{\lambda}\right) = 0 , \tag{8.10.286}$$

and from this result it is clear that the values $\varphi = \pm\mu/\sqrt{\lambda}$ are indeed extrema. The potential has the following form



(8.10.287)

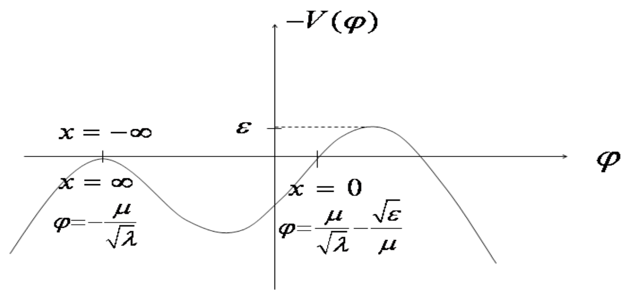
It vanishes at $\varphi = -\mu/\sqrt{\lambda}$ because we added the constant $\frac{2}{3}B(\mu/\sqrt{\lambda})^3$, but at $\varphi = \mu/\sqrt{\lambda}$ it is negative. Thus $\varphi = -\mu/\sqrt{\lambda}$ is the unstable vacuum and $\varphi = \mu/\sqrt{\lambda}$ is the stable vacuum. The value of the potential at the stable minimum is

$$V\left(\varphi = \mu/\sqrt{\lambda}\right) = -\epsilon = -\frac{4}{3}B\left(\mu/\sqrt{\lambda}\right)^3 . \tag{8.10.288}$$

There is a relative maximum a bit below the maximum of the symmetric potential $V(B=0,\varphi)$ at $\varphi = 0$; for small B it occurs at $\varphi \simeq -B/\lambda$ and its value is $\frac{1}{4}\mu^4/\lambda - \frac{1}{2}\epsilon + \mathcal{O}(B^2)$. These results are intuitively clear: if one pulls φ down at $\mu/\sqrt{\lambda}$ by an amount ϵ, then the maximum at $\varphi = 0$ is pulled down half as much, and moves of course a bit to the left.

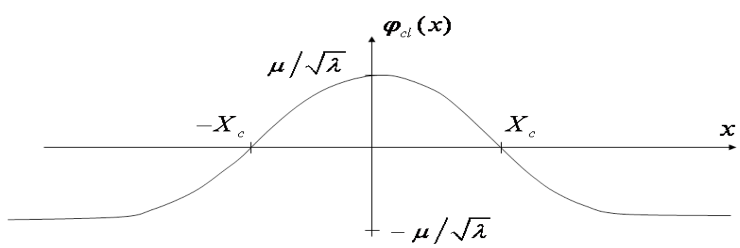
In addition to the three solutions of the classical field equations with constant $\varphi(\varphi = -\mu/\sqrt{\lambda}, \varphi = \mu/\sqrt{\lambda}$, and $\varphi \sim -B/\lambda)$, there is an exact kink-antikink solution.

This is clear by inspection of the inverted potential



(8.10.289)

A ball at rest at $\varphi = -\mu/\sqrt{\lambda}$ at $x = -\infty$ starts rolling down to the hill and up the other hill; it reaches the point where $V(\varphi) = 0$ at $x = 0$ and then returns and comes to rest at $\varphi = -\mu/\sqrt{\lambda}$ at $x = +\infty$. The classical solution $\varphi_{cl}(x)$ is thus a soliton of the following form



(8.10.290)

We approximate $\varphi_{cl}(x)$ by the following expression

$$\varphi_{cl}(x) = \frac{\mu}{\sqrt{\lambda}} \left[\tanh\left(\frac{m}{2}(x + X_c)\right) - \tanh\left(\frac{m}{2}(x - X_c)\right) - 1 \right] . \qquad (8.10.291)$$

This is a static soliton in $1+1$ dimensions, which can also be viewed as an instanton in x-space. (In the quantum mechanical models we considered previously, we dealth with instantons in Euclidean time). Near $x = -X_c$ the antikink is exponentially suppressed and the mass of the kink is M. Between the kink and antikink φ is equal to $\mu/\sqrt{\lambda}$ (up to exponentially suppressed corrections), and near $x = X_c$ we have an antikink with mass M. For large x we find the correct asymptotic value $\varphi_{cl}(x \to \pm\infty) = -\mu/\sqrt{\lambda}$. We fix the value of X_c such that the total energy of $\varphi_{cl}(x)$ (which is the energy of the ball rolling up and down the hills in (8.10.289)) vanishes

$$E = 2M - 2\epsilon X_c = 0 \ , \tag{8.10.292}$$

where $M = \frac{m^3}{3\lambda}$ is the classical mass of a single kink. Hence, the separation between the kink and antikink is given by $2X$ with $X = M/\epsilon$.

The exact solution begins at $V = 0$, climbs the hill, and comes down on the other side where it reaches the value $V = 0$, and then it returns, climbing the hill once more, and ending at $V = 0$. The approximate solution comes down to $V = -\epsilon$ after climbing the hill, but it has more energy in the kink (and antikink) region, such that in both cases the total energy is zero.

We now compute the transition amplitude from the unstable vacuum $\varphi = -\mu/\sqrt{\lambda}$ to the kink-antikink solution (the bubble). Once a bubble has formed, it will rapidly grow (the kink and antikink move increasingly fast away from each other, i.e., X exponentially increase).

This is a tunnelling process because classically it is forbidden but quantum mechanically allowed. If the field φ at $x = 0$ starts making a transition from the metastable vacuum to the stable vacuum, it must first climb the potential barrier, but when it comes down in the true vacuum energy density $-\epsilon$ is gained. However, as we already mentioned, it takes energy to distort the field in order to go from one vacuum to another; this is just the energy (mass) of a kink and of an antikink. These energies are located at the boundary of the bubble (around the centers of the kink and the antikink). Once in a while there occurs a quantum mechanical transition to a bubble which is large enough that $\epsilon 2X$ is larger than $2M$; in that case the bubble does not collapse but grows increasingly rapidly.

Note that we do not tunnel from the state $\varphi(x) = -\mu/\sqrt{\lambda}$ to the state $\varphi(x) = \mu/\sqrt{\lambda}$ because the energy difference of these states is infinite (namely ϵ times the volume of x-space, so $2L\epsilon$ with $L \to \infty$). When we discussed the unperturbed kink, the vacua $\varphi = \pm\mu/\sqrt{x}$ were exactly degenerate, and in such cases the true vacuum is a linear combination of these vacua which can be determined by tunnelling from one vacuum to another.

The intermediate configuration with the kink and antikink moving away from each other can be described by Lorentz boosting the kink to a velocity $-v$ and the antikink to a velocity $+v$

$$\varphi_{cl}(x,t) = \frac{\mu}{\sqrt{\lambda}}\left[\tanh\frac{m}{2}\left(\frac{x+X_c+vt}{\sqrt{1-v^2}}\right) - \tanh\frac{m}{2}\left(\frac{x-X-vt}{\sqrt{1^2-v^2}}\right) - 1\right] \quad (8.10.293)$$

For constant $\dot{X}$ the boost of the kink is again a solution because the field equation use relativistically invariant. However, since $\dot{X}$ itself is expected to change with time, we denote $X + \dot{X}t$ by $\lambda(t)$ and obtain then

$$\varphi_{cl}(x,t) = \frac{\mu}{\sqrt{\lambda}}\left[\tanh\frac{m}{2}\left(\frac{x+\lambda(t)}{\sqrt{1-\dot{\lambda}^2}}\right) - \tanh\frac{m}{2}\left(\frac{x-\lambda(t)}{\sqrt{1-\dot{\lambda}^2}}\right) - 1\right] \ . \quad (8.10.294)$$

The distance between the kink and antikink is now $2\lambda(t)$. The Lagrangian for this approximate solution is obtained by substituting φ_{cl} into the action. The calculation of the first two terms is straightforward. Taking twice the result for a single kink yields

$$\int_{-\infty}^{\infty}\left[\frac{1}{2}\dot{\varphi}^2 - \frac{1}{2}\left(\frac{\partial\varphi}{\partial x}\right)^2\right]\mathrm{d}x = 2\int_{-\infty}^{\infty}\mathrm{d}x\frac{1}{2}\frac{1}{\cosh^4\left(\frac{m}{2}\frac{x+\lambda}{\sqrt{1-\dot{\lambda}^2}}\right)}$$
$$\left[\frac{\mu^2}{\lambda}\frac{m^2}{4}\left(\frac{\dot{\lambda}}{\sqrt{1-\dot{\lambda}^2}} + \frac{\dot{\lambda}\ddot{\lambda}(x+\lambda)}{(1-\dot{\lambda}^2)^{3/2}}\right)^2 - \frac{\mu^2}{\lambda}\frac{m^2}{4}\frac{1}{1-\dot{\lambda}^2}\right] \ . \quad (8.10.295)$$

The calculation of the contribution from the nonderivative terms splits into two parts: from the region between the kink and antikink we obtain a term $\epsilon 2\lambda$, while from each of the two walls we find a term $\frac{1}{2}M\sqrt{1-\dot{\lambda}^2}$ as we now explain. Around $x=-\lambda$ and $x=+\lambda$, the integral $\int V(\varphi)\mathrm{d}x$ with $\varphi = \frac{\mu}{\sqrt{\lambda}}\tanh\frac{m}{2}\frac{x+\lambda}{\sqrt{1-\dot{\lambda}^2}}$ can be evaluated as follows. The integral $\int_{-\infty}^{\infty}\frac{1}{2}U^2(\varphi)\mathrm{d}x$ with $\varphi = \frac{\mu}{\sqrt{\lambda}}\tanh\frac{m}{2}(x^*+\lambda^*)$ with $x^* = \frac{x}{\sqrt{1-\dot{\lambda}^2}}$ and $\lambda^* = \frac{\lambda}{\sqrt{1-\dot{\lambda}^2}}$ is equal to $(\int \frac{1}{2}U^2(\varphi(x^*))\mathrm{d}x^*)\sqrt{1-\dot{\lambda}^2}$. From equipartion of energy for a static kink we know that the integral $\int\frac{1}{2}U^2(\varphi(y))\mathrm{d}y$ equals $\frac{1}{2}M$. Thus $\int\limits_{\text{around kink}}(\frac{1}{2}U^2)(\varphi_{cl})\mathrm{d}x = \frac{1}{2}M\sqrt{1-\dot{\lambda}^2}$.

Hence, neglecting term with $\ddot{\lambda}$, we find

$$\begin{aligned} L &= -\frac{m^4}{8\lambda}\left(\frac{2}{m}\sqrt{1-\dot{\lambda}^2}\int\limits_{-\infty}^{\infty}\frac{\mathrm{d}y}{\cosh^4 y}\right) + \epsilon 2\lambda - M\sqrt{1-\dot{\lambda}^2} \\ &= -2M\sqrt{1-\dot{\lambda}^2} + \epsilon 2\lambda \ . \end{aligned} \tag{8.10.296}$$

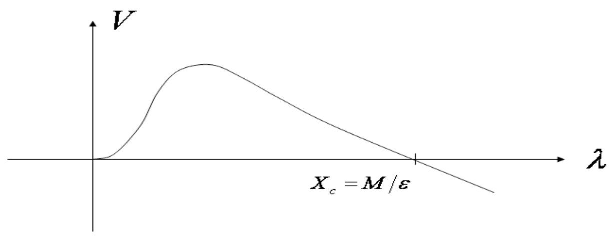
The Hamiltonian follows from $p = \frac{\partial L}{\partial\dot{\lambda}} = \frac{2M\dot{\lambda}}{\sqrt{1-\dot{\lambda}^2}}$ and reads

$$H = \frac{2M}{\sqrt{1-\dot{\lambda}^2}} - \epsilon 2\lambda = \sqrt{p^2+4M^2} - \epsilon 2\lambda \ . \tag{8.10.297}$$

We can split H into a kinetic term K and a potential term V

$$\begin{aligned} K &= \sqrt{p^2+4M^2} - 2M = \frac{1}{2}p^2/M + \mathcal{O}(p^4) \ , \\ V(\lambda) &= 2M - \epsilon 2\lambda \ . \end{aligned} \tag{8.10.298}$$

This formula for $V(\lambda)$ is valid when the bubble is reasonably large: when λ is larger than the kink size (when the bubble is larger than the thickness if its walls). For smaller x we expect that $V(\lambda)$ rises from 0 till a maximum value when the bubble is formed, and then decreases as the bubble gets larger



$$ \tag{8.10.299} $$

The value X_c corresponds to the classical solution, with energy $E = 0$ and constant X, corresponding to the ball rolling in the inverted potential. For this case, $p = 0$. Quantum fluctuations with $X < X_c$ produce only bubbles which collapse since their potential energy is positive, but bubbles with $X = X_c$ are metastable (they have constant $X = X_c$ so $p = 0$), while for $X > X_c$ the bubble expands.

We now treat H as the Hamiltonian of a point particle which sees the potential $V(\lambda)$ and has energy zero. We find with the WKB approximation for the tunneling amplitude

$$
\begin{aligned}
A &= \exp\Big[-\int_0^{X_c} |\, p \,| \, \mathrm{d}\lambda\Big] \\
&= \exp\Big[-\int_0^{X_c} \sqrt{4M^2 - (\epsilon 2\lambda)^2}\mathrm{d}\lambda\Big] ,
\end{aligned}
\tag{8.10.300}
$$

where we used that $H = 0 = \sqrt{p^2 + 4M^2} - \epsilon 2\lambda$. Since $X_c = \frac{M}{\epsilon}$, we have

$$
\begin{aligned}
A &= \exp\Big[-\frac{2M^2}{\epsilon}\int_0^1 \sqrt{1 - \frac{\epsilon^2}{M^2}\lambda^2}\mathrm{d}\left(\frac{\epsilon}{M}\lambda\right)\Big] \\
&= \exp\Big[-\frac{2M^2}{\epsilon}\int_0^1 \sqrt{1 - y^2}\mathrm{d}y\Big] = \exp(-\frac{\pi M^2}{2\epsilon}) .
\end{aligned}
\tag{8.10.301}
$$

Hence, the rate of the transition to the true vacuum is $\exp -\frac{\pi M^2}{\epsilon}$ per second and per unit volume. (To evaluate the integral we set $y = \cos\varphi$).

We end this section with a few comments. 1. The decay of the false vacuum per unit time and per unit volume is of the form $\Gamma/V = A\mathrm{e}^{-B/\hbar}(1+\mathcal{O}(\hbar))$. We computed B. For A see [74, 78].

2. We used energy conservation to determine how fast a bubble expands. However, we neglected radiation of mesons. In general, when the false vacuum collapses to the true vacuum, mesons will be created, and thus the bubble will expand less rapidly.

3. Above we considered the critical bubble: a static solution of the classical field equations which describes a bubble which has just the correct form and size that it is metastable. For larger sizes there is no static solution, but one can consider the creation at $t = 0$ of a large bubble which then expands. This is an initial value problem: $\varphi(x)$ is given and also $\frac{\partial\varphi}{\partial t} = 0$ at $t = 0$. One can define the size of a bubble for example as $Q = \int_{-\infty}^{\infty}(\varphi + \frac{\mu}{\sqrt{\lambda}})^2 \mathrm{d}x$. Far away, $\varphi = -\frac{\mu}{\sqrt{\lambda}}$, so Q is finite for bubbles. A problem we now want to solve is: given the size Q of a bubble, for which shape is its action minimal. (Minimal action in Euclidean space means maximal tunnelling rate). This will yield a one-parameter parametrization of bubbles; the parameter is a collective coordinate $\lambda(t)$, and having found the solution, we can then compare our ansatz in (8.10.291) and see how good the ansatz was. Mathematically, we can formulate this problem as a variational problem with a constraint. Introducing a constant Lagrange multiplier α we consider the action for the variational problem

$$\begin{aligned}\mathcal{L} = -\frac{1}{2}(\partial_x\varphi)^2 - \frac{\lambda}{4}\left(\varphi^2 - \frac{\mu^2}{\lambda}\right)^2 - B\left(\frac{1}{3}\varphi^3 - \frac{\mu^2}{\lambda}\varphi\right) \\ +\frac{2}{3}B\left(\frac{\mu}{\sqrt{\lambda}}\right)^3 + \frac{1}{2}\alpha\left(\varphi + \frac{\mu}{\sqrt{\lambda}}\right)^2 \ . \end{aligned} \tag{8.10.302}$$

The equation of motion

$$\frac{\partial}{\partial x}\frac{\partial\mathcal{L}}{\partial\varphi_x} - \frac{\partial\mathcal{L}}{\partial\varphi} = 0 \ , \tag{8.10.303}$$

has a first integral due to equipartition of energy

$$\frac{1}{2}\left(\frac{d\varphi}{dx}\right)^2 = \frac{\lambda}{4}\left(\varphi^2 - \frac{\mu^2}{\lambda}\right)^2 + B\left(\frac{1}{3}\varphi^3 - \frac{\mu^2}{\lambda}\varphi\right) - \frac{2}{3}B\left(\frac{\mu}{\sqrt{\lambda}}\right)^3 - \frac{1}{2}\alpha\left(\varphi + \frac{\mu}{\sqrt{\lambda}}\right)^2 . \tag{8.10.304}$$

The integration constant vanishes for bubbles. Introducing a field $\tilde{\varphi}$ which vanishes for large x

$$\tilde{\varphi} = \varphi + \frac{\mu}{\sqrt{\lambda}} , \qquad \varphi = \tilde{\varphi} - \frac{\mu}{\sqrt{\lambda}} , \tag{8.10.305}$$

we obtain

$$\frac{d\tilde{\varphi}}{dx} = \sqrt{\frac{\lambda}{2}}\tilde{\varphi}\sqrt{\left(\tilde{\varphi} - \frac{2\mu}{\sqrt{\lambda}}\right)^2 + \frac{4B}{\lambda}\left(\frac{1}{3}\tilde{\varphi} - \frac{\mu}{\sqrt{\lambda}}\right) - \frac{2\alpha}{\lambda}} . \tag{8.10.306}$$

For $\alpha = B = 0$ the solution is the kink, but for $\alpha \neq 0$ we get bubbles. One can actually solve this equation exactly by using (see Gradhstein and Resznik, page 84, 2.266)

$$\int \frac{\mathrm{d}y}{y\sqrt{a + by + cy^2}} = \frac{1}{\sqrt{a}}\operatorname{arc\,cosh}\frac{2a + by}{y\sqrt{-4ac}} , \tag{8.10.307}$$

which holds if $a > 0$ and $b^2 - 4ac > 0$. This corresponds to $0 < \alpha < 2\mu^2$. By writing the differential equation as

$$\int \mathrm{d}\left(\sqrt{\frac{\lambda}{2}}x\right) = \int \frac{\mathrm{d}\tilde{\varphi}}{\tilde{\varphi}\sqrt{\left(\frac{4\mu^2}{\lambda} - \frac{2\alpha}{\lambda} - \frac{4B\mu}{\lambda\sqrt{\lambda}}\right) + \left(-\frac{4\mu}{\sqrt{\lambda}} + \frac{4B}{3\lambda}\right)\tilde{\varphi} + \tilde{\varphi}^2}} , \tag{8.10.308}$$

we obtain for the bubble with fixed size and minimum action

$$\begin{aligned}
&\cosh\left(\sqrt{a}\sqrt{\frac{\lambda}{2}}x\right) = \frac{2a + b\tilde{\varphi}}{\tilde{\varphi}\sqrt{4ac - b^2}} \\
&\tilde{\varphi} = \frac{2a}{\sqrt{4ac - b^2}\cosh\left(\sqrt{a}\sqrt{\frac{\lambda}{2}}x - x_0\right) - b} \simeq \frac{\frac{2}{\sqrt{\lambda}}(2\mu^2 - \alpha)}{\sqrt{\alpha}\cosh(\sqrt{2}\mu(x - x_0)) + 2\mu} .
\end{aligned} \tag{8.10.309}$$

The constant α lies in the domain $0 < \alpha < 2\mu^2$. For $\alpha = 0$ we find $\tilde{\varphi} = \frac{2\mu}{\sqrt{\lambda}}$ $\left(\text{or } \varphi = \frac{\mu}{\sqrt{\lambda}}\right)$ while for $\alpha = 2\mu^2$ we find $\tilde{\varphi} = 0$ $\left(\text{or } \varphi = -\frac{\mu}{\sqrt{\lambda}}\right)$. In between, we have bubbles of finite extent; for small α the function $\tilde{\varphi}$ remains constant for a long time (the bubble) and then it falls rapidly off to zero (due to the cosh). This is the same behaviour as displayed by our ansatz in (8.10.291).

11 The strong CP problem

The vacua $|n>$ of Yang-Mills theory in Minkowski space with winding number n all have the same energy, namely zero (because they are vacua). We recall that at fixed time space was compactified to an S^3 which was mapped to the S^3 of the group manifold of $SU(2)$. Since there is tunnelling as we have discussed, the physical vacuum is a linear combination of all of them. Since they all appear on equal footing, we expect that the generator T for large gauge transformations which change the winding number, defined by $T|n>=|n+1>$, commutes with the Hamiltonian. Hence T maps the physical vacuum into itself. It follows that $T|\text{vac}>=\mathrm{e}^{i\varphi}|\text{vac}>$ with φ some phase. The solution of this equation is

$$| \text{ vac} >\equiv | \ \theta >= \sum_n \mathrm{e}^{in\theta}|n> \quad . \tag{8.11.310}$$

Indeed $T|\theta>=\sum \mathrm{e}^{in\theta} \mid n+1>= \mathrm{e}^{-i\theta} \mid \theta>$.

Instead of using the infinite set of states in (8.11.310), we can work with the ordinary vacuum $\mid 0>$ if at the same time we add a term

$$\mathcal{L}_\theta = -\theta_{\text{QCD}} \frac{g^2}{16\pi^2} \operatorname{tr} F_{\mu\nu} \ {}^*F_{\mu\nu} \tag{8.11.311}$$

to the action. This term yields a factor $\mathrm{e}^{in\theta}$ in the action $\mathrm{e}^{\frac{i}{\hbar}S}$ if one is in the vacuum with winding number n. We shall set $\hbar = 1$. Note that we are in Minkowski space and that $\mathcal{L}_\theta$ is hermitian.

Strictly speaking, we should first make a Wick rotation to Euclidean space because we can only discuss instantons in Euclidean space, but $\mathcal{L}_\theta$ has the same form in Euclidean space: one gets a factor i from d^4x and another factor i from F_{0i}. Together with the factor $\frac{i}{\hbar}$ in $\mathrm{e}^{\frac{i}{\hbar}S}$ one gets the same factor $\mathrm{e}^{in\theta}$ in Euclidean space. The θ-term is a total derivative, and usually one discards total derivatives in Lagrangians because fields vanish at infinity, but for instanton backgrounds one finds of course a nonvanishing contribution due to winding.

The θ-term clearly violates parity P. It conserves charge conjugation symmetry C, hence it violates CP. The strong interactions described by QCD are not supposed to violate P or PC, hence θ_{QCD} should be very small. However, the observed θ parameter contains more than only θ_{QCD}. There is a second origin for a θ-angle coming from the electroweak sector: the manipulations leading to the CKM matrix. Recall that the mass terms of the quarks in the Standard Model come from Yukawa couplings

$$\begin{aligned}\mathcal{L} = -\sum_{m,n}\Bigg[& g(qu)_{mn}\begin{pmatrix}\bar{q}_{L,m}\\ \bar{q}'_{L,m}\end{pmatrix}^T\begin{pmatrix}(h^0)^*\\ -(h^+)^*\end{pmatrix}q_{R,n}\\ &+g'(qu)_{mn}\begin{pmatrix}\bar{q}_{L,m}\\ \bar{q}'_{L,m}\end{pmatrix}^T\begin{pmatrix}h^+\\ h^0\end{pmatrix}q'_{R,n}\Bigg] + h.c.\ , \qquad (8.11.312)\end{aligned}$$

where $g(qu)$ are the Yukawa couplings to quarks, and h^+, h^0 are the two components of the complex $SU(2)$ Higgs doublet. Furthermore $m = 1, 2, 3$ labels the families, so q_1 denotes the up quark while q'_1 denotes the down quark. When h^0 gets a vacuum expectation value $< h^0 >= \frac{1}{\sqrt{2}}v$, one obtains mass matrices M for the (u, c, t) quarks and M' for the (d, s, b) quarks, where

$$M_{mn} = \frac{v}{\sqrt{2}}g_{mn}\ , \qquad M'_{mn} = \frac{v}{\sqrt{2}}g'_{mn}\ . \qquad (8.11.313)$$

These matrices are in general arbitrary complex 3×3 matrices. One diagonalizes them with 3×3 unitary matrices which are different for left- and right- handed quarks[48]

$$\begin{aligned}U_L M U_R^{-1} &= \ \mathrm{diag}(m_u, m_c, m_t) \equiv D\ ,\\ U'_L M' U'^{-1}_R &= \mathrm{diag}(m_d, m_s, m_b) \equiv D'\ . \qquad (8.11.314)\end{aligned}$$

[48]A complex matrix M can always be written as VH where V is unitary and H hermitian. This is the generalization to matrices of the decomposition $z = \mathrm{e}^{i\varphi}\rho$ of complex numbers. Then H can be diagonalized by a unitary matrix, $H = U_R D U_R^{-1}$, and U_L is given by VU_R.

The mass matrix for the quarks becomes then diagonal with real masses

$$\bar{q}_{L,m} M_{mn} q_{Rn} = \overline{(U_L q_L)} D (U_R q_R) \tag{8.11.315}$$

and similarly for $\bar{q}'_{L,m} M'_{mn} q'_{Rn}$. So, the physical quarks are $Q_L = U_L q_L$ and $Q_R = U_R q_R$, and similarly for Q'_L and Q'_R.

If one rescales q_L to Q_L, and q_R to Q_R, three things happen
(i) the quark mass terms are diagonalized as we have discussed, yielding real physical quark masses
(ii) a phase δ appears in the CKM matrix which describes electroweak CP violation
(iii) a new term is produced in the action by the Jacobian for these chiral rescalings. This new term is again proportional to $\int F_{\mu\nu}{}^* F_{\mu\nu} \mathrm{d}^4 x$ with a coefficient which we call $-\theta_{\mathrm{EW}}$. Hence, now the action contains the sum $\theta = \theta_{\mathrm{QCD}} + \theta_{\mathrm{EW}}$

$$\mathcal{L}_\theta = -(\theta_{\mathrm{QCD}} + \theta_{\mathrm{EW}}) \frac{g^2}{32\pi^2} \int (F^a_{\mu\nu}{}^* F^a_{\mu\nu}) \mathrm{d}^4 x \ . \tag{8.11.316}$$

There is no reason that $\theta_{\mathrm{strong}} = \theta_{\mathrm{QCD}} + \theta_{\mathrm{EW}}$ vanishes, yet, as we now discuss, this seems to be the case.

We can make a final chiral rescaling of the 3 light quarks (the u, d and s quarks) such that the θ-term is entirely removed. Rescaling the left-handed quarks by $U(1)$ factors $\mathrm{e}^{i\varphi_u}, \mathrm{e}^{i\varphi_d}$ and $\mathrm{e}^{i\varphi_s}$, the Jacobians for these rescalings yield a term

$$-(\varphi_u + \varphi_d + \varphi_s) \frac{g_s^2}{16\pi^2} \mathrm{tr}\, F_{\mu\nu}{}^* F_{\mu\nu} \ , \tag{8.11.317}$$

which cancels the θ-term if $\varphi_u + \varphi_d + \varphi_s = \theta_{\mathrm{strong}}$. Because the action is invariant except for the mass terms, only the transformation of the mass terms yields a new term in the action. In the diagonal mass term

$$m_u \bar{u} u + m_d \bar{d} d + m_s \bar{s} s \ , \tag{8.11.318}$$

the rescalings yield, to first order in $\varphi_u, \varphi_d, \varphi_s$, a new term in the action

$$\mathcal{L}_{\mathrm{CP\ violation}} = i\varphi_u m_u \bar{u}\gamma_5 u + i\varphi_d m_d \bar{d}\gamma_5 d + i\varphi_s m_s \bar{s}\gamma_5 s \ . \tag{8.11.319}$$

The φ's are only constrained by $\varphi_u + \varphi_d + \varphi_s = \theta_{\text{strong}}$, so we can still choose them such that this new term is $SU(3)_V$ invariant. Namely if $\varphi_u = \frac{\theta m_d m_s}{m_u m_d + m_u m_s + m_d m_s}$, and cyclically for φ_d and φ_s, then

$$\mathcal{L}_{\text{CP violation}} = \frac{i\theta_{\text{strong}} m_u m_d m_s}{m_u m_d + m_u m_s + m_d m_s} (\bar{u}\gamma_5 u + \bar{d}\gamma_5 d + \bar{s}\gamma_5 s) \ . \tag{8.11.320}$$

This term is hermitian and $SU(3)_V$ invariant, but it violates P, and since it conserves C, it also violates CP. The original θ-term in the action in (8.11.311) has been transformed into the masslike terms in (8.11.320). No longer does one have to deal with total derivatives, but an ordinary extra masslike term has appeared in the QCD action. There is no reason that θ_{strong} should be small, but one can compute the electric dipole moment of the neutron which is nonzero if θ_{strong} is nonzero, and since experimentally the electric dipole moment has a very small upper bound, one finds that θ_{strong} is incredibly small

$$\theta_{\text{strong}} < 10^{-9} \ . \tag{8.11.321}$$

The problem why θ_{strong} is so small is called the strong CP problem. Note that it has nothing to do with the CP violation due to the phase δ in the CKM matrix, which is an electroweak effect. Also the electroweak CP violation is very small; it can be parametrized by the area of the unitarity triangles (each of the 6 unitarity triangles has the same area $2J$ in the Standard Model)

$$J = (3.0 \pm 0.3) 10^{-5} \, . \tag{8.11.322}$$

12 The $U(1)$ problem

In this section we discuss an application of instantons in QCD.

In the 1960's, in the absence of a renormalizable theory of the strong interactions, current algebra was developed as a method to derive information about matrix elements of currents, mostly the vector and axial-vector Noether currents which correspond to the (approximate) rigid flavor symmetry of the up, down and strange quarks. In terms of modern QCD, the action for the strong interactions reads

$$\mathcal{L} = -\frac{1}{4}(F^a_{\mu\nu})^2 - \sum \bar{\psi}_i \, \bar{D\!\!\!/}\psi^i \ , \tag{8.12.323}$$

where $i = 1, ..., N_f$ labels the flavors. One can consider either two very light quarks (u and d), or three rather light quarks (u, d and s). Decomposing the massless quarks into left-handed and right-handed parts, their action becomes

$$\mathcal{L}(\text{quarks}) = \bar{\psi}_{i,L} \, \bar{D\!\!\!/}\psi^i_L - \bar{\psi}_{i,R} \, \bar{D\!\!\!/}\psi^i_R \ . \tag{8.12.324}$$

It has clearly a rigid $U_L(N_f) \times U_R(N_f)$ symmetry group, where U_L acts only on ψ^i_L and U_R only on ψ^i_R. Instead of U_L and U_R we consider the vector part $U_V(N_f)$ and the axial vector part $U_A(N_f)$. The vector part transforms ψ^i_L and ψ^i_R the same way, while they transform oppositely under U_A. The total number of symmetries and group parameters has not changed, but physically U_V and U_A are very different. The $SU(2)_V$ part of the symmetry is realized in Nature, and yields the $SU(2)$ classification scheme for quark hadroscopy. The $U(1)_V$ corresponds to baryon-number conservation which is also (very well) satisfied. The $SU(2)_A$ symmetry is spontaneously broken, and the corresponding Goldstone bosons form an $SU(2)$ multiplet of pseudoscalars (the pions and the η meson). One might be inclined to apply the same reasoning to the $U(1)_A$ symmetry, and argue that it, too, must be spontaneously broken because there is no doubling of multiplets with opposite parity observed in nature. However, the $U(1)_A$ symmetry is explicitly violated by the presence of instantons in QCD, leading to the instanton-induced six-fermion interaction in the effective action. This solves "the $U(1)$ problem" that no isoscalar Goldstone boson exists in Nature [80]. There is a pseudoscalar meson, the η with a mass of $478MeV$. It cannot be the Goldstone boson because from current algebra S. Weinberg has shown that the mass of such a Goldstone boson has to be smaller than $\sqrt{3}m_\pi$, far below the mass of the η

meson [81]. (The η meson can still be made of a quark and an antiquark, so the usual $SU(2)$ scheme is still applicable - only this η meson is not a Goldstone boson)[49].

The axial-vector isoscalar current associated with the $U_A(1)$ symmetry is $j^5_\mu = \sum_i^2 \bar{\psi}_i \gamma_5 \gamma_\mu \psi^i$. It has an Adler-Bell-Jackiw chiral anomaly

$$\partial^\mu j^{(5)}_\mu = N_f \frac{g_3^2}{32\pi^2} \epsilon^{\mu\nu\rho\sigma} F^a_{\mu\nu} F^a_{\rho\sigma} \ , \tag{8.12.325}$$

where $F^a_{\mu\nu}$ denotes the field strengths of the gluons, and g_3 is the QCD coupling constant. In a QCD instanton background, integration over spacetime yields

$$\int \frac{\partial}{\partial t} Q^5 \mathrm{d}t = N_f k \ , \tag{8.12.326}$$

where k is the winding number. To make sense of this equation one should first integrate in Euclidean space to obtain a non-vanishing expression for the right -hand side in terms of the winding number k of the QCD instanton, and then Wick-rotate so that the left-hand side can be written as $\int \mathrm{d}^4 x \partial_\mu j^\mu_5 \propto \int \frac{\partial}{\partial t} Q^5 \mathrm{d}t$. The conclusion is that Q^5 is not conserved because k can be different from zero. For further discussion of the $U(1)$ problem we refer to [80], [82,83], or to the lecture notes by Coleman in [**?**].

13 Baryon decay

In this section we present an application of instantons to the gauge fields of the electroweak sector of the Standard Model.

In an instanton background with winding number k, massless (or approximately massless) fermions in the fundamental representation of $SU(N)$ have $|k|$ zero modes, see (8.3.109). In the electroweak $SU(2)_w \times U(1)$ theory (the subscript w stands for weak), quarks and leptons are in the fundamental representation (doublets) of $SU(2)_w$. In Euclidean space the integration over zero modes of these quarks and

[49]One can extend this discussion to $U_L(3) \times U_R(3)$ with pions, kaons and η now 8 Goldstone bosons, and the η' with mass $958 MeV$ taking the place of η. This η' is an $SU(3)$ singlet.

leptons has dynamical consequences which we shall derive, but of course real quarks and leptons live in Minkowski space and not in Euclidean space. We assume that the Green functions in Minkowski spacetime can be obtained from those in Euclidean space by analytic continuation. Ideally we should prove that the Euclidean results give the main contribution to processes in Minkowski space in the same way as this was shown for tunnelling, but as far as we know this has not been done. Since processes involving electroweak instantons are suppressed by a factor $\exp\left(-\frac{1}{\hbar}\frac{8\pi^2}{g_2^2}|k|\right)$ with g_2 the electroweak $SU(2)$ coupling constant, we only consider instantons with $|k| = 1$ made from W^+, W^- and W^0 bosons. Then the left-handed quark doublets $\binom{u}{d'}$ and $\binom{c}{s'}$ each have 3 zero modes because there are 3 colors, while the lepton doublets $\binom{\nu_e}{e^-}_L$ and $\binom{\nu_\mu}{\mu^-}_L$ each have one zero mode. The primes on d' and s' denote Cabibbo-rotated quarks

$$\begin{aligned} d' &= d\cos\theta_c + s\sin\theta_c \\ s' &= s\cos\theta_c - d\sin\theta_c \end{aligned} \tag{8.13.327}$$

with $\theta_c = 13^0$ the Cabibbo angle. As we shall explain, this Cabibbo rotation makes it possible for a neutron and a proton (six quarks together) to decay into two antileptons [3]

$$p + n \to e^+ + \bar\nu_\mu \quad (\text{or} \quad \mu^+ + \bar\nu_e)\ . \tag{8.13.328}$$

In these instanton- induced processes, the electron number E, muon number M, up plus down number, and charm plus strangeness number change as follows

$$\Delta E = \Delta M = 1\ , \quad \Delta u + \Delta d' = 3\ , \quad \Delta c + \Delta s' = 3\ . \tag{8.13.329}$$

The decay of a proton with (u, u, d) and neutron with (u, d, d) quarks into e^+ and $\bar\nu_\mu$, or into μ^+ and $\bar\nu_e$, can be described by a local vertex operator with 3 up-quark fields with different colors from $\binom{u}{d'}$ doublets, and 3 down-quark fields also with different colors from $\binom{c}{s'}$ doublets, and further one field from each of the two lepton doublets. This operator is of course nonrenormalizable, but it can be used in effective field theories for phenomenological purposes. Although this efective operator is derived from field theory in the sector with an instanton, once it is obtained one can add it to the effective action and then forget about the existence of instantons. We now derive these results.

The $U(1)_A$ symmetry has at the perturbative level an anomaly. There are triangle graphs with an anomaly: one vertex of the triangle graph is given by $j_\mu^{(5)} = \sum_s \bar{\psi}^s \gamma_5 \gamma_\mu \psi^s$ (where $s = 1, \ldots, N_f$ and N_f is the number of flavors, 3 in our case if we restrict our attention to the lightest quarks u, d and s). The one-loop perturbative chiral anomaly is then given by

$$\partial^\mu j_\mu^{(5)} = iN_f \frac{g_2^2}{16\pi^2} G_{\mu\nu}^a {}^*G_{\mu\nu}^a \ , \tag{8.13.330}$$

where $G_{\mu\nu}^a$ is the W-boson field strength and g_2 the coupling constant of the $SU(2)$ weak interactions. (This is thus the abelian flavor $U(1)_A$ anomaly. The nonabelian anomaly for the rigid flavor group vanishes because it is proportional to the trace of T_a of the flavor group, which vanishes).

If one integrates over space and time, the anomaly equation becomes

$$\int \frac{\mathrm{d}}{\mathrm{d}t} Q^{(5)} \equiv \int dt \frac{\mathrm{d}}{\mathrm{d}t} \int \mathrm{d}^3x (i j_0^5) = 2N_f k \ . \tag{8.13.331}$$

The instanton number k counts the number of left-handed fermions minus the number of right-handed fermions, and in ordinary perturbation theory (with $k = 0$) for massless quarks, this difference is thus conserved. However, in an instanton background ($k \neq 0$), the chiral charge of the vacuum at $t = -\infty$ changes to a different chiral charge of the vacuum at $t = +\infty$: $\Delta Q^{(5)} = 2N_f k$. The conclusion is that the perturbative anomaly, and the violation of the axial charge which occurs when one tunnels from one vacuum to another, are related! Both are different aspects of the same chiral anomaly. The perturbative anomaly occurs when fields are small, so the winding vanishes and one is in the $k = 0$ sector. The nonperturbative anomaly is due to the same axial-vector current but now in the background of instantons which cannot be viewed as small and tending to zero at infinity, since they must produce winding.

One may at this point wonder whether the Higgs effect which gives the W-bosons a mass, in such a way that they vanish exponentially at large distances, does at the same time destroy the concept of winding. There is no contradiction. When we discussed the large instanton problem, we chose the regular gauge for the instanton to simplify the calculations. However, exponential fall-off only occurs in the singular gauge. In that case, the winding takes place at the origin, as we discussed in the introduction. In this section we use the regular gauge and then there is winding at infinity even in Higgs models.

To saturate the integrations over the Grassmann collective coordinates, one needs 6 chiral quark fields in a correlator (one for each zero mode). Each field has a mode expansion into a zero mode and all nonzero modes, but the integration measure d$\mathcal{K}$ over the Grassmann variable $\mathcal{K}$ in the mode expansion picks out only the zero mode. Then the integration over collective coordinates gives as result the product of the 6 zero mode functions. If we put one $SU(2)_w$ doublet with one up quark and one down quark at a point x_1, a second pair a x_2, and a third pair at x_3,[50] and the instanton is at x_0, we find from (8.4.144) for large separations ($x^2 >> \rho^2$) the factor[51]

$$\prod_{i=1}^{3} \frac{1}{(x_i - x_0)^6} \ . \tag{8.13.332}$$

So if one computes some correlator in a theory with instantons, six quark fields from

[50]A massless complex Dirac spinor contains two Weyl spinors which are decoupled from each other $\bar{\psi}_D \not{D} \psi_D = \bar{\psi}_L \not{D} \psi_L + \bar{\psi}_K \not{D} \psi_R$. Each has a zero mode. However, since only left-handed quarks couple to the W gauge fields, only left-handed quarks feel the presence of instantons, and so we neglect the right-handed quarks in this discussion.

[51]The down quark is contained in the s' of the doublet (c, s'). This is an $SU(2)_w$ doublet, and the instanton is an $SU(2)_w$ instanton. Although the c quark is heavier than the s quark, one can still view them as massless compared to the scale $250GeV$ of electroweak interactions. Massive spinors in an instanton background have no zero modes, as one may show by adding a mass term to (8.3.89) and (8.3.90). We assume that for such a broken $SU(2)_w$ doublet there still exists approximately a zero mode.

the correlator are needed to saturate the Grassmann integrals, and the remaining fields are then treated as in ordinary field theory (with propagators and vertices). Thus instantons induce a term proportional to $\prod_{i=1}^{3} \frac{1}{(x_i-x_0)^6}$ in the effective action which describes the annihilation of 6 quarks. Further there are σ matrices and other constants which are also due to the zero mode function.

One can now construct an effective local 6-quark vertex V at a point x_0 which yields the same results in a theory without instantons as one obtains in a theory with instantons if one integrates over the fermionic collective coordinates of the quarks. This vertex must contain 6 quark fields which contain the 6 different collective coordinates, hence it has the form $V = u_L^{\alpha,1} u_L^{\beta,2} u_L^{\gamma,3} d_L^{\delta,1} d_L^{\epsilon,2} d_L^{\zeta,3} T_{\alpha\beta\gamma\delta\epsilon\zeta}$ where T is a numerical tensor. Contraction of 6 "probe-quarks" at positions x_1, x_2, x_3 with V at x_0 using ordinary flat space propagators $\frac{1}{(x_i-x_0)^3}$ for massless quarks in a trivial vacuum precisely reproduces the result for the correlation function of the 6 probe-quarks in an instanton background centered around x_0, provided the form of T is correctly chosen.

These new vertices lead to anomalies in the baryon currents and lepton currents. In particular, the rigid $U(1)_A$ symmetry is **explicitly** broken by the presence of the interaction V in the action, and as we discussed in the previous section, this solves the $U(1)_A$ broken. As we already mentioned, a proton and a neutron (two baryons equal six quarks) may annihilate to form two antileptons (an e^+ or a μ^+, and an anti-neutrino). However, due to the incredibly small prefactor $\exp\left(-\frac{8\pi^2}{g_2^2}|k|\right)$, where g_2 is the $SU(2)$ weak coupling constant, these processes are not observable.

14 Discussion

In this chapter we have reviewed the general properties of single Yang-Mills instantons, and have given tools to compute non-perturbative effects in (non-) supersymmetric gauge theories. However, we have not discussed several other important or interesting topics:

• Perturbation theory around the instanton: the methods described here enable us to compute non-perturbative effects in the semi-classical approximation where the coupling constant is small. It is in many cases important to go beyond this limit, and to study subleading corrections that arise from higher order perturbation theory around the instanton [54,55]. Apart from a brief discussion about the one-loop determinants in section 7, we have not really addressed these issues.

• Multi-instantons: we have completely omitted a discussion of multi-instantons. These can be constructed using the ADHM formalism [29]. The main difficulty lies in the explicit construction of the collective coordinates in an instanton solution and of the measure of collective coordinates beyond instanton number $k = 2$. However, it was demonstrated that certain simplifications occur in the large N limit of $\mathcal{N} = 4$ SYM theories [23], where one can actually sum over all multi-instantons to get exact results for certain correlation functions. For reviews on the ADHM construction in super Yang-Mills theories, see e.g. [23,25,30]. The same techniques were later applied for $\mathcal{N} = 2, 1$ SYM [85,86], and it would be interesting to study the consequences of multi-instantons for large N non-supersymmetric theories. For a review on instantons in QCD, see for instance [87].

A Winding number

For a gauge field configuration with finite classical gauge action the field strength must tend to zero faster than x^{-2} at large x. For vanishing $F_{\mu\nu}$, the potential A_μ becomes then pure gauge, $A_\mu \stackrel{x\to\infty}{\longrightarrow} U^{-1}\partial_\mu U$. All configurations of A_μ which become pure gauge at infinity fall into equivalence classes, where each class has a definite winding number. As we now show, this winding number is given by

$$k = -\frac{1}{16\pi^2}\int \mathrm{d}^4x \, \mathrm{tr}\, {}^*F_{\mu\nu}F_{\mu\nu} \ , \qquad (8.A.333)$$

where ${}^*F_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F_{\rho\sigma}$ and T_a are the generators in the fundamental representation of $SU(N)$, antihermitean $N \times N$ matrices satisfying $\mathrm{tr}\, T_a T_b = -\frac{1}{2}\delta_{ab}$. This is the normalization we adopt for the fundamental representation. The key observation is that ${}^*F_{\mu\nu}F_{\mu\nu}$ is a total derivative of a gauge variant current[52]

$$\mathrm{tr}\, {}^*F_{\mu\nu}F_{\mu\nu} = 2\partial_\mu \mathrm{tr}\, \epsilon_{\mu\nu\rho\sigma}\left\{A_\nu\partial_\rho A_\sigma + \tfrac{2}{3}A_\nu A_\rho A_\sigma\right\} \,. \tag{8.A.334}$$

According to Stokes' theorem, the four-dimensional space integral becomes an integral over the three-dimensional boundary at infinity if one uses the regular gauge in which there are no singularities at the orgin. Since $F_{\mu\nu}$ vanishes at large x, one may replace $\partial_\rho A_\sigma$ by $-A_\rho A_\sigma$, and since A_μ becomes a pure gauge at large x, one obtains

$$k = \frac{1}{24\pi^2}\oint_{S^3(\text{space})} \mathrm{d}\Omega_\mu \epsilon_{\mu\nu\rho\sigma}\mathrm{tr}\,\left\{\left(U^{-1}\partial_\nu U\right)\left(U^{-1}\partial_\rho U\right)\left(U^{-1}\partial_\sigma U\right)\right\} \,, \tag{8.A.335}$$

where the integration is over a large three-sphere, S^3(space), in four-dimensional Euclidean space. To each point x^μ on this large three-sphere in space corresponds a group element U in the gauge group G. If $G = SU(2)$, the group manifold is also a three-sphere[53] S^3(group). Then $U(x)$ maps S^3(space) into S^3(group),[54] and as we now show, k is an integer which counts how many times S^3(space) is wrapped around S^3(group). Choose a parametrization of the group elements of $SU(2)$ in terms of group parameters[55] $\xi^i(x)$ $(i = 1, 2, 3)$. Then the functions $\xi^i(x)$ map x into $SU(2)$.

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[52]Note that ${}^*F_{\mu\nu}F_{\mu\nu}$ is equal to $2\epsilon_{\mu\nu\rho\sigma}\{\partial_\mu A_\nu \partial_\rho A_\sigma + 2\partial_\mu A_\nu A_\rho A_\sigma + A_\mu A_\nu A_\rho A_\sigma\}$ but the last term vanishes in the trace due to the cyclicity of the trace.

[53]The elements of $SU(2)$ can be written in the fundamental representation as $U = a_0\mathbb{1} + i\sum_k a_k\tau_k$ where τ_k are the Pauli matrices and a_0 and a_k are real coefficients satisfying the condition $a_0^2 + \sum_k a_k^2 = 1$. This defines a sphere S^3(group). (If the a's are not real but carry a common phase, one obtains the elements of $U(2)$).

[54]There is actually a complication. Far away $A_\mu = U^{-1}\partial_\mu U$ but in order that U be only a function on S^3(space) it should only depend on the 3 polar angles but not on the radius. Hence $A_r = U^{-1}\partial_r U$ should vanish. We can make a gauge transformation with a group element V such that $A'_r = V^{-1}(\partial_r + A_r)V$ vanishes. The V which achieves this is a path ordered integral along the radius from the origin, $V = P\exp - \int_0^r A_r dr$. Note that U is only defined for large r, but V must be defined everywhere, and $V \neq U$. In fact, V does not have winding since it can be continuously deformed to the unit group element. The winding number is computed in the text for UV, but since k in (8.A.333) is gauge invariant, k is also the winding number of the original gauge field A_μ.

[55]For example, Euler angles, or Lie parameters $U = a_0\mathbb{1} + i\sum_k a_k\tau_k$ with $a_0 = \sqrt{1 - \sum_k a_k^2}$.

Consider a small surface element of S^3(space). According to the chain rule

$$\begin{aligned} &\mathrm{tr}\left\{\left(U^{-1}\partial_\nu U\right)\left(U^{-1}\partial_\rho U\right)\left(U^{-1}\partial_\sigma U\right)\right\} \\ &= \frac{\partial \xi^i}{\partial x_\nu}\frac{\partial \xi^j}{\partial x_\rho}\frac{\partial \xi^k}{\partial x_\sigma}\mathrm{tr}\left\{\left(U^{-1}\partial_i U\right)\left(U^{-1}\partial_j U\right)\left(U^{-1}\partial_k U\right)\right\} \ , \end{aligned} \tag{8.A.336}$$

and using[56]

$$\Delta\Omega_\mu = \tfrac{1}{6}\epsilon_{\mu\alpha\beta\gamma}\Delta x_\alpha \Delta x_\beta \Delta x_\gamma \ , \tag{8.A.337}$$

with $\frac{1}{6}\epsilon_{\mu\nu\rho\sigma}\epsilon_{\mu\alpha\beta\gamma} = \delta^{\nu\rho\sigma}_{[\alpha\beta\gamma]}$ and $\Delta\xi^{[i}\Delta\xi^j\Delta\xi^{k]} = \epsilon^{ijk}\Delta^3\xi$, we obtain for the contribution Δk of the small surface element to k

$$\Delta k = \frac{1}{24\pi^2}\epsilon^{ijk}\mathrm{tr}\left\{\left(U^{-1}\partial_i U\right)\left(U^{-1}\partial_j U\right)\left(U^{-1}\partial_k U\right)\right\}\Delta^3\xi \ , \tag{8.A.338}$$

where $k = \oint_{S^3(\text{space})}\Delta k$. The elements $(U^{-1}(\xi)\partial_i U(\xi))$ lie in the Lie algebra, and define the group vielbein $e_i^a(\xi)$ by

$$\left(U^{-1}\partial_i U\right) = e_i^a(\xi) T_a \ . \tag{8.A.339}$$

With $\epsilon^{ijk}e_i^a e_j^b e_k^c = (\det e)\ \epsilon^{abc}$, we obtain for the contribution to k from a surface element $\Delta\Omega_\mu$

$$\Delta k = \frac{1}{24\pi^2}(\det e)\,\mathrm{tr}\left(\epsilon^{abc}T_a T_b T_c\right)\Delta^3\xi = -\frac{1}{16\pi^2}(\det e)\,\Delta^3\xi \ . \tag{8.A.340}$$

We used that for $SU(2)$ we have $[T_a, T_b] = \epsilon_{abc}T_c$. As we have demonstrated, the original integral over the physical space is reduced to one over the group with measure $(\det e)\,\mathrm{d}^3\xi$. The volume of a surface element of S^3(group) with coordinates $\mathrm{d}\xi^i$ is proportional to $(\det e)\,\mathrm{d}^3\xi$ (called the Haar measure). Since this expression is a scalar in general relativity,[57] we know that the value of the volume does not depend on which coordinates one uses except for an overall normalization. We fix this overall

[56] For example, if the surface element points in the x-direction we have $\Delta\Omega = \Delta y \Delta z \Delta\tau$ if $\epsilon_{1234} = 1$.

[57] Under a change of coordinates $\xi = \xi(\xi')$ at the point ξ, the vielbein transforms as $e_i^a(\xi) = \frac{\partial \xi'^j}{\partial \xi^i} e'^a_j(\xi(\xi'))$, hence $\det e(\xi) = \left(\det\frac{\partial \xi'}{\partial \xi}\right)\det(e'(\xi'))$, while $\mathrm{d}^3\xi$ is equal to $|\det\frac{\partial\xi}{\partial\xi'}|\mathrm{d}^3\xi'$. For small coordinate transformations $\det\partial\xi/\partial\xi'$ is positive, hence $\det e\ \mathrm{d}^3\xi$ is invariant.

normalization of the group volume such that near $\xi = 0$ the volume is $\Delta^3\xi$. Since $e_i^a = \delta_i^a$ near $\xi = 0$, we have there the usual Euclidean measure $\mathrm{d}^3\xi$. Each small patch on S^3(space) corresponds to a small patch on S^3(group), $\Delta k \sim \mathrm{Vol}(\Delta^3\xi)$. Since the U's fall into homotopy classes, integrating once over S^3(space) we cover S^3(group) an integer number of times. To check the proportionality factor in $\mathit{\Delta} k \sim$ Vol $(\Delta^3\xi)$, we consider the fundamental map

$$U(x) = ix_\mu\sigma_\mu/\sqrt{x^2}\ , \qquad U^{-1}(x) = -ix_\mu\bar{\sigma}_\mu/\sqrt{x^2}\ . \tag{8.A.341}$$

where σ_μ denotes the 2×2 matrices $(\vec{\sigma}, i)$ with $\vec{\sigma}$ the Pauli matrices, and $\bar{\sigma}_\mu = (\vec{\sigma}, -i)$. This is clearly a one-to-one map from S^3(space) to S^3(group) and should therefore yield $|k| = 1$. Direct calculation gives

$$U^{-1}\partial_\mu U = \frac{-x_\mu}{x^2} + \frac{x_\nu\bar{\sigma}_\nu}{x^2}\sigma_\mu = -\sigma_{\mu\nu}x_\nu/x^2\ , \tag{8.A.342}$$

where $\sigma_{\mu\nu}$ is defined in 8.B.351. Substitution into (8.A.335) leads to $k = -\frac{1}{2\pi^2}\oint \mathrm{d}\mathit{\Omega}_\mu$ $x_\mu/x^4 = -1$ making use of (8.B.361).[58] To obtain $k = 1$ one has to make the change $\sigma \leftrightarrow \bar{\sigma}$ in eq. (8.A.341).

Let us comment on the origin of the winding number of the instanton in the singular gauge. In this case A_μ^{sing} vanishes fast at infinity, but becomes pure gauge near $x = 0$. In the region between a small sphere in the vicinity of $x = 0$ and a large sphere at $x = \infty$ we have an expression for k in terms of a total derivative, but now for A_μ^{sing} the only contribution to the topological charge comes from the boundary near $x = 0$:

$$k = -\frac{1}{24\pi^2}\oint_{S^3_{x\to 0}(\mathrm{space})} \mathrm{d}\mathit{\Omega}_\mu \epsilon_{\mu\nu\rho\sigma} \mathrm{tr}\ \left\{\left(U^{-1}\partial_\nu U\right)\left(U^{-1}\partial_\rho U\right)\left(U^{-1}\partial_\sigma U\right)\right\}\ . \tag{8.A.343}$$

The extra minus sign is due to the fact that the normal to the S^3(space) at $x = 0$ points inward. Furthermore, $A_\mu^{\mathrm{sing}} \sim U^{-1}\partial_\mu U = -\bar{\sigma}_{\mu\nu}x_\nu/x^2$ near $x = 0$, while

[58] Only the commutator of the first two matrices in tr $(\sigma_{\nu\alpha}\sigma_{\rho\beta}\sigma_{\sigma\gamma})x^\alpha x^\beta x^\gamma$ contributes because the anticommutator is proportional to the unit matrix. In the result only the anticommutator gives a nonvanishing result, because the commutator yields term proportional to $\sigma_{\alpha\beta}$ whose trace vanishes.

$A_\mu^{\rm reg} \sim U\partial_\mu U^{-1} = -\sigma_{\mu\nu}x_\nu/x^2$ for $x \sim \infty$. There is a second extra minus sign in the evaluation of k from the trace of Lorentz generators. As a result $k_{\rm sing} = k_{\rm reg}$, as it should be since k is a gauge invariant object. The gauge transformation which maps $A_\mu^{\rm reg}$ to $A_\mu^{\rm sing}$ transfers the winding from a large to a small S^3(space).

B 't Hooft symbols and Euclidean spinors

In this appendix we give a list of conventions and formulae useful for instanton calculus. Let us first discuss the structure of Lorentz algebra $so(3,1)$ in Minkowski space-time. The generators can be represented by $L_{\mu\nu} = \frac{1}{2}(x_\mu\partial_\nu - x_\nu\partial_\mu)$ and form the algebra $[L_{\mu\nu}, L_{\rho\sigma}] = -\eta_{\mu\rho}L_{\nu\sigma} - \eta_{\nu\sigma}L_{\mu\rho} + \eta_{\mu\sigma}L_{\nu\rho} + \eta_{\nu\rho}L_{\mu\sigma}$, with the signature $\eta_{\mu\nu} = \mathrm{diag}(-,+,+,+)$. The spatial rotations $J_i \equiv \frac{1}{2}\epsilon_{ijk}L_{jk}$ and boosts $K_i \equiv L_{0i}$ satisfy the algebra $[J_i, J_j] = -\epsilon_{ijk}J_k, [J_i, K_j] = [K_i, J_j] = -\epsilon_{ijk}K_k$ and $[K_i, K_j] = \epsilon_{ijk}J_k$.

There exist two 2-component spinor representations, which we denote by λ^α and $\bar\chi_{\dot\alpha}$ ($\alpha = 1,2$ and $\dot\alpha = 1,2$). The generators for these spinor representations are $\sigma^{\mu\nu}$ and $\bar\sigma^{\mu\nu}$, where $\sigma_{\mu\nu} \equiv \frac{1}{2}(\sigma_\mu\bar\sigma_\nu - \sigma_\nu\bar\sigma_\mu), \bar\sigma_{\mu\nu} = \frac{1}{2}(\bar\sigma_\mu\sigma_\nu - \bar\sigma_\nu\sigma_\mu)$, with $\sigma_\mu^{\alpha\dot\beta} = (\vec\tau, I), \bar\sigma_{\mu\,\dot\alpha\beta} = (\vec\tau, -I), \mu = 1,2,3,0$, and I denotes the identity matrix. The matrices τ^i with $i = 1,2,3$ are the usual Pauli matrices. The Lorentz generators consist of $\sigma^{ij} = i\epsilon^{ijk}\tau^k$ and $\sigma^{0i} = \tau^i$ for λ^α, and $\bar\sigma^{ij} = i\epsilon^{ijk}\tau^k$ and $\bar\sigma^{0i} = -\tau^i$ for $\bar\chi_{\dot\alpha}$. The rotation generators σ^{ij} are clearly antihermitian, but the boost generators are hermitian.

Under a rotation or boost, both spinors simultaneously transform. Most importantly, the two spinor representations are complex. In fact, they are each other's complex conjugate up to a similarity transformation: $(\sigma^{\mu\nu})^* = \sigma_2\bar\sigma^{\mu\nu}\sigma_2$. The matrices $i\tau^k$ and τ^k form the 2×2 defining representation of the group $Sl(2,C)$, which is the covering group of $SO(3,1)$.

The situation differs for Euclidean space ($\delta_{\mu\nu} = \mathrm{diag}(+,+,+,+)$) with $SO(4)$ instead of the Lorentz group $SO(3,1)$. Now $[L_{\mu\nu}, L_{\rho\sigma}] = \delta_{\nu\rho}L_{\mu\sigma}$+3 terms, and $[J_i, J_j] =$

$-\epsilon_{ijk}J_k, [J_i, K_j] = -\epsilon_{ijk}K_k$ but $[K_i, K_j] = -\epsilon_{ijk}J_k$ where obviously $J_i \equiv \frac{1}{2}\epsilon_{ijk}L_{jk}$ and boosts $K_i \equiv L_{i4}$. The linear combinations of (ij) and $(4, i)$-plane rotations

$$M_i \equiv \frac{1}{2}(J_i + K_i) \ , \qquad N_i \equiv \frac{1}{2}(J_i - K_i) \ , \tag{8.B.344}$$

give the algebras of the two commuting $SU(2)$ subgroups of $SO(4) = SU(2) \times SU(2)$ in view of the anti-hermiticity $M_i^\dagger = -M_i$, $N_i^\dagger = -N_i$. We now denote the two spinor representations by λ^α and $\bar{\chi}_{\alpha'}$. Because M and N are represented by generators $i\vec{\sigma}_M$ and $i\vec{\sigma}_N$ which act in different spaces, one can transform λ^α while $\bar{\chi}_{\alpha'}$ stays fixed, or vice versa. The two spinor representations in Euclidean space are each pseudoreal: as we shall discuss $(\sigma^{\mu\nu})^* = \sigma_2\sigma^{\mu\nu}\sigma_2$ and $(\bar{\sigma}_{\mu\nu})^* = \sigma_2\bar{\sigma}_{\mu\nu}\sigma_2$.

It is an easy exercise to check that we can represent the operators M and N by

$$M_i = \bar{\eta}_{i\mu\nu} \ , \qquad \text{and} \qquad N_i = \eta_{i\mu\nu} \ , \tag{8.B.345}$$

where we introduced 't Hooft symbols [2]

$$\begin{aligned} &\eta_{a\mu\nu} \equiv \epsilon_{a\mu\nu} + \delta_{a\mu}\delta_{\nu 4} - \delta_{a\nu}\delta_{4\mu}, \text{ or } \eta_{aij} = \epsilon_{aij}, \eta_{aj4} = \delta_{aj} \\ &\bar{\eta}_{a\mu\nu} \equiv \epsilon_{a\mu\nu} - \delta_{a\mu}\delta_{\nu 4} + \delta_{a\nu}\delta_{4\mu}, \text{ or } \bar{\eta}_{aij} = \epsilon_{aij}, \bar{\eta}_{aj4} = -\delta_{aj} \end{aligned} \tag{8.B.346}$$

and $\bar{\eta}_{a\mu\nu} = (-1)^{\delta_{4\mu}+\delta_{4\nu}}\eta_{a\mu\nu}$. They form a basis of anti-symmetric 4 by 4 matrices and are (anti-)selfdual in vector indices ($\epsilon_{1234} = 1$)

$$\eta_{a\mu\nu} = \tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}\eta_{a\rho\sigma} \ , \qquad \bar{\eta}_{a\mu\nu} = -\tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}\bar{\eta}_{a\rho\sigma} \ . \tag{8.B.347}$$

The η-symbols obey the following relations

$$\begin{aligned} &\epsilon_{abc}\eta_{b\mu\nu}\eta_{c\rho\sigma} = \delta_{\mu\rho}\eta_{a\nu\sigma} + \delta_{\nu\sigma}\eta_{a\mu\rho} - \delta_{\mu\sigma}\eta_{a\nu\rho} - \delta_{\nu\rho}\eta_{a\mu\sigma} \ , \\ &\eta_{a\mu\nu}\eta_{a\rho\sigma} = \delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho} + \epsilon_{\mu\nu\rho\sigma} \ , \\ &\eta_{a\mu\rho}\eta_{b\mu\sigma} = \delta_{ab}\delta_{\rho\sigma} + \epsilon_{abc}\eta_{c\rho\sigma} \ , \\ &\epsilon_{\mu\nu\rho\tau}\eta_{a\sigma\tau} = \delta_{\sigma\mu}\eta_{a\nu\rho} + \delta_{\sigma\rho}\eta_{a\mu\nu} - \delta_{\sigma\nu}\eta_{a\mu\rho} \ , \\ &\eta_{a\mu\nu}\eta_{a\mu\nu} = 12 \ , \quad \eta_{a\mu\nu}\eta_{b\mu\nu} = 4\delta_{ab} \ , \quad \eta_{a\mu\rho}\eta_{a\mu\sigma} = 3\delta_{\rho\sigma} \ . \end{aligned} \tag{8.B.348}$$

The same holds for $\bar{\eta}$ except for the terms with $\epsilon_{\mu\nu\rho\sigma}$,

$$\begin{aligned}
\bar{\eta}_{a\mu\nu}\bar{\eta}_{a\rho\sigma} &= \delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho} - \epsilon_{\mu\nu\rho\sigma} \ , \\
\epsilon_{\mu\nu\rho\sigma}\bar{\eta}_{a\sigma\tau} &= -\delta_{\sigma\mu}\bar{\eta}_{a\nu\rho} - \delta_{\sigma\rho}\bar{\eta}_{a\mu\nu} + \delta_{\sigma\nu}\bar{\eta}_{a\mu\rho} \ .
\end{aligned} \tag{8.B.349}$$

Obviously $\eta_{a\mu\nu}\bar{\eta}_{b\mu\nu} = 0$ due to different duality properties. In matrix notation, we have

$$\begin{aligned}
[\eta_a, \eta_b] = -2\epsilon_{abc}\eta_c \ , &\qquad [\bar{\eta}_a, \bar{\eta}_b] = -2\epsilon_{abc}\bar{\eta}_c \ , \\
\{\eta_a, \eta_b\} = -2\delta_{ab} \ , &\qquad \{\bar{\eta}_a, \bar{\eta}_b\} = -2\delta_{ab} \ ,
\end{aligned} \tag{8.B.350}$$

and the two sets of matrices commute, i.e. $[\eta_a, \bar{\eta}_b] = 0$ (this is equivalent to the statement that the generators M and N commute).

The two inequivalent spinor representations of the Euclidean Lorentz algebra are given by

$$\sigma_{\mu\nu} \equiv \tfrac{1}{2}[\sigma_\mu\bar{\sigma}_\nu - \sigma_\nu\bar{\sigma}_\mu] \ , \qquad \bar{\sigma}_{\mu\nu} = \tfrac{1}{2}[\bar{\sigma}_\mu\sigma_\nu - \bar{\sigma}_\nu\sigma_\mu] \ , \tag{8.B.351}$$

in terms of Euclidean matrices

$$\sigma_\mu^{\alpha\beta'} = (\tau^a, i) \ , \qquad \bar{\sigma}_{\mu\,\alpha'\beta} = (\tau^a, -i) \ , \qquad \mu = 1,2,3,4 \ , \tag{8.B.352}$$

obeying the Clifford algebra $\sigma_\mu\bar{\sigma}_\nu + \sigma_\nu\bar{\sigma}_\mu = 2\delta_{\mu\nu}$. Since $\sigma_{\mu\nu}$ contains $\sigma_{ij} = \epsilon_{ijk}i\tau^k$ and $\sigma_{i4} = -i\tau_i$, while $\bar{\sigma}_{\mu\nu}$ contains $\bar{\sigma}_{ij} = \epsilon_{ijk}i\tau^k$ and $\bar{\sigma}_{i4} = i\tau_i$, they are not each others complex conjugate, contrary to the Minkowski case. Rather, they are pseudo-real, meaning that their complex-conjugates are related to themselves by a similarily transformation

$$\sigma^*_{\mu\nu} = \sigma_2\sigma_{\mu\nu}\sigma_2; (\bar{\sigma}_{\mu\nu})^* = \sigma_2\bar{\sigma}_{\mu\nu}\sigma_2 \ . \tag{8.B.353}$$

To prove these, and other, spinor relations, one needs some formulas which we now present. As in Minkowski space, also in Euclidean space σ_μ and $\bar{\sigma}_\mu$ are related by transposition

$${\sigma_\mu}^{\alpha\alpha'} = {\bar{\sigma}_\mu}^{\alpha'\alpha} \tag{8.B.354}$$

where $\bar{\sigma}_\mu{}^{\alpha'\alpha}$ is obtained from $\bar{\sigma}^\mu{}_{\beta'\beta}$ by raising indices

$$\bar{\sigma}_\mu{}^{\alpha'\alpha} \equiv \epsilon^{\alpha'\beta'}\epsilon^{\alpha\beta}\bar{\sigma}^\mu_{\beta'\beta} \tag{8.B.355}$$

We use everywhere the north-west convention for raising and lowering the spinor indices

$$\epsilon^{\alpha\beta}\xi_\beta = \xi^\alpha \ , \qquad \bar{\xi}^{\beta'}\epsilon_{\beta'\alpha'} = \bar{\xi}_{\alpha'} \ , \tag{8.B.356}$$

with $\epsilon_{\alpha\beta} = -\epsilon_{\alpha'\beta'}, \epsilon_{\alpha\beta} = \epsilon^{\alpha\beta}$, and $\epsilon_{\alpha'\beta'} = \epsilon^{\alpha'\beta'}$. However, the relation between σ_μ and $\bar{\sigma}_\mu$ under complex conjugation is different (as expected because $\sigma^0 = I$ but $\sigma^4 = iI$). In Minkowski space we have $(\sigma_\mu^{\alpha\dot{\beta}})^* = \bar{\sigma}_\mu^{\dot{\beta}\alpha}$, while in Euclidean space $(\sigma_\mu^{\alpha\beta'})^* = \bar{\sigma}_{\mu,\beta'\alpha} = \sigma_{\mu,\alpha\beta'}$ and $(\bar{\sigma}_{\mu,\alpha'\beta})^* = \sigma_\mu^{\beta\alpha'} = \bar{\sigma}_\mu^{\alpha'\beta}$. The result in Minkowski space is a direct consequence of the hermiticity of the matrices σ_μ, while the result in Euclidean space agrees with the usual rule for $SU(N)$ that $(v^i)^*$ transforms as v_i for vectors in the defining representation.

Let us now apply these formulas to give another proof that $\sigma_{\mu\nu}$ and $\bar{\sigma}_{\mu\nu}$ are pseudoreal in Euclidean space

$$\begin{aligned}
&((\sigma_{\mu\nu})^\alpha{}_\beta)^* = \frac{1}{2}(\sigma_\mu{}^{\alpha\beta'})^*(\bar{\sigma}_{\nu,\beta'\beta})^* - \mu \leftrightarrow \nu \\
&= \frac{1}{2}\sigma_{\mu,\alpha\beta'}\bar{\sigma}_\nu^{\beta'\beta} - \mu \leftrightarrow \nu = -\frac{1}{2}\sigma_{\mu,\alpha}{}^{\beta'}\bar{\sigma}_{\nu,\beta'}{}^\beta - \mu \leftrightarrow \nu \\
&= -\epsilon_{\gamma\alpha}(\sigma_{\mu\nu})^\gamma{}_\delta\epsilon^{\beta\delta} = -(-i\sigma_2)(\sigma_{\mu\nu})(-i\sigma_2) = \sigma_2\sigma_{\mu\nu}\sigma_2
\end{aligned} \tag{8.B.357}$$

and idem for $\bar{\sigma}_{\mu\nu}$.

The two spinor and vector representations of the $su(2)$ algebra are all given in terms of anti-hermitian 2x2 matrices $\sigma_{\mu\nu}, \bar{\sigma}_{\mu\nu}$ and $i\tau^a$ and they are related by the 't Hooft symbols,

$$\bar{\sigma}_{\mu\nu} = i\eta_{a\mu\nu}\tau^a \ , \qquad \sigma_{\mu\nu} = i\bar{\eta}_{a\mu\nu}\tau^a \ . \tag{8.B.358}$$

Furthermore, $\bar{\sigma}_{\mu\nu}$ is selfdual whereas $\sigma_{\mu\nu}$ is anti-selfdual. Some frequently used identities are

$$\begin{aligned}
&\bar{\sigma}_\mu\sigma_{\nu\rho} = \delta_{\mu\nu}\bar{\sigma}_\rho - \delta_{\mu\rho}\bar{\sigma}_\nu - \epsilon_{\mu\nu\rho\sigma}\bar{\sigma}_\sigma \ , \quad \sigma_\mu\bar{\sigma}_{\nu\rho} = \delta_{\mu\nu}\sigma_\rho - \delta_{\mu\rho}\sigma_\nu + \epsilon_{\mu\nu\rho\sigma}\sigma_\sigma \ , \\
&\sigma_{\mu\nu}\sigma_\rho = \delta_{\nu\rho}\sigma_\mu - \delta_{\mu\rho}\sigma_\nu + \epsilon_{\mu\nu\rho\sigma}\sigma_\sigma \ , \quad \bar{\sigma}_{\mu\nu}\bar{\sigma}_\rho = \delta_{\nu\rho}\bar{\sigma}_\mu - \delta_{\mu\rho}\bar{\sigma}_\nu - \epsilon_{\mu\nu\rho\sigma}\bar{\sigma}_\sigma \ .
\end{aligned} \tag{8.B.359}$$

The Lorentz generators are antisymmetric in vector and symmetric in spinor indices

$$\sigma_{\mu\nu\,\alpha\beta} = -\sigma_{\nu\mu\,\alpha\beta} \ , \qquad \sigma_{\mu\nu\,\alpha\beta} = \sigma_{\mu\nu\,\beta\alpha} \ , \tag{8.B.360}$$

and obey the algebra

$$\begin{aligned} [\sigma_{\mu\nu}, \sigma_{\rho\sigma}] &= -2\left\{\delta_{\mu\rho}\sigma_{\nu\sigma} + \delta_{\nu\sigma}\sigma_{\mu\rho} - \delta_{\mu\sigma}\sigma_{\nu\rho} - \delta_{\nu\rho}\sigma_{\mu\sigma}\right\} \ , \\ \{\sigma_{\mu\nu}, \sigma_{\rho\sigma}\} &= -2\left\{\delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho} - \epsilon_{\mu\nu\rho\sigma}\right\} \ . \end{aligned} \tag{8.B.361}$$

The same relations hold for $\bar{\sigma}_{\mu\nu}$ but with $+\epsilon_{\mu\nu\rho\sigma}$. In spinor algebra the following contractions are frequently used

$$\sigma_\mu^{\alpha\alpha'}\bar{\sigma}_{\mu\,\beta'\beta} = 2\delta_\beta^\alpha\delta_{\beta'}^{\alpha'} \ , \qquad \sigma_{\rho\sigma}{}^\alpha{}_\beta\sigma_{\rho\sigma}{}^\gamma{}_\delta = 4\left\{\delta_\beta{}^\alpha\delta_\delta{}^\gamma - 2\delta_\delta{}^\alpha\delta_\beta{}^\gamma\right\} \ . \tag{8.B.362}$$

Throughout the paper we frequently use the following integral formula

$$\int d^4x \frac{\left(x^2\right)^n}{\left(x^2+\rho^2\right)^m} = \pi^2\left(\rho^2\right)^{n-m+2}\frac{\Gamma(n+2)\Gamma(m-n-2)}{\Gamma(m)} \ , \tag{8.B.363}$$

which converges for $m - n > 2$.

C The volume of the gauge orientation moduli space

The purpose of this appendix[59] is to prove equation (8.5.161). Let us consider an instanton in $SU(N)$ gauge theory. Deformations of this configuration which are still self-dual and not a gauge transformation are parametrized by collective coordinates. Constant gauge transformations $A_\mu \to U^{-1}A_\mu U$ preserve self-duality and transversality but not all constant $SU(N)$ matrices U change A_μ. Those U which keep A_μ fixed form the stability subgroup H of the instanton, hence we want to determine the volume of the coset space $SU(N)/H$. If the instanton is embedded in the lower-right

[59]We thank R. Roiban for help in writing this appendix.

2×2 submatrix of the $N \times N$ $SU(N)$ matrix, then H contains the $SU(N-2)$ subgroup in the left-upper part, and a $U(1)$ subgroup with elements $\exp(\theta A)$ where A is the diagonal matrix

$$A = \frac{i}{2}\sqrt{\frac{N-2}{N}}\mathrm{diag}\left(\frac{2}{2-N}, \ldots, \frac{2}{2-N}, 1, 1\right) . \qquad (8.C.364)$$

All generators of $SU(N)$ (and also all generators of $SO(N)$ discussed below) are normalized according to $\mathrm{tr}\, T_a T_b = -\frac{1}{2}\delta_{ab}$, as in the main text.

At first sight one might expect the range of θ to be such that the exponents of all entries cover the range 2π an integer number of times. However, this is incorrect: only for the last two entries of $\exp(\theta A)$ we must require periodicity, because whatever happens in the other $N-2$ diagonal entries is already contained in the $SU(N-2)$ part of the stability subgroup. Thus all elements h in H are of the form [50]

$$h = \mathrm{e}^{\theta A} g, \qquad \text{with} \qquad g \infty SU(N-2) \qquad \text{and} \qquad 0 \leq \theta \leq \theta_{\max} = 4\pi\sqrt{\frac{N}{N-2}} . \qquad (8.C.365)$$

For $N = 3$ the range of θ is larger than required by periodicity of the first $N-2$ entries, for $N = 4$ it corresponds to periodicity of all entries, but for $N \geq 5$ the range of θ is less than required for periodicity of the first $N-2$ entries.[60] Thus $H \neq SU(N) \times U(1)$ for $N \geq 5$. The first $N-2$ entries of $\exp(k\theta_{\max} A)$ with integer k are given by $\exp\left(-ik\frac{4\pi}{N-2}\right)$ and lie therefore in the center Z_N of $SU(N-2)$. So, the $SU(N)$ group elements $h = \exp(\theta A)\, g$ with $0 \leq \theta \leq \theta_{\max}$ and g in $SU(N-2)$ form a subgroup H. We shall denote H by $SU(N-2) \times$ “$U(1)$” where “$U(1)$” denotes the part of the $U(1)$ generated by A which lies in H. We now use three theorems to

[60]For example, consider $SU(5)$ with $\exp[\frac{i\theta}{2}\sqrt{\frac{3}{5}}\ \mathrm{diag}\ (-\frac{2}{3}, -\frac{2}{3}, -\frac{2}{3}, 1, 1)]$. When θ runs from 0 to $\sqrt{\frac{5}{3}}4\pi$, last two entries repeat, but the first three entries only reach $\exp(-4\pi i/3)$. The first three entries form then an element of $SU(N-2) = SU(3)$, namely they yield an element z of the center Z_3. So when θ ranges beyond $\sqrt{\frac{5}{3}}4\pi$, these $SU(5)$ elements can be written as a product of z and $\exp i\theta A$ with θ smaller than $\sqrt{\frac{5}{3}}4\pi$. So, the range of θ is bounded by $\sqrt{\frac{5}{3}}4\pi$.

evaluate the volume of $SU(N)/H$:

$$\begin{aligned}
&\text{(I)} \qquad \text{Vol}\,\frac{SU(N)}{SU(N-2)\times \text{“}U(1)\text{”}} = \frac{\text{Vol}\;(SU(N)/SU(N-2))}{\text{Vol “}U(1)\text{”}}\;, \\
&\text{(II)} \qquad \text{Vol}\,\frac{SU(N)}{SU(N-2)} = \text{Vol}\,\frac{SU(N)}{SU(N-1)}\text{Vol}\,\frac{SU(N-1)}{SU(N-2)}\;, \\
&\text{(III)} \qquad \text{Vol}\,\frac{SU(N)}{SU(N-1)} = \frac{\text{Vol}\,SU(N)}{\text{Vol}\,SU(N-1)}\;.
\end{aligned} \tag{8.C.366}$$

It is, in fact, easiest to first compute Vol$\,(SU(N)/SU(N-1))$ and then to use this result for the evaluation of Vol $SU(N)/H$ (with Vol $SU(N)$ as a bonus).

In general the volume of a coset manifold G/H is given by $V = \int \prod_\mu dx^\mu \;\det\; e_\mu^m(x)$ where x^μ are the coordinates on the coset manifold and $e_\mu^m(x)$ are the coset vielbeins. One begins with “coset representatives” $L(x)$ which are group elements $g\infty G$ such that every group element can be decomposed as $g = L(x)h$ with $h\infty H$. We denote the coset generators by K_m and the subgroup generators by H_i. Then $L^{-1}(x)\partial_\mu L(x) = e_\mu^m(x)K_m + \omega_\mu^i(x)H_i$. We shall take the generators K_m and H_i in the fundamental representation of $SU(N)$: antihermitian $N\times N$ matrices. Under a general coordinate transformation from x^μ to $y^\mu(x)$, the vielbein transforms as a covariant vector with index μ but also as a contravariant vector with index m at $x=0$. Hence *V does (only) depend on the choice of the coordinates at the origin.* At the origin, $L^{-1}\partial_\mu L = e_\mu^m(0)K_m$, and we fix the normalization of K_m by $\operatorname{tr} K_m^2 = -\frac{1}{2}$ for K_m in the $N\times N$ matrix representation of $SU(N)$.

To find the volume of $SU(N)/SU(N-1)$ we note that the group elements of $SU(N)$ have a natural action on the space $\mathbf{C}^N$ and map a point $\left(z^1,\ldots,z^N\right)\infty\mathbf{C}^N$ on the complex hypersphere $\sum_{i=1}^N |z^i|^2 = 1$ into another point on the complex hypersphere. The “south-pole” $(0,\ldots,0,1)$ is kept invariant by the subgroup $SU(N-1)$, and points on the complex hypersphere are in one-to-one correspondence with the coset representatives $L(z)$ of $SU(N)/SU(N-1)$. We use as generators for $SU(N)$ the generators for $SU(N-1)$ in the upper-left block, and further the following coset generators: $N-1$ pairs T_{2k} and T_{2k+1} each of them containing only two non-zero elements

$$\begin{pmatrix} 0 & \dots & & 0 \\ & & & \cdot \\ \vdots & & & i/2 \\ & & \ddots & \vdots \\ 0 & i/2 & \dots & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & \dots & & 0 \\ & & & \cdot \\ \vdots & & & 1/2 \\ & & \ddots & \vdots \\ 0 & -1/2 & \dots & 0 \end{pmatrix}, \tag{8.C.367}$$

and further one diagonal generator

$$T_{N^2-1} = \frac{i}{2}\sqrt{\frac{2}{N(N-1)}}\mathrm{diag}\,(-1, \dots, -1, N-1)\,. \tag{8.C.368}$$

(For instance, for $SU(3)$ there are two pairs, proportional to the usual λ_4 and λ_5 and λ_6 and λ_7, and the diagonal hypercharge generator λ_8.) The idea now is to establish a natural one-to-one correspondence between points in $\mathbf{C}^N$ and points in $\mathbf{R}^{2N}$, namely we write all points $(x^1, \dots, x^{2N})$ in $\mathbf{R}^{2N}$ as points in $\mathbf{C}^N$ as follows: $(ix^1 + x^2, \dots, ix^{2N-1} + x^{2N})$. In particular the south pole $(0, 0, ..., 0, 1)$ in $\mathbf{R}^{2N}$ corresponds to the south pole $(0, 0, ..., 0, 1)$ in $\mathbf{C}^N$ and the sphere $\sum_{i=1}^{2N}(x^i)^2 = 1$ in $\mathbf{R}^{2N}$ corresponds to the hypersphere $\sum_{i=1}^{N}|z^i|^2 = 1$ in $\mathbf{C}^N$. Points on the sphere S^{2N-1} in $\mathbf{R}^{2N}$ correspond one-to-one to coset elements of $SO(2N)/SO(2N-1)$. The coset generators of $SO(2N)/SO(2N-1)$ are antisymmetric $2N \times 2N$ matrices A_I $(I = 1, \dots, 2N-1)$ with the entry $+1/2$ in the last column and $-1/2$ in the last row. The coset element $1 + \delta g = 1 + \mathrm{d}t^I A_I$ maps the south pole $s = (0, \dots, 0, 1)$ in $\mathbf{R}^{2N}$ to a point $s + \delta s$ in $\mathbf{R}^{2N}$ where $\delta s = 1/2(\mathrm{d}t^1, \dots, \mathrm{d}t^{2N-1}, 0)$. We know how points in $\mathbf{C}^N$ correspond to points in $\mathbf{R}^{2N}$, so we can ask which coset element in $SU(N)/SU(N-1)$ maps the south-pole in $\mathbf{C}^N$ to the point in $\mathbf{C}^N$ which corresponds to $s + \delta s$. In $\mathbf{C}^N$ the corresponding point is $s + \delta s$ with $\delta s = 1/2(i\mathrm{d}t^1 + \mathrm{d}t^2, \dots, i\mathrm{d}t^{2N-1})$. The coset generators of $SU(N)/SU(N-1)$ act in $\mathbf{C}^N$ as follows: $g = 1 + \mathrm{d}x^\mu K_\mu$ maps the south-pole s to $s + \delta s$ where now $\delta s = 1/2(i\mathrm{d}x^1 + \mathrm{d}x^2, \dots, i\sqrt{\frac{2(N-1)}{N}}\mathrm{d}x^{2N-1})$. We can cover $SO(2N)/SO(2N-1) = S^{2N-1}$ with small patches. Similarly we cover $SU(N)/SU(N-1)$ with small patches. Each patch of S^{2N-1} can be brought by the action of a suitable coset element to the south-pole, and then we can use the inverse of this group element to map this patch back into the manifold $SU(N)/SU(N-1)$. In this way both S^{2N-1} and $SU(N)/SU(N-1)$ are covered by patches which are in a one-to-one correspondence. Each pair of patches has the same ratio of volumes since both patches can be brought to the south pole by the same group element and at the south pole the ratio of their volumes is the same. To find the ratio of the volumes of S^{2N-1} and $SU(N)/SU(N-1)$, it is then sufficient to consider a small patch near the south pole. Near the south pole the vielbeins become unit matrices for coset manifolds, hence the volume of the patches near the south-pole is simply the product of the coordinates of these patches. Consider then a small patch at the south pole of S^{2N-1} with coordinates $(\mathrm{d}t^1, \ldots, \ \mathrm{d}t^{2N-1})$ and volume $\mathrm{d}t^1 \ldots \mathrm{d}t^{2N-1}$. The same patch at the south pole in $\mathbf{C}^N$ has coordinates $\mathrm{d}x^\mu$ where $\left(i\mathrm{d}t^1 + \mathrm{d}t^2, \ldots, i\mathrm{d}t^{2N-1}\right)$ $= \left(i\mathrm{d}x^1 + \mathrm{d}x^2, \ldots, i\sqrt{\frac{2(N-1)}{N}}\mathrm{d}x^{2N-1}\right)$. The volume of a patch in $SU(N)/SU(N-1)$ with coordinates $\mathrm{d}x^1, \ldots, \mathrm{d}x^{2N-1}$ is $\mathrm{d}x^1 \ldots \mathrm{d}x^{2N-1}$. It follows that the volume of $SU(N)/SU(N-1)$ equals the volume of S^{2N-1} times $\sqrt{\frac{N}{2(N-1)}}$ [61],

$$\text{Vol}\ \frac{SU(N)}{SU(N-1)} = \sqrt{\frac{N}{2(N-1)}}\ \text{Vol}\ S^{2N-1}\ . \tag{8.C.369}$$

From here the evaluation of Vol $SU(N)/H$ is straightforward. Using

$$\text{Vol}\ S^{2N-1} = \frac{2\pi^N}{(N-1)!}\, l\ , \tag{8.C.370}$$

where $l = 1$ if one uses the normalization $\operatorname{tr} K_m^2 = -2$, but $l = 2^{2N-1}$ with our normalization of $\operatorname{tr} K_m^2 = -\frac{1}{2}$, we obtain

$$\text{Vol}\ SU(N) = \sqrt{N} \prod_{k=2}^{N} \frac{\sqrt{2}\pi^k}{(k-1)!} 2^{2k-1}\ . \tag{8.C.371}$$

We assumed that $\text{Vol}\, SU(1) = 1$ which seems a natural value but must be, and will be, justified below. Then

$$\text{Vol}\ H = \text{Vol}\ SU(N-2)\text{Vol “}U(1)\text{”}\ , \qquad \text{Vol “}U(1)\text{”} = 4\pi\sqrt{\frac{N}{N-2}}\ ,$$

[61] This result yields the same answer for 8.5.161 as [50], but it yields $\pi^N/(N\,N!)$ for the volume of the complex projective space $CP(N) = SU(N+1)/(SU(N) \times U(1))$ which differs from the result $\text{Vol}[U(N+1)/(U(N) \times U(1))] = \text{Vol}S^{2N}$ given in [88].

$$\text{Vol } SU(N)/H = \tfrac{1}{2}\frac{\pi^{2N-2}}{(N-1)!(N-2)!}2^{2N-1}2^{2N-3}\ . \tag{8.C.372}$$

This then produces formula (8.5.161).

As an application and check of this analysis let us derive a few relations between the volumes of different groups. From now on till the end of this appendix we adopt the normalization $\mathrm{tr}(T_aT_b) = -2\delta_{ab}$ for the generators of all groups involved. Let us check that Vol $SU(2) = 2$Vol $SO(3)$, Vol $SU(4) = 2$Vol $SO(6)$ and Vol $SO(4) = \frac{1}{2}\left(\text{Vol } SU(2)\right)^2$ (the latter will follow from $SO(4) = SU(2) \times SU(2)/Z_2$). We begin with the usual formula for the surface of a sphere with unit radius (given already above for odd N)

$$\text{Vol } S^N = \frac{2\pi^{(N+1)/2}}{\Gamma\left(\frac{N+1}{2}\right)}\ . \tag{8.C.373}$$

In particular Vol $S^1 = 2\pi$ and

$$\begin{aligned} &\text{Vol } S^2 = 4\pi\ , \quad \text{Vol } S^3 = 2\pi^2\ , \quad \text{Vol } S^4 = \tfrac{8}{3}\pi^2\ , \\ &\text{Vol } S^5 = \pi^3, \quad \text{Vol } S^6 = \tfrac{16}{15}\pi^3\ , \quad \text{Vol } S^7 = \tfrac{1}{3}\pi^4\ . \end{aligned} \tag{8.C.374}$$

Furthermore Vol $SO(2) = 2\pi$ since the $SO(2)$ generator with $\mathrm{tr}\, T^2 = -2$ is $T = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and $\exp(\theta T)$ is an ordinary rotation $\left(\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix}\right)$ for which $0 \le \theta \le 2\pi$. The vielbein is unity for an abelian group, and thus the Haar measure is[62] simply $\mathrm{d}\theta$.

With $\text{Vol}\, SO(N) = \text{Vol}\, S^{N-1}\text{Vol}\, SO(N-1)$ we obtain $\text{Vol}\, SO(1) = 1$ and

$$\begin{aligned} &\text{Vol } SO(2) = 2\pi\ , \quad \text{Vol } SO(3) = 8\pi^2\ , \quad \text{Vol } SO(4) = 16\pi^4\ , \\ &\qquad \text{Vol } SO(5) = \tfrac{128}{3}\pi^6\ , \quad \text{Vol } SO(6) = \tfrac{128}{3}\pi^9\ . \end{aligned} \tag{8.C.375}$$

Now consider $SU(2)$. In the normalization $T_1 = -i\tau_1$, $T_2 = -i\tau_2$ and $T_3 = -i\tau_3$ (so that tr $T_aT_b = -2\delta_{ab}$) we find by direct evaluation[63] using Euler angles

[62] One clearly must specify the normalization of the generators T_a; for example by choosing $T_a = \begin{pmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix}$, the range of θ becomes $0 \le \theta \le 4\pi$, but the Haar measure is still $\mathrm{d}\theta$.

[63] Parametrize $g = \mathrm{e}^{\alpha T_3}\mathrm{e}^{\beta T_1}\mathrm{e}^{\gamma T_3}$, determine the range of α, β, γ and compute the group vielbeins.

Vol $SU(2) = 2\pi^2$. This also agrees with (8.C.369) and (8.C.371) for $N = 2$, justifying our assumption that Vol $SU(1) = 1$. For higher N we get

$$\text{Vol } SU(2) = 2\pi^2 \ , \qquad \text{Vol } SU(3) = \sqrt{3}\pi^5 \ , \qquad \text{Vol } SU(4) = \tfrac{\sqrt{2}}{3}\pi^9 \ . \tag{8.C.376}$$

The group elements of $SU(2)$ can also be written as $g = x^4 + i\vec{\tau}\cdot\vec{x}$ with $(x^4)^2 + (\vec{x})^2 = 1$ which defines a sphere S^3. Since near the unit element $g \approx 1 + i\vec{\tau}\cdot\delta\vec{x}$, the normalization of the generators is as before, and hence for this parametrization Vol $SU(2) = 2\pi^2$. This is indeed equal to Vol S^3. In the mathematical literature one finds the statement that Vol $SU(2)$ is twice Vol $SO(3)$ because $SU(2)$ is the double covering group of $SO(3)$. However, we have just found that Vol $SU(2) = \frac{1}{4}$Vol $SO(3)$. The reason is that in order to compare properties of different groups we should normalize the generators such that the structure constants are the same (the Lie algebras are the same, although the group volumes are not). In other words, we should use the normalization that the **adjoint** representations have the same tr T_aT_b. For $SU(2)$ the generators which lead to the same commutators as the usual $SO(3)$ rotation generators (with entries $+1$ and -1) are $T_a = \left\{-\frac{i}{2}\tau_1, -\frac{i}{2}\tau_2, -\frac{i}{2}\tau_3\right\}$. Then tr $T_aT_b = -\frac{1}{2}\delta_{ab}$. In this normalization, the range of each group coordinate is multiplied by 2, leading to Vol $SU(2) = 2^3 \cdot 2\pi^2 = 16\pi^2$. Now indeed Vol $SU(2) = 2$Vol $SO(3)$.

For $SU(4)$ the generators with the same Lie algebra as $SO(6)$ are the 15 antihermitean 4×4 matrices $\frac{1}{4}(\gamma_m\gamma_n - \gamma_n\gamma_m)$, $i\gamma_m/2$, $\gamma_m\gamma_5/2$ and $i\gamma_5/2$, where γ_m and γ_5 are the five 4×4 matrices γ_M obeying the Clifford algebra $\{\gamma_M, \gamma_N\} = 2\delta_{MN}$ [64]. Now, tr $T_aT_b = -\delta_{ab}$ (for example, tr $\left\{\left(\frac{1}{2}\gamma_1\gamma_2\right)\left(\frac{1}{2}\gamma_1\gamma_2\right)\right\} = -1$). Recall that originally we had chosen the normalization tr $T_aT_b = -2\delta_{ab}$. We must thus multiply the range of each coordinate by a factor $\sqrt{2}$, and hence we must multiply our original result for Vol $SU(4)$ by a factor $\left(\sqrt{2}\right)^{15}$. We find then indeed that the relation Vol $SU(4) = 2\,$Vol $SO(6)$ is fulfilled.

[64] As Dirac matrices in six dimensions we take $\gamma_m \otimes \tau_2, \gamma_5 \otimes \tau_2$ and $I \times \tau_3$.

Finally, we consider the relation $SO(4) = SU(2) \times SU(2)/Z_2$. (The vector representation of $SO(4)$ corresponds to the representation $\left(\frac{1}{2}, \frac{1}{2}\right)$ of $SU(2) \times SU(2)$, but representations like $\left(\frac{1}{2}, 0\right)$ and $\left(0, \frac{1}{2}\right)$ are not representations of $SO(4)$ and hence we must divide by Z_2. The reasoning is the same as for $SU(2)$ and $SO(3)$, or $SU(4)$ and $SO(6)$.) We choose the generators of $SO(4)$ as follows

$$T_1^{(+)} = \frac{1}{\sqrt{2}} (L_{14} + L_{23}) \ , \quad T_2^{(+)} = \frac{1}{\sqrt{2}} (L_{31} + L_{24}) \ , \quad T_3^{(+)} = \frac{1}{\sqrt{2}} (L_{12} + L_{34}) \ , \tag{8.C.377}$$

and the same but with minus sign denoted by $T_i^{(-)}$. Here L_{mn} equals $+1$ in the m^{th} column and n^{th} row, and is antisymmetric. Clearly tr $T_a T_b = -2\delta_{ab}$. The structure constants follow from

$$\left[\frac{1}{\sqrt{2}} (L_{12} + L_{34}) , \frac{1}{\sqrt{2}} (L_{14} + L_{23}) , \right] = - (L_{31} + L_{24}) \ , \tag{8.C.378}$$

thus

$$\left[T_i^{(+)}, T_j^{(+)}\right] = -\sqrt{2}\epsilon_{ijk} T_k^{(+)} \ , \quad \left[T_i^{(-)}, T_j^{(-)}\right] = -\sqrt{2}\epsilon_{ijk} T_k^{(-)} \ , \quad \left[T_i^{(+)}, T_j^{(-)}\right] = 0 \ . \tag{8.C.379}$$

We choose for the generators of $SU(2) \times SU(2)$ the representation

$$T_i^{(+)} = \frac{i\tau_i}{\sqrt{2}} \otimes \mathbb{1} \ , \qquad T_i^{(-)} = \mathbb{1} \otimes \frac{i\tau_i}{\sqrt{2}} \ . \tag{8.C.380}$$

Then we get the same commutation relations as for $SO(4)$ generators (8.C.379); however, the generators are normalized differently, namely tr $T_a T_b = -2\delta_{ab}$ for $SO(4)$ but tr $T_a T_b = -\delta_{ab}$ for $SU(2)$. With the normalization tr $T_a T_b = -2\delta_{ab}$ we found Vol $SU(2) = 2\pi^2$. In the present normalization we find Vol $SU(2) = 2\pi^2 \left(\sqrt{2}\right)^3$. The relation Vol $SO(4) = \frac{1}{2} \left(\text{Vol } SU(2)\right)^2$ is now indeed satisfied

$$\text{Vol } SO(4) = 16\pi^4 = \tfrac{1}{2} \left(\text{Vol } SU(2)\right)^2 = \frac{1}{2} \left(2\pi^2 \left(\sqrt{2}\right)^3\right)^2 \ . \tag{8.C.381}$$

D Zero modes and conformal symmetries

The bosonic collective coordinates obtained for gauge group $SU(2)$ and the one-instanton solution could all be identified with rigid symmetries of the action: a_μ with translations, ρ with scale transformations and θ^a with rigid gauge symmetries. Similarly, the fermionic collective coordinates for $SU(2)(\xi^\alpha$ and $\bar{\eta}_{\dot{\alpha}}$ with $\alpha, \dot{\alpha} = 1, 2)$ could be identified with ordinary supersymmetry and conformal supersymmetry. However, the full conformal algebra in 4 Euclidean dimensions is $SO(5, 1)$, and its generators are $P_\mu, K_\mu, D, M_{\mu\nu}$, so one might expect that the conformal boost transformations K_μ and the Lorentz rotations $M_{\mu\nu}$ produce further collective coordinates. As we now show, the transformations due to these symmetries can be undone by suitably chosen gauge transformations with constant gauge parameters [27]. So there are no further bosonic collective coordinates, as we already know from the index theorem discussed in the main text.

Consider first rigid Lorentz transformations. Here one should not forget that in addition to a spin part which acts on the indices of a field they also contain an orbital part that acts on the coordinates: $M_{\mu\nu} = \Sigma_{\mu\nu} + L_{\mu\nu}$. For example, for a spinor one has $\delta(\lambda_{mn})\psi = \frac{1}{4}\lambda_{mn}\gamma_{mn}\psi + (\lambda_{mn}x_m\partial_n)\psi$. One may check that only with this orbital part present the Dirac action is Lorentz invariant. In fact, starting with only the spin part or the orbital part, one can find the other part by requiring invariance of the action. We begin by considering the field strength $F_{\mu\nu} = 2\bar{\sigma}_{\mu\nu}\rho^2/(x^2 + \rho^2)^2$ for an instanton with $k = 1$ in the regular gauge. Under a Lorentz transformation with parameter $\lambda_{\mu\nu} = -\lambda_{\nu\mu}$ one has $\delta_M A_\mu = \lambda_{\mu\nu}A_\nu + \lambda_{mn}x_m\partial_n A_\nu$. (Note that coordinates transform opposite to fields: $\delta x^m = -\lambda_{mn}x^n$. One may check this transformation rule by showing that the Maxwell action is Lorentz invariant (use the Bianchi identities[65]), or just by writing down the transformation law for a covariant vector in general

[65] One has $\delta_M \frac{1}{4}F_{\mu\nu}^2 = F_{\mu\nu}\partial_\mu(\lambda_{\nu\rho}A_\rho) + F_{\mu\nu}\partial_\mu(\lambda_{mn}x_m\partial_n A_\nu) = F_{\mu\nu}\lambda_{mn}x_m\partial_\mu\partial_n A_\nu$. Replacing $\partial_\mu\partial_n A_\nu$ by $-\partial_n\partial_\nu A_\mu - \partial_\nu\partial_\mu A_n$ yields $\partial_\mu(\xi^\mu\mathcal{L})$.

relativity. The Lagrangian transforms into $\partial_\mu(\xi^\mu \mathcal{L})$, where $\xi^\mu = \lambda^{\rho\mu} x_\rho$.). The field strength of the instanton transforms as follows

$$\delta_M F_{\mu\nu} = \lambda_{\mu\rho} F_{\rho\nu} + \lambda_{\nu\rho} F_{\mu\rho} \tag{8.D.382}$$

There is no contribution from the orbital part because x^2 is Lorentz invariant. On the other hand, under a gauge transformation with parameter $\Lambda_{\rho\sigma}$ we obtain[66]

$$\delta_{\text{gauge}} F_{\mu\nu} = [\bar{\sigma}_{\mu\nu}, \tfrac{1}{4}\Lambda_{\rho\sigma}\bar{\sigma}_{\rho\sigma}](2\rho^2/(x^2+\rho^2)^2) = \Lambda_{\nu\sigma} F_{\mu\sigma} - \Lambda_{\mu\sigma} F_{\nu\sigma} \tag{8.D.383}$$

Thus $F_{\mu\nu}$ is invariant under combined Lorentz and gauge transformations with opposite parameters, $\Lambda_{\rho\sigma} = -\lambda_{\rho\sigma}$. Using $\bar{\sigma}_{\rho\sigma} = i\eta_{a\rho\sigma}\tau_a$, it is clear that the $SU(2)$ gauge parameter Λ_a is proportional to $\eta_{a\rho\sigma}\lambda_{\rho\sigma}$. Only the selfdual part of $\lambda_{\rho\sigma}$ contributes. For an anti-instanton we would have needed the anti-selfdual part of $\lambda_{\rho\sigma}$. So we have only proven that $F_{\mu\nu}$ is invariant under combined Lorentz and gauge transformation if the Lorentz parameter is self dual. However, the anti-self dual part of $\lambda_{\rho\sigma}$ leaves $F_{\mu\nu}$ separately invariant, without the need to add compensating gauge transformations. One can prove this directly, using that $\lambda_{\mu\rho}F_{\rho\nu} = -(*\lambda_{\mu\rho})(*F_{\rho\nu})$ and then working out the product of two ϵ-tensors and finally antisymmetrizing in $\mu\nu$, but it is already clear from the index structure: $F_{\mu\nu}$ is proportional to $(\bar{\sigma}_{\mu\nu})_{\alpha'}{}^{\beta'}$ while an anti-selfdual $\lambda_{\rho\sigma}$ has in spinor notation only undotted indices.

Let us now repeat this exercise for the gauge field A_μ. One finds for the combined Lorentz and gauge transformation

$$\delta A_\mu = \lambda_{\mu\nu} A_\nu + \lambda_{\rho\sigma} x_\rho \partial_\sigma A_\mu + [A_\mu, \tfrac{1}{4}\Lambda_{\rho\sigma}\bar{\sigma}_{\rho\sigma}] \tag{8.D.384}$$

The instanton field A_μ for $k=1$ in the regular gauge is given by $A_\mu = (-\bar{\sigma}_{\mu\nu}x^\nu)/(x^2+\rho^2)$. The orbital part with $\lambda_{\rho\sigma}$ now contributes, but there is no term $\partial_\mu \Lambda^a$ in the gauge transformation of A_μ since Λ^a is constant. One obtains

$$\delta A_\mu = \lambda_{\mu\nu} A_\nu + \frac{\bar{\sigma}_{\mu\nu}(\lambda_{\nu\rho} x_\rho)}{x^2+\rho^2} - \frac{(\Lambda_{\nu\sigma}\bar{\sigma}_{\mu\sigma} - \Lambda_{\mu\sigma}\bar{\sigma}_{\nu\sigma})x^\nu}{x^2+\rho^2} \tag{8.D.385}$$

[66] The usual form of an $SU(2)$ gauge transformation is $\delta F_{\mu\nu} = [F_{\mu\nu}, \Lambda^a(x)\frac{\tau^a}{2i}]$, but using $\eta_{a\mu\nu}\eta_{b\mu\nu} = 4\delta_{ab}$ and $\bar{\sigma}_{\rho\sigma} = i\eta_{a\rho\sigma}\tau^a$, this can be rewritten as $\delta F_{\mu\nu} = [F_{\mu\nu}, \frac{1}{4}\Lambda^{\rho\sigma}\bar{\sigma}_{\rho\sigma}]$ where $\Lambda_{\rho\sigma} = -\frac{1}{2}\eta_{a\rho\sigma}\Lambda^a$.

For $\Lambda_{\mu\nu} = -\lambda_{\mu\nu}$ all terms again cancel. Hence, Lorentz symmetry does not yield further zero modes.

In spinor notation these results are almost obvious. In general the selfdual part of a curvature reads in spinor notation

$$(F_{\mu\nu})^u{}_v(\bar{\sigma}_{\mu\nu})_{\alpha'}{}^{\beta'} \tag{8.D.386}$$

where u, v are the indices of $(\tau^a)^u{}_v$, and α', β' are the spinor indices. If we raise/lower indices by ϵ tensors, we get for the instanton solution

$$(F_{\mu\nu})^{uv}(\bar{\sigma}_{\mu\nu})_{\alpha'\beta'} \equiv F^{uv}_{\alpha'\beta'} \sim \delta^u_{\alpha'}\delta^v_{\beta'} + \delta^v_{\alpha'}\delta^u_{\beta'} \tag{8.D.387}$$

It is then clear that $F_{\mu\nu}$ is invariant under diagonal transformations of $SU(2)_R$ and $SU(2)_{\text{gauge}}$, and separately invariant under $SU(2)_L$. For an anti-instanton, the roles of $SU(2)_L$ and $SU(2)_R$ are interchanged.

We come now to the more complicated problem of conformal transformations. A conformal transformation of a field φ with constant parameter a^m is given by[67]

$$\begin{aligned}\delta(a^m K_m)\varphi &= (2a\cdot x\, x^m - a^m x^2)\partial_m\varphi + \delta(2a\cdot x D^{(\text{spin})})\varphi \\ &+ \delta(2a_m x_n M^{(\text{spin})}_{mn})\varphi\end{aligned} \tag{8.D.388}$$

where D^{spin} and M^{spin}_{mn} act only on $\varphi(0)$ and $\delta(a^m K_m)\varphi(x)$ is by definition $[\varphi(x), a^m K_m]$. As the notation indicates, only the spin parts of the dilatational generator D and the

[67]This formula follows from $\delta(a^m K_m)\varphi(x) = [\varphi(x), a^m K_m]$, and $\varphi(x) = e^{-P\cdot x}\varphi(0)e^{P\cdot x}$ with $[\varphi(0), P_\mu] = \partial_\mu\varphi(0)$. One may then use $e^{P\cdot x}K_m = (e^{P\cdot x}K_m e^{-P\cdot x})e^{P\cdot x}$ and $[K_m, P_n] = -2\delta_{mn}D - 2M_{mn}; [P_m, D] = P_m; [P_m, M_{rs}] = \delta_{mr}P_s - \delta_{ms}P_r$ and this yields (8.D.388). In the same way one may derive the Lorentz transformation rule for a spinor $\psi(x)$, with both spin and orbital parts, by using that the spin part is given by $[\psi(0), \frac{1}{2}\lambda_{mn}M_{mn}] = \frac{1}{4}\lambda_{mn}\gamma_{mn}\psi(0)$. One finds then the correct result: $\delta(\frac{1}{2}\lambda_{mn}M_{mn})\psi(x) = \frac{1}{4}\lambda_{mn}\gamma_{mn}\psi(x) + \lambda_{mn}x_m\partial_n\psi(x)$. Given the spin part of the transformation rule of the field at the origin, one derives in this way the orbital part. In this way one finds that the generators of the conformal algebra act as follows on the coordinates: $\delta(P_m)x^n = \delta_m{}^n, \delta(D)x^n = x^n, \delta(M_{st})x^m = x_s\delta_t{}^m - x_t\delta_s{}^m$ and $\delta(K_m)x^n = 2x_m x^n - x^2\delta_m{}^n$. Note that coordinates transform contragradiently to fields. For example, whereas $[\delta(K_m), \delta(P_n)]\varphi = -\delta([K_m, P_n])\varphi$ (by definition), one finds $[\delta(K_m), \delta(P_n)]x^s = \delta([K_m, P_n])x^s$.

Lorentz generators contribute. For example

$$\delta(D^{\text{spin}})A_\mu = [A_\mu, D^{(\text{spin})}] = A_\mu \ , \qquad \delta(\tfrac{1}{2}\lambda_{mn}M^{(\text{spin})}_{mn})A_\mu = \lambda_{\mu\nu}A_\nu \ . \quad (8.D.389)$$

Consider first $F_{\mu\nu}$. We obtain

$$\begin{aligned}\delta(a^m K_m)F_{\mu\nu} &= (2a\cdot x x^m - a^m x^2)\partial_m F_{\mu\nu} + 4a\cdot x F_{\mu\nu} \\ &+4\delta\left(\tfrac{1}{2}a_m x_n M^{(\text{spin})}_{mn}\right)F_{\mu\nu} \text{ with } F_{\mu\nu} = \frac{2\bar{\sigma}_{\mu\nu}\rho^2}{(x^2+\rho^2)^2} \ . \end{aligned} \quad (8.D.390)$$

We already know that the last term can be canceled by a suitable gauge transformation (there are no contributions from $M^{(\text{orb})}_{mn}$ because x^2 is Lorentz invariant). The first term gives $-4\frac{a\cdot x x^2}{x^2+\rho^2}F_{\mu\nu}$. The first and second term together produce then $\frac{4a\cdot x\rho^2}{x^2+\rho^2}F_{\mu\nu}$. But this is the opposite of a translation with parameter $a^m\rho^2$, namely

$$\delta(a^m\rho^2 P_m)F_{\mu\nu} = \frac{-4a\cdot x\rho^2}{x^2+\rho^2}F_{\mu\nu}; \delta(P_m)\varphi = \partial_m\varphi \ . \quad (8.D.391)$$

Thus the following combination of symmetry transformations leaves $F_{\mu\nu}$ **in**variant

$$a^m K_m + \rho^2 a^m P_m + \delta_{\text{gauge}}(\Lambda_{mn} = -2a_m x_n + 2x_m a_n) \quad (8.D.392)$$

Let us now check that also A_μ itself is invariant under this combination of symmetries. We find by direct evaluation, using $A_\mu = (-\bar{\sigma}_{\mu\nu}x^\nu)/(x^2+\rho^2)$ and (8.D.390) and (8.D.383)

$$\begin{aligned}\delta A_\mu &= \left(\frac{-2a\cdot x x^2}{x^2+\rho^2}A_\mu - \bar{\sigma}_{\mu\nu}\frac{(2a\cdot x x_\nu - a_\nu x^2)}{x^2+\rho^2}\right) + 2a\cdot x A_\mu \\ &+ (2a_\mu x_\nu A_\nu - 2x_\mu a_\nu A_\nu) + \left(-\frac{\rho^2 2a\cdot x}{x^2+\rho^2}A_\mu - \frac{\bar{\sigma}_{\mu\nu}a_\nu\rho^2}{x^2+\rho^2}\right) \\ &+ \partial_\mu(-a_\rho x_\sigma\bar{\sigma}_{\rho\sigma}) + [A_\mu, -a_\rho x_\sigma\bar{\sigma}_{\rho\sigma}] \ . \end{aligned} \quad (8.D.393)$$

As in the case of $F_{\mu\nu}$, the sum of the first, third and sixth term cancels. This takes care of the dilatation term and the denominator of A_μ. We are left with terms from the numerator, and Lorentz and gauge terms

$$\begin{aligned}&(2a\cdot x A_\mu + (\bar{\sigma}_{\mu\rho}a_\rho)\left(\frac{x^2}{x^2+\rho^2}\right) + (0 - 2x_\mu a_\nu A_\nu) + \\ &\left(\frac{-\bar{\sigma}_{\mu\nu}a_\nu\rho^2}{x^2+\rho^2}\right) + (\bar{\sigma}_{\mu\rho}a_\rho) + \frac{2x^\nu}{x^2+\rho^2}\begin{pmatrix} a_\nu x_\sigma\bar{\sigma}_{\mu\sigma} - a_\rho x_\nu\bar{\sigma}_{\mu\rho} \\ +0 + a_\rho x_\mu\bar{\sigma}_{\nu\rho}\end{pmatrix}\end{aligned} \quad (8.D.394)$$

The terms denoted by "0" vanish due to $x_\nu A_\nu = 0$. All other terms cancel in the following combinations
(i) the second, fourth, fifth, and seventh nonvanishing contributions sum up to zero. These are the terms with $\bar{\sigma}_{\mu\rho} a_\rho$.
(ii) the first and third-but-last nonvanishing term cancel each other. Here conformal boosts cancel a gauge term.
(iii) the remaining Lorentz term $-2x_\mu a_\nu A_\nu$ cancels the remaining gauge term $2x^\nu(x^2 + \rho^2)^{-1}(a_\rho x_\mu \bar{\sigma}_{\nu\rho})$.

Hence, conformal boosts do not lead to further zero modes either.

E Instantons at finite temperature

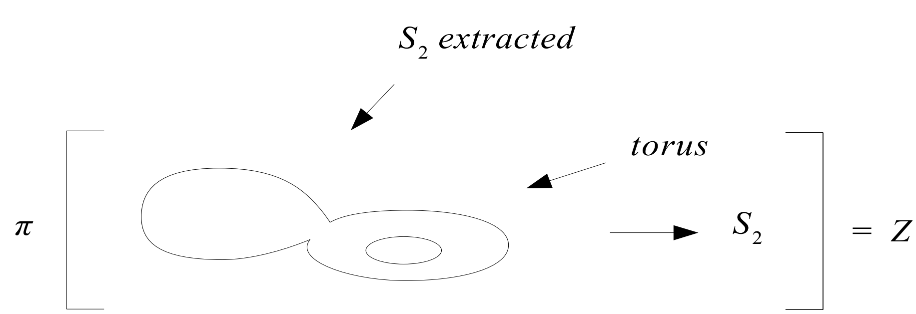
The instantons we have been considering so far live in a Euclidean space which can be obtained from Minkowski spacetime by a Wick rotation. This Euclidean space is not the physical space, but its instantons give a good approximation to the path integral in the physical Minkowski spacetime. There is a second case where Euclidean space has physical applications and in this case the Euclidean space is the physical space. This is the case of finite temperature field theory, where the Boltzmann factors $\exp -\beta E$ give the probabilities for states with energies E. These factors $\exp -\beta E$ can be used to define finite-temperature correlation functions, such as propagators $Tr(T\varphi(x)\varphi(y)e^{-\beta H})$. Converting such expressions into path integrals by inserting complete sets of x- and p-eigenstates as usual, one finds the Euclidean action in the exponent inside the path integral, together with the requirement that bosonic fields are periodic in Euclidean time τ with period β, while fermionic fields are antiperiodic. Thus in this case no Wick rotation is needed: one directly gets the Euclidean path integrals. The question now arises whether there also exist instantons which are periodic in τ because if they exist, they presumably will give a good approximation to the path integrals which describe processes at finite temperature in Minkowski spacetime.

In a classic paper, Harrington and Shepard found these instantons [89]. They have a finite action on the four-dimensional Euclidean space with x, y, z ranging from $-\infty$ to $+\infty$, but $0 \leq \tau \leq \beta$. The configurations with finite actions in finite temperature field theory must have curvatures $F_{\mu\nu}$ which vanish as the **three**-dimensional radius r tends to infinity. Thus

$$A_\mu(r \to \infty) \to g^{-1}(\theta, \varphi, \tau)\partial_\mu g(\theta, \varphi, \tau) \tag{8.E.395}$$

(As in the zero temperature case, the dependence of g on r can be removed by a gauge transformation with $U = P \exp \int_0^r A_r dr$.) We consider as usual the gauge group $SU(2)$, whose group manifold is the 3-sphere S_3. Since at fixed τ for large r we get a two-sphere S_2, and the periodicity of τ defines a one-sphere S_1, the maps from asymptotic space into the gauge group are the maps $S_2 \times S_1 \to S_3$. These maps divide into an infinite set of homotopy classes, with winding numbers $k = 0, \pm 1, \pm 2, \ldots$, just as for ordinary instantons which correspond to the map $S_3(\text{space}) \to S_3(\text{group})$.

In fact $\pi(S_{m_1} \otimes S_{m_2} \ldots \otimes S_{m_k} \to S_n) = Z$ for $m_1 + m_2 + \ldots + m_k = n$. (Better known results are $\pi_m(S_n) = 0$ if $m < n$ but $\pi_m(S_n)$ can be nonzero if $m > n$; for example, $\pi_2(S_1) = 0$ but $\pi_3(S_2) = Z$.) To get an intuitive understanding of this mathematical result, one may consider the simpler case of $S_1 \times S_1 \to S_2$, i.e., the continuous maps from the torus onto the sphere. One can deform the torus such that one extracts a surface which is topologically an S_2, and if one maps the right-hand side of this S_2 and the torus to one point of the other S_2, the homotopy of the rest of the first S_2 to the other S_2 is as usual Z.



(8.E.396)

Thus at finite temperature, topologically distinct periodic field configurations exist, just as in the zero temperature case [90].

These finite-temperature instantons were called calorons (from the latin word calor = heat) and their role in the thermodynamics of a Yang-Mills "gas" (a dilute gas of widely separated calorons and anti-calorons with arbitrary winding numbers was studied). We shall here derive the caloron solutions, but not enter into a discussion of their role in thermodynamics.

Consider $SU(2)$ Yang-Mills gauge theory in Euclidean space

$$\mathcal{L}_E = \frac{1}{2g^2} tr(F_{\mu\nu})^2, F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \tag{8.E.397}$$

where $A_\mu = g A_\mu^a (\sigma^a/2i)$ and σ^a are the Pauli matrices. The problem is to construct a solution of the field equations with finite action which is periodic in τ

$$A_\mu(\vec{x}, \tau) = A_\mu(\vec{x}, \tau + \beta) \tag{8.E.398}$$

We shall again look for solutions which are selfdual (the anti-selfdual case is similar), and make again the ansatz $A_\mu = \frac{1}{2}\sigma_{\mu\nu}\partial_\nu \ln\phi$. The solutions which correspond to this ansatz will be in the Lorentz gauge, $\partial_\mu A_\mu = 0$. Then ϕ is a solution provided $(\Box\phi)/\phi = 0$, and provided ϕ has no zeros. So far this is the same situation as for the usual instantons, but now we must construct a ϕ which is periodic, and whose action, when integrated only over $0 \le \tau \le \beta$, is finite.

Recall that 't Hooft found for zero-temperature instantons the following expression for ϕ

$$\phi = 1 + \sum_{i=1}^{N} \frac{\lambda_i^2}{(x - y_i)^2} \tag{8.E.399}$$

Jackiw, Nohl and Rebbi [27] generalized this to

$$\phi = \sum_{i=1}^{N+1} \frac{\lambda_i^2}{(x - y_i)^2} \tag{8.E.400}$$

Both solutions have winding number N, even though the second solution has $N+1$ poles. We shall show this later, but first we generalize (8.E.400) and (8.E.399) to calorons.

A periodic solution is given by

$$\phi = \sum_{k=-\infty}^{\infty} \frac{\lambda^2}{(\vec{x}-\vec{x}_0)^2 + (\tau - \tau_0 - k\beta)^2} \tag{8.E.401}$$

This series is absolutely convergent. It is clearly periodic in τ, and it is a solution because $\Box\phi = 0$ away from the poles, while ϕ is never zero and at the poles $\phi^{-1}\Box\phi$ still vanishes. One can perform the sum, using complex function theory with

$$\oint_{|z|=R} \frac{1}{(z-a)^2+y^2} \pi \cot \pi z \;\; dz \to 0 \text{ as } R \to \infty \tag{8.E.402}$$

where the contour is a circle with radius $z = (N+\frac{1}{2})\pi$. The solution is

$$\begin{aligned} \phi &= \frac{\pi\lambda^2}{\beta^2 y} \quad \frac{\cot \pi(a-iy) - \cot \pi(a+iy)}{2i} \\ a &= \frac{\tau-\tau_0}{\beta} \;\; ; \;\; y = \frac{|\vec{x}-\vec{x}_0|}{\beta} \end{aligned} \tag{8.E.403}$$

Using the identity

$$\frac{\cot \pi(a-iy) - \cot \pi(a+iy)}{2i} = \frac{\sinh 2\pi y}{\cosh 2\pi y - \cos 2\pi a} \tag{8.E.404}$$

(easy to check for $a=0$ or $y \to 0$), we find

$$\phi = \frac{\pi\lambda^2}{\beta|\vec{x}-\vec{x}_0|} \quad \frac{\sinh\left(\frac{2\pi}{\beta}|\vec{x}-\vec{x}_0|\right)}{\cosh\left(\frac{2\pi}{\beta}|\vec{x}-\vec{x}_0|\right) - \cos\left(\frac{2\pi}{\beta}(\tau-\tau_0)\right)} \tag{8.E.405}$$

This solution is real, periodic, and positive (never zero). However, in the zero-temperature limit ($\beta \to \infty$), ϕ reduces to

$$\phi \to \frac{\lambda^2}{(\vec{x}-\vec{x}_0)^2 + (\tau-\tau_0)^2} \tag{8.E.406}$$

This corresponds to a Yang-Mills field which is everywhere pure gauge[68], so this solution yields vanishing winding number and vanishing action. On the other hand,

[68] One finds $\frac{1}{2}\sigma_{\mu\nu}\partial_\nu \ln\phi = -\sigma_{\mu\nu}(x-x_0)^\nu (x-x_0)^{-2}$ which can be written as $U^{-1}\partial_\mu U$ with $U(x) = i\bar{\sigma}_\mu x_\mu / \sqrt{x^2}$ and $\bar{\sigma}_\mu = (\vec{\sigma}, -i)$. We showed this in section (3.2).

keeping β fixed, and integrating over $0 \leq \tau \leq \beta$ and $0 \leq |\vec{x}| < \infty$ yields a nonvanishing winding number as we shall see. Clearly, the limit $\beta \to \infty$ and the limit $|\vec{r}| \to \infty$ do not commute.

The simplest finite temperature solution which has nonvanishing winding number in the zero-temperature limit is obtained, following 't Hooft, by adding unity to ϕ. This yields a solution with winding number $k = 1$. To obtain solutions with winding number $k = N$, one may replace β in the solution by β/N; then there are N poles in the "fundamental domain" $0 \leq \tau - \tau_0 \leq \beta$. (Equivalently, one could keep β the same, but integrate over the larger domain $0 \leq \tau - \tau_0 \leq \beta N$.) We find then the N-caloron solution

$$\phi_N = 1 + \frac{\bar{\lambda}^2}{2\bar{r}} \frac{\sinh \bar{r}}{\cosh \bar{r} - \cos \bar{\tau}} \tag{8.E.407}$$

where

$$\bar{r} = \frac{2\pi N}{\beta} |\vec{x} - \vec{x}_0|; \quad \bar{\tau} = \frac{2\pi N}{\beta} (\tau - \tau_0); \quad \bar{\lambda} = \frac{2\pi N}{\beta} \lambda \tag{8.E.408}$$

More generally, one can add solutions with arbitrary size λ_j, pole location $x_{0,j}$, and winding number N_j

$$\phi = 1 + \sum_j \frac{\lambda_j^2}{2\bar{r}_j} \frac{\sinh \bar{r}_j}{\cosh \bar{r}_j - \cos \bar{\tau}_i} \tag{8.E.409}$$

This is all very similar to the usual zero-temperature Yang-Mills instantons.

Let us now determine the winding number of the solution ϕ_N in (8.E.407). In the zero-temperature case ϕ_N yields the singular instanton solution, whose winding is concentrated near the pole but not at large radius. Thus we expect for our periodic solution that the winding number comes from the region around the pole. Near the pole at $\bar{r} = \bar{\tau} = 0$ one has $\phi \sim 1 + \bar{\lambda}^2/\bar{x}^2$ with $\bar{x}^2 = \bar{r}^2 + \bar{\tau}^2$. The difference of ϕ_N and $1 + \bar{\lambda}^2/\bar{x}^2$ is regular, and satisfies $\Box(\phi_N - (1 + \bar{\lambda}^2/\bar{x}^2)) = 0$, hence we may drop it. For $\phi = 1 + \bar{\lambda}^2/\bar{x}^2$ we know that the winding number is unity. But since there are N poles, we get winding number N.

One can also calculate the winding number in an interesting, though more complicated way, as follows. We begin with the remarkable fact that the Lagrangian density can be written [69] as a four-fold derivative of $\ln\phi$ [27]

$$\mathcal{L}_E = -\frac{1}{2}trF_{\mu\nu}F_{\mu\nu} = -\frac{1}{2}\Box\Box\ln\phi \tag{8.E.410}$$

Using this expression for $\mathcal{L}_E$, we can compute the winding number k (defined as usual). Due to the periodicity in $\bar{\tau}$, we can drop total $\bar{\tau}$ derivatives. We also choose $[-\pi,\pi]$ as $\bar{\tau}$ integration region in order that the pole at $\bar{\tau}=0$ lies in the middle. This yields

$$\begin{aligned}
-16\pi^2 k &= \int\int_0^\beta trF_{\mu\nu}\,{}^*\!F_{\mu\nu}d^3x d\tau = \int\int_0^\beta \Box\Box\ln\phi d^3x d\tau \\
&= \int d^3x\nabla^2\nabla^2\left(\int_0^\beta \ln\phi\ d\tau\right) = 4\pi r^2\partial_r\nabla^2\int_0^\beta \ln\phi\ d\tau\Big|_{r=0}^{r=R} \\
&= 4\pi\bar{r}^2\partial_{\bar{r}}\bar{\nabla}^2\int_0^{2\pi N}\ln\phi\ d\bar{\tau}\Big|_{\bar{r}=0}^{\bar{r}=\bar{R}} = 4\pi N\bar{r}^2\partial_{\bar{r}}\bar{\nabla}^2\int_0^{2\pi}\ln\phi\ d\bar{\tau}\Big|_{\bar{r}=0}^{\bar{r}\to\infty} \\
&= 4\pi\int_{-\pi}^{\pi}d\bar{\tau}\left[\bar{r}^2\partial_{\bar{r}}\bar{\nabla}^2\ln\phi\right]_{\bar{r}=0}^{\bar{r}=\bar{R}}
\end{aligned} \tag{8.E.411}$$

There is no contribution from the term with $\bar{R}\to\infty$, as expected. To study the term with small $\bar{r}$, we approximate ϕ_N at small $\bar{r}$ by expanding the cosh to second order

$$\phi_N \simeq 1 + \frac{\bar{\lambda}^2}{2+\bar{r}^2-2\cos\bar{\tau}} \simeq \frac{\bar{\lambda}^2}{2+\bar{r}^2-2\cos\bar{\tau}}. \tag{8.E.412}$$

We must then evaluate the limit $\bar{r}\to 0$ in

$$-\bar{r}^2\partial_{\bar{r}}\left[\frac{1}{\bar{r}}\partial_{\bar{r}}^2\bar{r}\ln(2+\bar{r}^2-2\cos\bar{\tau})\right] \tag{8.E.413}$$

where we used $\nabla^2 = \frac{1}{r^2}\partial_r r^2\partial_r = \frac{1}{r}\partial_r^2 r$. Of course we expect only a contribution from the region where $\bar{\tau}=0$ or $\bar{\tau}=2\pi$, but for the moment we keep all $\bar{\tau}$. Straightforward

[69] The proof is tedious but straight forward. Outside the singularities one has $\Box\phi=0$, and then $\Box\Box\ln\phi = -\Box(\phi_{,\mu}/\phi)^2$ and this equals $-2(\phi_{,\mu\nu}/\phi)^2 + 8\phi_{,\mu\nu}\phi_{,\mu}\phi_{,\nu}\phi^{-3} - 6(\phi_{,\mu}/\phi)^4$. The same result is found for $trF_{\mu\nu}^2$ if one uses $A_\mu = \frac{1}{2}\sigma_{\mu\nu}\partial_\nu\ln\phi$ and $tr\sigma_{\mu\nu}\sigma_{\rho\sigma} = \frac{1}{2}tr\{\sigma_{\mu\nu},\sigma_{\rho\sigma}\} = 2(\delta_{\mu\sigma}\delta_{\nu\rho} - \delta_{\mu\rho}\delta_{\nu\sigma} + \epsilon_{\mu\nu\rho\sigma})$.

differentiation yields

$$\frac{12\bar{r}^3}{(2+\bar{r}^2-2\cos\bar{\tau})^2}-\frac{16\bar{r}^5}{(2+\bar{r}^2-2\cos\bar{\tau})^3} \tag{8.E.414}$$

We now turn to the integration over $\bar{\tau}$. Expanding the co-sine we obtain

$$\frac{12r^3}{(\bar{r}^2+\bar{\tau}^2+\ldots)^2}-\frac{16r^5}{(\bar{r}^2+\bar{\tau}^2+\ldots)^3} \tag{8.E.415}$$

Using that $\epsilon^3/(\epsilon^2+y^2)^2$ and $\epsilon^5/(\epsilon^2+y^2)^3$ are regularized delta function $\delta(y)$, we find that ϕ_N indeed has winding number N. At finite temperature the unity in (8.E.407) does not make a difference, but, as explained before, in the limit $\beta\to\infty$ the winding remains only if one includes this unity.

We finally show that the zero-temperature solution in $(8.E.400)$ with $N+1$ poles has winding number N. Putting little spheres around the poles, the $N+1$ poles give each a contribution $+1$ to the winding number, but the large sphere at infinity yields a contribution -1, so that $k=N$. One can give an elegant derivation of this result by multiplying ϕ by $N+1$ factors $(x-y_i)^2$ and an overall constant $(\sum\lambda_i^2)^{-1}$. Multiplication of ϕ by a factor $(x-y_i)^2$ or a constant does not change the winding number because, as we have seen, $\phi=\ln(x-y_i)^2$ corresponds to pure gauge. One finds then a manifestly nonsingular and positive expression for ϕ

$$\phi=\left[\sum_{j=1}^{N+1}\lambda_i^2\Pi_{j\neq i}(x-y_j)^2\right]\Big/\left(\sum_{j=1}^{N+1}\lambda_i^2\right) \tag{8.E.416}$$

Since this expression for ϕ tends to r^{2N} as r tends to infinity, we can easily calculate the winding number by using again Gausz' theorem, this time in 4 dimensions,

$$\begin{aligned} k &= -\frac{1}{16\pi^2}\int\Box\Box\ln\phi d^4x \\ &= \lim_{r\to\infty}-\frac{1}{16\pi^2}(2\pi^2)r^3\frac{\partial}{\partial r}\left(\frac{1}{r^3}\frac{\partial}{\partial r}r^3\frac{\partial}{\partial r}\right)\ln(r^{2N}+\cdots)=N \end{aligned} \tag{8.E.417}$$

where we used that the surface of S_3 is $2\pi^2$. Note that since the integrand is nonsingular, the use of Gausz' theorem for the evaluation of k is justified, and we obtained only a contribution from the sphere at infinity.

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Chapter 9

The anomalous magnetic moment of the electron and muon

The anomalous magnetic moment $a = \frac{1}{2}(g-2)$ of the electron (a_e) and muon (a_μ) have been measured and calculated over the years to extremely high precision. In early 1947, Nafe, Nelson and Rabi found that the hyperfine structure of the ground state of hydrogen and deuterium[1] deviated by 0.26% from theory [4]. Breit suggested that the electron might possess an anomalous contribution to its magnetic moment of the order of α times the value of the magnetic moment in Dirac theory. [3] Instigated by Rabi, Foley and Kusch [5] looked for similar effects in more complicated atoms and found a discrepancy of 0.1% between the measured value of the of the g factor in Na and Ga atoms and the theoretical value $g = 2$ as predicted by Dirac theory. A correction of 0.1% to the magnetic moment would explain both the deviations in the Na and Ga atoms, and also the hyperfine discrepancies because the electron and the nucleus contribute each a 0.1% correction. Furthermore, at about the same time Lamb and Retherford [6] found shifts in energy levels which should be degenerate according

[1] The magnetic moment of the proton was known at that time with 0.03% accuracy and that of the deuteron with 0.04% accuracy. The hyperfine structure was according to Pauli [1] due to the interaction of the nuclear and electronic magnetic moments. The theoretical result for the hyperfine splitting of S states was given by E. Fermi [2]. G. Breit calculated the corrections to Fermi's result due to nuclear motion [3]. The total uncertainty in the calculated values of the hyperfine splitting was 0.05%.

to Dirac theory. This started the modern era of quantum electrodynamics, where field quantization of the electrons supplants quantum mechanics based on the Dirac equation. In a $g-2$ experiment for muons at Brookhaven, the calculated one-loop **electroweak** corrections to a_μ are four times the expected experimental uncertainty, and as a consequence this $g-2$ experiment leads to another test of the electroweak sector of the Standard Model. It might even lead to a breakdown of the Standard Model and be an indication for supersymmetry [7]. We discuss the supersymmetric contributions to $g-2$ in Appendix C.

Before we begin our discussions of the field theoretical contributions to the anomalous magnetic moment, we recall that in 1928 the Dirac equation had given a firm theoretical derivation that the magnetic moment corresponds to $g=2$ for an electron. Classical electrodynamics predicted, of course, $g=1$. As one might expect, there was a time before 1928 when experiments yielded puzzling discrepancies between the measured value of the magnetic moment and the theoretical value with $g=1$. A little anecdote illustrates this confusion. (I thank E. Remiddi and V. Telegdi for providing me with this anecdote. See also A. Pais "Subtle is the Lord, The science and life of Albert Einstein", section 14b, page 245. A detailed account of the early experiments measuring g is given in Peter Gallison, "How experiments end", Univ. Chicago Press 1987, chapter 2.)

According to classical electrodynamics, a charged particle with angular momentum $\vec{M}$, charge e, and mass m carries a magnetic dipole moment $\vec{\mu}$ given by

$$\vec{\mu} = g\frac{e}{2mc}\vec{M} \tag{9.0.1}$$

with $g=1$. When it was discovered in the beginning of the 1900's that matter consists of charged particles with very different mass to charge ratio (the positive components have a charge whose absolute value is a small integer multiple of the electron[2] charge, while their masses are larger by a factor 2000 or more), it was

[2]The discovery of the electron is sometimes attributed, in addition to J.J. Thomson, to H.A. Lorentz

realized that a change in the magnetization of a bar should induce a change in its angular momentum and *vice versa*.[3] That fact became known as the Einstein-de Haas effect after the paper by A. Einstein and W.J. de Haas, "Experimenteller Nachweis der Ampèreschen Molekularströme", *Verh. d. Deutsch. Phys. Ges.* **17** (1915) 152 ("Nachweis" means proof in German.) This is perhaps the only experimental paper written by Einstein. Eq. (9.0.1) above, with $g = 1$, was written as

$$\vec{M} = \frac{2mc}{e}\vec{\mu} = \lambda\vec{\mu} \tag{9.0.2}$$

and an experiment was proposed to measure λ in order to obtain a new value for the charge to mass ratio of the lightest particle, the electron.

The theoretical value for λ which follows from (9.0.2) is $\lambda = 1.13 \times 10^{-7}$ in Gaussian units (the effect is small). They found $\lambda = 1.11 \times 10^{-7}$, with an agreement which was almost embarassing. Indeed the authors observed that even if the agreement was due to chance ("*auf Zufall beruhen*"), nevertheless even with a 10% uncertainty the effect was quantitatively established.

A related experiment on "Magnetization by Rotation" was carried out almost at the same time by S.J. Barnett, *Phys. Rev.* **6** (1915) 239 and **10** (1917) 7. His aim was to show that rotation can induce magnetization for explaining the magnetization of the earth in terms of its daily rotation[4]. When he became aware of the work of Einstein and de Haas, he presented his results as a measurement of the gyromagnetic ratio of the electron. For "electrons in slow motion" he expected from the theory 7.1×10^{-7} in the proper units, and obtained 3.1×10^{-7} in the first experiment, and results ranging

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and P. Zeeman who received the second Nobel prize in physics in 1902 "for their researches into the influence of magnetism upon radiation phenomena". Thomson received the 1906 Nobel prize "for his theoretical and experimental investigations on the conduction of electricity by gases".

[3]Applying a magnetic field to a bar, the small magnetic moments in the bar due to electrons in their orbits and (although unknown at that time) also due to the electron spins, become aligned, yielding a net, nonvanishing angular momentum. The bar must then counter-rotate to preserve angular momentum.

[4]If one gives an unmagnetized piece of iron an angular acceleration, the little permanent magnets inside it experience a torque that alignes them: rotation produces magnetism.

from 5.1×10^{-7} to 6.5×10^{-7} in the second experiment. He claimed satisfactory agreement with the Einstein model (but was not satisfied in other respects: "... Their paper contains no reference to the previous work of Maxwell, Schuster, Richardson, or myself").

Later on, the experiment was repeated by Emil Beck, "Zum experimentellen Nachweis der Ampèreschen Molekularströme", *Ann. d. Physik* **60** (1919) 109. He carried out three series of measurements, with final results $\lambda = 0.57 \times 10^{-7}, \lambda = 0.60 \times 10^{-7}, \lambda = 0.64 \times 10^{-7}$ "*sehr genau die Hälfte des zu erwartenden Wertes*" 1.13×10^{-7} (very precisely half of the expected value). He could not explain the disagreement with the Einstein-de Haas result, despite "*eine persönliche Unterredung mit Prof. Einstein*" (a personal discussion with Prof. Einstein), which was for him "*noch ganz besonders wertvoll*" (quite valuable).

We now know that Beck was right - the factor g in eq. (9.0.1) is equal to 2 for the electron, and slightly larger than 2 when QED radiative corrections are accounted for. (Only the spins of electrons contribute to (9.0.2), the orbital angular momenta cancel each other). This factor 2 should be in the denominator of the r.h.s. of eq. (9.0.2), implying a theoretical value of λ equal to $(1.13/2) \times 10^{-7} = 0.565 \times 10^{-7}$, very close to the values found by Beck. But at that time g was still equal to 1, and Beck could only get a job as high school teacher (de Haas continued his scientific career in Leiden).

When in 1925 Goudsmit and Uhlenbeck (both at Leiden, but unaware of de Haas's work) proposed $g = 2$ [8] to fit the experimental data on the anomalous Zeeman splitting of spectral lines[5], and the Dirac equation of 1928 gave a theoretical explanation, theory and experiment seemed for almost two decades in agreement as far as g was concerned. However, studies of radiative corrections in the 1930's and 1940's for var-

[5] A few months earlier, Pauli had put forward his exclusion principle [9], but he found that one can put **two** electrons in each state. The discovery of spin by Goudsmit and Uhlenbeck also explained this puzzling factor 2.

ious processes in QED gave infinities and the problem of eliminating these became a central issue. So when in 1947 Rabi and coworkers reported that experiments saw small but definitely nonvanishing departures of the value of the magnetic moment from Dirac theory, and Lamb and coworkers simultaneously reported similar deviations in the spectral lines for certain atomic energy transitions, theorists had their work cut out. In a few years, the full renormalizable theory of QED was established by Schwinger, Feynman, Dyson, Tomonaga, Kramers, Bethe, Breit, French, Weisskopf and others.

We present in what follows the theory of radiative corrections to the anomalous magnetic moment of the electron and muon. This is an excellent exercise in what is technically called on-shell renormalization of QED. The value of $g-2$ is a relatively simple S-matrix element. One must deal simultaneously with ultraviolet and infrared divergences, and also take into account that the external fermions are on-shell. In Appendix A we discuss these issues further. We calculate the one-loop correction of Schwinger in Appendix B, but also discuss in detail the two-loop corrections. Next we discuss 3-loop and higher-loop corrections. Then we discuss a recent experiment on $g-2$ for the muon, and calculate the contributions due to the weak interactions. In Appendix C, we discuss the predictions of the minimally susy Standard Model (the MSSM) for $g-2$. For a review of the status of QED, see [10].

At the one-loop level, Schwinger's famous result from end 1947 [11] states that the magnetic moment of charged leptons (electron, muon, and since the late 1970's also the tau lepton) is related to its spin by $\vec{\mu} = g\frac{e\hbar}{2mc}\frac{\vec{\sigma}}{2} = (1+a)\frac{e}{mc}\vec{s}$ where $\vec{s} = \frac{\hbar}{2}\vec{\sigma}$ and a at the one-loop level is given by

$$\begin{aligned} &g = 2(1+a) \\ &a_e = a_\mu = a_\tau = \tfrac{\alpha}{2\pi} = 0.001\ 161\ 409\ 7 \end{aligned} \tag{9.0.3}$$

We shall repeat this calculation below; only the one-loop vertex corrections contribute, and the radiative corrections to a are both ultraviolet (UV) and infrared (IR) finite, as well as independent of the gauge chosen. The physical picture behind this calculation is quite simple[6]: an electron dissociates part of the time into an electron and a photon, during which time the electron has a different four-momentum and during this time it couples differently to the magnetic field. This resolved the problem Rabi and coworkers had found, but it also raised the question whether theory and experiment agree at the two-loop level.

Since a_μ and a_e should be dimensionless, and UV and IR convergent, they can only contain mass-independent terms and terms proportional to the ratios of masses, but they cannot depend on the renormalization scale. In the QED sector with electrons (e), muons (μ) and tau-leptons (τ) one has thus [10]

$$\begin{aligned} a_e &= a_e(\text{no } m) + a_e\left(\frac{m_e}{m_\mu}\right) + a_e\left(\frac{m_e}{m_\tau}\right) + a_e\left(\frac{m_e}{m_\mu}, \frac{m_e}{m_\tau}\right) \\ a_\mu &= a_\mu(\text{no } m) + a_\mu\left(\frac{m_\mu}{m_e}\right) + a_\mu\left(\frac{m_\mu}{m_\tau}\right) + a_\mu\left(\frac{m_\mu}{m_e}, \frac{m_\mu}{m_\tau}\right) \end{aligned} \tag{9.0.4}$$

The 2-loop m-independent terms due to QED were first calculated by R. Karplus and N.M. Kroll in 1949 [13], and a small error was corrected by A. Petermann [14] and C. Sommerfeld [15]. The result is

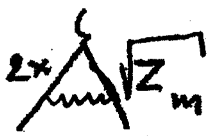
$$\begin{aligned} a_e^{QED}(\text{no m}) = a_\mu{}^{QED}\ (\text{no m}) &= \left(\frac{\alpha}{\pi}\right)^2 \left[\frac{197}{144} + \frac{\pi^2}{12} + \frac{3}{4}\zeta(3) - \frac{1}{2}\pi^2 \ln 2\right] \\ &= -0.328\ 478\ 965\ldots \left(\frac{\alpha}{\pi}\right)^2 = -.000\ 001\ 772 \end{aligned} \tag{9.0.5}$$

There are five graphs which contribute to the anomalous magnetic moment of the electron: a ladder and a crossed box graph with two virtual photons, and further 1-loop vertex corrections and 1-loop fermion selfenergy corrections and a 1-loop photon selfenergy correction inserted into Schwinger's one-loop graph. Furthermore, there are 1-loop graphs with an insertion of a 1-loop counter term. We display the graphs, and quote below in each column the contribution to (9.0.5), as obtained from dimensional

[6]This idea is due to Wick who explained in this way why the magnetic moment of the proton is very different from the value Dirac theory predicts. [12]

regularization, omitting an overall factor $(\frac{\alpha}{\pi})^2$.

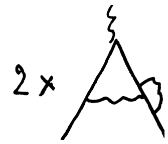
$\left(-\frac{3}{4}\frac{1}{d-4}+\frac{107}{48}+\frac{1}{18}\pi^2\right)$



$\left(\frac{3}{2}\frac{1}{d-4}-1\right)\times \frac{1}{2}(1-2(d-4))$

$\underbrace{}_{\left(\frac{11}{48}+\frac{1}{18}\pi^2\right)} = 0.778$

$\underbrace{}_{\left(\frac{1}{6}+\frac{13}{36}\pi^2-\frac{5}{6}\pi^2\ln 2+\frac{5}{4}\zeta(3)\right)} = -0.467$



$\left(-\frac{1}{2}\frac{1}{d-4}-\frac{19}{24}+\frac{1}{18}\pi^2+\frac{1}{3}\pi^2\ln 2-\frac{1}{2}\zeta(3)\right)$

$\left(-\frac{1}{d-4}+\frac{5}{24}-\frac{1}{18}\pi^2\right)$

$\left(\frac{3}{2}\frac{1}{d-4}-\frac{7}{4}\right)$

$\underbrace{}_{\left(-\frac{7}{3}+\frac{1}{3}\pi^2\ln 2-\frac{1}{2}\zeta(3)\right)} = -0.654$

$\underbrace{}_{\left(\frac{119}{36}-\frac{1}{3}\pi^2\right)} = 0.016$ (9.0.6)

As one can see, the contributions from different graphs cancel each other a good deal, and as a consequence the two-loop corrections are a factor 1000 smaller than the one-loop correction.

These results were obtained using dimensional regularization. [16] In the works of Petermann and Sommerfeld, the IR divergences of individual graphs were regulated by giving the photon a small mass[7] λ, and one finds then that $\frac{1}{d-4}$ is replaced by $\ln \lambda^2/m_e^2$. For higher-loop calculations (3 loops and 4 loops), dimensional regularization is far

[7] If one regulates QED by giving the photon a small mass, one should sum over 3 rather than 2 polarizations. This is crucial for the Lamb shift [17]. Schwinger and Feynman who initially overlooked this subtlety got an incorrect result. French and Weisskopf who got the correct result for the Lamb shift by using noncovariant methods, delayed publication because their calculations gave a result which differed from Schwinger's and Feynman's. For the anomalous magnetic moment one can safely ignore the subtleties introduced by a longitudinal polarization of the massive photon because $k_\mu k_\nu/m^2$ terms in the photon propagator cancel due gauge invariance.

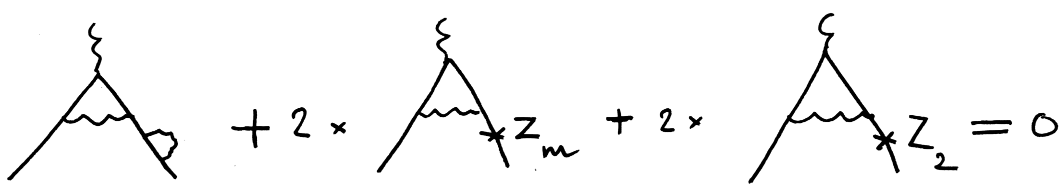
simpler than any other scheme, and hence we shall base our discussion of the two-loop corrections on ordinary 't Hooft-Veltman dimensional regularization.

Before dimensional regularization became the universally preferred regularization scheme, the method of dispersion relations was widely used for higher-loop calculations. Using dispersion relations and a particular regularization scheme (Pauli-Villars for example), the subtraction procedure leads to finite dispersion integrals, and the subtraction constants are fixed by the renormalization procedure. In this way one obtains renormalized quantities in terms of subtracted dispersion relations without the need for any explicit knowledge of counter terms (Z factors). In the dispersion approach, one first treats selfenergies and vertex corrections with the dispersion method, and then one uses the results as building blocks in larger diagrams. On the other hand, in dimensional regularization one evaluates separately the diagrams with and without counter term insertions, and only at the end one adds their contributions.

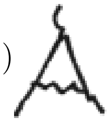
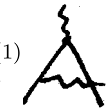
Other approaches that have been used include a partial wave expansion in 4-dimensional Euclidean space, and, of course, various numerical methods. We do not discuss there approaches but refer to the article by Kinoshita in [10].

Because the residue of the renormalized (finite) fermion propagator is unity according to on-shell renormalization, the usual correction factors $(\sqrt{\text{residue}}\,)^{-1}$ for external lines in the definition of the S-matrix are just unity, and all corrections on external fermion lines cancel. For example

2×



Because $Z_1 = Z_2$ in QED, only one factor Z_1 contributes to the 2-loop graphs

 + 2×  + 2×  $= (3Z_1^{(1)} - 2Z_2^{(1)})$  $= Z_1^{(1)}$  (9.0.7)

(The minus sign in front of $-Z_2^{(1)}$ can only be understood by an explicit calculation of the third graph, keeping track of all factors i.) However, since $Z_1^{(1)} = Z_m^{(1)}$[8] one can also write the contribution with $Z_1^{(1)}$ as $Z_m^{(1)}$ times the one-loop graph, and this has been done in the first column of the 2-loop graphs. As a result, one only finds counter terms with Z_m and Z_3, but none with Z_1 and Z_2. Also at higher loop it is believed that all contributions to $g-2$ only need the counter terms Z_m and Z_3, but not Z_1 or Z_2.

The product of the one-loop graph and $Z_m^{(1)}$ does not only yield a contribution $2(\sqrt{Z_m} - 1)\frac{\alpha}{2\pi}$, where $\frac{\alpha}{2\pi}$ is the one-loop correction to $\frac{1}{2}(g-2)$, but because there is a pole in Z_m, one must also calculate the 1-loop correction to order $d-4$.[9] The on-shell mass renormalization correction can be found in textbooks [19], and reads

$$Z_m = 1 + \frac{\alpha}{\pi}\left(\frac{3}{2}\frac{1}{d-4} - 1 + \frac{3}{4}\gamma_E - \frac{3}{4}\ln\frac{4\pi\mu^2}{m^2}\right) + \mathcal{O}\left(\frac{\alpha}{\pi}\right)^2 \tag{9.0.8}$$

On the other hand, the one-loop correction to the magnetic moment in d dimensions to order $d-4$ will be derived in Appendix B. Dropping the terms with the Euler

[8] At the one-loop level, in the Lorentz gauge and using dimensional regularization, $Z_m = 1 - \frac{\alpha}{4\pi}(\frac{3}{\epsilon} + 3\{\frac{4}{3} - \gamma_E + \ln\frac{4\pi\mu^2}{m^2}\})$ and $Z_1 = 1 - \frac{\alpha}{4\pi}(\frac{1}{\epsilon} - 2(-\frac{1}{\epsilon}) + 4 - 3\gamma_E + 3\ln\frac{4\pi\mu^2}{m^2})$ where $\epsilon = \frac{1}{2}(4-d)$ and the pole $\frac{1}{\epsilon}$ in Z_1 is due to an ultraviolet divergence but the pole indicated by $(-\frac{1}{\epsilon})$ in Z_1 is due to an infrared divergence [19]. Clearly the total Z_1 and Z_m satisfy $Z_1^{(1)} = Z_m^{(1)}$. The equality of the γ_E and $\ln(4\pi\mu^2/m^2)$ terms is not surprising because one can multiply diagrams by overall factors with Γ functions, which lead to the γ_E and logarithms, but the equality of the pole terms and the finite terms is surprising. The 3-loop on-shell Z factors can be found in [18], and there one can see that Z_m is no longer equal to Z_1 at higher loop levels.

[9] The need of terms of higher order in $d-4$ is perhaps the only disadvantage of dimensional regularization. The continuous dimensional regularization method is much more convenient than the Pauli-Villars method together with a small photon mass.

constant γ_E, with $\ln 4\pi$ and with $\ln(\mu^2/m_e^2)$ one obtains [20]

$$a^{(1)}(d \text{ dims}) = \frac{\alpha}{2\pi}(1 - 2(d-4)) \tag{9.0.9}$$

The product of (9.0.8) and (9.0.9) yields the result for the second graph in the first column of (9.0.6).

The crossed box is UV and IR finite by itself. There are no IR and UV subdivergences as one easily checks either by power counting or by letting the momenta of one or both photons tend to zero, and the overall divergence of this graph contributes only to the charge renormalization.

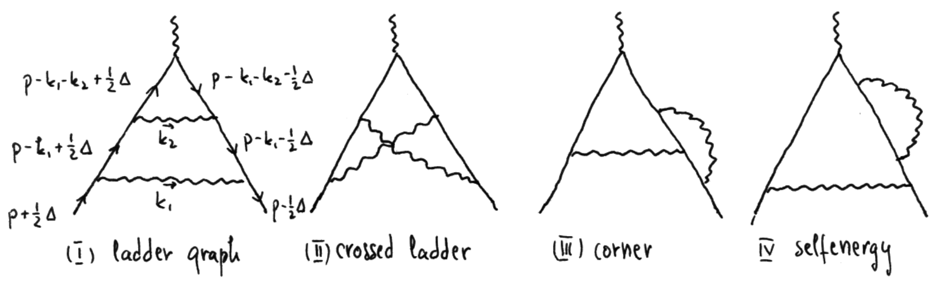
The "corner graphs" with vertex corrections on the side and the "selfenergy graphs" with an electron selfenergy are each divergent. It is natural to combine the selfenergy graph with the graph with a mass renormalization counter term for the internal electrons, but there remain divergences as one can see. However, the sum of the corner graphs, selfenergy graphs, and the graph with an internal mass renormalization is UV and IR finite. (The contributions from the counter terms with Z_1 and Z_2 to these vertex corrections and fermion selfenergy cancel separately). The contribution from the Z_m counter term is not simply Z_m times the one-loop result

 $Z_m \neq Z_m$  (9.0.10)

Rather, the graph with a Z_m counter term insertion contains 3 instead of 2 fermion propagators, and must be calculated separately.

Finally, the vacuum polarization graph and the counter term with Z_3 produce together an IR and UV finite part which is also gauge-choice independent. This contribution should be UV and IR finite and gauge-choice independent because the imaginary part of the vacuum polarization graph yields the cross section for e^+e^- annihilation into an e^+e^- pair. We calculate this contribution in Appendix B by using a dispersion integral.

We now give some more details of the two-loop corrections to $g-2$. Higher-loop radiative corrections have become heavily dependent on algebraic manipulation computer programs, such as Schoonschip, Reduce and Ashmedai in the past, and Form at present. [21] In addition, each person or group of persons develops its own computer software, and often also a whole array of theoretical tricks. For these reasons it is impossible to give a comprehensive review, but one can work through an example, and our example is $g-2$ at two-loop order. We shall discuss four graphs together, the ladder graph (I), the crossed ladder graph (II), the graph with the vertex correction on the corner (III) and the graph with a self energy of the fermion (IV). The vacuum polarization graph (Schwinger's graph but with an extra fermion loop inserted in the virtual photon) will be evaluated separately by a dispersion integral, as we already mentioned.



(9.0.11)

We discuss these four graphs together because one can choose the momenta in all of them such that only the fermion lines carry the momentum $\Delta = p_1 - p_2$, where p_1 is the incoming and p_2 the outgoing momentum. We define $\Delta^2 = -t, p = \frac{1}{2}(p_1+p_2), p_1 = p + \frac{1}{2}\Delta$ and $p_2 = p - \frac{1}{2}\Delta$. The figure illustrates the kinematics for the ladder graph. We want to compute $F_2(t=0)$, where $F_2(t)$ is defined in (9.0.12) but one cannot set $\Delta = 0$ in the graphs themselves for two reasons

(i) these graphs also contain F_1, and F_1 is IR divergent at $t=0$

(ii) to extract $F_2(t)$ we use a projection operator but this operator has a pole at $t=0$.

We first extract $F_2(t)$ from a given graph for nonvanishing $t = -(p_1-p_2)^2$, and then expand $F_2(t)$ sufficiently far in terms of Δ that singularities in the projection operator used to project out F_2 cancel. We use standard 't Hooft-Veltman dimensional regularization. (In QED there are no problems with γ^5 or $\epsilon^{\mu\nu\rho\sigma}$ so there is no need to use dimensional reduction). Let a given vertex graph be parametrized by

$$V^\mu = F_1(t)\gamma^\mu + F_2(t)\gamma^{\mu\nu}\frac{(p_1 - p_2)_\nu}{2im} \tag{9.0.12}$$

where $\gamma^{\mu\nu} = \frac{1}{2}[\gamma^\mu, \gamma^\nu]$ and $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$. Using the Gordon identities (which follow from the Dirac equation $\not{p}_1 u_1 = imu_1$ and $\bar{u}_2\not{p}_2 = \bar{u}_2 im$)

$$\begin{aligned} im\bar{u}_2\gamma^\mu u_1 &= \bar{u}_2 p_1^\mu u_1 + \bar{u}_2\gamma^{\mu\nu}p_{1\nu}u_1 \\ im\bar{u}_2\gamma^\mu u_1 &= \bar{u}_2 p_2^\mu u_1 - \bar{u}_2\gamma^{\mu\nu}p_{2\nu}u_1 \end{aligned} \tag{9.0.13}$$

one finds the identity

$$2im\ \bar{u}_2\gamma^\mu u_1 = \bar{u}_2(p_1 + p_2)^\mu u_1 + \bar{u}_2\gamma^{\mu\nu}u_1(p_1 - p_2)_\nu \tag{9.0.14}$$

Hence, the vertex can also be written as

$$\begin{aligned} V^\mu &= (F_1 + F_2)\gamma^\mu - F_2\frac{(p_1 + p_2)^\mu}{2im} \\ &= F_1\frac{(p_1 + p_2)^\mu}{2im} + (F_1 + F_2)\gamma^{\mu\nu}\frac{(p_1 - p_2)_\nu}{2im} \end{aligned} \tag{9.0.15}$$

The term with $(p_1 + p_2)^\mu$ in the last line contains the convection current, while the term with $\gamma^{\mu\nu}$ contains the spin current. These terms correspond to an interaction $ieA_\mu V^\mu = ieA_\mu[-F_1\ \bar{\psi}\ \overleftrightarrow{\partial^\mu}\ \psi/2m - (F_1 + F_2)\partial_\nu(\bar{\psi}\gamma^{\mu\nu}\psi)/2m]$ and after partial integration the second term gives an interaction energy density formula

$$\mathcal{H}_{\rangle\backslash\sqcup} = -\mathcal{L}_{\rangle\backslash\sqcup} = -\rceil(\mathcal{F}_\infty + \mathcal{F}_\in)\vec{\mathcal{B}}\cdot\psi^\dagger\vec{\sigma}\psi/(\in\Updownarrow) \tag{9.0.16}$$

The anomalous magnetic moment is then according to (9.0.1)

$$a = F_2(0) \tag{9.0.17}$$

because on-shell renormalization implies $F_1(t = 0) = 1$. To obtain the contribution from a given graph to $\frac{1}{2}(g - 2)$, one may cast V^μ into the form $V^\mu = A\gamma^\mu + B(p_1 + p_2)^\mu/2im$ and identify then $F_2(0) = -B(0)$.

To project out the terms with $F_2(t)$ from a given graph, we use a d-dimensional projection operator. We set

$$V^\mu = (F_1(t) + F_2(t))\,\gamma^\mu - F_2(t)\frac{(p_1+p_2)^\mu}{2i\,m} \tag{9.0.18}$$

and define

$$\Pi_\mu = (-i\not{p}_1 + m)[c_1\gamma_\mu + c_2\frac{(p_1+p_2)_\mu}{2im}](-i\not{p}_2 + m) \tag{9.0.19}$$

where the first and last factor implement the mass-shell conditions $\not{p}_1 = im$ and $\not{p}_2 = im$ for the fermions. If we then choose

$$c_1 = \frac{-1}{d-2}\frac{2m^2}{t}\frac{1}{t-4m^2}; c_2 = \frac{-1}{d-2}\frac{2m^2}{t}\frac{1}{t-4m^2}\frac{(d-2)t+4m^2}{t-4m^2} \tag{9.0.20}$$

one may check by direct calculation that

$$tr\Pi_\mu\gamma^\mu = 0;\;\; tr\Pi_\mu\frac{(p_1+p_2)^\mu}{(-2im)} = 1 \tag{9.0.21}$$

Thus

$$tr\,\Pi_\mu V^\mu = F_2(t) \tag{9.0.22}$$

Contractions are taken in d dimensions, so for example $\gamma^\mu\gamma_\mu = d$ and $\gamma^\mu\gamma^\rho\gamma_\mu = (2-d)\gamma^\rho$, but the trace of the unit matrix is defined to be equal to 4.

Even though we have regulated the UV and IR divergences by using dimensional regularization, we still cannot set $\Delta = 0$ in $tr\,\Pi_\mu\,V^\mu$ because of the occurrence of terms with $\frac{1}{t}$ in Π_μ. However, $F_2(t)$ should be nonsingular as $t \to 0$; in other words, these $1/t$ singularities are spurious. So one expands each of the four fermion propagators in terms of Δ, using $\frac{1}{A+B} = \frac{1}{A}(1 - \frac{B}{A+B})$ and iterating as many times as necessary. One only needs to expand until one has obtained two extra powers of Δ in the numerator because this is enough to cancel the pole at $t = 0$ in Π_μ. Thus the expansions we use are as follows

$$\frac{1}{(p-k_1+\frac{1}{2}\Delta)^2+m^2} \to \frac{1}{(p-k_1)^2+m^2}$$
$$\left[1 - \frac{\frac{1}{4}\Delta^2}{(p-k_1)^2+m^2} - \frac{(p-k_1)\cdot\Delta}{(p-k_1)^2+m^2} + \frac{\{(p-k_1)\cdot\Delta\}^2}{\{(p-k_1)^2+m^2\}^2}\right] \tag{9.0.23}$$

Since $F_2(-\Delta^2)$ does not depend on the orientation of Δ, we average over Δ, keeping Δ orthogonal to p because $\Delta \cdot p = (p_1 - p_2) \cdot (p_1 + p_2) = 0$. Thus in the expression for $\Pi_\mu V^\mu$ with every fermion propagator expanded to second order, we select the terms quadratic in Δ^μ and make the following replacement

$$\Delta_\mu \Delta_\nu \to \int \frac{d\Omega}{4\pi} \Delta_\mu \Delta_\nu = \frac{\Delta^2}{d-1}(\delta_{\mu\nu} - p_\mu p_\nu / p^2) \tag{9.0.24}$$

(Equivalently, we go to the center of mass frame, and then replace $\vec{\Delta}_i \vec{\Delta}_j$ by $\frac{1}{d-1}\vec{\Delta}^2 \delta_{ij}$). One finds at this point that all terms with poles in t cancel, and in the remainder we set $\Delta = 0$. As a result of these manipulations, one finds an expression for the contribution of the ladder graph to $F_2(0)$ in d dimensions of the following form

$$\begin{aligned} F_2 \quad = \quad & \int dk_1 dk_2 N[d, k_1^2, k_2^2, k_1 \cdot k_2, p \cdot k_1, p \cdot k_2, m^2] \\ & [(p-k_1)^2 + m^2]^{-\alpha} [(p - k_1 - k_2)^2 + m^2]^{-\beta} [(p-k_1)^2 + m^2]^{-\gamma} \\ & [(p - k_1 - k_2)^2 + m^2]^{-\delta} k_1^{-2} k_2^{-2} \end{aligned} \tag{9.0.25}$$

where $\alpha, \beta, \gamma, \delta$ can each range from 1 (no expansion at all) to 3 (twice expanded). If one propagator has been expanded twice, none of the others need to be expanded; for example if $\delta = 3$ then $\alpha = \beta = \gamma = 1$. In this way one finds a result for the ladder graph with four different propagators (including the photon propagators in this counting) and $\alpha + \beta + \gamma + \delta \leq 6$.

The selfenergy graph also leads to 4 different propagators, but the crossed box and the corner graphs lead to 5 different propagators. In the numerators, one may replace $k_i \cdot p_j$ by $-\frac{1}{2}\{(k_i - p_j)^2 + m^2 - k_i^2\}$ and in this way one cancels many propagators (or one reduces the power of a given propagator). For the ladder graph one cannot express all 5 inner products $p \cdot k_1, p \cdot k_2, k_1^2, k_2^2$ and $k_1 \cdot k_2$ into the 4 propagators, since there is one inner product left for which we take $k_1 \cdot k_2$. Also for the selfenergy graph we are left with a factor $k_1 \cdot k_2$ to some power, but for the crossed box graph and the corner graph no factors are left in the numerator because these graphs have 5 independent propagators.

The ladder graph is now given by an expression of the form

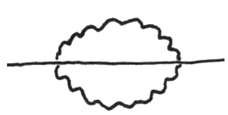
$$F(d, \alpha, x_1, x_2, x_3, x_4) =$$
$$= \int dk_1 dk_2 (k_1 \cdot k_2)^\alpha [(k_1 - p)^2 + m^2]^{-x_1} [(k_1 + k_2 - p)^2 + m^2]^{-x_2} (k_1^2)^{-x_3} (k_2^2)^{-x_4} \tag{9.0.26}$$

with $x_3 = x_4 = 1$. To compute such momentum integrals one derives a set of relations between F's with different $\alpha, x_1, x_2, x_3, x_4$. The basic strategy is to use that in dimensional regularization one may integrate by parts, dropping boundary terms. Thus

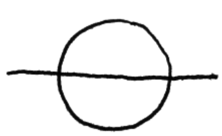
$$\int dk_1 dk_2 \frac{\partial}{\partial l_\mu} (v_\mu F) = 0 \tag{9.0.27}$$

where l can be equal to k_1 or k_2, and v can be equal to p, k_1, or k_2. So a given integrand F generates a set of 6 relations between different F's. It turns out that one can express all F's into a set of "master integrals" which have propagators with $x_j = 0$ or $x_j = 1$, and all have $\alpha = 0$. Each of the four graphs can be expressed in terms of the following set of master integrals

$$= \int dk [k^2 + m^2]^{-1} \tag{9.0.28}$$



$$= \int dk_1 dk_2 [(k_1 + k_2 - p)^2 + m^2]^{-1} k_1^{-2} k_2^{-2} \tag{9.0.29}$$



$$= \int dk_1 dk_2 [(k_1 + k_2 - p)^2 + m^2]^{-1} [k_1^2 + m^2]^{-1} [k_2^2 + m^2]^{-1} \tag{9.0.30}$$

The first two integrals are special cases of (9.0.25) but the last integral is generated by the crossed ladder graph. The results for each of the four graphs in (9.0.11) in d dimensions is given in figure (9.0.6). Adding the renormalization counter terms the final answer for $a = \frac{1}{2}(g-2)$ is both IR and UV finite. This concludes our discussion of the m-dependent two-loop corrections to $\frac{1}{2}(g-2)$.

For historical interest we mention that Karplus and Kroll [13] wrote the ladder graph, corner graph and selfenergy graphs, together with all graphs with renormalization factors, as the lowest order graph with 1-loop renormalized propagators and vertices. For example they wrote the crossed box graph as follows

$$\begin{aligned}
&\bar{\psi}(p_2)\gamma_\nu \left[m + \frac{i\gamma}{2}\frac{\partial}{\partial p}\int_{m^2}^{\infty} d\mu\right]\gamma_\lambda \left[m + \frac{i\gamma}{2}\frac{\partial}{\partial p'}\int_{m^2}^{\infty} d\mu'\right]\gamma^\mu \\
&\left[m + \frac{i\gamma}{2}\frac{\partial}{\partial p''}\int_{m^2}^{\infty} d\mu''\right]\gamma^\nu \left[m + \frac{i\gamma}{2}\frac{\partial}{\partial p'''}\int_{m^2}^{\infty} d\mu'''\right]\gamma^\lambda\psi(p_1) \\
&\int \frac{d^4k d^4k'}{k^2(k')^2}\frac{1}{(p-k)^2+\mu}\,\frac{1}{(p'-k-k')^2+\mu'}\,\frac{1}{(p''-k-k')^2+\mu''}\,\frac{1}{(p'''-k')^2+\mu'''}
\end{aligned} \tag{9.0.31}$$

where at the end one must set all μ's equal to m^2 and $p = p' = p_2$, $p'' = p''' = p_1$, and where $\gamma\frac{\partial}{\partial p}$ stands for $\gamma^\mu\frac{\partial}{\partial p^\mu}$. First they evaluated the scalar integral with six (!) propagators, using Feynman auxiliary variables, and then they acted on the result with the operator $\bar{\psi}(p_2)\cdots\psi(p_1)$ in front. They made small mistakes in the crossed box graph and in the corner graph.

The 3-loop m-independent corrections due to QED were completed by S. Laporta and E. Remiddi [20]. There are 40 different Feynman graphs, and each individual graph may still contain $\frac{1}{d-4}$ and $\frac{1}{(d-4)^2}$ divergences. In the sum all divergences cancel, and the total analytical result for the sum of all 40 graphs reads

$$\begin{aligned}
a_e^{QED}(\text{no } m) &= \left(\frac{\alpha}{\pi}\right)^3\left\{\frac{83}{72}\pi^2\zeta(3) - \frac{215}{24}\zeta(5) + \frac{100}{3}\left[\left(a_4 + \frac{1}{24}\ln^4 2\right) - \frac{1}{24}\pi^2\ln^2 2\right]\right. \\
&- \left.\frac{239}{2160}\pi^4 + \frac{139}{18}\zeta(3) - \frac{298}{9}\pi^2\ln 2 + \frac{17101}{810}\pi^2 + \frac{28259}{5184}\right\} \\
&= \left(\frac{\alpha}{\pi}\right)^3 1.181\,241\,456\ldots
\end{aligned} \tag{9.0.32}$$

where $\zeta(p) = \sum_{n=1}^{\infty} n^{-p}$ and $a_4 = \sum_{n=1}^{\infty}\frac{1}{2^n n^4}$.

The only m-dependent contribution at the 2-loop level comes from the vacuum polarization. This is clear if one looks for a moment at the graphs in (9.0.6). For

a_e it comes from the insertion of a muon-loop or tau-loop into the virtual photon of Schwinger's graph for the electron but it is too small to be detected. Namely, the correction due to a μ loop is $\left(\frac{\alpha}{\pi}\right)^2 \frac{1}{45}\left(\frac{m_e}{m_\mu}\right)^2 = 2.8 \cdot 10^{-12}$ (see Appendix B) and for a τ-loop it is $(\frac{\alpha}{\pi})^2 \frac{1}{45}\left(\frac{m_e}{m_\tau}\right)^2 = 0.1 \cdot 10^{-13}$ whereas experimentally $\Delta a_e^{exp} = 4.3 \cdot 10^{-12}$. The hadronic and electroweak corrections are $-1.6\ 10^{-12}$, again not yet measurable.[10] So a_e is a purely QED affair and only involves electrons. Up to 4 loops contribute in the QED sector, and the contributions from the 891 Feynman diagrams which contribute at this order have been numerically calculated by Kinoshita and coworkers [22]. The most recent value is $a_e^{4\text{ loop}} = (-1.5098(384))(\frac{\alpha}{\pi})^4$. In the final result for a_e the theoretical uncertainty in the QED sector is slightly larger than the experimental uncertainty[11]

$$\begin{aligned}
1 + a_{e^-}{}^{\text{exp}} &= 1\,001\,159\,652\,188.4\ (4.3)\ 10^{-12} \\
1 + a_{e^+}{}^{\text{exp}} &= 1\,001\,159\,652\,187.9\ (4.3)\ 10^{-12} \\
1 + a_e{}^{th} &= 1 + c_1 \frac{\alpha}{\pi} + c_2 \left(\frac{\alpha}{\pi}\right)^2 + c_3 \left(\frac{\alpha}{\pi}\right)^3 + c_4 \left(\frac{\alpha}{\pi}\right)^4 + \ldots + \delta a_e + \Delta a_e \\
& \quad c_1 = 0.5; c_2 = -0.328\,478\,965\ldots; c_3 = 1.181241456\ldots; c_4 = -1.5098 \\
\delta a_e &= -1.66\ 10^{-12}
\end{aligned} \tag{9.0.33}$$

The theoretical uncertainty δa_e is mostly due to the numerical evaluation of c_4. In fact, the dominant contribution to the theoretical uncertainty does not come from numerical evaluation of complicated integrals for 3-loop and 4-loop graphs, but from the value of the fine structure constant α, which gives an uncertainty Δa_e of the order of $27 \cdot 10^{-12}$. If one uses the value for α obtained from the quantum Hall effect (α_{QHE})

[10] The electroweak effects are negligible because they are proportional to $G_F m_e^2$ and for the muon (with a factor $G_F m_\mu^2$) the contribution to a_μ at the one-loop level is $1.95 \cdot 10^{-9}$, as we shall calculate, while $(m_e/m_\mu)^2 \sim (207)^{-2}$. The hadronic corrections for the muon are of the order of 6.10^{-8}, and multiplication by $(m_e/m_\mu)^2$ yields $1.6 \cdot 10^{-12}$.

[11] The magnetic moment of the electron (and positron) is nowadays best measured by trapping an electron in a "Penning trap" [10], an enclosure with a homogeneous magnetic field and an electrostatic quadrupole field. The whole system can be treated as an atom, with the electromagnetic fields replacing the nucleus.

one finds

$$1 + a_e{}^{th} = 1\ 001\ 159\ 652\ 153.5\ (1.2)\ (28.0)\ 10^{-12} \tag{9.0.34}$$

where the first error is due to the numerical error in c_4 and c_3, while the second is the error in α as obtained from the quantum Hall effect. Thus for a_e, experiment and theory agree within 1.3 standard deviations

$$a_e^{\exp} - a_e^{th} = 35(28)10^{-12} \tag{9.0.35}$$

but further progress must await a better determination of α. One can also assume that the results for $a_e^{\exp}$ and a_e^{th} are exactly equal, and use this to determine the value of α. These two methods to determine α give the following results

$$\begin{aligned} \alpha_{QHE}^{-1} &= 137.036\ 003\ 7(33) \\ \alpha_{a_e}^{-1} &= 137.035\ 999\ 58\ (14)\ (50) \end{aligned} \tag{9.0.36}$$

The first error in the last line is due to c_4 and c_3, while the second error is from $a_e^{\exp}$. This concludes our discussion of the anomalous magnetic moment of the electron.

For a_μ things are very different. The mass-independent terms are the same as for the electron, but the mass dependent terms yield quite different contributions, as we discuss below (9.0.41) in more detail. One finds already a measurable contribution to $a_\mu(\frac{m_\mu}{m_e})$ at the two-loop level,

(9.0.37)

Inserting an electron loop into the virtual photon in Schwinger's graph for a_μ yields [23]

$$a_\mu{}^{QED}\left(\frac{m_\mu}{m_e}\right) = \left(\frac{\alpha}{\pi}\right)^2 \left[\frac{1}{3}\ln\frac{m_\mu}{m_e} - \frac{25}{36}\right.$$

$$+\frac{\pi^2}{4}\frac{m_e}{m_\mu} - 4\left(\frac{m_e}{m_\mu}\right)^2 \ln\frac{m_\mu}{m_e} + 3\left(\frac{m_e}{m_\mu}\right)^2 + \mathcal{O}\left(\frac{m_e}{m_\mu}\right)^3\Bigg]$$
$$= 1.09426\left(\frac{\alpha}{\pi}\right)^2 \tag{9.0.38}$$

which is a factor -3 larger than the m-independent two-loop contributions in (9.0.5)! This is due to the enhancement factor $\ln\frac{m_\mu}{m_e} = 5.3$ in a_μ instead of the suppression factor $(\frac{m_e}{m_\mu})^2$ in a_e.[12] The insertion of a tau loop gives a correction to a_μ which is of the same form as the correction to a_e due to a muon loop [24]

$$a_\mu{}^{QED}\left(\frac{m_\mu}{m_\tau}\right) = \left(\frac{\alpha}{\pi}\right)^2\left[\frac{1}{45}\left(\frac{m_\mu}{m_\tau}\right)^2 + \mathcal{O}\left(\frac{m_\mu}{m_\tau}\right)^4 \ln\frac{m_\mu}{m_\tau}\right] \simeq 7.79\cdot 10^{-5}\left(\frac{\alpha}{\pi}\right)^2 \tag{9.0.39}$$

The sum of these two theoretical contributions is of order $6\cdot 10^{-7}$.

The last a_μ experiment was the 1977 CERN experiment which gave

$$1 + a_\mu{}^{\text{exp}} = 1\ 001\ 165\ 923\ 00(840)10^{-11} \tag{9.0.40}$$

Clearly, the two-loop corrections of (9.0.38) and (9.0.39) do contribute here. In fact, 3-loop graphs from pure QED with a running $\alpha(QED)$ due to e, μ and τ loops and hadronic corrections contribute.[13] The theoretical result had an error of $76\cdot 10^{-11}$, so much less than the experimental error, and theory and experiment agreed with these errors.

The new Brookhaven experiment increases the experimental resolution by a factor 20: $\Delta a_\mu^{\text{exp}} = \pm 40\cdot 10^{-11}$. This is still a factor 100 less accurate than for the electron. Protons from the Brookhaven AGS hit a target and produce pions which decay in

[12] To illustrate how an electron loop in a muon propagator can give a much larger correction than a muon loop in an electron propagator, consider (as a simplified example) the function $\ln\frac{M^2+m^2}{M^2}$. If $M = m_\mu$ and $m = m_e$ one finds $\sim m_e^2/m_\mu^2$, but in the reverse case one finds $\sim \ln m_\mu^2/m_e^2$.

[13] The terms with powers of $\ln(m_\mu/m_e)$ in a_μ coming from vacuum polarization loops can be accounted for by the running of $\alpha\,(QED)$ [25]. However, there are also terms with $\ln m_\mu/m_e$ in a_μ coming from electron loops in light-by-light scattering graphs which are not accounted for by the running of $\alpha(QED)$ [26].

flight into 3 GeV muons (μ^+) which are 97% polarized. [14] These muons are introduced into a ring of $R = 7$ meter radius, with a homogeneous magnetic field pointing upwards which is due to superconducting magnets. Since $m_\mu = 105 MeV$ and $\tau_\mu = 2.2 \cdot 10^{-6}$ sec, $\gamma = \frac{3000}{105}$ and the muons live on average 65 μsec and make $\frac{3000}{105}\tau_\mu \frac{c}{2\pi R} \simeq 400$ revolutions before they decay. The muon spin precesses in the horizontal plane with a precession rate $\Omega = \frac{d\phi}{dt}$ which is a factor $\gamma \times a = 30 \times \alpha/2\pi = 3/86$ times the orbital angular velocity. Hence every revolution the muon spin precesses 12^0, and every 4.4 μsec it makes a complete turn. The muons decay into positrons (and neutrinos) which are mostly emitted in the direction of the muon spin. Those emitted forward have higher energy than those emitted backward, hence if one only measures positrons with an energy above the average positron energy, and plots the number of detected positrons versus time (in bins of 0.25 μsec), one finds an oscillation with frequency 4.4 μsec. A given batch of muons can then easily produce $65/4.4 \simeq 15$ oscillations. One imposes the results of many batches of muons, but because the sum of a harmonic functions is again a harmonic function, one need not be precise in joining the results of different batches into one oscillation curve. Hence by measuring the higher-energy positrons over long times, these oscillation curves are known to high precision ($\Delta a_\mu^{\mathrm{exp}} \simeq 40 \cdot 10^{-11}$ and $a_\mu \sim 1/860$ corresponds to knowing the difference of the maximum of the first and the last oscillation of the curve to about $5 \cdot 10^{-7}$). Because $\Delta a_\mu^{\mathrm{exp}}/a_\mu \sim \ 4 \cdot 10^{-7}$ one must measure the magnetic field with the same

[14] There is a good reason for selecting muons with this energy. One needs electric fields $\vec{E}$ to collimate the beam. A travelling muon sees then a magnetic field $\vec{B}\,' = -\gamma \vec{\beta} \times \vec{E}$ according to special relativity. This field $\vec{B}'$ leads to an additional precession of the muon spin. In addition, there is a second additional precession of the muon spin due to the Thomas effect: the electric field $\vec{E}$ accelerates the muon a little bit, and in an accelerated frame the muon spin precesses (this is the reason for the famous Thomas factor 1/2 in the spin-orbit coupling of the hydrogen atom). These two additional precessions cancel each other for a particular velocity (hence energy) of the muon. Namely, according to the BMT equation [27] the longitudinal polarization $\hat{\beta} \cdot \vec{s}$ of the muon spin $\vec{s}$ in the direction of the muon velocity $\vec{\beta}$ changes in time as follows $\frac{d}{dt}(\hat{\beta} \cdot \vec{s}) = -\frac{e}{mc}\,\vec{s}_\perp \cdot \left[a_\mu \hat{\beta} \times \vec{B} + \left(a_\mu - \frac{1}{\gamma^2-1}\right)\beta \vec{E}\right]$ where $\vec{s}_\perp$ is the component of the muon spin perpendicular to $\vec{\beta}$. For $a_\mu = (\gamma^2 - 1)^{-1}$ the effects of the electric quadrupole fields on the $g-2$ experiment cancel, and this corresponds to muons with energy 3 GeV.

precision, but this can be done using magnetic resonance probes and this leads to a very precise measurement of the anomalous magnetic moment a_μ.

Theoretically one finds the following contributions

$$a_\mu{}^{\text{theory}} = a_\mu{}^{QED} + a_\mu{}^{\text{had}} + a_\mu{}^{\text{EW}} \tag{9.0.41}$$

which we now discuss separately.

In $a_\mu{}^{QED}$ one needs all QED corrections for e, μ and τ leptons up to 5 loops. (For the 5 loop effects only the leading terms are needed and these "trivially enhanced terms" are estimated by using the renormalization group. Note that for a_e one needs only 4-loop QED effects but for a_μ one needs 5-loop QED effects due to the enhancement factors discussed before.) They yield

$$\begin{aligned} a_\mu{}^{QED} &= \frac{\alpha}{2\pi} + 0.76\ldots\left(\frac{\alpha}{\pi}\right)^2 + 24.0\ldots\left(\frac{\alpha}{\pi}\right)^3 \\ &+ 126.0\ldots\left(\frac{\alpha}{\pi}\right)^4 + 930(170)\left(\frac{\alpha}{\pi}\right)^5 \\ &= 1\,001\,165\,847\,06(2)10^{-11} \end{aligned} \tag{9.0.42}$$

The increasing value of the coefficients in the α/π expansion is due to the presence of large $\ln m_\mu/m_e \sim 5.3$ terms. Here the theoretical error is much less than the anticipated experimental error of $4 \cdot 10^{11}$.

In $a_\mu{}^{\text{had}}$ one finds effects of hadronic loops in photon propagators and hadronic corrections to light-by-light scattering. In the latter case, three virtual photons couple the muon line to a hadronic blob, and the external photon couples also to the hadronic blob.[15]

(9.0.43)

[15] Contributions from hadronic blobs which couple with 2 photons to the muon line vanish due to Furry's theorem. This in turn proves that the graphs with light-by-light scattering are gauge invariant, and hence infrared finite.

The effects of hadronic vacuum polarization can be estimated by using the data for $e^+e^- \to$ hadrons and dispersion relations. The dominant contribution comes from the kinematical region of low-lying $\pi - \pi$ resonances such as the ρ and below.[16] The contribution is estimated to be $6831(59)(20)10^{-11}$. [29] For comparison we note that the muon loop contributes less than 1% to the total two loop contribution, the electron loop contributes 300%, and hadron loops contribute 4% . Note that the theoretical uncertainty in a_μ^{had} due to the data for $e^+e^- \to$ hadrons is twice $\Delta a_\mu{}^{exp}$ (!). One clearly needs a factor 4 reduction in the experimental uncertainties in the low energy process of $e^+e^- \to$ hadrons. It has been claimed that extrapolating from known data on τ decay from CLEO at Cornell and using CVC one may achieve this.[17] (A τ^- lepton decays into ν_τ and W^-; the W^- decays into hadrons. This vertex has weak-isospin 1, and final states with an even number of pions have G parity +1, which means that only the vector (but not the axial vector) current contributes. According to CVC, the weak-interaction isovector Lorentz-vector vertex is equal to the QED vertex.) Higher-order QED corrections to hadronic vacuum polarization must also be taken into account, but here the theoretical uncertainties are fortunately smaller than $\Delta a_\mu^{\mathrm{exp}}$.

The graphs for light-by-light scattering by hadrons in (9.0.43) pose a more serious problem. There is no general method to compute these graphs; instead one has to use models. Recent estimates[18] based on "realistic models" find $a = 89.6\ (15.4) \cdot 10^{-11}$.

[16] Because the muon lies much closer to the ρ meson than the electron ($m_\mu/m_\rho \gg m_e/m_\rho$), the hadronic contribution to a_μ is much larger than that to a_e.

[17] R. Alemany, M. Davier and A. Höcker, *Eur. Phys. J.* **C2** (1998) 123. The absolute error in the hadronic vacuum polarization from the ρ meson (with isospin 1) is 1.7% while the error due to ω (with isospin 0) is 0.2% and the J/ψ and Y contribute 0.1% and 0% respectively. The major error in the vacuum polarization comes thus from the ρ meson whose contribution can be determined with better accuracy using CVC and τ decays since it has isospin 1. (See CLEO collaboration, *Phys. Rev. D* **61** (2000) 112002). Another important source of experimental uncertainty comes from hadrons in the final state of a few GeV, see S. Eidelman and F. Jegerlehner, *Z. Phys. C* **67** (1995) 585, in particular table 3a. The error is 1.8% and here Frascati and Beijing (and perhaps Novosibirsk) are expected to improve the data.

[18] Initially a sign error in the contraction of two ϵ-tensors performed with FORM gave a negative contribution, and with this incorrect sign the Brookhaven experiment deviated from QED by 2.4 σ. With the

Let us only mention that if one were to match the hadronic corrections to the vacuum polarization to $g-2$ by a massive quark loop, one finds that this quark has a mass $160 MeV$. Using this same quark in a loop for light-by-light scattering, one would predict that the hadronic corrections are $a = 150 \cdot 10^{-11}$. [30] The total hadronic contribution is $a_\mu(hadr) = 6739\,(67)\,10^{-11}$.

A very interesting contribution to a_μ comes from the electroweak sector of the SM. At the one-loop level one finds [31]

$$\begin{aligned} a_\mu{}^{EW} &= \frac{G_F m_\mu{}^2}{8\sqrt{2}\pi^2}\left[\frac{10}{3} + \left\{\frac{-5}{3} + \frac{1}{3}(1-4\sin^2\theta_w)^2\right\} + \mathcal{O}\left(\frac{m_\mu{}^2}{m_h{}^2}\right)\right] \\ &= 195(10)\ 10^{-11} \text{ (reduced to } 151(4)\,10^{-11} \text{ at two-loops)} \end{aligned} \tag{9.0.44}$$

where the $\frac{10}{3}$ comes from two triangle graphs with W^+, W^-, ν_μ and W^+, h^-, ν_μ and W^-, h^+, ν_μ in the loop, while the terms within curly brackets come from the triangle graphs with a Z and $Im\,h^0$ exchanged, and the term denoted by $\left(\frac{m_\mu^2}{m_h^2}\right)$ comes from the exchange of a Higgs boson.[19] (The fields h^+, h^- and $Im\,h^0$ are the would-be Goldstone bosons). The graph with h^+, h^- does not contribute to $g-2$ because it does not have enough γ matrices. Unfortunately, the first two terms partially cancel

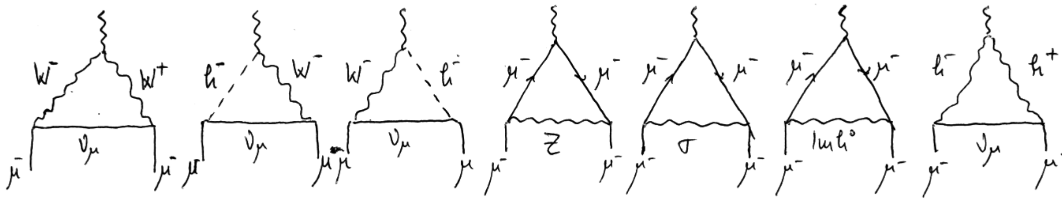
correct sign this is reduced to 1.6 σ [28].

[19] In fact, a full 2-loop calculation in the electroweak sector has been achieved. If one of the two-loops is electromagnetic there is again a logarithmic enhancement and these effects reduce the one-loop result in (9.0.44) by 20%

$$a_\mu^{EW}(2-loop) = 151\ (4)\ 10^{-11}\ ;\ a_\mu{}^{EW}(2-loop)/a_\mu^{EW}(1-loop) = -97\frac{\alpha}{\pi}$$

The contributions depending on the top mass m_t are a small part of this so totally unobservable. Also the effects of m_H are unobservable. Notice that (9.0.45) is not of order one times $\frac{\alpha}{\pi}$, rather of order 100, due to large logarithms of m_μ/m_e. This 2-loop calculation is a nice confirmation of the renormalizability of the SM, which does not allow divergences proportional to $\bar\psi\gamma^{\mu\nu}\psi F_{\mu\nu}$. Goldstone bosons, Faddeev-Popov ghosts, 't Hooft gauges, etc. all enter the calculations, and one must also deal with finite renormalizations here.

each other.



(9.0.45)

In the unitary gauge the graphs with h^- and $Im\ h^0$ vanish, and the triangle graph with a Higgs (σ) exchanged is proportional to $\frac{m_\mu{}^2}{m_h{}^2}$ and can be neglected.[20] Clearly, the one-loop electroweak corrections are five times the expected experimental uncertainty, so *the muon anomalous magnetic moment yields a realistic test of the electroweak sector of the SM.* The factor m_μ^2 in front of (9.0.44) shows that the muon anomalous magnetic moment restricts new physics (or the fantasy of theorists) more than the electron anomalous magnetic moment: although a_e is experimentally better known than a_μ by a factor 100 ($\Delta a_e^{\rm exp} \sim 4 \cdot 10^{-12}$ and $\Delta a_\mu^{\rm exp} \sim 40 \cdot 10^{-11}$), the enormous enhancement factor $(m_\mu/m_e)^2 \sim 42000$ turns the tide in the favour of the muon.

We summarize the results for a_e and a_μ. For the electron and positron one has

$$\begin{aligned} a_e^{\rm exp} &= 1\,159\,652\,188.4\,(4.3)\,10^{-12} \\ a_e^{\rm th} &= 1\,159\,652\,153.5\,(1.2)\,(28.0) \cdot 10^{-12} \end{aligned} \tag{9.0.46}$$

The first error in the theoretical result is due to the numerical evaluation of 3-loop

[20]Most calculations are nowadays performed not using the unitary gauge but rather renormalizable gauges. Then one also needs the would-be Goldstone bosons which in earlier days were absent because they were eaten by the to W and Z bosons in the unitary gauge. In the unitary gauge, there were routing ambiguities which were fixed (before the advent of dimensional regularization) by taking the ξ-regulated vector-boson propagator of T.D. Lee and C.N. Yang, *Phys. Rev.* **128** (1962) 885, and then taking the limit $\xi \to \infty$. This propagator is precisely the same as due to the Higgs effect (although proposed long before the Higgs effect)

$$\begin{aligned} D_{\mu\nu} &= \frac{\eta_{\mu\nu} - k_\mu k_\nu/k^2}{k^2+m^2} + \frac{\xi k_\mu k_\nu/k^2}{k^2+\xi m^2} = \frac{\eta_{\mu\nu} + k_\mu k_\nu/m^2}{k^2+m^2} - \frac{k_\mu k_\nu/m^2}{k^2+\xi m^2} \\ &= \left[\eta_{\mu\nu} + (\xi-1)k_\mu k_\nu/(k^2+\xi m^2)\right]/(k^2+m^2) \end{aligned}$$

In the limit $\xi \to \infty$, one finds the propagator in the unitary gauge while the would-be Goldstone bosons decouple. This explains why "the ξ-limiting procedure" gave the correct results.

and 4-loop graphs while the second error is due to the uncertainty in the value of α. Using the results for a_e to fix α, one uses this α (which one might call α_{QED}) in the results for a_μ.

For the muon and antimuon we can only quote current results because the Brookhaven experiment is still ongoing.

$$\begin{aligned}
a_\mu^{\exp} &= 1\,165\,920\,23\,(160)\,10^{-11} && (9.0.47)\\
a_\mu^{th} &= 1\,165\,915\,97\,(67)\,10^{-11} && (9.0.48)\\
a_\mu^{QED} &= 1\,165\,847\,05.7\,(2.9)\,10^{-11} && (9.0.49)\\
a_\mu^{\mathrm{had}}\ (\text{vac. pol}) &= 69\,24\,(62)\,10^{-11} && (9.0.50)\\
a_\mu^{\mathrm{had}}\ (\text{light by light}) &= 89.6(15.4)10^{-11} && (9.0.51)\\
a_\mu^{\mathrm{EW}} &= 1\,52\,(4)\,10^{-11} && (9.0.52)
\end{aligned}$$

Theory and experiment agree with 1.6 standard derivation

$$a_\mu^{\exp} - a_\mu^{th} = 426\,(165)\,10^{-11}\ . \tag{9.0.53}$$

A more conservative approach to a_μ^{had} (vac. pol) which ignores τ data and uses QCD input as little as possible gives $a_\mu^{had} = 6789\,(108)\,10^{-11}$, and from this one gets $a_\mu^{th} = 1\,165\,916\,47\,(108)\,10^{-11}$.

A On-shell renormalization of QED

The cancellation of IR and UV divergences in the anomalous magnetic moment and, more generally, in the form factor $F_2(Q^2)$ of QED is an excellent exercise in practical quantum field theory, in particular on-shell renormalization theory. Since the overall divergences of vertex graphs only contribute to $F_1(Q^2)$, one need only study and remove the subdivergences if one is only interested in $F_2(0)$. Before renormalization, only the ladder graphs (one electron vertex with a series of parallel photons exchanged between the two legs of the vertex) are IR divergent. The UV counter terms Z_1 and Z_2 (but not Z_3 and Z_m) introduce further IR divergences. However, as we already discussed, in QED at least, one may omit the Z factors for vertex renormalization (Z_1) and electron wave function renormalization (Z_2) because these contributions cancel separately (due to $Z_1 = Z_2$ in QED). This cancellation even occurs inside subsets of graphs which are gauge invariant (an example is the sum of the graphs III and IV in (9.0.11)). One still needs mass renormalization (Z_m) and photon wave function renormalization (Z_3) but, as we already noted, these do not introduce IR divergences.

Consider as an example the following one-loop graphs

(9.A.1)

In the second and fourth graph the cross denotes $\frac{1}{2}$ times both a Z_2 and a Z_m counter term, and in the first and third graph one has $\frac{1}{2}$ times the selfenergy. It is clear that only the vertex graph (the fifth graph) can contribute to $F_2(Q^2)$ because all other graphs are proportional to γ^μ and hence contribute only to $F_1(Q^2)$. Furthermore, the contribution of this vertex graph to $F_2(Q^2)$ is UV finite because to construct the tensor structure $\bar{u}[\gamma^\mu, \gamma^\nu]u(p_1 - p_2)_\nu$ one must extract one momentum which turns the UV logarithmically divergent integral into a convergent integral. Finally $F_2(Q^2)$ is also IR finite because an IR divergence can only occur when the four-momentum of the virtual photon tends to zero, but in this case the vertex graph factorizes into an IR factor $\int d^4k[k^2\,(p \cdot k)\,(p' \cdot k)\,]^{-1}$ times the lowest order vertex γ^μ, which shows that also the IR divergence only occurs in $F_1(Q^2)$. The limit $Q^2 \to 0$ in $F_2(Q^2)$ exists; for example, there are no terms proportional to $\ln Q^2$ in $F_2(Q^2)$ for small Q^2 because $F_2(Q^2)$ is UV finite. Thus the one-loop contribution to $g - 2$ is indeed UV and IR finite, and one may restrict one's attention to the vertex graph, as Schwinger did.

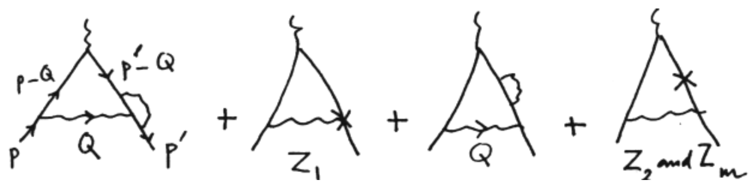
Next we make a few comments concerning $F_1(Q^2)$ at the one-loop level. The sum of the first four graphs vanishes because the residue of the fermion selfenergy corrections vanishes in on-shell renormalization. On-shell renormalization of the fermion selfenergy and the photon selfenergy is possible in QED, but not in QCD where infrared divergences due to the coupling of massless gluons to each other make on-shell renormalization impossible. (The precise renormalization condition is $\frac{1}{i\not{p}+m}\Gamma_{el}^{(2),\text{ren}}(\not{p},m)u(p)=u(p)$. This fixes the finite parts both in the electron wave function renormalization constant Z_2 and in the mass renormalization constant Z_m. As a result $\Gamma^{(2),\text{ren}}(\not{p},m)u=(i\not{p}+m)u=0$. One sometimes writes this as $\Gamma^{(2),\text{ren}}(\not{p},m)=(i\not{p}+m)+\cdots$ or $<0|\psi(p)\bar{\psi}(p)|0>=(i\not{p}+m)^{-1}+\cdots$. This shows that the physical pole is at m^2 and that the residue at the pole is unity.) At $Q_\mu=0$ also the sum of the last two graphs vanishes in on-shell renormalization. (This fixes the finite parts in Z_1. Note, however, that although $F_2(Q^2)\bar{u}[\gamma^\mu,\gamma^\nu]uQ_\nu$ vanishes at $Q_\nu=0$, the value of $F_2(0)$ is nonvanishing). By power counting, the vertex graph is both IR and UV divergent. At $Q_\mu\neq 0$, the counter term Z_1 of the last graph removes the UV divergence from the vertex graph, but in the sum an IR divergence remains proportional to γ^μ. (Both Z_1 and the vertex graph are IR divergent, and in their sum the IR divergence only vanishes at $Q=0$. For $Q\neq 0$, this IR divergence is proportional to γ^μ, hence absent from $F_2(Q^2)$, and is canceled in the cross section by Bremsstrahlung graphs according to the Bloch-Nordsieck mechanism.)

The renormalization condition $F_1^{\text{ren}}(Q^2=0)=1$ shows that one only needs to subtract the graph at $Q=0$ from the vertex graph. The incoming and outgoing fermion are on-shell, hence this renormalization condition is due to on-shell renormalization. The complete renormalized vertex correction can be written as

$$\Gamma^{(\text{ren})}(p,p',m)=\int\left[\frac{[-i(\not{p}'+\not{k})+m]}{(p'+k)^2+m^2}\frac{\gamma^\mu}{k^2}\frac{[-i(\not{p}+\not{k})+m]}{(p+k)^2+m^2}\right.$$
$$\left.-\frac{[-i(\not{p}+\not{k})+m]}{(p+k)^2+m^2}\frac{\gamma^\mu}{k^2}\frac{[-i(\not{p}+\not{k})+m]}{(p+k)^2+m^2}\right]\frac{d^4k}{(2\pi)^4} \tag{9.A.2}$$

The last term is an UV and IR divergent constant. (In dimensional regularization the sum of the UV and IR poles does not cancel. The UV divergence is directly found from the k integral for $n < 4$, and cancels in the sum of both terms, but the IR divergence shows up in the integral over Feynman parameters and for its evaluation one needs to continue n to $n > 4$).

Next consider the two-loop corrections to $g-2$. Here gauge invariant subsets are separately UV and IR convergent. The first subset consists of



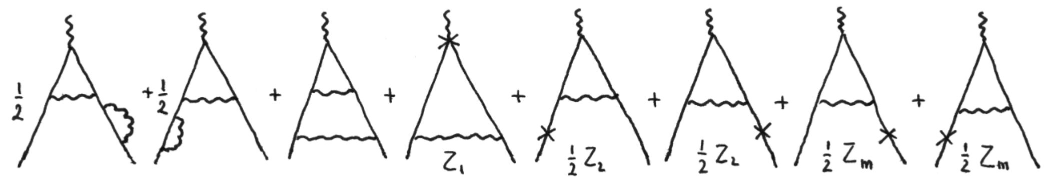
(9.A.3)

If one detaches in the first graph the horizontal photon from the fermion on the right, and lets this photon attach in all possible ways to the two fermions, one obtains a gauge invariant set which contains in addition to the first and third graph also graphs with wave renormalizations of the outgoing or ingoing fermions. The latter can be disregarded in on-shell renormalization as we have discussed.

The first and third graph give an IR finite but UV divergent contribution to $g-2$. There is no IR divergence when the loop momentum Q_μ tends to zero. (If this loop momentum tends to zero and one drops all Q's in all numerators, one finds an on-shell vertex correction at the vertex on the right-hand side which is proportioned to $-Z_1$ and which cancels against a similar on mass - shell renormalization of the fermion in the third graph, which is proportional to Z_2). The counter terms remove the UV divergences but introduce IR divergences in each graph.[21] However, the wave function renormalizations cancel ($Z_1 = Z_2$) and the mass renormalization Z_m is IR finite. Hence there are only UV divergences in each of the remaining graphs, and the sum of the subset of four graphs shown above indeed yields an IR and UV finite contribution to $g-2$, but note that mass renormalization is needed.

[21]The reason that the IR divergences in $F_2(0)$ from the first two graphs do not cancel is that $F_2(0)$ is the coefficient of $\bar{u}[\gamma^\mu, \gamma^\nu]uQ_\nu$, so one cannot set everywhere in the first graph $Q_\nu = 0$.

Another example is the following set of two-loop graphs



(9.A.4)

The sum of the first two and last four graphs vanishes in on-shell renormalization. In the third graph one finds an IR divergence equal to Z_1 times the one-loop vertex graph if the lower photon goes on-shell. The graph with the counter term Z_1 removes both the IR divergence and the UV divergence from $F_2(0)$. (The UV divergence is due to the triangle graph with the upper photon). Hence, the ladder graph together with the fourth graph contributes a finite amount to $g-2$.

The crossed graph is IR finite and has no UV subdivergences, hence it contributes an IR and UV finite amount to $g-2$. Finally, the vacuum polarization graph gives only an UV divergent contribution to $g-2$, but these divergences only occur in $F_1(Q^2)$, hence this graph yields also an IR and UV finite contribution to $g-2$. (An easy way to see this is to write the vacuum polarization graph after renormalization as a dispersion integral with massive photons. In the one-loop vertex graph one makes then the replacement

$$\frac{1}{k^2-i\epsilon} \to \alpha \int_{4m^2}^{\infty} \frac{1}{k^2+M^2-i\epsilon}\rho(M^2)\frac{dM^2}{M^2} \tag{9.A.5}$$

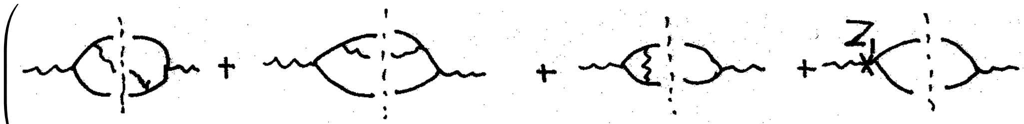
We discuss this in Appendix C.)

On-shell renormalization of the photon selfenergy requires that $\Pi_{\mu\nu}^{\mathrm{ren}} = (k^2\eta_{\mu\nu} - k_\mu k_\nu)(1+\Pi^{\mathrm{ren}}(k^2))$ where $\Pi^{\mathrm{ren}}(k^2) = \mathcal{O}(k^2)$. Hence, $\Pi_{\mu\nu}^{\mathrm{ren}}$ does not contribute to the residue of the photon propagator, and all renormalization contributions cancel also for external on-shell photons.

As a final example consider the two-loop vacuum polarization graphs in QED

(9.A.6)

These graphs were first calculated by Kallen and Sabry [32]. One may again ignore wave function and vertex renormalization because the contributions cancel (there are as many Z_1 as Z_2). Mass renormalization is needed, but does not introduce IR divergences. A particular simple way (at least conceptually simple) to evaluate these diagrams is again to use dispersion relations. Due to on-shell renormalization of the electrons, one only needs to evaluate the following cut graphs [32]

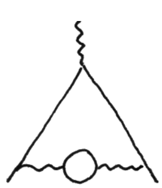
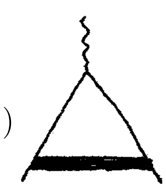
$2 \times \Bigg($  $\Bigg)$

(9.A.7)

Clearly there are no UV divergences in $Im\Pi(k^2)$, but also the IR divergences in $Im\Pi(k^2)$ cancel according to the Bloch-Nordsieck mechanism.

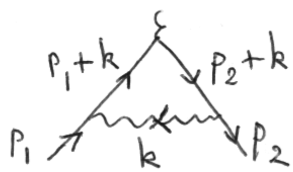
B The vacuum polarization

We explicitly calculate the order α^2 vacuum polarization corrections to $g-2$. They are obtained by inserting a photon self-energy into the one-loop vertex correction and in order to be able to use a dispersion integral, we calculate first the order α contribution due to a massive photon with mass M^2. We perform the calculation in d dimensions in order to obtain the order $(d-4)$ correction to the one-loop result which is needed to obtain the correct result for the 2-loop as we discussed in the main text. The order α^2 correction will then be obtained schematically as follows

 $+$  $= \int \rho(M^2)$  dM^2

(9.B.1)

We thus consider the one-loop vertex correction to $g-2$ with a massive propagator $(k^2+M^2-i\epsilon)^{-1}$ instead of the usual massless propagator $(k^2-i\epsilon)^{-1}$. We compute



$$= \int \frac{d^dk}{(2\pi)^4}\frac{N^\mu}{D} \tag{9.B.2}$$

where N^μ is the same as for massless photons but D contains $[k^2+M^2-i\epsilon]^{-1}$ and the two fermion propagators are $[2p_1\cdot k+k^2]^{-1}$ and $[2p_2\cdot k+k^2]^{-1}$. The incoming and outgoing spinors are on-shell, hence $\not{p}_1=\not{p}_2=im$.

The denominator D becomes[22] after using Feynman variables as in $[ABC]^{-1} = 2\int_0^1 dx\int_0^{1-x} dy[xA+(1-x-y)B+yC]^{-3}$

$$\begin{aligned} D &= \int[(k+p_1x+p_2y)^2+M^2(1-x-y)-(p_1x+p_2y)^2]^3 \\ &= \int[\kappa^2+M^2(1-x-y)+m^2(x^2+y^2)-2p_1\cdot p_2xy]^3 \\ &= \int[\kappa^2+M^2(1-x-y)+m^2(x+y)^2+(p_1-p_2)^2xy]^3 \end{aligned} \tag{9.B.3}$$

where $\int = 2\int_0^1 dx\int_0^{1-x} dy$ and $\kappa = k+p_1x+p_2y$. For the anomalous magnetic moment we only need to keep track of terms linear in $p_1^\mu-p_2^\mu$. Setting $p_1=p_2$ in the denominator, it only depends on $x+y$. The numerator[23] simplifies upon using the Dirac equation $\not{p}_1=\not{p}_2=im$

$$\begin{aligned} N^\mu &= \gamma_\alpha(-i(\not{p}_2+\not{k})+m)\gamma^\mu(-i(\not{p}_1+\not{k})+m)\gamma^\alpha = \\ &\quad (-2ip_{2,\alpha}-i\gamma_\alpha\not{k})\gamma^\mu(-2ip_1^\alpha-i\not{k}\gamma^\alpha) \end{aligned} \tag{9.B.4}$$

We decompose N^μ into terms with γ^μ and terms with $(p_1+p_2)^\mu$. Dropping all terms proportional to γ^μ since we are only interested in $F_2(0)$, we obtain in n dimensions

$$-(2\not{p}_1\not{k}\gamma^\mu+2\gamma^\mu\not{k}\not{p}_2+(2-d)\not{k}\gamma^\mu\not{k}) = 2\not{k}\not{p}_1\gamma^\mu+2\gamma^\mu\not{p}_2\not{k}-2(2-d)k^\mu\not{k}$$

[22]As explained in the main text, we could first project out F_2 and already set $p_1=p_2$ before introducing Feynman parameters. This would lead to a small algebraic simplification.

[23]In principle one should use $\eta_{\alpha\beta}+k_\alpha k_\beta/M^2$ for the numerator of the propagator of the massive photon, but the $k_\alpha k_\beta/M^2$ term does not contribute for the same reason that the gauge parameter ξ in the massless propagator $\eta_{\mu\nu}+(\xi-1)k_\mu k_\nu/k^2$ does not contribute.

$$
\begin{aligned}
&= -2im(\not{k}\gamma^\mu + \gamma^\mu \not{k}) + 4\not{k}(p_1^\mu + p_2^\mu) - 2(2-d)k^\mu \not{k} \\
&= -4imk^\mu + 4\not{k}(p_1^\mu + p_2^\mu) - 2(2-d)k^\mu \not{k} \qquad (9.B.5)
\end{aligned}
$$

Replacing k by $\kappa - p_1 x - p_2 y$, using again $\not{p}_1 = \not{p}_2 = im$, dropping terms with κ because they either vanish using symmetric integration or do not contribute to $F_2(0)$, and using the symmetry of the denominator in x and y to replace single factors x and y in the numerator by $\frac{1}{2}(x+y)$, one arrives at

$$
\begin{aligned}
N^\mu &= 4im\frac{1}{2}(x+y)(p_1+p_2)^\mu - 4(x+y)im(p_1+p_2)^\mu \\
&\quad -2(2-d)(p_1^\mu + p_2^\mu)\frac{1}{2}(x+y)(im)(x+y) \\
&= -im[2z + (2-d)z^2](p_1^\mu + p_2^\mu) \qquad (9.B.6)
\end{aligned}
$$

where $z = x + y$. Using

$$
2\int_0^1 dx \int_0^{1-x} dy = 2\int_0^1 z dz \quad \text{and} \quad \int d^d k [k^2 + M^2]^{-3} = \frac{1}{2} i\pi^2 M^{-2+(d-4)} \qquad (9.B.7)
$$

we finally obtain from the definition of F_2 in (9.0.15)

$$
F_2(0, M^2) = \frac{\alpha}{\pi}\int_0^1 dz \frac{(z^2 - z^3)(1 + (d-4)\frac{1}{2}\ln\left\{\frac{M^2}{m^2}(1-z) + z^2\right\}) - \frac{1}{2}(d-4)z^3}{z^2 + \frac{M^2}{m^2}(1-z)} \qquad (9.B.8)
$$

where $\alpha = e^2/(4\pi)$.

For $M^2 = 0$ we recover Schwinger's result, but now in d dimensions

$$
F_2(0) = \frac{\alpha}{2\pi}(1 - 2(d-4)) \qquad (9.B.9)
$$

The numerator N^μ contributes a correction $-\frac{1}{2}(d-4)$ while the denominator contributes a correction $-\frac{3}{2}(d-4)$.

For the contribution of the vacuum polarization to $g-2$ we need the case with $M^2 \neq 0$ but $d = 4$. For $M^2 \gg m^2$ we obtain

$$
F_2(0, M^2) = \frac{\alpha}{3\pi}\frac{m^2}{M^2} \qquad (9.B.10)
$$

For $4m^2 \leq M^2$ we find by using $z^2 = [z^2 + (M^2/m^2)(1-z)] - (M^2/m^2)(1-z)$

$$\begin{aligned} F_2(0, m^2, M^2) &= \frac{\alpha}{\pi}\left[\frac{1}{2} - \frac{M^2}{m^2} - \frac{M^2}{m^2}\left(1 - \frac{M^2}{2m^2}\right)\ln\frac{M^2}{m^2}\right. \\ &\qquad \left. + \left(-\frac{M^2}{m^2}\right)\left(1 - \frac{2M^2}{m^2} + \frac{M^4}{2m^4}\right) J\right] \end{aligned} \tag{9.B.11}$$

where

$$J = \int_0^1 dz \frac{1}{z^2 + (M^2/m^2)(1-z)} . \tag{9.B.12}$$

Completing squares and using $\int \frac{dy}{y^2 - a^2} = \frac{1}{2a} \ln \frac{a-y}{a+y}$ we find

$$J = \frac{m^2}{M^2} \frac{1}{\sqrt{(1 - \frac{4m^2}{M^2})}} \ln\left(\frac{1 + \sqrt{1 - 4m^2/M^2}}{1 - \sqrt{1 - 4m^2/M^2}}\right) \simeq \frac{m^2}{M^2} \ln \frac{M^2}{m^2} + \cdots \tag{9.B.13}$$

Expanding in terms of $\frac{m^2}{M^2}$, one may check that indeed the leading term in $F_2(0)$ is of order m^2/M^2, which is a good (but tedious) check of the algebra. In fact, one can simplify this expression by taking the argument of the logarithm as a new variable

$$\begin{aligned} x &= \frac{1 + \sqrt{1 - 4m^2/M^2}}{1 - \sqrt{1 - 4m^2/M^2}} \\ \frac{x-1}{1+x} &= \sqrt{1 - 4m^2/M^2} \Rightarrow \frac{4m^2}{M^2} = 1 - \left(\frac{1-x}{1+x}\right)^2 = \frac{4x}{(1+x)^2} \end{aligned} \tag{9.B.14}$$

One finds then $F_2(0, m^2, M^2)$ as a function of x.

The photon selfenergy from the one-loop graph

$$= (k^2\eta_{\mu\nu} - k_\mu k_\nu)\Pi(k^2) \tag{9.B.15}$$

is most easily obtained from a dispersion integral, see below. A direct calculation with minimal subtraction yields

$$\begin{aligned} \Pi(k^2) = \frac{-e^2}{6\pi^2 k^2} &\left[2m^2 + \frac{1}{2}k^2\left(\ln\frac{m^2}{\mu^2} - 5/3\right)\right. \\ &+ \frac{1}{2}(k^2 - 2m^2)\sqrt{1 + 4m^2/k^2}\ln\left(\frac{1 + \sqrt{1 + 4m^2/k^2}}{1 - \sqrt{1 + 4m^2/k^2}}\right) \text{ for } k^2 > 4m^2 \end{aligned} \tag{9.B.16}$$

On the interval $-4m^2 < k^2 < 0$ one needs an arctg function. For $-k^2 \geq 4m^2$ there is a cut and the logarithim develops an imaginary part

$$\ln \frac{1+\sqrt{1+4m^2/k^2}}{1-\sqrt{1+4m^2/k^2}} = i\pi\theta(-k^2-4m^2) \text{ for } -k^2 \geq 4m^2 \tag{9.B.17}$$

On-shell renormalization implies that $\Pi^{\text{ren}}(k^2) = 0$ at $k^2 = 0$. Hence, to impose this renormalization condition one must remove the first two terms in $\Pi(k^2)$ while in the last term one should drop the contributions of order k^{-2} and k^0. It is easier to perform the renormalization in the dispersion integral for $\Pi(k^2)$; one should just subtract the value at $k^2 = 0$. To obtain $\Pi(k^2)$ from a dispersion integral we first need its imaginary part.[24] One obtains then the following dispersion integral for the renormalized $\Pi(k^2)$

$$\Pi(k^2) = \int_{4m^2}^{\infty} \frac{e^2}{12\pi^2}\left(1+\frac{2m^2}{M^2}\right)\sqrt{1-\frac{4m^2}{M^2}}\left(\frac{1}{k^2+M^2}-\frac{1}{M^2}\right) dM^2 \tag{9.B.18}$$

The factor $1+\frac{2m^2}{M^2}$ is due to the numerator of fermion loop after putting both fermions in the fermion loop on-shell. The factor $(1-4m^2/M^2)^{1/2}$ is due to the phase space integral. As a check one may verify that for small k^2

$$\Pi(k^2) = -\frac{e^2}{60\pi^2}\frac{k^2}{m^2} + \mathcal{O}(k^4) \tag{9.B.19}$$

which is the Uehling limit and which is used in the calculation of the Lamb shift.

For the 2-loop correction to $g-2$ we must multiply $\Pi_{\mu\nu}$ by two factors k^{-2}. The terms with $k_\mu k_\nu$ in the photon propagators $(\eta_{\mu\nu} - k_\mu k_\nu/k^2)k^{-2}$ cancel because $\Pi_{\mu\nu}$ is transversal. Similarly the terms with $k_\mu k_\nu/k^2$ in $\Pi_{\mu\nu}$ do not contribute because of

[24] The imaginary part of $\Pi(k^2)$ follows from the optical theorem, and the phase space integral

$$\int \frac{d^3p_1 d^3p_2}{2\omega_1 2\omega_2}\delta^4(p_1+p_2-k) = \frac{\lambda^{1/2}(s,m_1^2,m_2^2)}{8s} = \frac{\sqrt{[s-(m_1+m_2)^2][s-(m_1-m_2)^2]}}{8s}$$

where $s = -k^2 \geq 4m^2$ and $\lambda(x,y,z) = x^2+y^2+z^2-2xy-2xz-2yz$ is Kallen's function.

the Ward identity

(9.B.20)

One obtains then in the dispersion integral the following factor

$$\frac{1}{k^2}\left(\frac{1}{k^2+M^2}-\frac{1}{M^2}\right)=-\frac{1}{M^2(k+M^2)} \tag{9.B.21}$$

The photon propagator is thus modified as follows

$$\frac{1}{k^2}\rightarrow\frac{1}{k^2}+\alpha\int\limits_{4m^2}^{\infty}\rho(m^2,M^2)\,\frac{1}{k^2+M^2}\,dM^2\ . \tag{9.B.22}$$

The factor $(k^2+M^2)^{-1}$ yields then the anomalous magnetic moment for a massive photon.

Thus the 2-loop contribution to $F_2(0)$ due to the photon selfenergy is given by

$$F_2(0)=\int\limits_{4m^2}^{\infty}\rho(m^2,M^2)F_2(0,\bar{m}^2,M^2)dM^2 \tag{9.B.23}$$

where

$$\rho(m^2,M^2)=\frac{e^2}{12\,\pi^2}\left(1+\frac{2m^2}{M^2}\right)\sqrt{1-\frac{4m^2}{M^2}}\,\frac{1}{M^2} \tag{9.B.24}$$

and $\bar{m}$ is the mass of the external fermion while m is the mass of the fermions in the vacuum polarization loop. Evaluation of this integral yields the contribution in the last column of (9.0.6).

This same method can be used to compute the contribution to $g-2$ due to hadronic corrections to the photon propagator. One writes again $\Pi(k^2)$ as a dispersion integral over $Im\Pi(k^2)$, but now one expresses $Im\Pi(k^2)$ into the total cross section for $e^+e^-\rightarrow$ hadrons, using the optical theorem. We give some details.

The total cross section for $e^+e^- \to \mu^+\mu^-$ is given by[25]

$$\sigma(s, m_\mu, m_e) = \frac{4\pi\alpha^2}{3s}\sqrt{\frac{s-4m_\mu^2}{s-4m_e^2}}\left[1+2\frac{(m_\mu^2+m_e^2)}{s}+\frac{4m_\mu^2 m_e^2}{s^2}\right] \tag{9.B.25}$$

For massless electrons this reduces to

$$\sigma(s, m_\mu, 0) = \frac{4\pi\alpha^2}{3s^2}\sqrt{\frac{s-4m_\mu^2}{s}}(s+2m_\mu^2) \tag{9.B.26}$$

From (12.5.25) we find for $Im\Pi(k^2)$ the following expression

$$Im\Pi(k^2=-s) = \frac{e^2}{6\pi s}\frac{1}{2}(s+2m_\mu^2)\sqrt{1-\frac{4m_\mu^2}{s}} \tag{9.B.27}$$

Hence

$$Im\Pi(k^2=-s) = \frac{16s\sigma(s,m_\mu^2,0)}{e^2} \tag{9.B.28}$$

The hadronic corrections to the vacuum polarization are then related to the total cross section of massless e^+e^- annihilation into hadrons by

$$Im\Pi_{\text{hadr}}(k^2=-s) = \frac{s\sigma(e^+e^- \to \text{hadrons})}{4\pi\alpha} \tag{9.B.29}$$

Substituting this expression for $Im\Pi_{\text{hadr}}$ into the dispersion integral in (9.B.23) yields the correction to $g-2$.

C Susy contributions to $g-2$

One explanation for a possible discrepancy between the theoretical and experimental values for the anomalous magnetic moment of the muon is supersymmetry. In the

[25] One has $\sigma = (\frac{1}{4}\sum_{\text{spins}})\int \frac{d^3k_1 d^3k_2}{(2\pi)^3 2E_1(2\pi)^3 2E_2}|M|^2(2\pi)^4\delta^4(k_1+k_2-p_{e^-}-p_{e^+})\frac{1}{(2E_{e^-})(2E_{e^+})v_{\text{rel}}}$ where $v_{\text{rel}} = |\vec{p}_{e^-}/E_{e^-} - \vec{p}_{e^+}/E_{e^+}|$. The flux factor $(2E_{e^-})(2E_{e^+})v_{\text{rel}}$ is relativistically invariant, and given by $4(\vec{p}_{e^-}^{\,2}s)^{1/2}$ where $s=(E_{e^-}+E_{e^+})^2$. The center of mass momentum $\vec{p}_{e^-}$ follows from equating $2p_{e^-}\cdot p_{e^+} = -2\vec{p}_{e^-}^{\,2} - 2\sqrt{\vec{p}_{e^-}^{\,2}+m_{e^-}^2}\sqrt{\vec{p}_{e^-}^{\,2}+m_{e^-}^2}$ to $2p_{e^-}\cdot p_{e^+} = (p_{e^-}+p_{e^+})^2 + m_{e^-}^2 + m_{e^+}^2 = -s + m_{e^-}^2 + m_{e^+}^2$. One finds then that $|\vec{p}_{e^-}| = \frac{1}{2}\lambda(s, m_{e^-}^2, m_{e^+}^2)^{1/2}s^{-1/2}$ and thus $2E_{e^-}2E_{e^+}v_{\text{rel}} = 2\lambda(s, m_{e^-}^2, m_{e^+}^2)^{1/2}$. The integral over $\vec{k}_2$ is trivial, and the integral over E_2 uses up the last delta function.

minimally supersymmetric standard model (MSSM) the contribution from susy particles is $a_\mu(\text{susy}) = (\text{sgn}\ \mu)\ 130 \cdot 10^{-11}(\frac{100GeV}{\tilde{M}})^2\ tg\beta$ where $\tilde{M}$ is the mass of the susy particles (taken to be the same for all susy particles for simplicity), and $tg\beta$ is the ratio of vev's of the two Higgs scalars in the MSSM, usually taken between 3 and 40. The infamous μ parameter can have either sign. One can explain a discrepancy of 2-3 σ if $\tilde{M}$ is of order $100 - 400GeV$, which are typical masses for susy theories which can barely be detected at Fermilab and easily at the LHC.

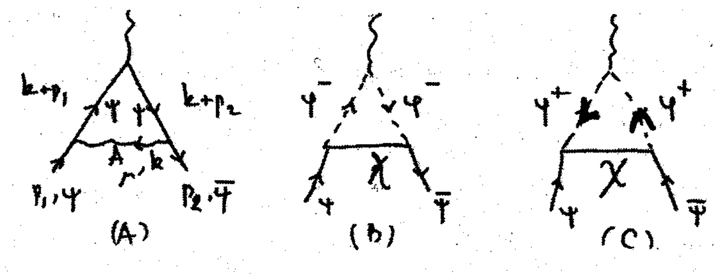
The susy additions to $g-2$ come from graphs with the susy partners of the photon and muon, and other particles in the MSSM. In susy QED with one common mass m for all matter, $g-2$ vanishes due to cancellations between contributions from bosonic and fermionic particles [33]. We explain this. Susy QED contains a photon A_μ and real photino χ coupling to the Dirac electron ψ and two complex scalars φ^+ and φ^- with opposite electric charge [26]. The action reads

$$\begin{aligned}\mathcal{L} = &-|D_\mu\varphi^+|^2 - |D_\mu\varphi^-|^2 - \bar{\psi}\gamma^\mu D_\mu\psi - \frac{1}{4}F_{\mu\nu}^2 - \frac{1}{2}\bar{\chi}\partial\!\!\!/\chi \\ &-m^2(|\varphi^+|^2 + |\varphi^-|^2) - m\bar{\psi}\psi + ce\bar{\chi}\left[\varphi_-^*\frac{(1-\gamma_5)}{2} + \varphi^+\frac{(1+\gamma_5)}{2}\right]\psi \\ &\quad +c^*e\bar{\psi}\left[\varphi_+^*\frac{(1-\gamma_5)}{2} + \varphi^-\frac{(1+\gamma_5)}{2}\right]\chi \end{aligned} \tag{9.C.1}$$

where $D_\mu\varphi^\pm = \partial_\mu\varphi^\pm \pm ieA_\mu\varphi^\pm$, $D_\mu\psi - ieA_\mu\psi$ and $|\,\varphi^+\,|^2 = \varphi^+\varphi_+^*$ with $\varphi_+^* = (\varphi^+)^*$. Susy requires that the complex electron spinor ψ couples to φ^+ and φ_-^* with coupling constant e, and we can choose the phases of φ^+ and φ_-^* such that the relative coupling constant is unity. Hermiticity then yields the last term. The over all constant c is fixed by susy ($c = 1$), but we leave it temporarily free to see how $(g-2)$ depends on it.

[26] One vector multiplet (A_μ, χ) of $N = 1$ susy couples to two scalar multiplets (φ_1, ψ_1) and (φ_2, ψ_2) of $N = 1$ susy. One needs two scalar multiplets to construct a complex Dirac spinor which can minimally couple to the photon.

There are three graphs to compute in susy QED which contribute to $g-2$



(9.C.2)

Actually, the contributions of the last two graphs are equal. The denominators of all graphs are equal, and , as we now show, the sum of the numerators cancel. The numerator is proportional to

$$N^{\nu} = \gamma_{\nu}[-i(\not{k}+\not{p}_2)+m]\gamma^{\mu}[-i(\not{k}+\not{p}_1)+m]\gamma^{\nu} - 2\mid c\mid^2 (p_1+p_2+2k)^{\mu}\not{k} \quad (9.C.3)$$

The crucial relative minus sign between the contributions from the first and latter two graphs is due to the absence of a factor i in the Yukawa couplings. Combining the denominators with Feynman parameters leads to

$$D = [(k+p_1x+p_2y)^2 - (p_1x+p_2y)^2]^3 \quad (9.C.4)$$

One may then replace k in N^{μ} by $\kappa - p_1x - p_2y$ and integrate over $\kappa = k + p_1x + p_2y$.

Dropping all terms proportional to γ^{μ} and retaining only the terms with $(p_1+p_2)^{\mu}$ yields the magnetic form factor $F_2((p_1-p_2)^2)$. We set $p_1 = p_2 = p$ to obtain $F_2(0)$. When $p_1 = p_2$ the Gordon identities (9.0.13) reduce to $2im\bar{u}_2\gamma^{\mu}u_1 = 2p_1^{\mu}\bar{u}_2u_1$, but we set $p_1 = p_2$ in N^{ν} and do not contract with $\bar{u}_2$ and u_1. The last term in N^{μ} yields then

$$N^{\mu}(B+C) = -2\mid c\mid^2 2(1+x+y)(x+y)im(p_1+p_2)^{\mu} \quad (9.C.5)$$

where we used the Dirac equation $\not{p}_1u(p_1) = imu(p_1)$. The first term can also be simplified by using the Dirac equation.

$$[\gamma_{\nu}(-i\not{k}) - 2ip_{2,\nu}]\gamma^{\mu}[-i\not{k}\gamma^{\nu} - 2ip_1^{\nu}]$$

$$\begin{aligned}
&= -[\gamma_\nu \not{k}\gamma^\mu \not{k} + 2\not{p}_1 \not{k}\gamma^\mu + 2\gamma^\mu \not{k}\not{p}_2 + \mathcal{O}(\gamma^\mu)] \\
&= 2\not{k}\gamma^\mu \not{k} + 2\not{k}\not{p}_1\gamma^\mu + 2\gamma^\mu \not{p}_2 \not{k} + \mathcal{O}(\gamma^\mu) \\
&= 4k^\mu \not{k} - 2\not{k}\gamma^\mu im - 2im\gamma^\mu \not{k} + \mathcal{O}(\gamma^\mu) \\
&= 4k^\mu \not{k} - 4imk^\mu + \mathcal{O}(\gamma^\mu) \qquad (9.C.6)
\end{aligned}$$

Setting again $k = \kappa - p_1 x - p_2 y$ we obtain

$$N^\mu(A) = 4im[(x+y)^2 + (x+y)]\,(p_1+p_2)^\mu \qquad (9.C.7)$$

Clearly, for $|c|^2 = 1$ all contributions to $g-2$ cancel. This is indeed the value of $|c|^2$ given by susy.

Let us now study the susy contributions to $g-2$ in more detail. We restrict our attention to the minimal susy Standard Model (MSSM), with all susy breaking terms added which are soft (do not introduce quadratic divergences). The relevant terms in the action contain the following superfields: a muon-neutrino doublet $L^i (i = 1, 2)$, a muon singlet μ_R^c, and two Higgs doublets H_1^i and H_2^i. One often views the MSSM as a special case of models with two Higgs doublets, and one denotes the two Higgs doublets in the general case by ϕ_1^i and ϕ_2^i. Then one has the following superfield content: the gauge fields and gauginos, and the matter superfields

$$L = \begin{pmatrix} \nu_\mu \\ \mu_L \end{pmatrix}; \quad \mu_R^c; \quad H_2 = \begin{pmatrix} H_2^+ \\ H_2^0 \end{pmatrix}; \quad H_1 = \begin{pmatrix} H_1^0 \\ H_1^- \end{pmatrix} \qquad (9.C.8)$$

The vacuum expectation values of the Higgs fields define the important parameter $tg\beta$

$$< H_1 >= \begin{pmatrix} v_1/\sqrt{2} \\ 0 \end{pmatrix}, < H_2 >= \begin{pmatrix} 0 \\ v_2/\sqrt{2} \end{pmatrix}; tg\beta = v_2/v_1 \qquad (9.C.9)$$

It can be shown that the phases of v_1 and v_2 both vanish at the minimum of the potential so that there is no CP violation in the Higgs sector. The terms in the action which are relevant for $g-2$ are susy preserving terms (from the kinetic actions for the superfields and from the superpotential W), and susy-breaking terms $\mathcal{L}$ (soft).

The terms in the superpotential with preserve R parity are the terms which are even in the number of susy partners, and are given by

$$W = \epsilon_{ij}[Y_{(\mu)}\mu_R^c L^i H_1^j + \mu_H H_2^i H_1^j] \qquad \text{(superfields)} \tag{9.C.10}$$

Here $Y_{(\mu)}$ is the Yukawa coupling constant for the muons, and μ_H is the (in) famous "μ-parameter" which gives the Higgs fields a gauge-invariant mass. The relevant soft[27] susy-breaking terms are masses for the scalars (the smuons, the sneutrinos and the 5 Higgs scalars), masses for the gauginos (winos and bino), and trilinear couplings of chiral scalars

$$\begin{aligned}
&\mathcal{L}_{\text{soft}} = -m_1^2 \mid H_1^i \mid^2 - m_2^2 \mid H_2^i \mid^2 - (m_{12}^2 \epsilon_{ij} H_1^i H_2^j + c.c.) \\
&-m_L^2(\tilde{\mu}_L^* \tilde{\mu}_L + \tilde{\nu}_{\mu,L}^* \tilde{\nu}_{\mu,L}) - m_R^2 \tilde{\mu}_R^{c,*} \tilde{\mu}_R^c \\
&-\left(\frac{1}{2} m_W \tilde{W}^a \tilde{W}^a + \frac{1}{2} m_B \tilde{B}\tilde{B} + A_{(\mu)}\, \epsilon_{ij}\, \tilde{\mu}_R^c\, \tilde{\mu}_L^i H^j + c.c.\right)
\end{aligned} \tag{9.C.11}$$

We have written the gaugino mass terms in terms of two-component spinors.

In the limit of unbroken susy but broken electroweak symmetry, the potential for $\langle H_1 \rangle$ and $\langle H_2 \rangle$ reduces to

$$V = \frac{1}{2}\, \mu_H^2\, (v_1^2 + v_2^2) + \frac{1}{32}\, (g_1^2 + g_2^2)\, (v_1^2 - v_2^2)^2 \tag{9.C.12}$$

where g_1 and g_2 are the coupling constants for U_1 and SU(2).

Hence, for exact susy,

$$\mu = 0, \qquad v_1 = v_2 \ . \tag{9.C.13}$$

The physical charged fermions in the gaugino and Higgs sector are called charginos. There are four charginos:

$$\textbf{charginos}: \quad \tilde{W}^+,\ \tilde{W}^-,\ \tilde{H}_1^-,\ \tilde{H}_2^+ \tag{9.C.14}$$

[27]Soft means not introducing quadratic divergences. The latter would bring back the hierarchy problem, whose solution has become the main motivation for susy.

The mass terms for the charginos are

$$\mathcal{L}^{\text{mass}}_{\text{charg}} = -(\tilde{W}^+ \tilde{H}_2^+) \begin{pmatrix} m_W & g_2 < H_1 > \\ -g_2 < H_2 > & \mu \end{pmatrix} \begin{pmatrix} \tilde{W}^- \\ \tilde{H}_1^- \end{pmatrix} + h.c. \qquad (9.C.15)$$

In the limit of unbroken susy ($m_W = 0$, also $\mu = 0$, see above), the charginos get the same mass as the W boson.[28]

Further, the physical uncharged fermions in the gauginos-Higgs sector are called neutralinos. There are also four neutralinos.

$$\textbf{neutralinos} \ : \tilde{W}^3, \tilde{B}, \tilde{H}_1^0, \tilde{H}_2^0 \qquad (9.C.16)$$

The mass terms for the neutralinos are

$$\mathcal{L}^{\text{neutr}}_{\text{mass}} = -\frac{1}{2} \begin{pmatrix} i\tilde{B} \\ i\tilde{W}^3 \\ \tilde{H}_1^0 \\ \tilde{H}_2^0 \end{pmatrix}^T \left(\begin{array}{cc|cc} -m_B & 0 & -g_1 v_1 & g_1 v_2 \\ 0 & -m_W & g_2 v_1 & -g_2 v_2 \\ \hline -g_1 v_1 & g_2 v_1 & 0 & \mu \\ g_1 v_2 & -g_2 & \mu & 0 \end{array} \right) \begin{pmatrix} i\tilde{B} \\ i\tilde{W}^3 \\ \tilde{H}_1^0 \\ \tilde{H}_2^0 \end{pmatrix} + h.c. \qquad (9.C.17)$$

In the limit of unbroken susy, the mass matrix becomes purely off diagonal, and its eigenvalues follow from

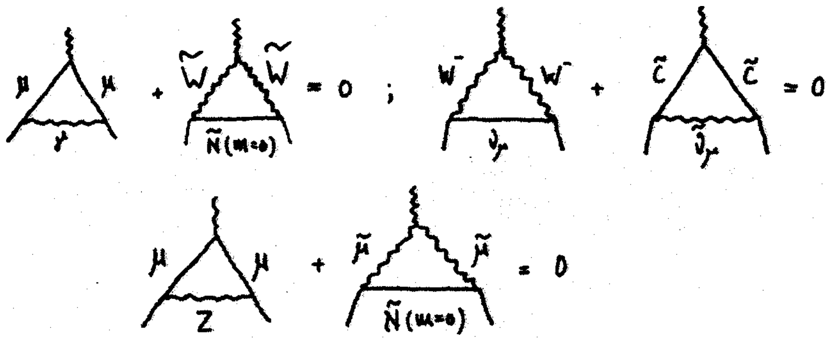
$$\begin{aligned} &\det(M - \lambda I_4) = \det(\lambda I_2 - M^T \lambda^{-1} M) \det(\lambda I_2) \\ &= \lambda^4 - \lambda^2 tr m^T m + \det m^T m \text{ with } m = \begin{pmatrix} -g_1 v_1 & g_1 v_2 \\ g_2 v_1 & -g_2 v_2 \end{pmatrix} \end{aligned} \qquad (9.C.18)$$

Since $\det m = 0$, there are two eigenvalues $\lambda = 0$, and two eigenvalues given by $\lambda^2 = (g_1^2 + g_2^2)(v_1^2 + v_2^2)$. Hence, in the limit of exact susy, two neutralinos are massless (the photino and one Higgsino), and two other neutralinos have the mass of the Z boson (the zino and the other Higgsino).

We can now understand why and how in the MSSM without susy breaking $g - 2$ vanishes exactly: the two massless neutralinos cancel the 1-loop graph with a photon

[28] $\tilde{H}_1^-$ and its complex conjugate yield one Majorana spinor. Idem $\tilde{H}_2^+$. Together $\tilde{H}_1^-$ and $\tilde{H}_2^+$ yield one Dirac spinor with 4 states, the same number of states as given by the charged W bosons.

(Schwinger's result),



(9.C.19)

the two charginos cancel the contribution from the graph with a (W, ν_μ) in the loop, and finally the two massive neutralinos cancel the contribution from the graph with a (μ, Z) in the loop.

When susy is broken, the graph with the charginos becomes proportional to $tg\beta$, but the graphs with the neutralinos remain of order unity. Hence, for large $tg\beta$, the contribution of susy to $g-2$ can become large as $tg\beta$ becomes large [34].

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Chapter 10

The Dirac formalism and Hamiltonian path integrals

Instead of the Lagrangian BRST quantization method, one may also use the Hamiltonian BRST quantization method. The advantage of the former is its manifest Lorentz covariance (for Lorentz-invariant gauge fixing terms). While the Hamiltonian approach is not manifestly Lorentz covariant because space and time play a different role, it has the advantage that the Hamiltonian BRST charge, Q_H, is gauge-choice independent. As we shall see, in the Hamiltonian approach one fixes the gauge at the very end, which is to be contrasted with the Lagrangian BRST approach, where one begins by fixing the gauge and then constructs the corresponding ghost action. The gauge independence of the BRST charge Q_H has great advantages at the formal level. Furthermore, the Hamiltonian approach is needed if one wants to use operator methods and it is the natural framework to discuss states. In particular, it allows one to define physical states in a clear explicit way. For these reasons we discuss Hamiltonian path integrals in this chapter.

As we already mentioned, in the Lagrangian BRST approach, both the ghost action and the BRST charge, Q_L, depend on the gauge. Since they are functions of q and $\dot{q}$, one expects that by replacing $\dot{q}$ by p, the charge Q_L goes over into Q_H, but this "replacing $\dot{q}$ by p" is ambiguous for gauge theories because in the Hamiltonian approach there are "constraints", according to which we may replace some canonical variables by a combination of others. The reverse way, eliminating p in terms of $\dot{q}$, is unambiguous and this would seem to guarantee that Q_H reverts then to Q_L. However, the quantum action in the Hamiltonian BRST formalism is not directly obtained by merely constructing $p\dot{q} - H$; rather, one extracts from the classical Lagrangian action certain ingredients (H_0 and φ_α, see below), from which one constructs the quantum action of the Hamiltonian formalism and the charge Q_H. This raises the possibility that the quantum action in the Hamiltonian formalism is not equal (after elimination of momenta) to the quantum action in the Lagrangian approach. Yet, at least in all known cases, one recovers the action and BRST charge of the Lagrangian formalism. Thus, in this sense, the Lagrangian and Hamiltonian approach to quantization are equivalent. One can, however, also eliminate a combination of p's and q's instead of all p's, and in this case one ends up with a Lagrangian which looks different, but should still be physically equivalent to the original Lagrangian. We shall give an example later.

To deal with systems with constraints, Dirac has constructed a general formalism, now known as the Dirac formalism. [1] This formalism deals primarily with the classical theory. At the end of his book [1], Dirac also indicated how his formalism could be used in quantum field theory but he used an operator approach, and the modern approach to quantum field theory uses instead path integrals. However, Dirac's classical formalism forms the basis for the Hamiltonian BRST approach, and for this reason we give a detailed account of "Dirac theory". It first divides constraints into primary, secondary, tertiary, etc., and then into first-class and second-class constraints. First-class constraints generate gauge transformations on the phase-space variables, and we shall trace their relation to the gauge transformations in configuration space theories. (In Yang-Mills theory, there are no second class constraints, and the first class constraints are the Gauss operators $D^i E_i^a$ as we shall show). Second-class constraints are eliminated by replacing Poisson brackets by so-called Dirac brackets.

The Hamiltonian BRST formalism is an extension of the Dirac formalism where one introduces in addition to the variables q_i and p^i of classical phase space, new variables: ghosts, antighosts, Lagrange multipliers, and conjugate momenta for **all** of them. (For Yang-Mills theory, the q_i correspond to A_j^a and the p^i correspond to $E_j^a = D_0 A_j^a$ while the Lagrange multipliers correspond to A_0^a.) This larger phase space we shall call extended phase space. By integrating over the extra variables in the Hamiltonian path integral, one recovers the usual Lagrangian path integral.

In the Hamiltonian approach of Dirac, one starts with a Lorentz-invariant classical action in order to incorporate special relativity from the start. Then one constructs a Hamiltonian H_0 which only depends on p^i and q_i, but not on Lagrange multiplier fields. (For Yang-Mills theory, this H_0 equals the space integral of $\frac{1}{2}(E_i^a)^2 + \frac{1}{2}(B_i^a)^2$ as we shall show). Next one determines the algebra of H_0 and the first-class constraints, second class constraints being taken care of by using Dirac brackets instead of Poisson brackets. Also the first-class constraints depend only on p^i and q_i. Following the generalization of Dirac's work by Fradkin and Vilkovisky [2], one extracts from the algebra of first-class constraints the structure functions ${}^{(n)}U_{b_1 \ldots b_{n+1}}{}^{a_1 \ldots a_n}$, which depend on the canonical variables p^i and q_i. (For Yang-Mills theory, only the rank-one structure functions, namely those with $n = 1$, are nonvanishing, and ${}^{(1)}U_{b_1 b_2}{}^{a_1} = g f_{b_1 b_2}{}^{a_1}$). Using the structure functions and the canonical variables of the extended phase space, one then constructs a BRST charge Q_H which is nilpotent

$$\{Q_H, Q_H\} = 0 \tag{10.0.1}$$

In addition one constructs a BRST invariant Hamiltonian $H_{BRST} = H_0 + \ldots$

$$\{H_{BRST}, Q_H\} = 0 \tag{10.0.2}$$

This relation states both that Q_H is conserved, and that H_{BRST} is BRST invariant.[1]

[1]In fact, given any classical observable A_0 (a function of the p^i and q_i which is gauge-invariant, has

The final quantum action S^{qu} reads in general

$$S^{qu} = \int [\dot{q}_i p^i + \dot{\lambda}^\mu \pi_\mu + \dot{\eta}^a p_a - H_{BRST} + \{\psi, Q_H\}] dt \tag{10.0.3}$$

where λ^μ are the Lagrange multipliers, π_μ their conjugate momenta, η^a denotes both the ghosts C^α and antighost-momenta $p(B)^\alpha$, and p_a denotes the ghost-momenta $p(C)_\alpha$ and minus the antighosts B_α, while ψ is the "gauge-fixing fermion" which is an arbitrary imaginary anti-commuting function of the variables of the extended phase-space. Repeated indices include an integration over space coordinates since we are dealing with field theories. The crucial property of the phase space path-integral

$$Z = \int D\mu \, e^{\frac{i}{\hbar} S^{qu}} \tag{10.0.4}$$

where μ are all variables in the extended phase space, is that it is independent of ψ (for suitable boundary conditions). From this property we then prove in the final section that the S-matrix is gauge-choice independent. We shall also discuss the definition of physical states, and apply it to both QED and QCD.

We claim that the Lagrangian BRST methods or Hamiltonian BRST methods are entirely equivalent: anything one can do in one of them, one can also do in the other. The reason we nevertheless discuss both methods is that there exist systems which at present seem very hard to quantize covariantly. An example is the superstring and heterotic string. In such cases different ways of looking at the same problem may be helpful.

We begin this chapter with the example of the Hamiltonian BRST approach to Yang-Mills theory to avoid becoming too abstract. Next we discuss Dirac theory, and give as examples Yang-Mills theory, the Dirac action and the bosonic string. Then

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even statistics and is conserved, i.e., its Poisson (or Dirac) bracket with the first-class constraints and H_0 is proportional to any first-class constraints weakly vanishing), one can construct a corresponding BRST invariant extension $A = A_0 + \cdots$ which in general depends on the structure functions ${}^{(n)}U$ and the ghosts and antighost coordinates and momenta. The Hamiltonian H_{BRST} is just an example of this construction.

we present the general treatment of the Hamiltonian BRST method. There exists a quantization method which combines the virtues of the Lagrangian BRST method and the Hamiltonian BRST method: the antifield formalism of Batalin-Vilkovisky (BV formalism). This method deals with actions, and is manifestly Lorentz covariant (for Lorentz-invariant gauge-fixing terms), while it also fixes the gauge at the very end. It does so by adding for each field an "antifield", which can be considered as a conjugate momentum with opposite statistics. In this approach, the action is equal (!) to the BRST charge. For a true understanding of this BV formalism, one has to be familiar both with the usual Lagrangian BRST formalism and with the Hamiltonian quantization method. This is yet another reason to discuss the Hamiltonian approach.

1 Yang-Mills theory

We begin this section by discussing the Hamiltonian path integral for ordinary gauge fields. As we go along we shall introduce the various ideas. This has the advantage that readers who are, for the time being, only interested in the results for Yang-Mills theory, find here what they are looking for, while for others this section may serve as a warming-up exercise for the more general treatment that follows later. Beginning with the most obvious example one gets to appreciate more the reasons for introducing various extra fields which are not present in the Lagrangian formulation.

The classical gauge action is given by

$$S_{cl} = \int -\frac{1}{4}(G_{\mu\nu}{}^a)^2 d^4x, \quad G_{\mu\nu}{}^a = \partial_\mu A_\nu{}^a - \partial_\nu A_\mu{}^a + g f^a{}_{bc} A_\mu{}^b A_\nu{}^c \tag{10.1.1}$$

Decomposing the indices μ into $i = 1,2,3$ and $\mu = 0$, and using the Minkovski metric $\eta_{\mu\nu} = (1,1,1,-1)$, we obtain for the Lagrangian

$$L_{cl} = \int \left[-\frac{1}{4}(G_{ij}{}^a)^2 + \frac{1}{2}(\dot{A}_i{}^a - D_i A_0^a)^2 \right] d^3x \tag{10.1.2}$$

where $D_i A_0{}^a = \partial_i A_0{}^a + g f^a{}_{bc} A_i{}^b A_0{}^c$.

The momenta conjugate to $A_i{}^a$ are denoted by $p_a{}^i$ and satisfy the equal-time Poisson brackets

$$\left\{p_b{}^j(\vec{y},t), A_i{}^a(\vec{x},t)\right\} = -\delta^j{}_i\delta_b{}^a\delta^3(\vec{x}-\vec{y}) \tag{10.1.3}$$

They are given by

$$p_i{}^a = G_{0i}{}^a = \dot{A}_i{}^a - D_iA_0{}^a \tag{10.1.4}$$

Defining $E_a{}^i = p_a{}^i$ and $B_i{}^a = \frac{1}{2}\epsilon_{ijk}G_{jk}{}^a$, the classical action reads $S_{cl} = \int[\frac{1}{2}(E_a{}^i)^2 - \frac{1}{2}(B_i{}^a)^2]d^4x$ and can be rewritten in Hamiltonian form as follows

$$\begin{aligned} S_{cl} &= \int\left[\dot{A}_i{}^aE_a{}^i - H_0 + A_0{}^a(D_iE_a{}^i)\right]d^3xdt \\ H_0 &= \int\frac{1}{2}\left\{(E_i{}^a)^2 + (B_i{}^a)^2\right\}d^3x \end{aligned} \tag{10.1.5}$$

Note that H_0 differs from the naive (canonical) Hamiltonian $p\cdot q - L \equiv H_L$ by the term with $A_0^a(D_iE_a^i)$.

$$H_L = H_0 - \int A_0{}^a\left(D_iE_a{}^i\right)d^3x \tag{10.1.6}$$

Further note that the field A_0^a is a Lagrange multiplier: it appears without derivatives, and hence its conjugate momentum vanishes, $p(A_0^a) \equiv p_0^a = 0$. These are the primary constraints. Requiring that the primary constraints are maintained in time, $\{H_L, p(A_0^a)\} = 0$, leads to the secondary constraints $D_iE_a^i = 0$. The secondary constraints themselves are maintained in time as we now show, hence in Yang-Mills theory there are only primary and secondary constraints.

The original configuration space is spanned by the q's and $\dot{q}$'s, where the q's are the $A_\mu^a(x)$ at all μ, a, x. Phase space is spanned by all p's and all q's. In phase space all constraints together define a hypersurface. In our example, on this surface $p_0^a = 0$, but away from it p_0^a is nonvanishing, and we can compute commutators involving p_0^a as usual.

First class constraints φ_α satisfy by definition $\{\varphi_\alpha, H_0\} = V_\alpha{}^\beta\varphi_\beta$ and $\{\varphi_\alpha,\varphi_\beta\} = f_{\alpha\beta}{}^\gamma\varphi_\gamma$. We claim that $p(A_0^a)$ and $D_iE_a{}^i$ satisfy the Poisson brackets of first-class constraints. Indeed, at equal times one has

$$\begin{aligned}
&\left\{H_0, D_i E_a{}^i\right\} = 0 \;;\; \{H_0, p(A_0^a)\} = 0 \\
&\left\{D_i E_a{}^i(x), D_j E_b{}^j(y)\right\} = g f_{ab}{}^c D_j E_c{}^j(x)\delta(\vec{x} - \vec{y}) \\
&\{p(A_0^a), p(A_0^b\} = 0 \;;\; \{p(A_0^a), D_i E_a^i\} = 0
\end{aligned} \tag{10.1.7}$$

Thus $V_\alpha{}^\beta = 0$ and $f_{\alpha\beta}{}^\gamma$ is equal to $g f_{ab}{}^c$ or zero. Hence, the complete set of first class constraints is $p(A_0^a) \equiv p_a^0 = 0$ and $D_i E_a^i = 0$. (To prove this, it is useful to begin with $\int \lambda^a(x) D_i E_a{}^i(x) d^3x$, partially integrate, evaluate the brackets, and finally integrate back. This shows that the coefficient of the right-hand side of the $[DE, DE]$ bracket is one, and not two, as one might perhaps naively expect).

The charge $\int \lambda^a(x) D_i E_a{}^i(x) d^3x$ generates gauge transformations on A_i^a and E_a^i. Namely $\delta A_i{}^a = D_i \lambda^a$ while $\delta p_i{}^a$ transforms homogeneously, $\delta p_i{}^a = f^a{}_{bc} p_i{}^b \lambda^c$. For the invariance of the action in we also need $\delta A_0^a = D_0 \lambda^a$, but in a true Hamiltonian formalism no time derivatives can appear. Clearly we need a conjugate momentum for A_0^a if we want to obtain the transformation law δA_0^a from a commutator with a charge and this charge should depend on $p(A_0^a)$. We also expect a term $\dot{A}_0^a p_a^0$ in the quantum action because all fields should appear on equal footing after gauge fixing.

We now take a gigantic leap, and require that the quantum action starts with a term of the form $\dot{Q}_I p^I$, but where I ranges not only over the classical fields A_i^a and A_0^a but also over the ghost C^α and antighosts B_α. This may seem too much at first sight, because in Yang-Mills theory, the ghost action for the gauge $\partial^\mu A_\mu = 0$ reads $(\partial^\mu B) D_\mu C$, so that $p(C) \sim \dot{B}$ and $p(B) \sim \dot{C}$. Another example is the axial gauge $A_3 = 0$ which yields $B D_3 C$, and then $p(C) = p(B) = 0$. However, by introducing independent fields $p(C)$ and $p(B)$ (which later will be eliminated from the path integral) we cover the most general case. Hence we begin with

$$\mathcal{L}(p, Q) = \dot{A}_i^a E_a^i + \dot{A}_0^a p_a^0 + \dot{C}^a p(C)_a + \dot{B}_a p(B)^a \tag{10.1.8}$$

To this expression we should add a Hamiltonian H because of the generic relation

$L = p\dot{q} - H$, but the precise form of H needs to be discussed. Furthermore, one still expects to need a gauge fixing and a ghost term in the action, for the same reasons as in the Lagrangian approach.

At this point the concept of BRST invariance enters. In the Lagrangian approach one writes BRST transformations as $\delta A^a_\mu = (D_\mu c)^a \Lambda$, but one cannot write this result as $\{A^a_\mu, Q_{BRST}\}\Lambda$ because in the Lagrangian approach there are no momenta and commutators. In the Hamiltonian approach which we are developing, on the other hand, we can and actually should require that for any field

$$\delta_{BRST} \text{ field } = \{ \text{ field}, \ Q_H\}\Lambda \tag{10.1.9}$$

where Q_H is the Hamiltonian BRST charge. It should be nilpotent,

$$\{Q_H, Q_H\} = 0 \tag{10.1.10}$$

(Since the Poisson bracket of two anticommuting objects is symmetric in these objects, the relation $\{Q_H, Q_H\} = 0$ is nontrivial).

Moreover, it should be conserved. Since conservation of a charge means time independence, there should be a generator H_{BRST} which commutes with Q_H

$$\{H_{BRST}, Q_H\} = 0 \tag{10.1.11}$$

Our first problem is now to determine Q_H and H_{BRST}. Our second problem is to construct the quantum action.

We start with the latter problem, and, by analogy with the Lagrangian case, we try to find an action which is BRST invariant. We claim that the following expression satisfies this requirement.

$$L_H = L(p, Q) - H_{BRST} + \{\psi, Q_H\} \tag{10.1.12}$$

The function $\psi(p, Q)$ is called the gauge fermion to indicate that it is an anticommuting function of all canonical variables p^I and Q_I. Since L_H is real, and Q_H is real, and the bracked $\{\psi, Q_H\}$ is symmetric in ψ and Q_H, ψ must be imaginary. We shall discuss some useful choices for ψ, in particular of course those ψ which reproduce the most commonly used gauge fixing terms in the Lagrangian formalism.

The last two terms are separately BRST invariant: H_{BRST} because of (10.1.11), and the term $\{\psi, Q_H\}$ because of the Jacobi identities

$$\begin{aligned} 0 = \{Q_H, \{\psi, Q_H\}\} + \{\psi, \{Q_H, Q_H\}\} \\ +\{Q_H, \{Q_H, \psi\}\} = 2\{Q_H, \{\psi, Q_H\}\} \end{aligned} \tag{10.1.13}$$

(we used that $\{Q_H, \psi\} = \{\psi, Q_H\}$. Finally the time-integral of the first term, $S(p, Q) = \int L(p, Q)dt = \int \dot{Q}_I p^I d^4x$, is also BRST invariant as the following little calculation shows

$$\begin{aligned} &\left\{Q_H, \int (\dot{Q}_I p^I) d^4x\right\} = -\int \{Q_H, Q_I\}\dot{p}^I d^4x - \int \dot{Q}_I \{p^I, Q_H\} d^4x \\ &= -\int \partial Q_H/\partial p^J \{p^J, Q_I\}\dot{p}^I d^4x - \int \dot{Q}_I \{p^I, Q_J\} \frac{\partial}{\partial Q_J} Q_H d^4x \\ &= \int \frac{d}{dt} Q_H d^4x = 0 \end{aligned} \tag{10.1.14}$$

We partially integrated the derivative $\frac{\partial}{\partial t}$ in the first step, and we used the fact that the Poisson bracket $\{p, Q\}$ of p and Q is antisymmetric if p and/or Q are commuting, and symmetric if both p and Q are anticommuting. In the last step we used that the basic Poisson bracket is

$$\{p^I, Q_J\} = -\delta^I_J \tag{10.1.15}$$

both for commuting and anticommuting variables. We discuss later that the definition in (10.1.15) is needed if one generalizes the term $\dot{q}_i p^i$ to $\dot{Q}_I p^I$ for all variables, but at this point we view (10.1.15) as just a definition. Note that $\int \dot{Q}_I p^I d^4x$ commutes with **any** charge because nowhere in (10.1.14) did we use any properties of Q_H.

We now must construct Q_H and H_{BRST}. In general Q_H is quite complicated, but nevertheless there is a direct construction of Q_H in the general case. For Yang-Mills theory, however, Q_H is rather simple

$$Q_H = \int \Big[C^a(D_iE^i_a) - p(B)^a p^0_a - -\frac{1}{2}gC^bC^c f_{cb}{}^a p(C)_a\Big] d^3x \tag{10.1.16}$$

It is rather easy to check that Q_H is nilpotent. The last term is nilpotent by itself due to the Jacobi identities for the structure constants. The second term is not needed for nilpotency but it plays a crucial role in obtaining the correct BRST transformation rules. Since $D_iE^i_a$ and p^0_a are all first class constraints, it is natural to expect a term with p^0_a if there is a term with $D_iE^i_a$, and p^0_a can only be multiplied by $p(B)^a$ since only $p(B)^a$ has the same ghost number as C^a. By a suitable rescaling of p^0_a (and a corresponding counter-rescaling of A^a_0) one can fix the coefficient of the $p(B)p^0_a$ term to minus unity.

Yang-Mills theory is a theory with only nonvanishing first-order structure functions (to be explained) and for such theories the general form of Q_H reads

$$Q_H = \eta^a G_a - \frac{1}{2}\eta^c\eta^b f_{bc}{}^a p_a(-)^c \tag{10.1.17}$$

(for Yang-Mills theory $(-)^c = 0$ but for commuting η^c one has $(-)^c = -1$). In our case $\eta^a = \{C^a, -p(B)^a\}, p_a = \{p(C)_a, -B_a\}$ and $G_a = \{D_iE^i_a, p^0_a\}$. We take η^a as real and p_a as imaginary. Then Q_H is real. Since all constraints in G_a have the same ghost number (zero), all fields in η^a should have the same ghost number. Both C^a and $p(B)^a$ have ghost number one, and $p(C)_a$ and B_a have ghost number minus one. Note that G_a are all first constraints: they satisfy

$$\{G_a, G_b\} = f_{ab}{}^c G_c, [G_a, H_0] = V_a{}^b G_b \tag{10.1.18}$$

In our case, $V_a{}^b = 0$ and the group structure constants $f_{\alpha\beta}{}^\gamma$ for the constraints $D_iE_a{}^i \equiv \partial_i E_a{}^i + gf^a{}_{bc}A_i{}^b E_c{}^i$ are given by $gf_{ab}{}^c$, see (10.1.7).

The BRST transformation rules for any field follow from $\delta(\text{field}) = \Lambda\{Q_H, \text{field}\} = \{\text{ field}, Q_H\}\Lambda$. We get

$$\delta A_i{}^a \;=\; -D_iC^a\Lambda \;;\; \delta A_0{}^a = -p(B)^a\Lambda$$

$$\begin{aligned}
\delta E_a{}^i &= -g f_a{}^b{}_c E_b{}^i C^c \Lambda \; ; \; \delta p_a{}^0 = 0 \\
\delta C^a &= -\frac{1}{2} g f^a{}_{bc} C^b C^c \Lambda ; \delta p(B)^a = 0 \\
\delta p(C)_a &= (-D_i E_a^i + g p_b(C) f^b{}_{ac} C^c) \Lambda ; \delta B_a = p_a{}^0 \Lambda
\end{aligned} \tag{10.1.19}$$

where we used the following Poisson brackets

$$\{C^a, p(C)_b\} = -\delta_a^b ; \left\{B_a, p(B)^b\right\} = -\delta_a{}^b \tag{10.1.20}$$

Taking Λ to be imaginary, these rules preserve the reality properties of all fields.

One may verify by direct computation, that the rules in (10.1.19) are nilpotent. Note that these extended phase-space transformation rules are independent of the gauge-fermion ψ and do not contain time derivatives. If one eliminates the momenta, time derivatives appear, and also dependence on ψ will in general creep in since $\{\psi, Q_H\}$ will contribute in general to the relations of momenta in terms of velocities. In the Lagrangian formalism we saw that we could find BRST laws which were independent of the gauge choices, provided we kept the BRST-auxiliary field d_a in the theory. Hence, we expect that d_α is one of the momenta in the Hamiltonian formalism. The transformation law $\delta B_a = p_a^0 \Lambda$ shows that p_a^0 is to be identified with d_a.

Next we must construct H_{BRST}. It must commute with Q_H, and extend H_0. Again in general H_{BRST} is a rather complicated expression, containing not only the classical variables but also ghosts, antighosts and their canonical momenta. However, again for Yang-Mills theory H_{BRST} is very simple

$$H_{BRST} = H_0 = \int \left[\frac{1}{2} (E_i^a)^2 + \frac{1}{2} (B_i^a)^2 \right] d^3x \tag{10.1.21}$$

To prove this, we must show that $[H_{BRST}, Q_H] = 0$. It is clear that we only need consider the terms $C^a(D^i E_i^a) - p(B)^a p_a^0$ in Q_H. The latter commutes with H_0 since H_0 does not depend on the Lagrange multiplier A_0^a, and the former contains the operator $D^i E_i^a$ which rotates E_j^b and B_j^b as vectors, leaving H_0 invariant.

The complete quantum action for Yang-Mills theory in the Hamiltonian formalism is thus given by

$$\begin{aligned} S_{qu} &= \int \Big[E_a{}^i \dot{A}_i{}^a + p_a{}^0 \dot{A}_0{}^a - p(C)_a \dot{C}^a - p(B)^a \dot{B}_a \\ &- \frac{1}{2} \left\{ (E_a{}^i)^2 + \frac{1}{2} (G_{ij}{}^a)^2 \right\} + \{\psi, Q_H\} \,\Big]\, d^4x \end{aligned} \tag{10.1.22}$$

We recall that

$$Q_H = \int \left[C^a (D_i E_a{}^i) - p(B)^a p_a{}^0 - \frac{1}{2} g C^b C^c f_{cb}{}^a p(C)_a \right] d^3x \tag{10.1.23}$$

Let us consider a special useful gauge-fermion

$$\psi = \int \left(B_a \chi^a - p(C)_a A_0{}^a \right) d^3x \tag{10.1.24}$$

where χ^a is real and determines the gauge chosen. Note that ψ is antihermitian, as it should be in order that $\{\psi, Q_H\}$ be hermitian.[2] Then we find for the gauge fixing term

$$\begin{aligned} \{\psi, Q_H\} &= \int \Big[p_a{}^0 \chi^a + B_a \{\chi^a, Q_H\} + A_0{}^a (D_i E_i{}^a) \\ &- g p(C)_a f^a{}_{bc} A_0{}^b C^c + p(C)_a p(B)^a \,\Big]\, d^3x \end{aligned} \tag{10.1.25}$$

If χ^a only depends on $A_i{}^a$, the term $\{\chi^a, Q_H\}$ is independent of momenta (and, of course, independent of velocities, since in the Hamiltonian formalism there are momenta but no velocities). One can then integrate over $E_a{}^i, p(C)_a, p(B)^a$ and $p_a{}^0$; these are three Gaussian integrals and one integral which gives a Dirac delta function. One finds then that $p(B)^a = D_0 C^a$ and $p(C) = -\dot{B}$, hence $\delta A_0^a = -D_0 C^a \Lambda$, while

$$Z = \int DA_i{}^a DA_0{}^a DC^a DB_a \delta \left[\chi^a + \dot{A}_0{}^a \right] \exp\left(S_{cl} + S_{Fp} \right) \tag{10.1.26}$$

[2] We work in this section with Poisson brackets, not yet (anti) commutators. At the quantum level, the commutators acquire an extra factor $i\hbar$, and then one would have to write $i\{\psi, Q_H\}$ where $\{\psi, Q_H\}$ denotes an anticommutator.

where S_{cl} is the classical action of the Lagrange formalism and S_{FP} is just the Faddeev-Popov action.

$$\begin{aligned} S_{Fp} &= \int \left[B_a \left\{ \chi^a, Q_H \right\} + \dot{B}_a D_0 C^a \right] d^4x \\ &= \int B_a \delta_{BRST} \left(\chi^a + \dot{A}_0^a \right) \end{aligned} \tag{10.1.27}$$

Choosing for χ a **Coulomb**-like expression, $\chi^a = -\partial_k A_k{}^a$, this becomes $S_{Fp} = -\int (\partial^\mu B_a)(D_\mu C^a) d^4x$ which is the familiar expression for the ghost action in the **Lorentz** gauge. Incidentally, in order that $\chi^a + \dot{A}_0^a$ be the invariant expression $\partial^\mu A_\mu^a$ and S_{Fp} be also Lorentz invariant, the relative sign in ψ gets fixed.

A somewhat more general class of gauges is obtained by choosing $\chi^a = -\frac{1}{2}\xi p^{o,a} - \partial^k A_k{}^a$. Then one finds an extra term $\frac{1}{2}\xi (p^{o,a})^2$ in the action, and integrating only over $E_a{}^i$ one finds for the action

$$\begin{aligned} S_{qu} &= \int d^4x \,[\, -\frac{1}{4}(F_{\mu\nu}{}^a)^2 + \frac{1}{2}\xi (p^{o,a})^2 - p_a{}^o (\partial^\mu A_\mu{}^a) \\ &- \; p(C)_a D_o C^a - p(B)^a \dot{B}_a + p(C)_a p(B)^a + B_a \partial^k D_k C^a \,] \end{aligned} \tag{10.1.28}$$

This identifies the conjugate momentum p_a^0 with the auxiliary field of the Lagrangian BRST formalism. Furthermore, for $\xi = 1$ we find the Feynman gauge, $\xi = 0$ leads to the Landau gauge and $\xi \to \infty$ yields the unitary gauge.

Integration over $p(C)$ and $p(B)$ leads to the usual ghost action, but let us consider what happens if one integrates over C and B instead. One finds then $\delta(\dot{p}(B) - \partial^k D_k C) = |\det \partial^k D_k| \delta((\partial D)^{-1} \dot{p}(B) - C)$ leading to a nonlocal ghost action $\mathcal{L}_{gh} = -p(C) D_0 (\partial^k D_k)^{-1} \dot{p}(B) + p(C) p(B)$ and an extra factor $|\det \partial^k D_k|$ in the measure. Introducing new integration variables $\dot{p}(B) \to C$ and $p(C) \to \dot{B}$, the Jacobian is unity, and then changing $C \to -\partial^k D_k C$ all Jacobians with $\partial^k D_k$ also cancel, yielding back the usual ghost action. One finds in this way new noncovariant and nonlocal BRST transformation laws (for example, $\delta A_0^a = \frac{1}{\partial_0} \partial^k D_k C^a$) which leave the quantum action invariant [3], but they are still nilpotent. On-shell they coincide with the usual local BRST laws.

A more general gauge is obtained by putting $\chi^a = \frac{1}{\beta}\hat{\chi}^a\left(A_k^b, E_c{}^j\right)$. In that case we rescale $p_a^0 = \beta \hat{p}_a{}^0$ and $B_a = \beta \hat{B}_a$ (the Jacobians cancel) and find

$$\begin{aligned}
\mathcal{L}_{qu} &= E_a{}^i \dot{A}_i{}^a - p(C)_a \dot{C}^a - \frac{1}{2}\left(E^2 + \frac{1}{2}G_{ij}{}^2)\right) \\
&+ \hat{p}_a{}^0 \hat{\chi}^a + \hat{B}_a \{\hat{\chi}^a, Q_H\} + A_0{}^a (D_i E_i{}^a) \\
&+ p(C)_a p(B)^a - g p(C)_a f^a{}_{bc} A_0{}^b C^c \\
&+ \beta \left\{\hat{p}_a{}^0 \dot{A}_0{}^a - p(B)^a \dot{B}_a\right\} \qquad (10.1.29)
\end{aligned}$$

For β tending to zero, the integration over $\hat{p}^0{}_a$ gives $\delta(\hat{\chi}^a)$, while the $p(C)p(B)$ term, the $p(C)\dot{C}$ term and the $p(C)gA_0C$ term vanish after integration over $p(C)$ and $p(B)$. The integral over $A_0{}^a$ gives then $\delta(G_a)$. Finally, the integral over $\hat{B}_a$ and C^a yields $\det\{\hat{\chi}^a, G_b\}$. Hence from this gauge one obtains

$$Z = \int DA_i{}^a Dp_a{}^j \prod_{a,x} \delta(G_a(x))\delta(\hat{\chi}^a) \det\{\hat{\chi}^a, G_b\} \exp\frac{i}{\hbar}\int d^4x \left[E_a{}^i \dot{A}_i{}^a - H^0\right] \qquad (10.1.30)$$

This is the expression Faddeev [4] obtained in his canonical analysis. Since in this formulation one only integrates over physical coordinates and momenta, it is a good starting point for proving unitarity.

Leaving A_0^a and $\hat{p}_a^0$ in (10.1.29) and integrating over $p(C)$ and $p(B)$, one recognizes the classical action in the form (10.1.15), together with the gauge fixing term $\hat{p}_a^0\hat{\chi}^a$ and the ghost action for $\hat{\chi}^a$. This is the quantum action which the Lagrangian approach yields for the unweighted gauge $\hat{\chi}^a = 0$.

2 The Dirac formalism

Classical gauge actions can often be **rewritten** in the following Hamiltonian form

$$S_{cl} = \int \left(\dot{q}_i p^i - H_0 + \lambda^\alpha \varphi_\alpha\right) dt \qquad (10.2.1)$$

where q_i and p^i are pairs of conjugate momenta, λ^α are Lagrange multiplier fields (fields which appear in the action without time derivatives), $\varphi_\alpha(q,p)$ are the set of all "first-class constraints", and $H_0(q,p)$ is a classical Hamiltonian. First class constraints satisfy by definition the following algebra

$$\begin{aligned}\{H_0,\varphi_\alpha\} &= V_\alpha{}^\beta\varphi_\beta \\ \{\varphi_\alpha,\varphi_\beta\} &= f_{\alpha\beta}{}^\gamma\varphi_\gamma\end{aligned} \tag{10.2.2}$$

where the structure functions $V_\alpha{}^\beta$ and $f_{\alpha\beta}{}^\gamma$ may depend on q^i and p_i and may even contain derivatives (as in the case of gravity).

The brackets in (20.0.5) are Poisson brackets (or Dirac brackets when second-class constraints are present.) To define Poisson brackets we first need to define the canonical commutation relations between coordinates and conjugate momenta. For the usual variables of ordinary quantum mechanics we define $\{q,p\} = 1$ or, equivalently, $\{p,q\} = -1$. However, for anticommuting variables the brackets are symmetric in p and q, $\{p,q\} = \{q,p\}$, hence it makes a difference whether one defines $\{q,p\} = 1$ or $\{p,q\} = -1$. The reader who does not care why we choose a certain set of definitions may skip the following discussion, and proceed to (10.2.12).

We want to obtain a set of definitions which is **uniformly** valid (for commuting as well as anticommuting variables) and which coincides with the usual definitions of ordinary quantum mechanics. There are two compatibility requirements: the BRST charge Q_H should be nilpotent and the Heisenberg equations of motion should hold. As we shall later discuss in detail, the first two terms in the BRST charge read

$$Q_H = c^\alpha G_\alpha + \frac{1}{2}\lambda c^\beta c^\alpha f_{\alpha\beta}{}^\gamma p_\gamma + \dots \tag{10.2.3}$$

where λ is a constant which may depend on α,β,γ and on whether c^β are commuting or anticommuting, the G_α satisfy $\{G_\alpha,G_\beta\} = f_{\alpha\beta}{}^\gamma G_\gamma$ and the Poisson bracket $\{\ ,\}$ is still to be defined. Requiring that the terms quadratic in c^α cancel in $\{Q_H,Q_H\}$ leads to[3]

$$(-)^{\beta+1}c^\beta c^\alpha\{G_\alpha,G_\beta\} + \lambda c^\beta c^\alpha f_{\alpha\beta}{}^\gamma\{p_\gamma,c^\delta\}G_\delta = 0 \tag{10.2.4}$$

[3]For commuting G_β we define $(-)^\beta = 1$, while for anticommuting G_β we define $(-)^\beta = -1$. Note that $c^\beta G_\beta$ is always anticommuting.

whose solution is $\lambda\{p_\gamma, c^\delta\} = (-)^\beta \delta^\delta_\gamma$. Hence

$$Q_H = c^\alpha G_\alpha + \frac{1}{2}\sigma c^\beta c^\alpha f_{\alpha\beta}{}^\gamma p_\gamma (-)^\beta + \dots \tag{10.2.5}$$

where the sign σ is defined by

$$\{p_\gamma, c^\delta\} = \sigma \delta_\gamma{}^\delta \tag{10.2.6}$$

Note that σ is not fixed by requiring nilpotency of Q_H.

Next we consider the Heisenberg equations of motion of quantum fields. With the Poisson brackets for p_α and c^β given, the quantum commutators are now uniformly given by $[p_\gamma, c^\delta\} = -i\hbar\delta_\gamma{}^\delta$ where the bracket now denotes a commutator or an anticommutator. Then, for any Heisenberg field $\varphi(x)$ we must find

$$\dot{\varphi}(x) = \frac{i}{\hbar}[H, \varphi] \tag{10.2.7}$$

Given a Lagrangian $L = \int \mathcal{L} d^3x$, one can define the canonically conjugate momenta either by left- or by right-differentiation: $p_\ell = \frac{\partial}{\partial \dot{q}} L$ or $p_r = \partial L / \partial \dot{q}$. For fermionic variables this makes a difference. For example, $\mathcal{L} = \alpha \dot{b}\dot{c}$ with α a constant and b, c anticommuting, leads to $p(c)_\ell = -\alpha\dot{b}$ and $p(c)_r = \alpha\dot{b}$, and similarly $p(b)_\ell = \alpha\dot{c}$ and $p(b)_r = -\alpha\dot{c}$. The Hamiltonian can be written in two ways

$$\begin{aligned} H &= \dot{q}\left(\frac{\partial}{\partial \dot{q}} L\right) - L = (\partial L/\partial \dot{q})\dot{q} - L \\ &= \dot{q} p_\ell - L = p_r \dot{q} - L \end{aligned} \tag{10.2.8}$$

In either way of writing H, in the variations of H the terms with $\delta\dot{q}$ cancel. Other ways of writing H, such as $\dot{q}\partial L/\partial\dot{q}$ do not have this property and must be rejected. One finds for the example

$$\begin{aligned} H &= \dot{c}(-\alpha\dot{b}) + \dot{b}(\alpha\dot{c}) - \alpha\dot{b}\dot{c} = \alpha\dot{b}\dot{c} \\ &= \frac{-1}{\alpha} p(c)_\ell p(b)_\ell = -\frac{1}{\alpha} p(c)_r p(b)_r \end{aligned} \tag{10.2.9}$$

Hence, the Hamiltonian in terms of left-variables is the same as in terms of right-variables.

The Heisenberg equations of motion fix the signs in the Poisson brackets of ghosts and antighosts. We claim that if one uses left-derivatives to define conjugate momenta, the Poisson brackets are $\{P_I, Q^J\} = -\delta_I{}^J$ for all variables, but if one uses right-derivatives, one has instead $\{Q^J, P_I\} = \delta^J{}_I$. One may check this in the example: the basic (anti) commutator $\{p^\alpha, q_\beta\} = -\delta^\alpha_\beta$ is only compatible with the Heisenberg equations for left-derivatives. We use quantum anticommutators and find then for the Heisenberg equation of motion of the ghost field

$$\begin{aligned} \dot{c} = \frac{i}{\hbar}[H, c] &= \frac{i}{\hbar}\left(\frac{1}{\alpha}\right)\{c, p(c)\}p(b) \\ &= \frac{i}{\hbar}(\frac{1}{\alpha})\{c, p(c)_\ell\}\alpha\dot{c} = \frac{i}{\hbar}\left(\frac{1}{\alpha}\right)\{c, p(c)_r\}(-\alpha\dot{c}) \end{aligned} \tag{10.2.10}$$

Hence for consistency the anticommutator for c and its conjugate momentum is given by

$$\{c, p(c)_\ell\} = -i\hbar, \{c, p(c)_r\} = i\hbar \tag{10.2.11}$$

Since the brackets for fermionic variables are symmetric, we have $\{p(c)_\ell, c\} = -1$ but $\{c, p(c)_r\} = \hbar$.

We can now fix the sign σ in Q_H which was still left unfixed. If one uses left-derivatives nilpotency of Q shows that it is given by (10.2.5) with $\sigma = +1$, but if one uses right-derivatives one may write Q in terms of $G_\alpha c^\alpha$ and $p_\gamma f^\gamma{}_{\alpha\beta} c^\beta c^\alpha$.

To summarize: a consistent set of definitions which we follow is $Q_H = c^\alpha G_\alpha + \cdots$ and

$$\begin{aligned} p^i &= \frac{\partial}{\partial \dot{q}_i} L, H = \dot{q}_i p^i - L, \\ \{p^\alpha, q_\beta\}_P &= -\delta^\alpha{}_\beta \text{ (classically)}, \ [\, p^\alpha, q_\beta \,\} = -i\hbar\delta^\alpha{}_\beta \text{ (quantum)} \end{aligned} \tag{10.2.12}$$

The symbol $\{\, ,\}_P$ denoted Poisson brackets and $[\, ,\}$ denotes a commutator or an anticommutator, depending on whether p_α and q^β denote commuting or anticommuting variables, respectively. The p^i and q_j denote only the classical variables but p^α and q_β may also be (commuting or anticommuting) ghosts.

We define Poisson brackets for commuting or anticommuting functions f and g of the canonical variables p_α and q^β as follows

$$\{f, g\} = -(\partial f/\partial p^\alpha)\left(\frac{\partial}{\partial q_\alpha} g\right) + (-)^{fg}(\partial g/\partial p^\alpha)\left(\frac{\partial}{\partial q_\alpha} f\right) \tag{10.2.13}$$

When f and g are equal to p's and q's this reproduces our result $\{p^i, q_j\} = -\delta^i{}_j$. Moreover, the bracket is symmetric or antisymmetric in f, g depending on whether f and g are both anticommuting or not, respectively. Finally, this bracket coincides with the usual definition of Poisson brackets in classical mechanics when p and q, and also f and g, are commuting objects.

We return to gauge theories, and, following Dirac, we define the naive Hamiltonian H_L by

$$H_L \equiv \dot{q}_i p^i - L \tag{10.2.14}$$

(One could also call this the canonical Hamiltonian). It follows that H_L is only a function of p^i and q_i, but not of $\dot{p}^i$ and/or $\dot{q}_i$, since

$$\begin{aligned} \delta H_L &= \dot{q}_i \delta p^i + \delta\dot{q}_i p^i - \delta q_i \frac{\partial}{\partial q_i} L - \delta\dot{q}_i \frac{\partial}{\partial \dot{q}_i} L \\ &= \dot{q}_i \delta p^i - \delta q_i \frac{\partial}{\partial q_i} L \end{aligned} \tag{10.2.15}$$

Hence $\frac{\partial}{\partial q_i} H_L = -\frac{\partial}{\partial q_i} L$, while $\partial H_L/\partial p^i = \dot{q}^i$. On-shell, $\frac{\partial}{\partial q_i} L = \frac{d}{dt}\, \frac{\partial}{\partial \dot{q}_i} L = \dot{p}^i$ so that a dynamical variable $F(q, p)$ evolves in time as follows

$$\frac{dF}{dt} = \dot{q}_i \frac{\partial}{\partial q_i} F + \partial F/\partial p^i \dot{p}^i = \{F, H_L\} \tag{10.2.16}$$

The object H_L is in general not equal to H_0, as is clear from the example of Yang-Mills theory.

The first-class constraints generate gauge transformations of the classical action. They are defined by

$$\delta p^i = \left\{\epsilon^\alpha \varphi_\alpha, p^i\right\} = \frac{\partial}{\partial q_i}(\epsilon^\alpha \psi_\alpha)$$

$$
\begin{aligned}
\delta q_i &= \{\epsilon^\alpha \varphi_\alpha, q_i\} = -\epsilon^\alpha \partial\varphi_\alpha/\partial p^i \\
\delta\lambda^\alpha &= \dot{\epsilon}^\alpha - \epsilon^\beta V_\beta{}^\alpha - \lambda^\gamma \epsilon^\beta f_{\beta\gamma}{}^\alpha
\end{aligned}
\tag{10.2.17}
$$

where ϵ^α is an arbitrary local parameter independent of q_i and p^i. These transformation laws leave the classical action in (20.0.5) invariant. The proof of the gauge invariance follows from the following intermediate results

$$
\begin{aligned}
\delta \int \dot{q}_i p^i dt &= \int \epsilon^\alpha \dot{\varphi}_\alpha dt = -\int \dot{\epsilon}^\alpha \varphi_\alpha dt \\
\delta H_0 &= \{\epsilon^\alpha \varphi_\alpha, H_0\} = -\epsilon^\alpha V_\alpha{}^\beta \varphi_\beta \\
\delta\varphi_\alpha &= \left\{\epsilon^\beta \varphi_\beta, \varphi_\alpha\right\} = \epsilon^\beta f_{\beta\alpha}{}^\gamma \varphi_\gamma
\end{aligned}
\tag{10.2.18}
$$

We neglected here contributions from the boundary, but in a later section we shall give a detailed discussion of boundary conditions.

In general, one introduces canonical momenta for all fields, also for the Lagrange multipliers. When the relations $p^i = \frac{\partial}{\partial \dot{q}_i} L$ cannot be solved for $\dot{q}_i$, one has **primary constraints**, which Dirac denotes by

$$
\varphi_m(q,p) = 0 \;\; \text{(primary constraints)} \tag{10.2.19}
$$

The Hamiltonian H_L, when written as a function of p and q, is then ambiguous, and one only knows that the true Hamiltonian is contained in the set of functions $H_T \equiv H_L + u^m \varphi_m$, where the u^m may depend on p and q. (For example, in the case of the Dirac action considered below, there is a constraint $p_A - i\psi^*{}_A = 0$, and hence one may either use p_A or $i\psi^*{}_A$ in H_L).

We now require that if the constraints are imposed at a given time t, they should hold also at other times. This requirement goes under the name "consistency of the constraints". Consistency of the constraints $\varphi_m = 0$ leads to the requirement $\{H_T, \varphi_m\} \approx 0$, or written out in detail

$$
\{H_L, \varphi_m\} + u^{m'} \{\varphi_{m'}, \varphi_m\} \approx 0 \tag{10.2.20}
$$

where the symbol ≈ 0 means that the right-hand side may be proportional to constraints ("weakly zero"). This equation may lead to further (secondary) constraints and/or fix (part of) the u^m. We shall give examples of all cases.

In Yang-Mills theory, we could have proceeded in the same way and first constructed H_T. One finds then as primary constraints that the fields ${A_0}^a$ have vanishing conjugate momenta ${p_a}^0$. Hence the primary constraints $\varphi_m(p,q)$ read in this case

$$ {\varphi_0}^a = {p_0}^a \tag{10.2.21} $$

Consistency requires that $\{H_L + \int u_b {p_0}^b d^3x, {p_0}^a\} \approx 0$, with H_L given by $H_0 - \int A_0^a$ $D_i E_a^i d^3x$ with H_0 in (20.0.17). This example does not lead to any condition on the u_b, but it leads to secondary constraints

$$ D_i {E_a}^i = 0 \tag{10.2.22} $$

From (10.1.7) we see that there are no further constraints, as $\{H_T, D_i {E_a}^i\} \approx 0$. Since ${p_0}^a$ commute with themselves and with $D_i {E_b}^i$, all constraints are first-class. Hence, we should have used (as we did) Poisson brackets. In this case we have $H_T = H_0 - \int ({A_0}^a D_i {E_a}^i - u_a {p_0}^a)\, d^3x$. It is clear that $D_i {E_a}^i$ generates gauge transformations on ${A_i}^a$ and ${p_a}^i$, but the primary first class constraint $u_a {p_0}^a$ only transforms ${A_0}^a$, and $\delta {A_0}^a = u^a$ is not the gauge transformation which leaves the classical action invariant. We already gave in (20.0.12) the correct result for $\delta\lambda^a = \delta {A_0}^a$ but this result did not follow from a charge.

Motivated by this example, and by the wish to isolate $H_0(q,p)$ from H_T, Dirac introduces at this point first an "extended Hamiltonian" H_E which equals H_T plus an arbitrary linear combination of **all first class** constraints, thus not only the primary first-class constraints

$$ H_E = H_T + u^\alpha \varphi_\alpha \;, \varphi_\alpha \;\text{ all first-class constraints} \tag{10.2.23} $$

It describes a time evolution which is due to H_T while simultaneously the system undergoes gauge transformations. It is clear that by redefinition of the u^α we can remove the term $\lambda^\alpha \varphi_\alpha$ with the Lagrange multiplier fields (if present) from H_L, and in this case H_L is reduced to the "bare Hamiltonian" H_0 which only depends on p^i and q_i.

More generally, Dirac defines first-class dynamical variables $R(q,p)$ as variables whose Poisson brackets with all first-class constraints φ_α vanish weakly

$$\{R, \varphi_\alpha\} = c_\alpha{}^I \varphi_I \tag{10.2.24}$$

where φ_I denote all constraints, first class or second class. In terms of Dirac brackets this reduces to $\{R, \varphi_\alpha\} = c'_\alpha{}^\beta \varphi_\beta$. Clearly, H_0 and φ_α themselves are first-class. (The φ_α are first-class because their Dirac brackets in (20.0.6) imply the Poisson brackets in (10.2.24).) The consistency condition for the Hamiltonian $H_T = H_L + u^m \varphi_m$ (with φ_m all primary constraints), which required that all constraints are maintained in time, is the statement that H_T must be first-class.

$$\{H_L, \phi_I\} + u^m \{\varphi_m, \phi_I\} \approx 0 \tag{10.2.25}$$

(where ϕ_I denote all constraints, first class or second class). Viewed as a set of equations for u^m, the general solution is $u^m = U^m + v^a V_a{}^m$ (with U^m a particular solution, and $V_a{}^m$ the general homogeneous solution and v^a arbitrary functions of x). Then

$$H_T = H' + v^a \varphi_a; H' = H_L + U^m \varphi_m; \varphi_a = V_a{}^m \varphi_m \tag{10.2.26}$$

It is clear that both the φ_a and H' are first class (because $\{\varphi_a, \varphi_I\} \approx 0$, and with $\{H_T, \phi_I\} \approx 0$ this implies that $\{H', \phi_I\} = 0$). If one then drops the Lagrange multiplier term $\lambda^\alpha \varphi_\alpha$ from H' one obtains H_0.

Example 1. Maxwell theory. We already discussed the Hamiltonian BRST approach to Yang-Mills theory; we needed to introduce ghosts and obtained the quantum action. We split H_L into a part H_0 depending on p_i and q^i, and a rest which depends on Lagrange multiplier fields (the Gausz term $-\int A_0^a D^i E_i^a d^3x$). From H_0 one constructs H_{BRST} while the Gausz term reappears in the final quantum action in the term $\{\psi, Q_H\}$.

The Dirac formalism does not introduce ghosts, and one works with H_L. The classical action is

$$S_{cl} = \int -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} d^4x \tag{10.2.27}$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, and is invariant under $\delta A_\mu = \partial_\mu \lambda$. There is one primary constraint and one secondary constraint; both are first-class

$$\varphi_1 = \pi_0 \ , \ \varphi_2 = \partial_i \pi^i \tag{10.2.28}$$

where $\pi_i = F_{0i} = \partial_0 A_i - \partial_i A_0 = -E_i$. The extended Hamiltonian is $H_L + \lambda^1 \varphi_1 + \lambda^2 \varphi_2$ where $H_L = H_0 - \int A_0 \partial_i \pi^i d^3x$, and λ^1 and λ^2 are new Lagrange multiplier fields. The extended action thus reads

$$S_E = \int [\dot{A}_i \pi^i + \dot{A}_0 \pi^0 - H_L - \lambda^1 \varphi_1 - \lambda^2 \varphi_2] d^4x \tag{10.2.29}$$

The local gauge transformations under which S_E is invariant are given by (20.0.11), so we must first determine the ${V_\beta}^\alpha$ in

$$[H_L, \varphi_j] = \tag{10.2.30}$$

Example 2. The Dirac action. The Dirac action

$$S_{cl} = \int (-\psi^\dagger i \gamma^0 \gamma^\mu \partial_\mu \psi) d^4x \quad , \quad \{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \tag{10.2.31}$$

contains eight independent coordinates, ψ^A and ${\psi^*}_A \equiv (\psi^A)^*$, with $A = 1, 4$. The conjugate momenta are $\frac{\partial}{\partial \dot{\psi}^A} L = p_A = -i{\psi^*}_A$ and $p^{*A} = 0$. Hence there are eight primary constraints

$$\varphi_{1A} = p_A + i{\psi^*}_A \quad , \quad {\varphi_2}^A = p^{*A} \tag{10.2.32}$$

The naive Hamiltonian $H_L = \dot{q}_i p^i - L$ becomes in this case

$$\begin{aligned} H_L &= \int [\dot{\psi}^A p_A + \dot{\psi}^*_A p^{*A} + \psi^\dagger i \gamma^0 \gamma^\mu \partial_\mu \psi] d^3x \\ &= -\int p_A \gamma^0 \gamma^k \partial_k \psi d^3x \end{aligned} \tag{10.2.33}$$

With $\mathcal{H}_T = -p\gamma^0\gamma^k\partial_k\psi + u^A\varphi_{1A} + v_A{\varphi_2}^A$, consistency of the primary constraints yields $u^A = 0$ and $v_A = -i\partial_k(p\gamma^0\gamma^k)_A$, hence all u^m are fixed and no secondary constraints are found. The Hamiltonian becomes

$$H_T = -\int p\gamma^0\gamma^k\partial_k(\psi - ip^*)d^3x \tag{10.2.34}$$

Since all u^m are fixed, $H_T = H'$.

The constraints commute with H_0 by construction, and the set (φ_{1A}) have vanishing brackets among themselves, as do the set (φ_2^A), but the φ_{1A} and ${\varphi_2}^A$ have a nonvanishing Poisson bracket

$$\left\{\varphi_{1A}(\vec{x},t), \varphi_2^B(\vec{y},t)\right\} = -i\delta_A^B\delta(\vec{x}-\vec{y}) \tag{10.2.35}$$

Hence, none of the eight constraints are first class, and $H_T = H_E$. Since there are no terms with $\lambda^\alpha\psi_\alpha$ in H_L, in this example $H' = H_0$. Hence, for the Dirac action, $H_E = H_T = H' = H_0$.

In general, constraints which are not first class are called second class constraints. Second class constraints, denoted henceforth by χ_I, have the property that the (super) matrix $\{\chi_I, \chi_J\}$ is invertible [–]. This allowed Dirac to generalize the concept of the Poisson bracket to what is now called the Dirac bracket. It is defined by

$$\{A,B\}_D = \{A,B\} - \{A,\chi_I\}\left(\{\chi,\chi\}^{-1}\right)^{IJ}\{\chi_J,B\} \tag{10.2.36}$$

and has the property that

$$\{A,\chi_I\}_D = 0 \tag{10.2.37}$$

for any $A(p,q)$. Hence, it is really a projection operator onto the space of functions orthogonal to the second class constraints. Clearly, if one uses the Dirac bracket, one can forget about second class constraints as they drop out of the brackets.

Returning to the Dirac action, if we use from now on Dirac brackets, we may put all $p^* = 0$ and replace all $i\psi^*$ by $-p$. Then H_T becomes equal to

$$H_T = -\int(p\gamma^0\gamma^k\partial_k\psi)d^3x \tag{10.2.38}$$

and $\left\{\psi^A, p_B\right\}_D = \{\psi^A, p_B\} = -\delta_A{}^B$. The action takes then the standard form in (20.0.5) with $H_0 = H_T$ and without a $\lambda^\alpha\varphi_\alpha$ term. Note that the Dirac fermions ψ^A and p_B in this example are complex.

If one would have used Majorana fermions, one would have the reality condition

$$\bar{\psi} = \psi^T C \ , \ \mathcal{L} = -\frac{1}{2}\psi^T C\gamma^\mu\partial_\mu\psi \ , \ C\gamma^\mu C^{-1} = -\gamma^{\mu,T} \tag{10.2.39}$$

In that case there are only four independent (possibly complex) coordinates ψ^A. Their conjugate momenta are $p(\psi) = \frac{1}{2}\psi^T C\gamma^0$. They constitute 4 primary constraints which do not commute

$$\begin{aligned} \varphi_A &= p_A - \left(\frac{1}{2}\psi^T C\gamma^0\right)_A \\ \{\varphi_A, \varphi_B\} &= \left(C\gamma^0\right)_{AB} \end{aligned} \tag{10.2.40}$$

With $H_T = \int\left(\frac{1}{2}\psi^T C\gamma^k\partial_k\psi + \varphi_A u^A\right)d^3x$, the consistency condition for $\varphi_A = 0$ yields

$$\left(C\gamma^0\right)_{AB}u^B + \partial_k\left(\psi^T C\gamma^k\right)_A \approx 0 \tag{10.2.41}$$

whose solution fixes u^A. Then all $\psi\partial\psi$ terms cancel and H_T takes on the same form as for Dirac fermions

$$H_T = \int p_A u^A d^3x = \int p\gamma^0\gamma^k\partial_k\psi d_3x = H_0 \tag{10.2.42}$$

Now, however, the Dirac bracket differs from the Poisson bracket

$$\left\{\psi^A, p_B\right\}_D = -\frac{1}{2}\delta^A{}_B\delta\left(\vec{x} - \vec{y}\right) \tag{10.2.43}$$

Although this factor $\frac{1}{2}$ follows rigorously from (20.0.29), it can to some extent be explained as follows: the Poisson bracket$\{\psi, p\}$ is either minus one (by its definition), or zero (if one replaces p by $\frac{1}{2}\psi^T C\gamma^0$). The Dirac bracket yields the compromise: a factor 1/2.

We end this section with an example, which illustrates what to do when the φ_α seem to depend on Lagrange multipliers.

Example 3: the bosonic string. As action we take

$$S_{cl} = \int (-\frac{1}{2}\tilde{g}^{\mu\nu}\partial_\mu \vec{X}\, \partial_\nu \vec{X}) d\sigma dt \tag{10.2.44}$$

where $\tilde{g}^{\mu\nu} = \sqrt{-g}g^{\mu\nu}$ has determinant -1. As independent Lagrange multipliers we take λ^0 and λ^1 where

$$\tilde{g}^{00} \equiv \lambda^0 \, , \tilde{g}^{01} \equiv \lambda^1 \quad \text{hence } \tilde{g}^{11}\lambda^0 - (\lambda^1)^2 = -1. \tag{10.2.45}$$

In flat space $\lambda^0 = -1$ and $\lambda^1 = 0$ but we consider curved space with arbitrary λ^0 and λ. There are two primary constraints

$$\pi_\mu(\lambda) = 0 \, , \; \mu = 0, 1 \tag{10.2.46}$$

where $\pi_\mu(\lambda)$ is the momentum conjugate to λ^μ, and $\vec{p}\,(X) = -\lambda^0 \frac{d\vec{X}}{dt} - \lambda^1 \partial_\sigma \vec{X}$. Hence

$$H_L = \int -\frac{1}{2\lambda^0}\left[p(X)^2 + (\partial_\sigma X)^2 + 2\lambda^1\partial_\sigma X \cdot p(X)\right] d\sigma \tag{10.2.47}$$

The consistency conditions $\{\pi_\mu(\lambda), H_T\} \approx 0$ give two secondary constraints $\varphi_\mu = 0$ where

$$\begin{aligned} \varphi_0 &= -\frac{1}{2(\lambda^0)^2}\left[p(X)^2 + (\partial_\sigma X)^2 + 2\lambda^1\partial_\sigma X \cdot p(X)\right] \\ \varphi_1 &= \frac{1}{\lambda^0}\partial_\sigma X \cdot p(X) \end{aligned} \tag{10.2.48}$$

There are no tertiary constraints. The problem is now that these φ_μ depend on Lagrange multipliers. By taking suitable linear combinations of the constraints one can always get rid of the Lagrange multipliers.[–] The remaining constraints are sometimes called **"physical constraints"**. In our case these linear combinations are $(\lambda^0)^2[\varphi_0 + \frac{\lambda^1 \mp 1}{\lambda^0}\varphi_1]$, and the physical constraints are

$$\varphi_+ = \frac{1}{2}(p(X) + \partial_\sigma X)^2 \qquad , \quad \varphi_- = \frac{1}{2}(p(X) - \partial_\sigma X)^2 \tag{10.2.49}$$

The Poisson brackets yield the following first-class algebra

$$\begin{aligned} \{\varphi_\mu(\sigma,\tau), \varphi_\nu(\sigma',\tau)\} &= f_{\mu\sigma,\nu\sigma'}{}^{\rho\sigma''}\varphi_\rho(\sigma'',\tau) \text{ with } \mu,\nu = +,- \\ f_{+\sigma,+\sigma'}{}^{+\sigma''} &= [\delta(\sigma-\sigma'') + \delta(\sigma'-\sigma'')]\,\partial_\sigma\delta(\sigma-\sigma') \\ f_{-\sigma,-\sigma'}{}^{-\sigma''} &= -f_{+\sigma,+\sigma'}{}^{+\sigma''} \end{aligned} \tag{10.2.50}$$

Hence all constraints are first-class, and since $H_L = \int \lambda^0 \varphi_0 d\sigma$, we find $H_0 = 0$. Since $H_0 = 0$, also $V_\alpha{}^\beta = 0$ in this case. This algebra looks very much like that of general relativity, see ref. [ ].

This concludes our brief discussion of the Dirac formalism. We have discussed that from any classical action one can extract a bare Hamiltonian $H_0(p^i, q_i)$ and first-class constraints $\varphi_\alpha(p^i, q_i)$ whose (Poisson or Dirac) bracket algebra closes

$$\begin{aligned} \{H_0, \varphi_\alpha\} &= V_\alpha{}^\beta \varphi_\beta \\ \{\varphi_\alpha, \varphi_\beta\} &= f_{\alpha\beta}{}^\gamma \varphi_\gamma \end{aligned} \tag{10.2.51}$$

The constraints $\varphi_\alpha(p^i, q_i)$ are physical constraints: they do **not** depend themselves on Lagrange multipliers λ^μ. Therefore, the conjugate momenta of the Lagrange multipliers, $\pi(\lambda)_\mu$, are also first-class constraints. Hence the total set of first-class constraints is $G_a = \{\pi_\mu(\lambda), \varphi_\alpha\}$.

3 Structure functions

For the construction of the BRST charge Q_H and the BRST-invariant Hamiltonian H_{BRST} we need the structure functions

$$^{(n)}U_{b_1 \ldots b_{n+1}}{}^{a_1 \ldots a_n}\left(q_i, p^i\right) \tag{10.3.1}$$

which, as indicated, may depend on the canonical variables q_i and p^i. They are later to be contracted with $(n+1)$ ghost fields and n conjugate momenta of the ghost fields, hence the structure functions have the symmetry of ghosts fields. For example, if a pair of adjacent variables a_i and a_j refers to bosonic constraints G_{a_i} and G_{a_j}, then the structure function $^{(n)}U$ is antisymmetric in them, but if one or both refer to a fermionic constraint, then $^{(n)}U$ is symmetric in them. If the indices are not adjacent, one gets a sign, due to pulling the fermionic indices from their original position to their final position. In formula one has

$$ {}^{(n)}U_{b_1\ldots b_{n+1}}{}^{a_1\ldots a_k\ldots a_l\ldots a_n} = (-)^{\sigma}\ {}^{(n)}U_{b_1\ldots b_{n+1}}{}^{a_1\ldots a_l\ldots a_k\ldots a_n} \tag{10.3.2} $$

where

$$ \sigma = [\sigma(a_l)+1]\left[\sum_{p=k}^{l-1}\{\sigma(a_p)+1\}\right] + [\sigma(a_k)+1]\left[\sum_{p=k+1}^{l-1}\{\sigma(a_p)+1\}\right] \tag{10.3.3} $$

As mentioned before, we shall usually omit the symbols σ and write "a_k" instead of $\sigma(a_k)$ to avoid too cumbersome notation. The b-indices have the same symmetry. We shall call this symmetry "ghost-symmetry".

The lowest order structure constants are by definition just the constraints themselves

$$ {}^{(0)}U_a = G_a = \left\{\pi_\mu(\lambda), \varphi_\alpha(q_i, p^i)\right\} \tag{10.3.4} $$

The first-order structure constants ${}^{(1)}U$ are proportional to the structure constants of the (Poisson or Dirac) brackets of the constraints

$$ {}^{(1)}U_{b_1 b_2}{}^{a_1} = -\frac{1}{2} f_{b_1 b_2}{}^{a_1}(-)^{b_2} \tag{10.3.5} $$

The factor $-1/2$ is conventional and only introduced in order that the BRST charge will have a very simple expression in terms of the ${}^{(n)}U$, but the factor $(-)^{b_2}$ is needed in order that ${}^{(1)}U$ have ghost-symmetry

$$ {}^{(1)}U_{b_1 b_2}{}^{a_1} = (-)^{(b_1+1)(b_2+1)}\quad {}^{(1)}U_{b_2 b_1}{}^{a_1} \tag{10.3.6} $$

To verify (10.3.6) use that the symmetry of the ordinary structure constants is given by

$$ f_{b_1 b_2}{}^{a_1} = -(-)^{b_1 b_2} f_{b_2 b_1}{}^{a_1} \tag{10.3.7} $$

Namely, they are antisymmetric, except when both b_1 and b_2 are fermionic.

It may help understanding this sign factor $(-)^{b_2}$, and other sign factors below, if we write the bracket of constraints with gauge parameters ${\epsilon_1}^a, {\epsilon_2}^b$ as follows

$$ \left\{{\epsilon_1}^a G_a, {\epsilon_2}^b G_b\right\} = -{\epsilon_1}^b {\epsilon_2}^a f_{ab}{}^c G_c \tag{10.3.8} $$

The $\epsilon_i{}^a G_a$ are commuting objects. If we take for $\epsilon_i{}^a$ the product $\Lambda_i C^a$, where Λ_i are constant anticommuting BRST parameters and C^a are ghost fields, the right-hand side becomes

$$-\Lambda_1\Lambda_2 C^b C^a(-f_{ab}{}^c(-)^b)G_c \tag{10.3.9}$$

This shows why the sign factor $(-)^b$ is needed for contractions of ${}^{(1)}U$ with ghosts. [Actually, if we would have put $\epsilon_i{}^a = C^a\Lambda_i$, we would have gotten as sign factor $(-)^a$, and thus also $f_{b_1b_2}{}^{a_1}(-)^{b_1}$, will have ghost-symmetry in b_1b_2. We leave this for the reader to check. We continue with the former choice of sign factor].

The second order structure constants ${}^{(2)}U$ are obtained from the Jacobi identities

$$\left\{\left\{\epsilon_1{}^a G_a, \epsilon_2{}^b G_b\right\}, \epsilon_3{}^c G_c\right\} + \text{ cyclic in } 1,2,3 \equiv 0 \tag{10.3.10}$$

Replacing $\epsilon_i{}^a$ again by $\Lambda_i C^a$, we find that the Jacobi identities can be written as $\Lambda_1\Lambda_2\Lambda_3 C^c C^b C^a$ times the somewhat unusual, but for our purposes useful, form

$$\left\{\{G_a, G_b\}(-)^b, G_c\right\} + \text{ ghost-cyclic in a, b, c} = 0 \tag{10.3.11}$$

The expression $\{G_a, G_b\}(-)^b$ has ghost-symmetry, as we already discussed in (10.3.8) and (10.3.9). Since G_aG_b has the same statistics as C^aC^b, one does not get a sign factor $(-)^c$ if one pulls the third ghosts C^c in $\epsilon_3{}^c$ all the way to the far left, across $\{G_a, G_b\}$ and C^bC^a. Readers who find this trick of replacing $\epsilon_i{}^a$ by $\Lambda_i C^a$ in order to deduce the correct identities confusing, may directly verify that in (10.3.11) all terms cancel pairwise. Written out in full horror, the left-hand side of (10.3.11) reads

$$\begin{aligned}
&\left[(-)^b G_aG_b - (-)^{b+ab}G_bG_a\right]G_c - (-)^{c(a+b)}G_c\,[\text{same}\,] \\
&+(-)^{(a+1)(b+c)}\left\{\left[(-)^c G_bG_c - (-)^{c+bc}G_cG_b\right]G_a - (-)^{a(b+c)}G_a[\text{same}]\right\} \\
&+(-)^{(c+1)(a+b)}\left\{\left[(-)^a G_cG_a - (-)^{a+ac}G_aG_c\right]G_b - (-)^{b(a+c)}G_b[\text{same}]\right\}
\end{aligned} \tag{10.3.12}$$

In this expression, the two sign factors in front of $\{\}$ are the signs due to ghost-cyclicity, while in the first line G_aG_b stands for the first term of the Poisson bracket, and G_bG_a for the second term. The reader may verify that all terms indeed cancel pairwise.

The Jacobi identities in (10.3.11) can now be written as follows. **Before** adding ghost-cyclic terms, the first term can be written (up to a factor -2) as

$$^{(1)}D_{b_1b_2b_3}{}^{a_1}G_{a_1}(-)^{a_1} \tag{10.3.13}$$

The Jacobi identity states that **after** adding ghost-cyclic terms, this expression vanishes

$$\left(^{(1)}D_{b_1b_2b_3}{}^{a_1}\right)_A G_{a_1}(-)^{a_1} \equiv 0 \tag{10.3.14}$$

The symbol A denotes the ghost-symmetry in the indices b_1, b_2, b_3. Explicitly, one has

$$\begin{aligned} &^{(1)}D_{b_1b_2b_3}{}^{a_1}G_{a_1}(-)^{a_1} = \left\{^{(1)}U_{b_1b_2}{}^{c}G_c, G_{b_3}\right\} \\ = \quad &^{(1)}U_{b_1b_2}{}^{c} \quad ^{(1)}U_{cb_3}{}^{a_1}G_{a_1}(-2)(-)^{b_3} + (-)^{a_1b_3}\left\{^{(1)}U_{b_1b_2}{}^{a_1}, G_{b_3}\right\}G_{a_1} \end{aligned} \tag{10.3.15}$$

Hence

$$\begin{aligned} ^{(1)}D_{123}{}^{a_1} &= -2(-)^{a_1+b_3}\ {}^{(1)}U_{b_1b_2}{}^{c}\ {}^{(1)}U_{cb_3}{}^{a_1} \\ &+ (-)^{a_1+a_1b_3}\left\{^{(1)}U_{b_1b_2}{}^{a_1}, {}^{(0)}U_{b_3}\right\} \end{aligned} \tag{10.3.16}$$

We have normalized $^{(1)}D$ such that if we contract all b-indices with ghosts fields

$$\begin{aligned} ^{(1)}D^{a_1} &\equiv c^{b_3}c^{b_2}c^{b_1}\,{}^{(1)}D_{123}{}^{a_1} \\ ^{(1)}U^{a_1} &\equiv c^{b_2}c^{b_1}\,{}^{(1)}U_{b_1b_2}{}^{a_1} \\ ^{(0)}U &\equiv c^{b_3}\ {}^{(0)}U_{b_3} \end{aligned} \tag{10.3.17}$$

then

$$^{(1)}D^{a_1} = \left\{^{(1)}U^{a_1}, {}^{(0)}U\right\} + \dots \tag{10.3.18}$$

[Later we shall construct objects ${}^{(n)}D$ which are normalized to ${}^{(n)}D^{a_1\ldots a_n} = \left\{{}^{(n)}U^{a_1\ldots a_n}, {}^{(0)}U\right\} + \ldots$.]

Let us pause for a moment, and take stock. We have defined structure functions ${}^{(0)}U_b$ and ${}^{(1)}U_{b_1b_2}{}^{a_1}$, and found an object ${}^{(1)}D_{123}{}^{a_1}$ satisfying the (Jacobi) identity

$$\left({}^{(1)}D_{123}{}^{a_1}\right)_A G_{a_1}(-)^{a_1} = 0. \tag{10.3.19}$$

We shall presently show that the general solution of this equation for ${}^{(1)}D$ is given by

$$\left({}^{(1)}D_{123}{}^{a_1}\right)_A = 2\,{}^{(2)}U_{123}{}^{a_1a_2}G_{a_2} \tag{10.3.20}$$

(Again,the factor 2 is added to get a simple BRST charge Q_H in the end.) The function ${}^{(2)}U$ has ghost symmetry both in b_i **and in** a_i. These ${}^{(2)}U$ are the second order structure functions.

From here on, the general scheme is as follows. One works iteratively. Given at level n that ${}^{(n)}D$ can be written as

$$\left({}^{(n)}D_{b_1\ldots b_{n+2}}{}^{a_1\ldots a_n}\right)_A = (n+1)\,{}^{(n+1)}U_{b_1\ldots b_{n+2}}{}^{a_1\ldots a_{n+1}}G_{a_{n+1}} \tag{10.3.21}$$

we take the bracket of ${}^{(n)}D - (n+1)\,{}^{(n+1)}UG$ with $G_{b_{n+3}}$. The result for this bracket vanishes even when we do not antisymmetrize in all $b_1, \ldots, b_{n+3}$ (of course, antisymmetrization in $b_1, \ldots, b_{n+2}$ is present since ${}^{(n)}D$ and ${}^{(n+1)}U$ have this property). The bracket would then tell us that there exists a function ${}^{(n+1)}D$ which vanishes when contracted with a G, but this ${}^{(n+1)}D$ would not be antisymmetric in its a_j indices, hence we could not use the lemma to find a corresponding ${}^{(n+1)}U$. However, if we begin by antisymmetrizing in all $b_1, \ldots, b_{n+3}$, then the corresponding ${}^{(n+1)}D$ has ghost-symmetry w.r.t. its a-indices, and a corresponding ${}^{(n+2)}U$ with ghost symmetry can be found. So we ghost-symmetrize the bracket of ${}^{(n)}D - (n+1)\,{}^{(n+1)}UG$ with G in all indices $b_1, \ldots b_{n+3}$ and after certain manipulations, the result assumes the form

$$\begin{aligned}&\left\{{}^{(n)}D - (n+1)\,{}^{(n+1)}UG, G_{b_{n+3}}\right\}_A = \\ &\left({}^{(n+1)}D_{b_1\ldots b_{n+3}}{}^{a_1\ldots a_{n+1}}\right)_A G_{a_{n+1}}(-)^{a_{n+1}} = 0\end{aligned} \tag{10.3.22}$$

The general solution is then ${}^{(n+1)}D = (n+2)^{(n+2)}UG$, etc. This defines the sequence ${}^{(n)}U$.

We shall show in an example below that the coefficient ${}^{(n+1)}D$ has indeed ghost symmetry in its upper indices, as well as in its lower indices. Quite generally, ${}^{(n)}D$ contains two kinds of terms: brackets of two U functions without any contraction $\left\{{}^{(p)}U_{\dots}{}^{\dots}, {}^{(n-p)}U_{\dots}{}^{\dots}\right\}$ with $p = 0, \dots n$ and products of two U functions with exactly one contraction ${}^{(p+1)}U_{\dots}{}^{\dots k}\ {}^{(n-p)}U_{\dots k}{}^{\dots}$ with $p = 0, \dots, n-1$. This structure is indeed present in (10.3.16) for $n = 1$.

[One can even consider the case $n = 0$. Take ${}^{(0)}D_{b_1 b_2} = -\frac{1}{2}\{G_{b_1}, G_{b_2}\}(-)^{b_2}$ and if we define that contraction of ${}^{(0)}D_{b_1 b_2}$ with G_{b_3} vanishes (which is a reasonable definition: since ${}^{(0)}D$ has no upper indices, it cannot be contracted), then the lemma goes through since ${}^{(0)}D_{b_1 b_1} = \left({}^{(0)}D_{b_1 b_2}\right)_A$ is indeed equal to ${}^{(1)}U_{b_1 b_2}{}^{a} G_a$. From here we proceed again as follows: we take ${}^{(0)}D -{}^{(1)} UG$ and take the bracket with G_{b_3}. Antisymmetrizing in $b_1, \dots, b_3$, the terms from $\{{}^{(0)}D, G_{b_3}\}$ vanish identically, and we reach the point where (10.3.14) and (10.3.16) hold.]

Let us now first prove that the general solution of an equation

$$\left({}^{(n)}D_{b_1 \dots b_{n+2}}{}^{a_1 \dots a_n}\right)_A G_{a_n} (-)^{a_n} = 0 \tag{10.3.23}$$

is indeed given by

$$\left({}^{(n)}D_{b_1 \dots b_{n+2}}{}^{a_1 \dots a_n}\right)_A = (n+1)^{(n+1)} U_{b_1 \dots b_{n+2}}{}^{a_1 \dots a_{n+1}} G_{a_{n+1}} \tag{10.3.24}$$

It is clear that this is a solution, since the G's have ordinary symmetry, while the U's have ghost symmetry. (For example, if the two G's are commuting, the function U is antisymmetric, and the double contraction then obviously vanishes). It is not so obvious that this solution is the general solution. To prove this, requires a little mathematical detour. This detour will occupy the rest of this section.

Consider a function $F^{a_1 \dots a_n}$ with ghost symmetry. Define two maps δ_2 and $\delta_2{}'$

which map an n-tensor into an $(n-1)$ tensor

$$\begin{aligned} \delta_2 \ &: \ F^{a_1 \ldots a_n} \to F^{a_1 \ldots a_n} G_{a_n} (-)^{a_n} \\ \delta_2' \ &: \ F^{a_1 \ldots a_n} \to F^{a_1 \ldots a_n} G_{a_n} \end{aligned} \tag{10.3.25}$$

It is rather clear that $\delta_2 \delta_2' = 0$ and also $\delta_2' \delta_2 = 0$ because the symmetry of F and that of two G functions is opposite. [An explicit check proceeds as follows. One has

$$(\delta_2' \delta_2 F)^{a_1 \ldots a_{n-2}} = F^{a_1 \ldots a_{n-1} a_n} G_{a_n} (-)^{a_n} G_{a_{n-1}} \tag{10.3.26}$$

Interchanging the indices a_{n-1} and a_n in F gives a sign $(-)^\sigma$ with $\sigma = (a_{n-1}+1)(a_n+1)$. Interchanging G_{a_n} and $G_{a_{n-1}}$ gives a sign $(-)^\rho$ with $\rho = a_n a_{n-1}$. Relabelling $a_{n-1}(\leftrightarrow)a_n$, one finds that $\delta_2 \delta_2' F$ equals minus itself.]

There are now the following theorems[4]

Theorem I: if $\delta_2 F = 0$ then $F = \delta_2' K$

Theorem II: if $\delta_2' F = 0$ then $F = \delta_2 K$

To prove these theorems, consider a change of the phase space variables q_i, p^i to a new set y_a, x_α where $y_a = G_a(q,p)$ and x_α are some further coordinates. This is possible as long as the rectangular matrix $\partial G_a / \partial(q_i, p^j)$ has maximal rank (rank n if there are n functions G_a. If the rank is smaller than n, we are in the case of a reducible system, which we discuss in chapter 4. We restrict ourselves here to the irreducible case).

Define now an operator δ by

$$\delta y_a = 0 \quad , \quad \delta x_\alpha = 0 \quad , \quad \delta p_a = y_a \quad , \quad \delta q_i = \delta p^j = 0 \tag{10.3.27}$$

where p_a are conjugate to the ghost-like variables η^a. Any ghost-symmetric function $F^{a_1 \ldots a_n}$ can be contracted with p's, and, conversely, any polynomial in p's corresponds

[4] We work locally. Actually, these results hold globally as follows: if the BRST charge is $Q(x,y)$ in one patch, and $Q'(x',y')$ in another patch, then in the intersection where (x,y) and (x',y') both correspond to (p,q), one can use the freedom in ${}^{(n)}U$ to achieve that $Q(p,q) = Q'(p,q)$. We thank M. Henneaux for explaining this to us, and the proof of these theorems.

to a ghost-symmetric $F^{a_1 \dots a_n}$. The operation δ corresponds then indeed to contraction of $F^{a_1 \dots a_n}$ with G_{a_n}. Furthermore, if $f(p,q)$ has no a-indices (is independent of p_a), we require that $\delta f = 0$ (because contraction with G_a is not possible) and this is equivalent to requiring $\delta q_i = \delta p_i = 0$. We define δ on $F^{a_1 \dots a_n}(q_i, p^j) p_{a_n} \dots p_{a_1}$ by imposing the Leibniz rule:

$$\delta(AB) = (\delta A)B + (-)^A A \delta B \tag{10.3.28}$$

We now want to prove

Theorem II: if $F \equiv F^{a_1 \dots a_n} p_{a_n} \dots p_{a_1}$ satisfies $\delta' F = 0$, then $F = \delta M$. We shall assume that F has at least one p_a (F has at least one upper index a). So, we consider F with antighost number ≥ 1.

Proof: introduce operators σ and δ' such that

$$\sigma y_a = p_a (-)^a \quad , \quad \sigma p_a = 0 \quad , \quad \sigma x_\alpha = 0 \quad , \quad \sigma q_i = \sigma p^j = 0 \quad ,$$

$$\delta' y_a = 0 \quad , \quad \delta' p_a = y_a (-)^a \quad , \quad \delta' x_\alpha = 0 \quad , \quad \delta' q_i = \delta' p^i = 0 \quad . \tag{10.3.29}$$

Furthermore, we require that the Leibniz rule holds also for σ. Then $\sigma\delta' + \delta\sigma$ vanishes on x_α, q_i, p^j but on y_a and p_a it is unity. Hence, $\sigma\delta' + \delta\sigma$ counts the "degree" of F, i.e., the total number of y_a and p_a present in F. It follows that $\sigma\delta' + \delta\sigma$ commutes with δ' (because δ' does not change the degree of F). Thus we can decompose F into parts with different degrees

$$F = \sum_p F^{(p)} \tag{10.3.30}$$

Hence $\delta' F = 0$ implies $\delta' F^{(p)} = 0$, and $\delta' F^{(p)} = 0$ for all (p), implies $\delta F = 0$. Now, given a $F^{(p)}$ we can write the following sequence of obvious facts

$$F^{(p)} = \frac{p}{p} F^{(p)} = (\sigma\delta' + \delta\sigma) \frac{1}{p} F^{(p)} = \delta\sigma p \frac{1}{p} F^{(p)} \tag{10.3.31}$$

Clearly, if $\delta' F^{(p)} = 0$, $F^{(p)} = \delta G^{(p)}$, namely $G^{(p)} = \frac{1}{p}\sigma G^{(p)}$. This proves the theorem. In a similar manner one proves theorem I.

[In fact, δ acts like the exterior derivative acts on x: d maps x^μ into dx^μ, and the dx^μ are anticommuting when the x^μ are commuting. Furthermore, σ maps dx^μ into x^μ. One may check that $\delta\sigma + \sigma\delta = I$ on x^μ and dx^μ.]

As a corollary one then has

Lemma: $\delta_2 F = H$ has a solution for F if and only if $\delta_2' H = 0$. (Idem for $\delta_2' F$).

Lemma: the general solution of $\delta_2 F = H$ is given by $F^{(0)} + \delta_2' K$, where $F^{(0)}$ is a particular solution and K is arbitrary. (Idem for $\delta_2' F = H$).

If all constraints are bosonic, $\delta_2 = \delta_2'$, and one has $\delta_2\delta_2 = 0$. Hence, δ_2 is like the Poincaré exterior derivative "d", and one might wonder why $\delta_2 F = 0$ does not imply $F = \delta_2 G +$ "harmonic term". The reason there are no "harmonic terms" is that one can prove the theorem in the space of polynomials in the G_a, and there the topology is trivial. We return to the structure functions, and see that $\delta_2{}^{(n)}D = 0$, so that indeed ${}^{(n)}D = \delta_2'\,{}^{(n+1)}U$. Let us now consider an example, with bosonic constraints so that all sign factors disappear. In particular, we want to verify the claim that ${}^{(2)}D$ contains only terms with two U functions.

4 Example: nonlinear Lie algebras

Consider the following Poisson brackets

$$\{T_a, T_b\} = f_{ab}{}^c T_c + V_{ab}{}^{cd} T_c T_d \tag{10.4.1}$$

The generators T_a are bosonic. Later we shall require that they have a coset decomposition, meaning that one can split them into two sets,one set $H_i$which generate an ordinary Lie subalgebra, and another set K_α with the following brackets

$$\begin{aligned}
\{H_i, H_j\} &= f_{ij}{}^k H_k \quad \text{(subalgebra)} \\
\{H_i, K_\alpha\} &= f_{i\alpha}{}^\beta K_\beta \quad \text{(reductivity)} \\
\{K_\alpha, K_\beta\} &= f_{\alpha\beta}{}^i H_i + V_{\alpha\beta}{}^{ij} H_i H_j
\end{aligned} \tag{10.4.2}$$

The symbols ${V_{\alpha\beta}}^{ij}$ are symmetric in (ij). For the time being, however, we shall not impose this coset structure, and work with arbitrary ${f_{ab}}^c$ and ${V_{ab}}^{cd}$. We shall consider the T_a as an example of first class constraints G_a.

The lowest order structure functions are, by definition,

$$ {}^{(0)}U_a = T_a \quad , \quad {}^{(1)}{U_{ab}}^c = -\frac{1}{2}({f_{ab}}^c + {V_{ab}}^{cd}T_d) \tag{10.4.3} $$

The Jacobi identities, ${}^{(1)}{D_{[abc]}}^d T_d = 0$, see (10.3.14), yield three conditions which ensure the vanishing of all terms with one, two or three generators (see (10.3.15)

$$ \begin{aligned} {f_{[b_1b_2}}^k {f_{b_3]k}}^a &= 0 \\ {f_{[b_1b_2}}^k V^{a_1a_2}_{b_3]k} &= 2{f_{k[b_1}}^{(a_1} V^{a_2)k}_{b_2b_3]} \\ V^{k(a_1}_{[b_1b_2} V^{a_2a_3)}_{b_3]k} &= 0 \end{aligned} \tag{10.4.4} $$

The function ${}^{(1)}{D_{123}}^a$ is obtained from $\{\{T_{b_1}, T_{b_2}\}, T_{b_3}\}$ by extracting one generator T_a. One obtains from (10.3.16)

$$ \begin{aligned} {}^{(1)}{D_{123}}^a = \frac{1}{2} \Big[& {f_{b_1b_2}}^k {f_{b_3k}}^a + (\, {f_{b_1b_2}}^k V^{at}_{b_3k} + {f_{b_3k}}^a {V_{b_1b_2}}^{kt} \\ & - {f_{kb_3}}^t V^{ak}_{b_1b_2} \,) \, T_t + \left(V^{ks}_{b_1b_2} V^{at}_{b_3k} - V^{ak}_{b_1b_2} V^{st}_{kb_3} \right) T_s T_t \, \Big] \end{aligned} \tag{10.4.5} $$

If we antisymmetrize in b_1, b_2 and b_3, the ff terms and the fV terms vanish due to the Jacobi identities (the fV terms vanish because they happen already to be symmetric in a, t after antisymmetrization in b_i). However, the VV terms contribute

$$ {}^{(1)}{D_{[123]}}^{a_1} = \frac{1}{2} \left(V^{ks}_{[b_1b_2} V^{a_1t}_{b_3]k} + V^{ka_1}_{[b_1b_2} V^{st}_{b_3]k} \right) T_s T_t \tag{10.4.6} $$

Of course, if we contract this with T_{a_1} it vanishes identically (because that is the content of the Jacobi identity), but this expression as it stands is **not totally symmetric** in s, t, a_1, and hence does not vanish in general. (Contraction with T_{a_1} projects out its totally symmetric part.) The general theory we discussed states that it should be possible to write this as

$$ \left({}^{(1)}{D_{123}}^{a_1} \right)_A = 2 \; {}^{(2)}{U_{123}}^{a_1a_2} T_{a_2} \tag{10.4.7} $$

where ${}^{(2)}U$ is **antisymmetric** in a_1a_2. Hence, we extract a T_{a_2} in the most general way

$$ {}^{(1)}D_{[123]}{}^{a_1} = 2\left[\frac{1}{4}\left(\alpha V^{ka_2}_{[b_1b_2}V^{a_1t}_{b_3]k} + (1-\alpha)V^{kt}_{[b_1b_2}V^{a_1a_2}_{b_3]k} + V^{ka_1}_{[b_1b_2}V^{a_2t}_{b_3]k}\right)T_t\right]T_{a_2} \tag{10.4.8} $$

and try to fix α so that the expression between square brackets becomes antisymmetric in a_1a_2. Using the VV Jacobi identity one finds that $\alpha = 1/3$ and

$$ {}^{(2)}U_{123}{}^{a_1a_2} = \frac{1}{6}V^{k[a_1}_{[b_1b_2}V^{a_2]t}_{b_3]k}T_t \tag{10.4.9} $$

The reader may feel that we made things needlessly complicated by extracting a T_a from ${}^{(1)}D_{123}{}^aT_a$ such that the resulting ${}^{(1)}D$ was nonvanishing. He might argue that in the TTT sector one can extract a T_a in such a way that the VV expression in (10.4.7) is symmetric in a_1, s, t and would therefore vanish. In that case, ${}^{(1)}D$ would be zero, and ${}^{(2)}U$ would vanish, too. The answer to this criticism is that we removed the particular T_a which is left when we write the result abstractly, in terms of ${}^{(1)}U$ and ${}^{(0)}U$, as in (10.3.16)

$$ {}^{(1)}D_{123}{}^a = 2\,{}^{(1)}U_{b_1b_2}{}^k\,{}^{(1)}U_{b_3k}{}^a + \left\{{}^{(1)}U_{b_1b_2}{}^a, G_{b_3}\right\} \tag{10.4.10} $$

In this way we do not obtain total symmetry in the indices a_1, s, t in (10.4.7). (Incidentally, we normalize the functions ${}^{(n)}D$ such that the term $\{{}^{(n)}U, T_a\}$ has coefficient $(-)^{n+1}$, and we normalize ${}^{(n)}U$ such that ${}^{(n)}D = (n+1)\ {}^{(n+1)}U$).

To go on, we take the bracket of ${}^{(1)}D - 2\,{}^{(2)}UT \equiv 0$ with another T_{b_4}. If we antisymmetrize in all four b_i, the result is, as we shall show,

$$ {}^{(2)}D_{[b_1b_2b_3b_4]}{}^{a_1a_2}T_{a_2} = 0 \tag{10.4.11} $$

where

$$ \begin{aligned} {}^{(2)}D^{a_1a_2}_{b_1b_2b_3b_4} &= -\left\{{}^{(2)}U^{a_1a_2}_{b_1b_2b_3}, T_{b_4}\right\} - \frac{1}{2}\left\{{}^{(1)}U^{a_1}_{b_1b_2},\ {}^{(1)}U^{a_2}_{b_3b_4}\right\} \\ &-3\,{}^{(1)}U_{b_1b_2}{}^k\ {}^{(2)}U^{a_1a_2}_{b_3b_4k} - 4\ {}^{(2)}U^{a_1k}_{b_1b_2b_3}\ {}^{(1)}U^{a_2}_{b_4k} \end{aligned} \tag{10.4.12} $$

As one may check, all ${}^{(n)}D$ contain only bracket terms of the form $\{{}^{(p)}U,\ {}^{(n-p)}U\}$ with $p = 0, \ldots, n$ or contracted terms of the form ${}^{(p+1)}U_{\cdots}^{\cdots k} \quad {}^{(n-p)}U_{\cdots k}^{\cdots}$ with $p = 0, \ldots, n-1$. To see how terms with a product of 3 U-functions are eliminated, we take a closer look at the derivation of ${}^{(2)}D$. We must evaluate

$$\left\{{}^{(1)}D_{b_1b_2b_3}{}^{a_1}, G_{b_4}\right\} - 2\left\{{}^{(2)}U_{b_1b_2b_3}{}^{a_1a_2}G_{a_2}, G_{b_4}\right\} = 0 \tag{10.4.13}$$

In the second term we always get a result proportional to ${}^{(0)}U$

$$\left[-4\ {}^{(2)}U_{b_1b_2b_3}{}^{a_1k}\ {}^{(1)}U_{b_4k}{}^{a_2} - 2\left\{\ {}^{(2)}U_{b_1b_2b_3}{}^{a_1a_2},\ {}^{(0)}U_{b_4}\right\}\right]G_{a_2} \tag{10.4.14}$$

These terms are of the form one expects for ${}^{(n)}D$, see above.

In the first term in (10.4.13), we must do some work to achieve this.

We substitute

$${}^{(1)}D_{123}{}^{a_1} = \left\{{}^{(1)}U_{b_1b_2}{}^{a_1}, {}^{(0)}U_{b_3}\right\} + 2\ {}^{(1)}U_{b_1b_2}{}^{k}\ {}^{(1)}U_{b_3k}{}^{a_1} \tag{10.4.15}$$

into $\left\{{}^{(1)}D, G\right\}$, and obtain then from the first term on the right-hand side of (10.4.15), using the Jacobi identities, the following structures:

$$\begin{aligned}\left\{\{{}^{(1)}U, {}^{(0)}U\}, {}^{(0)}U\right\} &= -\frac{1}{2}\left\{\{{}^{(0)}U, {}^{(0)}U\}, {}^{(1)}U\right\} \\ &= \left\{{}^{(1)}U, {}^{(1)}U\right\}G_k + {}^{(1)}U\left\{{}^{(0)}U_k, {}^{(1)}U\right\}\end{aligned} \tag{10.4.16}$$

From the second term in (10.4.15) we get

$$-2\left\{{}^{(1)}U_{b_1b_2}{}^{k}, {}^{(0)}U_{b_3}\right\}{}^{(1)}U_{b_4k}{}^{a_1} + 2{}^{(1)}U_{b_1b_2}{}^{k}\left\{{}^{(1)}U_{b_3k}{}^{a_1}, {}^{(0)}U_{b_4}\right\} \tag{10.4.17}$$

We now observe that the sum of the two terms of the form $\left\{{}^{(1)}U_{\cdot\cdot},\ {}^{(0)}U_{\cdot}\right\}\ {}^{(1)}U_{b_1b_2}{}^{k}$ in (10.4.16) and (10.4.17) are actually totally antisymmetric in the three indices indicated by a dot. The same holds, of course, for the term $\left\{{}^{(1)}U_{b_1b_2}{}^{k}, {}^{(0)}U_{b_3}\right\}\ {}^{(1)}U_{b_4k}{}^{a_1}$. We can now replace $\left\{{}^{(1)}U, {}^{(0)}U\right\}$ in either case by ${}^{(1)}D - 2{}^{(1)}U{}^{(1)}U$ (see the definition of ${}^{(1)}D$ in (**??**)), and then replace ${}^{(1)}D$ by $2{}^{(2)}UG$. At this point all terms in

$$\left\{{}^{(1)}D - 2\ {}^{(2)}UG,\ G\right\} \tag{10.4.18}$$

are proportional to a G, and of the form expected, except two terms which are each of the form ${}^{(1)}U\ {}^{(1)}U\ {}^{(1)}U$. They are given by

$$
\begin{aligned}
&6\ {}^{(1)}U_{..}{}^{k}\ {}^{(1)}U_{[..}{}^{l}\ {}^{(1)}U_{k]l}{}^{a_1} \\
&4^{(1)}\ U_{..}{}^{k}\ {}^{(1)}U_{.\ell}{}^{k}\ {}^{(1)}U_{.k}{}^{a_1}
\end{aligned}
\tag{10.4.19}
$$

where the 4 dots indicate the indices b_i. **These terms cancel** (in the first term the combination $2\ {}^{(1)}U_{..}{}^{k}\ {}^{(1)}U_{..}{}^{\ell}\ {}^{(1)}U_{kl}{}^{a_1}$ vanishes separately, and the rest cancels). Hence, the bracket of ${}^{(1)}D - 2^{(2)}UG \equiv 0$ with another G is indeed proportional to ${}^{(2)}DG$, with ${}^{(2)}D$ given in (10.4.12).

To keep the algebra under control, we now impose the coset structure on the constants V. Hence, only $V_{\alpha\beta}{}^{ij}$ is nonzero, and all VV contractions vanish. In that case ${}^{(2)}U$ in (10.4.9) vanishes and

$$
\begin{aligned}
{}^{(2)}D_{[b_1b_2b_3b_4]}{}^{a_1a_2} &= -\frac{1}{8}V_{[b_1b_2}{}^{a_1s}V_{b_3b_4]}{}^{a_2t}\left(f_{st}{}^{a_3}+V_{st}{}^{a_3u}T_u\right)T_{a_3} \\
&= 3\left[{}^{(3)}U_{b_1\ldots b_4}{}^{a_1\ldots a_3}\right]T_{a_3}
\end{aligned}
\tag{10.4.20}
$$

where

$$
{}^{(3)}U_{b_1\ldots b_4}{}^{a_1\ldots a_3} = -\frac{1}{24}V_{[b_1b_2}{}^{a_1s}V_{b_3b_4]}{}^{a_2t}f_{st}{}^{a_3}
\tag{10.4.21}
$$

One may prove that this expression is indeed antisymmetric in a_1, a_2, a_3 by using the Jacobi identities twice. [The details are as follows

$$
\begin{aligned}
&V_{b_1b_2}{}^{a_1s}V_{b_3b_4}{}^{t(a_2}f_{st}{}^{a_3)} = \\
&\quad V_{b_1b_2}{}^{a_1s}\left[-2V_{sb_3}{}^{t(a_2}f_{b_4t}{}^{a_3)}+f_{b_3b_4}{}^{t}V_{st}{}^{a_2a_3}+2f_{b_3s}{}^{t}V_{b_4t}{}^{a_2a_3}\right] \\
&\quad \sim \left(V_{b_1b_2}{}^{a_1s}f_{b_3s}{}^{t}\right)V_{b_4t}{}^{a_2a_3} \sim \left(f_{b_1b_2}{}^{s}V_{b_3s}{}^{a_1t}\right)V_{b_4t}{}^{a_2a_3} = 0
\end{aligned}
\tag{10.4.22}
$$

where we used the second line in (10.4.4), and then the third line.]

There are no further structure functions. For example, ${}^{(3)}D$ vanishes, as it contains terms proportional to

$$
\left\{{}^{(0)}U,\ {}^{(3)}U\right\},\left\{{}^{(1)}U,\ {}^{(2)}U\right\}
\tag{10.4.23}
$$

which vanish since ${}^{(3)}U$ are constants, and terms proportional to

$$ {}^{(1)}U^{\cdots k}_{\cdots}\ {}^{(3)}U^{\cdots}_{\cdots k} \quad \text{and} \quad {}^{(3)}U^{\cdots k}_{\cdots} \quad {}^{(1)}U^{\cdots}_{\cdots k} \tag{10.4.24} $$

The terms without T's should vanish (since contracting them with a T should yield zero according to ${}^{(3)}DT \equiv 0$). The terms with one T vanish since they contain always a VV contraction.

Before we close this example, let us mention that all conditions which follow from ${}^{(n)}DT = 0$ (which one might call "higher order Jacobi identities") follow from the lowest order Jacobi identities in (10.4.4).

5 The Hamiltonian BRST charge Q_H

In order to construct a function Q_H whose bracket with itself vanishes, we introduce new canonical pairs. Corresponding to

$$ G_a = \{\pi_\mu(\lambda), \varphi_\alpha\} \tag{10.5.1} $$

we have "ghosts"

$$ \eta^a = \{\sigma p^\mu(B), C^\alpha\} \tag{10.5.2} $$

and "antighosts"

$$ p_a = \{B_\mu, p_\alpha(C)\} \tag{10.5.3} $$

The reason we must put $p^\mu(B)$ instead of B_μ in η^a is that the former has the same ghost number as C^α, see below. The factor $\sigma = +1$ for anticommuting B_μ and (-1) for commuting B_μ, in order that (11.1.8) and (10.2.12) hold.

Since in some important applications, such as string theory, the antighosts have a different index structure than the ghosts (although the number of antighost field components is the same as the number of ghost field components) we distinguish their indices, and use μ for the former and α for the latter.

Actually, $p^\mu(B)$ are the conjugate momenta of the antighosts, which appear in the same multiplet (η^a) as C^α. Similar remarks apply to $p_\alpha(C)$. Hence, the bracket of two η's vanishes,as does the bracket of two p's.

We define also a ghost number: it is $+1$ for η^a and -1 for p_a (hence B has ghost number -1 and $p(B)$ has $+1$, just as expected).

In order to obtain a Q_H satisfying $\{Q_H, Q_H\} = 0$ and having ghost number $+1$, we claim that the following expression is a solution.

$$\begin{aligned} Q_H &= \eta^a G_a + \sum_{n\geq 1} \eta^{b_{n+1}} \ldots \eta^{b_1}\ {}^{(n)}U_{b_1 \ldots b_{n+1}}{}^{a_1 \ldots a_n} p_{a_n} \ldots p_{a_1} \\ &\equiv \sum_{n\geq 0} {}^{(n)}U \end{aligned} \tag{10.5.4}$$

The term with $n = 1$ is given by

$$\eta^{b_2}\eta^{b_1}\ {}^{(1)}U_{b_1 b_2}{}^a p_a = -\frac{1}{2}\eta^{b_2}\eta^{b_1} f_{b_1 b_2}{}^a p_a (-)^{b_2} \tag{10.5.5}$$

The structure functions ${}^{(n)}U$ are defined by relations which shall turn out also to be the conditions for $\{Q_H, Q_H\} = 0$. Rather than give the abstract (tedious) proof, it is more useful for applications to check the first few terms. This we shall now do.

The terms with two ghosts in $\{Q_H, Q_H\}$ come from two sources

$$\left\{\eta^a G_a, \eta^b G_b\right\} = (-)^{b+1}\eta^b\eta^a f_{ab}{}^c G_c \tag{10.5.6}$$

$$2\left\{\eta^a G_a, \eta^{b_2}\eta^{b_1}\ {}^{(1)}U_{b_1 b_2}{}^c p_c\right\} = 2\eta^{b_2}\eta^{b_1}(-\frac{1}{2}) f_{b_1 b_2}{}^c \left\{p_c, \eta^a\right\} G_a (-)^{b_2} + \ldots \tag{10.5.7}$$

It is clear that these terms cancel if we define the bracket of the ghosts by

$$\left\{p_a, \eta^b\right\} = -\delta_a{}^b. \tag{10.5.8}$$

Conversely, given this definition, the relation ${}^{(1)}U_{b_1 b_2}{}^a = -\frac{1}{2} f_{b_1 b_2}{}^a (-)^{b_2}$ follows.

The terms with three ghosts come from various sources. To begin with, there is a left-over from (10.5.7); it reads

$$2\eta^a\eta^{b_2}\eta^{b_1}\left\{{}^{(1)}U_{b_1 b_2}{}^c, G_a\right\} p_c (-)^{ac} \tag{10.5.9}$$

Then there is a contribution from the leading term in

$$\begin{aligned} &2\left\{\eta^{b_3}\eta^{b_2}\eta^{b_1}\ {}^{(2)}U_{123}{}^{a_1a_2}p_{a_2}P_{a_1}, \eta^a G_a\right\} \\ &= -4\eta^{b_3}\eta^{b_2}\eta^{b_1}\ {}^{(2)}U_{123}{}^{a_1a_2}p_{a_2}G_{a_1} + \dots \end{aligned} \tag{10.5.10}$$

And finally there is the diagonal term with two ${}^{(1)}U$ functions

$$\begin{aligned} &\left\{\eta^{b_2}\eta^{b_1}\ {}^{(1)}U_{b_1b_2}{}^{c}p_c, \eta^{d_2}\eta^{d_1}\ {}^{(1)}U_{d_1d_2}{}^{e}p_e\right\} = \\ &\qquad -4\eta^{b_2}\eta^{b_1}\ {}^{(1)}U_{b_1b_2}{}^{d_2}\eta^{d_1}\ {}^{(1)}U_{d_1d_2}{}^{e}p_e \end{aligned} \tag{10.5.11}$$

In order for these terms to cancel, we clearly need a relation of the form ${}^{(2)}UG + \left\{{}^{(1)}U, G\right\} +{}^{(1)} U{}^{(1)}U = 0$. Recalling the definition of ${}^{(2)}U$, namely ${}^{(1)}D = 2{}^{(2)}UG$, one sees that the condition for nilpotency of Q_H is the same as the definition of the structure functions in (10.3.20) and (10.3.16).

One could directly determine Q_H, by putting $Q_H = c^aG_a + \dots$ and requiring $\{Q_H, Q_H\} = 0$. Order by order in the number of ghosts (or the number of antighosts) one would find the structure functions ${}^{(n)}U$.

A last property one may define for Q_H is **reality**. Assuming that all G_a are real, and defining the η^a to be real

$$(\psi_\alpha)^* = \psi_\alpha, (\pi(\lambda)^\mu)^* = \pi(\lambda)^\mu, (C^\alpha)^* = C^\alpha,\ p(B)^{\mu\ *} = p(B)^\mu \tag{10.5.12}$$

we see that η^aG_a is real (η^a and G_a commute with each other as Q_H is anticommuting) and reality of Q_H requires that $\eta^{b_2}\eta^{b_1}\ {}^{(1)}U_{b_1b_2}{}^{c}$ and p_c acquire the same sign under the $*$ operation (they commute with each other as Q is anticommuting). The bracket $\left\{p_a, \eta^b\right\} = -\delta_a{}^b$ gives, using $\left\{p_a, \eta^b\right\} = -(-)^a\left\{\eta^b, p_a\right\}$, the reality condition $(p_a)^* = (-)^{a+1}p_a$, or, in components $B_\mu{}^* = (-)^{\mu+1}B_\mu$, , $p(C)_\alpha{}^* = (-)^{\alpha+1}p(C)_\alpha$. (It is easiest to consider the quantum (anti) commutation relations with the factor i, and to take the ordinary hermitian conjugation). So, for example, the coordinate antighost, and the coordinate ghost momentum are antihermitian. (In bosonic string theory, the ghost action is $B_{++}\partial_- C^+$, and is clearly hermitian.) For the classical variables q_i, p^i we take q_i to be real and then p^i is real (imaginary) if it is commuting (anticommuting). With these rules we find that the Poisson bracket in (10.2.13) satisfies the following reality condition.

$$\{f, g\}^* = -\{g^*, f^*\} \tag{10.5.13}$$

As an example, note that invariance undere the * operation of $\{\psi_\alpha, \psi_\beta\} = f_{\alpha\beta}{}^\gamma \psi_\gamma$ shows that the structure constants are real except when both ψ_α and ψ_β are anticommuting. Hence, also the second term in Q_H is real. Also for higher n one can then prove that the structure functions ${}^{(n)}U$ have those reality properties which guarantee that Q is real.

For abelian theories one has $Q_H = \eta^a G_a$. For gauge theories with a closed gauge algebra with field-independent structure constants one has

$$Q_H = \eta^a G_a - \frac{1}{2}\eta^{b_2}\eta^{b_1} f_{b_1 b_2}{}^a p_a (-)^{b_2} \tag{10.5.14}$$

For example, in Yang-Mills theory one has

$$Q_H = \int \left[c^a D_i E_i{}^a + p(B)^a \pi_0{}^a(\lambda) - \frac{1}{2} g C^b C^c f_{cb}{}^a p_a(C)\right] d^3x \tag{10.5.15}$$

while for the bosonic string one finds from (10.2.49) and (10.2.50)

$$\begin{aligned} Q_H = \int \Big[&C^+\varphi_+ + C^-\varphi_- + p(B)^\mu \pi_\mu(\lambda) \\ &-\partial_\sigma C^+ C^+ p_+(C) + \partial_\sigma C^- C^- p_-(C)\Big] d\sigma \end{aligned} \tag{10.5.16}$$

6 The BRST invariant Hamiltonian

A gauge-invariant dynamical variable A_0 is by definition a real bosonic function A of the variables q_i, p^i satisfying

$$\{A_0, G_a\} = W_a{}^b G_b \tag{10.6.1}$$

According to this definition, H_0 is gauge invariant.

There exists always an extension of A_0 whose bracket with the BRST charge Q_H vanishes

$$\{A, Q_H\} = 0 \tag{10.6.2}$$

To prove this, one may start with

$$A = A_0 + \sum_{n\geq 1} \eta^{b_n} \ldots \eta^{b_1} A_{b_1 \ldots b_n}{}^{a_1 \cdot\cdot a_n} p_{a_n} \cdots p_{a_1} \tag{10.6.3}$$

Using $Q_H = \eta^a G_a + \cdots$, one finds easily the first term

$$A = A_0 + \eta^a W_a{}^b p_b + \cdots \tag{10.6.4}$$

However, the higher-order terms in A are as difficult to construct as the higher-order terms in Q_H, and hence it would be nice if the work done for Q_H would already be sufficient to obtain A. This is indeed the case. The brackets

$$\{Q_H, Q_H\} = 0, \{Q_H, A\} = 0 \tag{10.6.5}$$

suggest to introduce a superfield

$$S = Q_H + c_0 A \tag{10.6.6}$$

where c_0 is a new real anticommuting ghost whose ghost number is $+1$. Then

$$\{S, S\} = \{Q_H, Q_H\} + 2c_0 \{A, Q_H\} = 0 \tag{10.6.7}$$

and thus S is a BRST charge in a larger space containing c^0. For consistency we also need its conjugate momentum p_0 satisfying by definition

$$\left\{p_0, c^0\right\} = -1 \quad , \quad (c^0)^* = c^0 \quad , \quad p_0^* = -p_0 \tag{10.6.8}$$

If we enlarge the space of constraints and define

$$G_A = \{G_a, A_0\} \quad ; \quad C^A = \{\eta^a, c^0\} \quad ; \quad p_A = \{p_a, p_0\} \tag{10.6.9}$$

then

$$\{G_A, G_B\} = f_{AB}{}^C G_C, \tag{10.6.10}$$

and $f_{0b}{}^c = W_b{}^c$ while $f_{00}{}^c = f_{AB}{}^0 = 0$.

For anticommuting A_0 one would have to introduce a new commuting ghost c_0, and then one would find that S is given by a power series in c_0, and one would find many constraints contained in $\{S, S\} = 0$. We restrict our attention to **commuting** A_0.

We may now repeat the construction of a BRST invariant charge S, starting with G_A. It is possible to show that S is independent of p_0. For the first two terms this is clear

$$\begin{aligned} S &= C^A G_A + C^B C^A \, {}^{(1)}U_{AB}{}^C p_C + \dots \\ &= \eta^a G_a + c^0 A_0 + \eta^b \eta^a \, {}^{(1)}U_{ab}{}^c p_c + 2c^0 \eta^a \, {}^{(1)}U_{a0}{}^b p_b + \cdots \end{aligned} \tag{10.6.11}$$

In general

$${}^{(n)}U_{B_1 \cdots B_{n+1}}{}^{A_1 \cdots A_n} \tag{10.6.12}$$

vanishes if at last one of the superscripts A_i equals zero. This follows by induction. If the term with ${}^{(n)}U_{ao}{}^b$ would contain a p_0, then at the level of n ghosts in the bracket $\{S, S\} = 0$ there would be only one contribution containing a p_0, namely the contribution from the bracket between $c^a G_a$ and the term with ${}^{(n)}U_{a0}{}^b$. Since the Poisson brackets never produce a p_0, the term with ${}^{(n)}U_{a0}{}^b$ must vanish.

For the Hamiltonian H_0, the BRST extension reads, with (20.0.6)

$$H_{BRST} = H_0 + \eta^a V_a{}^b p_b + \dots \tag{10.6.13}$$

In most applications, H_0 commutes with G_a, and in these cases H_{BRST} is just equal to H_0.

7 The quantum action

We come now to the main problem: the construction of the quantum action S_{qu}. It is here that we shall need the ingredients prepared in the previous sections. We shall use as input only the knowledge that the classical system is characterized by

$$\begin{aligned}
\{H_0, \varphi_\alpha\} &= V_\alpha{}^\beta \varphi_\beta \\
\{\varphi_\alpha, \varphi_\beta\} &= f_{\alpha\beta}{}^\gamma \varphi_\gamma \\
G_a &= \{\varphi_\alpha, \pi_\mu\}
\end{aligned} \tag{10.7.1}$$

In other words, the dynamics is given by $H_0(q, p)$, the gauge symmetries by the first class constraints $\varphi_\alpha(q, p)$, while the existence of π_μ indicates that there are further fields λ^μ which were Lagrange multipliers in the classical action.

What properties should the quantum action possess? We have seen in (20.0.11) that the classical action in Hamiltonian form has gauge invariances, which correspond to the gauge invariances of the classical action in Lagrangian form. As in the Lagrangian formalism, we thus expect to need gauge-fixing terms in order that propagators be non-singular in the quantum theory. (Without gauge-fixing terms, we expect kinetic matrices to be singular, in general. More generally, gauge-invariance of the classical theory implies that the path-integral would be divergent, as the gauge volume will be infinite for local symmetries. The arguments we discuss here for the Hamiltonian case are the same as in the Lagrangian case.)

Being a true Hamiltonian believer for the duration of this chapter, the reader should require that the theory contains for every field a canonical momentum. In some models one finds in the ghost action a kinetic term $B\dot{C}$ (in string theory, for example) and one might wish to identify B as the momentum conjugate to C, and discard B as an independent coordinate in the Hamiltonian altogether. However, there are also models where the action contains $\dot{B}\dot{C}$ (for example, Yang-Mills theory), and thus the correct way to proceed is to have momenta $p(C)$ and $p(B)$ for both the ghosts C and the antighosts B. In models where the original ghost action is of the form $B\dot{C}$ one should then find that after integrating out $p(B)$ and $p(C)$ one recovers $B\dot{C}$.

We also expect from analogy with the Lagrangian case, that at the quantum level a residue is left of the classical gauge invariance, namely a nilpotent rigid symmetry, the BRST symmetry. However, whereas in the Lagrangian case one must specify the transformation rules of the various fields like $\delta C^a = -\frac{1}{2} f^a{}_{bc} C^c \Lambda C^b$, in the Hamiltonian case **all** transformation rules should follow from the bracket with the BRST generator

$$\delta(\text{field}) = \Lambda \{Q_H,\ \text{field}\ \} = \{\text{field},\ Q_H\}\, \Lambda \tag{10.7.2}$$

where the expected nilpotency of the transformation rules implies

$$\{Q_H, Q_H\} = 0 \tag{10.7.3}$$

We know more about Q_H. For the classical fields q_i and p^i, the BRST transformation rules should be gauge transformations in which the gauge parameter ξ^α is replaced by $C^\alpha\Lambda$, with C^α the corresponding ghost fields. Hence, Q_H must start with a term $C^\alpha\varphi_\alpha$ since φ_α generates gauge transformations of q^i, p_i

$$Q_H = C^\alpha \varphi_\alpha + \ldots \tag{10.7.4}$$

Moreover, also the Lagrange multipliers λ^μ transform in the Lagrangian case with a result that is linear in ghosts, hence we expect in Q_H also a term with $\pi_\mu(\lambda)$ times some conjugate momentum which becomes the BRST law if we integrate out this momentum in the path-integral. (Since $\delta A_0{}^a = \partial_0 \lambda^a + \ldots$, and terms with $\partial/\partial t$ are forbidden in the Hamiltonian formalism, $\pi(\lambda)_\mu$ must multiply an object which subsequently must be integrated out). Since $\pi(\lambda)_\mu$ has ghost number zero, the conjugate momentum which multiplies it in Q_H must have ghost number +1, so it must be $p(B)^\mu$. This suggest to combine the term $C^\alpha\psi_\alpha$ and $p(B)^\mu\pi(\lambda)_\mu$

$$Q = \eta^a G_a \tag{10.7.5}$$

where $G_a = \{\varphi_\alpha, \pi_\mu\}$ and $\eta^a = \{C^\alpha, p(B)^\mu\}$.

The quantum action, then, is expected to be of the general form "$p\dot{q}$"–"H" $+L$ (fix), where "$p\dot{q}$" stands for a sum over **all** fields, H is some Hamiltonian, and L(fix) takes care of the gauge-fixing. The simplest way this action could possibly be BRST invariant is by requiring that each term be **separately** invariant. As we shall see, $\int$ "$p\dot{q}$" is always an invariant (if time runs from $-\infty$ to $+\infty$, or if suitable boundary conditions are imposed). The proof is the same as the canonical invariance of $p\dot{q}$ in classical mechanics: these terms vary into a total time derivative. For "H" we also have an obvious choice: the BRST invariant extension of H_0 of the previous section.

That leaves us with L (fix) which should by BRST invariant by itself, and depend on arbitrary functions, the gauge choices (as many as there are gauge invariances, i.e., as many as there are φ_α's). Also here there is an obvious choice

$$L\text{ (fix) } = \{\psi, Q_H\} \tag{10.7.6}$$

since $\{L\text{ (fix)}, Q_H\} = 0$ due to the Jacobi identities and (11.1.35). The anticommuting function ψ is called the gauge fermion, and it is usually taken as $\psi = B_\mu \xi^\mu + \cdots$ where ξ^μ corresponds to the gauge choices made in the Lagrangian formalism. However, in principle any choice for ψ is allowed (within certain restrictions to be discussed).

We shall thus obtain a BRST invariant quantum action in this manner (details follow), but this is not enough: we also wish that the S-matrix does not depend on the choice of the gauge ψ. This we shall prove by making a change of integration variables in the path-integral which has the form of a BRST transformation, except that the BRST constant parameter Λ is replaced by $\int(\psi - \psi')dt$, where ψ' is another gauge choice. Since ψ and ψ' depend on fields, the Jacobian now does not vanish, and it precisely cancels the change in the action due to replacing ψ by ψ'.

As BRST invariant action in Hamiltonian form, we claim that one can take the following expression

$$
\begin{aligned}
S_{qu} &= \int [\dot{q}_i p^i + \dot{\lambda}^\mu \pi_\mu + \dot{\eta}^a p_a - H_{BRST} + \{\psi, Q_H\}] dt \\
\dot{\eta}^a p_a &= \dot{C}^\alpha p(C)_\alpha + \dot{p}(B)^\mu B_\mu
\end{aligned}
\tag{10.7.7}
$$

Note that according to (11.1.14) the action is hermitian when ψ is antihermitian. The BRST transformation rules of all phase space variables are generated by the BRST charge Q_H. So

$$
\begin{aligned}
\delta p^i &= \{p^i, Q_H\}\Lambda = -\partial/\partial q_i Q_H \Lambda \\
\delta q_i &= \Lambda\{Q_H, q_i\} = -\Lambda \partial Q_H / \partial p^i
\end{aligned}
\tag{10.7.8}
$$

Clearly

$$
\delta \int \dot{q}_i p^i dt = \int \left\{ -\dot{q}_i \frac{\partial}{\partial q_i} Q_H \Lambda + \Lambda\, \partial Q_H / \partial p^i \dot{p}^i + \frac{d}{dt}(\delta q_i p^i) \right\} dt \tag{10.7.9}
$$

The term $\int_{t_0}^{t_1} \frac{d}{dt}(\delta q_i p^i) = \delta q_i p^i \mid_{t_0}^{t_1}$ vanishes when $q_i(t_0)$ and $q_i(t_1)$ are fixed by the boundary conditions, so that $\delta q_i = 0$ at t_0 and t_1. When $p^i(t_0)$ and $p^i(t_1)$ are fixed, one finds in the action a term $-q_i \dot{p}^i$, and now the total derivative become $\frac{d}{dt}(-q_i \delta p^i)$ and this boundary term cancels for the same reasons. The same analysis holds for the terms $\dot{\lambda}^\mu \pi(\lambda)_\mu$ and $\dot{\eta}^a p_a$. Hence the "kinetic" terms in S_{qu} transform into $\Lambda \int \frac{dQ_H}{dt} dt$ which vanishes for suitable boundary conditions. There are further restrictions on the boundary conditions which we discuss in section 9, but they do not invalidate the proof that the action is BRST invariant.

Since all terms in S_{qu} are separately BRST invariant, we have found a BRST invariant quantum action. Moreover, all terms are real.

8 Boundary conditions and gauge-choice independence

The transition amplitude to go from a state $|\psi_0>$ at $t = t_0$ to a state $|\psi_1>$ at $t = t_1$, is given in the path-integral formalism by

$$Z = \int D\mu \ \exp \frac{i}{\hbar} \int_{t_0}^{t_1} \mathcal{L}_{qu} d^4x \tag{10.8.1}$$

where $D\mu$ is the measure $Dq_i \ldots DC^\alpha DP(C)_\alpha$, and the paths all start at $t = 0$ and end at t_1 such that "at $t = t_0$ the boundary conditions specify the state $|\psi_0>$ and at $t = t_1$ the state $|\psi_1>$". What is meant by the phrase in quotation marks?

One determines a maximal set of commuting variables, diagonalizes these operators, and chooses definite eigenvalues for each. This maximal set contains half as many variables as there are in extended phase space because for each pair of conjugate variables one must fix one of them; in the path integral one integrates then over the other variable. For example, in quantum mechanics we can study the transition element

$$< x = z, t = t_0 \mid \exp -\frac{i}{\hbar} \hat{H} t_0 \mid x = y, t = 0 > \tag{10.8.2}$$

or one can specify the initial and final states by giving the momenta $p(t_0)$ and $p(t = 0)$; one can also use coherent states to specify the initial and final states (the so-called holomorphic representation). At the quantum level the variables which are fixed become the variables on which the wave function φ depends, while the other variables are represented as derivatives w.r.t. to the variables which appear in the wave function. Denoting collectively all variables which are specified by Z_α, and all variables which are left unfixed by Y^α, we have $< Z \mid \psi >= \varphi(Z)$, and $< Z \mid \hat{Y}^\alpha \mid \psi >= \frac{\partial}{\partial Z_\alpha} \varphi(Z)$. For an anticommuting variable, fixing means setting it equal to a Grassmann variable. (For example $C^a(t_0) = \theta^a$ with $(\theta^a)^2 = 0$, or $C^a(t_0) = 0$).

For gauge theories there are unphysical modes as well as physical modes. The former correspond to the gauge degrees of freedom and the ghosts in Fock space.

Hence, physical states must be further specified, and in the BRST formalism this is done very simply by requiring that they are annihilated by the BRST operator $\hat{Q}_H$. Hence, $\hat{Q}_H|\psi_0>= 0$ and $\hat{Q}_H|\psi_1>= 0$ in our case. Inserting complete sets of Z and Y eigenstates, one has for a ket vector with eigenvalues Z_0

$$\int < Y \mid \hat{Q}_H \mid Z >< Z \mid \psi > dZ = Q_H(Y, Z_0)\varphi(Z_0) = 0 \tag{10.8.3}$$

Hence we find that the BRST charge Q_H, which is a **function** of Y and Z, must vanish if one substitutes the boundary values $Z(t_0)$ or $Z(t_1)$ into $Q_H(Z, Y)$, and this should hold for any Y.

We must therefore choose the boundary conditions on the fields at t_0 such that $Q_H(t_0)$ vanishes. To focus our ideas, recall the form of Q_H

$$Q_H = p(B)^\mu \pi_\mu(\lambda) + C^\alpha \varphi_\alpha - \frac{1}{2} C^\beta C^\alpha f_{\alpha\beta}{}^\gamma p(C)_\gamma (-)^\beta + \dots \tag{10.8.4}$$

Let us begin by choosing the boundary conditions $C^\alpha(t_0) = 0$. In order that $Q_H(t_0) = 0$, we then can take either $p(B)^\mu(t_0) = 0$ and/or $\pi_\mu(t_0) = 0$. To completely specify the physical states, one must further specify $B_\mu(t_0)$ if we do not use $p(B)^\mu(t_0)$, or $\lambda^\mu(t_0)$ if we do not take $\pi_\mu(t_0) = 0$. As boundary conditions on the physical variables q_i and p^i one may choose any condition (for example, fixed q_i, or p^i, or $\psi_\alpha(q, p) = 0$) because the term $C^\alpha \psi_\alpha$ already vanishes due to $C^\alpha = 0$. We denote these conditions on q_i and p^i by $F(q, p) = 0$. This set of boundary conditions for physical states reads then

$$\begin{aligned} C^\alpha(t_0) &= 0, p(B)^\mu(t_0) = 0, \pi_\mu(t_0) \text{ or } \lambda^\mu(t_0) \text{ fixed }, F(p, q) = 0 \\ C^\alpha(t_0) &= 0, \pi_\mu(t_0) = 0, p(B)^\mu(t_0) \text{ or } B_\mu(t_0) \text{ fixed }, F(p, q) = 0 \end{aligned} \tag{10.8.5}$$

Another set of boundary conditions is obtained by starting from $p(C)_\alpha(t_0) = 0$. In order that $Q_H(t_0) = 0$ we must then again take either $p(B)^\mu(t_0) = 0$ or $\pi_\mu(t_0) = 0$, or both. Since C^α is now nonvanishing, we must impose $\psi_\alpha = 0$, which means that the physical states are gauge invariant. Hence we now arrive at

$$\begin{aligned} p(C)^\alpha(t_0) &= 0, p(B)^\mu(t_0) = 0, \pi_\mu(t_0) \text{ or } \lambda^\mu(t_0) \text{ fixed }, \psi^\alpha(p,q) = 0 \\ p(C)^\alpha(t_0) &= 0, \pi_\mu(t_0) = 0, p(B)^\mu(t_0) \text{ or } B_\mu(t_0) \text{ fixed, } \psi^\alpha(p,q) = 0 \end{aligned} \tag{10.8.6}$$

These boundary conditions lead to technical complications when φ_α contains both momenta and coordinates, because at the quantum level (with operators denoted by hats, and not using Poisson brackets anymore), the operator conditions $\hat{\varphi}_\alpha|\psi_0>=0$ may become inconsistent due to operator ordering. (Namely, if $[\hat{\varphi}_\alpha, \hat{\varphi}_\beta]$ is no longer proportional to $\hat{\varphi}_\gamma$, one gets further constraints on the physical states. For example, in general relativity, no consistent ordering of the p^i and q_i in $\hat{\varphi}_\alpha$ is known.)

Historically, Dirac did not introduce ghosts, but worked only in the space of q_i, p^j, and defined physical states by $\hat{\varphi}_\alpha|\psi>=0$. The BRST formalism allows one to escape the aforementioned inconsistencies by adding ghosts and using a more general definition of physical states, namely $Q_H|\psi>=0$.

The set of boundary conditions $p(C)^\alpha = \pi_\mu = B_\mu = \psi^\alpha(p,q) = 0$ not only leads to $Q_H = 0$, but it has another property: this set of conditions is itself invariant under BRST transformations. For example, $\delta\psi^\alpha \sim \{\psi^\alpha, Q_H\}$ vanishes, and also $\{p(C)^\alpha, Q_H\} \sim \psi^\alpha + \mathcal{O}(p(C) \sim 0$ and $\{\pi_\mu, Q_H\} \sim 0$ while $\{p(B), Q_H\} = 0$ and $\{B_\mu, Q_H\} \sim \pi_\mu \sim 0$. One may find other boundary conditions which (i) satisfy $Q_H = 0$ and (ii) are BRST invariant.[Henneaux]. On the other hand, the set in (10.8.5) is not BRST invariant for general $F(p,q)$, but only for $F(p,q) = \psi(p,q)$. Hence, both sets (10.8.5) and (10.8.6) are BRST invariant if one fixes the classical canonical variables by requiring that physical states are gauge invariant. In Dirac's approach to quantum mechanics and quantum field theory without any ghosts, physical states satisfy $\psi^\alpha(p,q) = 0$. Hence one suspects that there are further conditions which the boundary conditions must satisfy, and that as a result of these further conditions one finds that physical states must also satisfy $\psi^\alpha = 0$. Do the extra boundary conditions only apply to physical states, or also to arbitrary initial and final states in the path integral? What is the origin of these further conditions?

There are two further principles which restrict the boundary conditions. In order that the path integral describes a quantum **extension** of the given physical system, one should require that the set of solutions of the field equations of the full quantum action coincides with the set of solutions of the original classical action. Classical solutions correspond to tree graphs, and if there were more (or less) tree graphs in the quantum theory than in the classical theory, the quantum action would correspond to a different physical system. Of course, in the quantum action one has fixed the gauge, but in the classical theory the solutions are obtained from the gauge invariant action, hence the proposed requirement of equality of solutions needs further analysis.

A second condition on the boundary condition holds both for Green's functions and for S-matrix element, hence for any initial or final state, not only those satisfying $Q_H = 0$. Solutions of the field equations are satisfied in the bulk of spacetime, and on the boundary these fields should satisfy the boundary conditions. One should then require that at the boundary the field equations are compatible with the boundary conditions, i.e., substituting the boundary conditions into the field equations should not lead to inconsistencies. We have assumed so far that we are dealing with "Dirichlet boundary conditions" (conditions on the fields ϕ, not on their derivatives), hence $\frac{d\phi}{dt}$ is unspecified, and one does not find any inconsistency at the initial **or** the final time. However, in general there may not be a solution of the quantum field equations which interpolates between the boundary conditions on the initial and final states. Whereas this does not constitute an inconsistency by itself, it makes evaluation of the path integral (for example by the background field formalism) very difficult, and when a classical solution exists but a quantum solution does not, we are no longer dealing with an extension of the original classical system.

Hence we propose that boundary conditions must always satisfy the following criteria

(i) there exists a solution of the quantum field equations with these boundary conditons,

(ii) physical states must satisfy in addition $Q_H = 0$.

It remains now to be seen whether these conditons imply others which have been proposed in the literature without the motivation we have given.

The matrix elements of the time evolution operator $\exp -\frac{i}{\hbar}\hat{H}$, called the transition elements, are the S-matrix elements if the initial and final states are physical states. This statement is far too simple: one must renormalize in the case of field theories, and also apply the operation of truncation of Green's functions. Nevertheless, the usual proof of the independence of the S-matrix on the choice of gauge fixing terms in quantum field theory has a corresponding proof in the context of phase space path integrals which we now give. The original proof was given in as early as 1974 by Lee and Zinn-Justin, while the proof based on BRST symmetry in phase space is sometimes called the Fradkin-Vilkovisky theorem. In both proofs one makes a change of integration variables. (In the 1974 proof, one uses a nonlocal gauge transformation which leaves the product of the measure $dA_\mu{}^a$ and the Faddeev-Popov determinant invariant. In extended phase space one makes a BRST change of variables).

Suppose one wants to show that physical transition elements due to a gauge choice ψ are the same as those due to ψ'. Consider then a BRST transformation with parameter $\Lambda = \epsilon \int_{t_0}^{t_1} (\psi' - \psi) dt$ with ϵ a small constant ordinary number. This BRST transformation will change ψ into $\psi + \epsilon(\psi' - \psi)$ as we shall show. By exponentiation (repeatedly making infinitesimal BRST transformations) we shall then end up with ψ'.

Under the infinitesimal transformation $\delta\phi = \{\phi, Q\}$ of the fields, the action is invariant up to total derivatives, and those total derivatives vanish when the boundary conditions which specify the physical states are BRST invariant. The total derivatives come from

$$\delta(\dot{q}p) = \dot{q}\{p, Q\}\Lambda + \frac{d}{dt}(\delta qp) - \Lambda\{Q, q\}\dot{p} \tag{10.8.7}$$

and lead to $\Lambda\frac{dQ}{dt} + \frac{d}{dt}(\{q, Q\}\Lambda p)$. Of course Λ is field-dependent, and depends on t_0 and t_1, but it does not depend on t, hence $\frac{d}{dt}\Lambda = 0$, and all extra terms are really total time derivatives. We already showed that these extra terms cancel due to the boundary conditions.

Only the measure is not invariant under these infinitesimal BRST transformations, because Λ is field dependent. One finds for the Jacobian $J = 1 + str\partial\delta\phi/\partial\phi$ where ϕ denotes all variables in extended phase space. Since $\delta\phi = \Lambda\{Q, \phi\}$, there are terms due to differentiating $\{Q, \phi\}$ and terms due to differentiating Λ. The former vanish; for example from $\partial\delta p^i/\partial p^i = \partial(\Lambda\{Q, p^i\})/\partial p^i$ one finds a term $(-)^{i+1}\Lambda\partial^2 Q/\partial q_i\partial p^i$, while from $\partial\delta q_i/\partial q_i = \partial(\Lambda\{Q, q_i\})/\partial q_i$ one finds a term $\Lambda\partial Q/\partial p^i\partial q_i$. These terms cancel. However, the terms involving derivatives of Λ yield a nonvanishing result. Hence the measure transforms as follows

$$\delta(d\mu) = d\mu[1 + str(\{p^i, Q\}\partial\Lambda/\partial p_i + \{q_i, Q\}\partial\Lambda/\partial q^i + \ldots)] \tag{10.8.8}$$

where the sum sums over all variables in extended phase space. More explicitly, $\partial\delta q/\partial q$ stands for $\delta q(t)/\partial q(t')$ and thus $\{p^i, Q\}\partial\Lambda/\partial p_i = -(\partial/\partial q_i(t)Q)\,\partial(\int\psi' - \psi)/\partial p^i(t')$.

The Jacobian is then seen to be equal to the Poisson bracket of the gauge condition with the BRST charge

$$\delta(d\mu) = d\mu(1 + \epsilon str\int_{t_0}^{t}\{(\psi' - \psi), Q_H\}dt) \tag{10.8.9}$$

Exponentiation leads to an extra term in the quantum action

$$(d\mu)' = (d\mu)\exp\frac{i}{\hbar}\int_{t_0}^{t_1}\{\psi' - \psi, Q_H\}dt \tag{10.8.10}$$

When this term is added to the gauge fixing term $\{\psi, Q_H\}$, it clearly replaces ψ by ψ'. This shows that physical matrix elements are independent of the choice of gauge fermion ψ.

As we have already mentioned, there are many loopholes in this proof: renormalization and truncation of Green's functions of the quantum field theories are overlooked. A serious problem is regularization: to properly define the path integral, one should discretize it, but, as well-known in quantum mechanics, one needs at the discretized level N+1 complete set of momentum eigenstates but only N complete sets of coordinate eigenstates. Similar results hold in extended phase space. This discretized measure is then no longer BRST invariant. However, for Yang-Mills gauge theories the Jacobian associated with the $(\mathrm{N}+1)^{\mathrm{th}}$ integration variable is $\partial(\delta p_{N+1})/\partial p_{N+1}$, and this leads to a contribution proportional to the trace of the structure constants, which vanishes. Probably, it also vanishes in other cases (since they have already been cured in the usual approach to quantum field theory). The main virtue of the Hamiltonian path integral approach is that it gives a unified derivation of various quantum actions, and rather than try to duplicate results of Feynman graph quantum field theory, one should use it in the operator approach to quantum field theory.

We conclude with a few comments.

(1) Since the action in (11.1.39) has a rigid BRST symmetry, there is a Noether current j^μ, and we expect that $Q_H = \int j^0 d^3x$. This is indeed the case, because only the kinetic terms are not invariant when Λ becomes spacetime dependent, see (11.1.31) and (11.2.2), and vary into $-\int \dot{\Lambda} j^0 d^3x$ where $\int j^0 d^3x = Q_H$.

(2) In classical mechanics, $\{H, A\}$ only equals $\frac{dA}{dt}$ if one uses the Hamilton equations. We have constructed an H_{BRST} which satisfies $\{H_{BRST}, Q_H\} = 0$, but this does not mean that $dQ_H/dt = 0$ when the fields in the path-integral are "off-shell", i.e., they do not satisfy any field equations. As usual, the charge is conserved only on-shell, but the Hamiltonian is BRST invariant also off-shell.

(3) One may check the BRST transformation rules by showing that the BRST variation of the canonical brackets vanishes. This checks, for example, the factor $\frac{1}{2}$ in δC^α.

(4) The gauge fermion ψ should in general satisfy some conditions, for example to avoid the Gribov problem. This is just as in the Lagrangian formalism, hence one may look there for further discussions of these conditions on the gauge-fixing terms.

(5) Because the Hamiltonian approach is very algebraic, see for instance the construction of the structure functions, it is particularly well-suited for group theoretical methods. We refer to articles on Hamiltonian reduction, coadjoint orbits, etc.

(6) One could use the action with $\psi = 0$ for a lattice approach, since in the lattice approach one does not need to construct propagators for perturbation theory. For $\psi = 0$, the action has a **local** gauge invariance, see the next chapter, but that is no problem in the lattice approach.

(7) The Hamiltonian approach is, of course, needed for an operator formalism. In string theory, this is much used.

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Chapter 11

The antifield formalism

The Lagrangian BRST quantization we discussed in chapter I and the Hamiltonian BRST quantization we discussed in chapter VIII can be combined, keeping the good features of each and avoiding the bad features. The result is a formalism which we shall call the antifield formalism. It is an extension of the BRST formalism with the external sources for BRST variations which Zinn-Justin used to derive the Ward identity $(\Gamma, \Gamma) = 0$ for the effective action. The external sources ${K_a}^\mu$ and L_a are the antifields for the gauge fields ${A_\mu}^a$ and ghost fields c^a. A more general approach was worked out by Batalin and Vilkovisky, who introduced also antifields for the antighosts and BRST auxiliary fields, and gave a very general quantization scheme which can for example also be applied to theories with ghosts-for-ghosts and extra ghosts, and/or with open gauge algebras, leading to theories in which the BRST sources sometimes appear nonlinearly. This more general scheme is also called the Batalin-Vilkovisky formalism, or just the BV formalism. [1]

For physicists whose interest does not reach beyond Yang-Mills theory, the antifield formalism is not needed. The antifield formalism is rather formal, but physicists who have a good understanding of the canonical Hamiltonian approach to classical mechanics, should have no difficulty following the discussions in this chapter. As a first introduction to the antifield formalism we suggest reading this introduction, the first two pages of section 1, and sections (3.1) and (5.1).

One of the physical reasons we are interested in a BRST invariant action is that it leads to Ward identities from which one may prove unitarity and renormalizability. Among the various BRST approaches, the antifield formalism has the advantage of treating all quantum systems (with/without open algebra's, with/without ghosts for ghosts) in a uniform manner. This brings out the essential features more clearly, and that, in turn, might be helpful in quantizing systems, such as the heterotic string or closed-string field theory, which have hitherto defied covariant quantization attempts.

In the Lagrangian and Hamiltonian approaches the detailed form of the classical action is not important. All that matters is that it is gauge invariant. Hence one can add, for example, topological terms. The same is true for the BV approach. In the Hamiltonian approach one introduces for **every** field a canonically conjugate momentum but one loses manifest Lorentz covariance, because the $\dot{\phi}^i$ are replaced by p^i and for example A_0 is replaced by a Lagrange multiplier λ. In the BV formalism one keeps all ϕ^i, and manifest Lorentz covariance is preserved. Instead, for each field ϕ^i an antifield ${\phi^*}_i$ is introduced which plays the role of a kind of covariant canonical momentum. For example, for a Maxwell field A_μ, the antifield is $A^{*\mu}$.

The BV formalism is a Lagrangian formalism; one can maintain manifest Lorentz invariance since at no stage one need decompose spacetime indices into space parts and time parts. On the other hand, it keeps the idea of fundamental brackets between "momenta" and coordinates of the Hamiltonian formalism, but these "BV momenta" are Lorentz-covariant. As in the Hamiltonian case, all transformation rules, of fields and antifields, are obtained by taking the bracket with the BRST generator. Also gauge-fixing is treated in the Hamiltonian way: at the very end of the quantization procedure. This allows one to trace questions such as gauge dependence in a simpler way. However, by far the greatest virtue of the BV formalism is that it puts the BRST generator in a very central position: **the antifield-extended action is the**

BRST generator!

Looking back at the usual BRST Lagrangian quantization of chapter I, we can already see traces of this idea: in the action one adds extra terms of the form "(source) times (BRST variation of fields)", where the fields are either classical fields ϕ^I or ghosts C^α. There were two such terms

$$K_I \delta_{BRST} \phi^I / \Lambda + L_\alpha \delta_{BRST} C^\alpha / \Lambda \tag{11.0.1}$$

(The symbol $\delta_{BRST} A/\ \Lambda$ indicates that we remove the BRST parameter Λ in δA from the right. One often introduces the symbol s for the result, so $K_I s\phi^I + L_\alpha s C^\alpha$ in (11.0.1)). If we postulate a new bracket (,) such that $(\phi^I, K_J) = \delta^I{}_J$ and $(C^\alpha, L_\beta) = \delta^\alpha{}_\beta$, then we see that we can rewrite the BRST transformation rules as follows

$$\delta_{BRST} \phi^I = (\phi^I, S\Lambda), \delta_{BRST} C^\alpha = (C^\alpha, S\Lambda) \tag{11.0.2}$$

where S is the quantum action including the source terms for BRST variations. Clearly, for the classical fields ϕ^I and the ghosts C^α, the action S plays the role of BRST charge provided one introduces this new bracket.

This idea, that the action is at the same time the BRST generator, has been taken by BV as the starting point of their quantization method. (Already in string theory, one had taken the BRST charge as action in some cases). Let us write more formally

$$\delta_{BRST} \phi^A = (\phi^A, S\Lambda) \tag{11.0.3}$$

where ϕ^A denotes at this point both the classical fields ϕ^I and the ghosts C^α. Later it will denote all fields, for example also ghosts-for-ghosts, BRST auxiliary fields, antighosts, extra ghosts, etc.

At first sight, it seems impossible that the action is also the BRST charge, since an action is commuting but the BRST charge anticommuting. However, all one really knows is that the combination of the BRST generator and the bracket should be anticommuting. In the Hamiltonian case one takes the BRST generator (Q_H) anticommuting and the bracket (Poisson or Dirac bracket) as commuting, but in the BV scheme one does things just opposite: the charge (=action S) is commuting and **the bracket is now anticommuting**. It is called the “antibracket”. As suggested by the Lagrangian BRST formalism (recall our proposal $(\phi^I, K_J) = \delta^I{}_J$), we therefore introduce for each field ϕ^A an antifield ϕ^*_A with opposite statistics, and with bracket

$$(\phi^A, \phi^*_B) = \delta^A{}_B \tag{11.0.4}$$

The $\delta^A{}_B$ includes a 4-dimensional delta function $\delta^4(x-y)$. In addition we make the natural further requirements

$$(\phi^A, \phi^B) = 0, (\phi^*_A, \phi^*{}_B) = 0 \tag{11.0.5}$$

We shall now first discuss this antibracket in more detail and then construct the quantum action.

1 The antibracket and the quantum action

Since BRST transformations for classical fields are obtained by replacing the gauge parameter ξ^α by $C^\alpha\Lambda$, it follows that Λ has ghost number -1 if C^α has ghost number $+1$ and hence the ϕ^*_A (for example, the K_I in chapter II) have ghost number -1 if the ϕ^A have ghost number zero. Those ϕ^A which themselves have ghost number $+1$ (namely the ghosts C^α) have a corresponding antifield C^*_α (for example the L_α in chapter II) with ghost number -2. **In general the sum of the ghost numbers of ϕ^A and ϕ^*_A is -1.**

The antibracket for functions f and g depending on fields and antifields is defined by

$$(f, g) = \partial f/\partial\phi^A \, \frac{\partial}{\partial\phi^*_A} g - \partial f/\partial\phi^*_A \, \frac{\partial}{\partial\phi^A} g \tag{11.1.1}$$

When both f and g are bosonic the bracket is symmetric in f and g. An example is the Ward identity $(\Gamma,\Gamma)=0$ of chapters II and III (where the sources K and L played the role of ϕ^*_A). In fact, the definition in (11.1.1) is derived from the expression for (Γ,Γ). In all other cases, the antibracket is antisymmetric in f and g. To prove this recall that the left-derivative of a bosonic field (or combination of fields) w.r.t. an anticommuting field is equal to minus the right-derivative. Thus if ϕ^A is commuting then the left- and right-derivatives are related by $\frac{\partial}{\partial\phi^*_A}f=(-)^{f+1}\,\partial f/\partial\phi^*_A$ since ϕ^*_A is then anticommuting. (The symbol $(-)^f$ equals $+1$ if f is commuting and -1 if f is anticommuting). Hence

$$(f,g)=(-)^{f+g+fg}(g,f)=-(-)^{(f+1)(g+1)}(g,f) \tag{11.1.2}$$

One may understand this sign as expressing the fact that if f and g are interchanged, one obtains the usual sign factor $(-)^{fg}$, whereas due to the anticommuting nature of the bracket, pulling f from left to right across ϕ^A and ϕ^*_A in (11.1.1) gives a sign $(-)^f$, and similarly for g. The reader may check that using this symmetry requirement in the first term in (11.1.1) one reproduces the second term. We define the bracket as in (11.1.1), that is we interchange ϕ and ϕ^* in the second term but keep f and g in the same position, because then no extra signs factors are needed. One can also write the bracket in a more Poisson-like fashion as

$$(f,g)=\partial f/\partial\phi^A\ \frac{\partial}{\partial\phi^*_A}\ g-(-)^{(f+1)(g+1)}\ \partial g/\partial\phi^A\ \frac{\partial}{\partial\phi^*_A}f \tag{11.1.3}$$

but then one gets the extra signs in the second term.

From its definition, the antibracket clearly satisfies

$$(\phi^A,\phi^*_B)=-(\phi^*_B,\phi^A)=\delta^A{}_B \tag{11.1.4}$$

This also agrees with the fact that the bracket is antisymmetric if f and g have opposite statistics. Because the antibracket is realized in terms of derivatives, it satisfies the Jacobi identities

$$(A,(B,C))+(B,(C,A))(-)^{(A+1)(B+C)}+(C,(A,B))(-)^{(C+1)(A+B)}=0 \tag{11.1.5}$$

where the sign factors are determined by requiring that all terms cancel pairwise.

Again one may understand the signs as follows: the sign in front of the second term is due to pulling A to the right, yielding $(-)^{A(B+C)}$ (because it passes two brackets) and then pulling B and C to the left across a bracket, yielding $(-)^B$ and $(-)^C$, respectively. The same reasoning applies of course to the last term.

Actually, since the bracket is defined in terms of derivatives, it satisfies the stronger property of the Leibniz rule

$$(A, BC) = (A, B)C + (-)^{B(A+1)}B(A, C) \tag{11.1.6}$$

The signs have again the interpretation given below (11.1.1). We have already argued that BRST transformations are generated by the action: $\delta_{BRST}\phi^A = (\phi^A, S\Lambda)$. We now also require that **always**, no matter how complicated the system, the BRST transformations are nilpotent. Using the Jacobi identities $(S, (S, \phi^A)) + (S, (\phi^A, S))(-)^\phi + (\phi^A, (S, S)) = 0$ and $(\phi^A, S) = (S, \phi^A)(-)^\phi$, this implies

$$(S, (S, \phi^A)) = 0 \Leftrightarrow (\phi^A, (S, S)) = 0 \tag{11.1.7}$$

Hence, if the "**master equation**" $(S, S) = 0$ holds, the BRST transformations defined by $\delta\phi^A = (\phi^A, S)\Lambda$ are nilpotent.[1]

As a simple example, consider $S = \phi^*\eta + \eta^*\phi$ with commuting ϕ and anticommuting η. Then $(S, \phi) = (\phi, S) = \eta$ and $(S, \eta) = -(\eta, S) = -\phi$ while $(S, S) = 2\phi^*\phi + 2\eta^*\eta$. Hence, in this example the BRST transformations $\delta\phi = \eta\Lambda$ and $\delta\eta = \phi\Lambda$ are not nilpotent. If one introduces instead N commuting fields ϕ^i with $i = 1, N$ and the classical action is given by $S_{cl} = \phi^i\phi^i$, then the classical gauge invariance

[1] The converse, $(S, S) = 0$, would follow if the BRST transformations are nilpotent on ϕ^A **and** on ϕ^*_A, namely if $(S, (S, \phi^A)) = 0$ and also $(S, (S, \phi^*_A)) = 0$. However, as we shall later see, the antifields ϕ^*_A are eliminated at some moment as independent fields, and are replaced by expressions in terms of ϕ^B. Then the final BRST laws in general are only nilpotent if the full quantum field equations for ϕ^A hold, even if before eliminating ϕ^* the action $S(\phi, \phi^*)$ does satisfy $(S, S) = 0$.

$\delta\phi^i = \lambda^a(x)(T_a)^i{}_j\phi^j(x)$ with real antisymmmetrix $(T_a)^i{}_j$ leads to corresponding nilpotent BRST rules, and one may construct a nilpotent S for this system.

We start now our discussion of the action in the antifield formalism. In chapter I, we have seen that the ordinary BRST transformation rules split into two sectors, a sector with gauge fields and ghosts, and a sector with the antighosts and the auxiliary field. These sectors do not transform into each other, and in each sector the BRST charge is nilpotent. In a similar manner the action S in the antifield formalism consists of two parts: a minimal part $S^{\min}$ and a nonminimal action S^{nonmin}. The minimal $S^{\min}$ depends only on ϕ^I, C^α and ghosts-for-ghosts if present, as well as their antifields, but not on antighosts or antighost - antifields or auxiliary fields or extra ghosts (see below). We first discuss the minimal action.

The antifields appear in the minimal action as

$$S^{\min} = S_{cl}(\phi^I) + \phi^*_A\ (\delta_{BRST}\phi^A)/\Lambda + \ldots \tag{11.1.8}$$

where the terms with ... indicate terms with two and more antifields, and ϕ^I denote the classical fields. The classical action is obtained by putting all antifields equal to zero

$$S_{cl}(\phi^I) = S^{\min}(\phi^*_A = 0, \phi^A) \tag{11.1.9}$$

This condition states that if one puts all antifields to zero, no ghosts C^α remain. Since the ghost number of the action is zero and all ghosts have positive ghost number, while in the BV formalism the antighosts are introduced at a later stage (through the nonminimal action) this "classical correspondence limit" is automatically satisfied. In the Hamiltonian formalism things are different: there one does not obtain the classical action by putting all momenta equal to zero. Rather, one must eliminate them through their field equation.

One may be a little puzzled why we impose BRST invariance of the action **before** gauge fixing. After all, historically BRST symmetry was used for simplifying the construction of ghost actions from gauge fixing terms, but it had no a priori classical role. From our present perspective, we view the BRST transformation rather as a symmetry which one can extend beyond the gauge invariance of the classical action, even before gauge fixing. In this respect, the situation is just as in the Hamiltonian case, where the quantum action with ghosts but without the gauge fixing term $\int \{\psi, Q\} dt$ is BRST invariant by itself.

The master equation $(S, S) = 0$ implies a hierarchy of equations, obtained by expanding S in terms of the number of antifields

$$S = S_0 + S_1 + S_2 + S_3 + S_4 + \dots \tag{11.1.10}$$

From (S, S) one then finds

$$\begin{aligned}
&(S_0, S_0) = 0, \quad (S_0, S_1) = 0 \\
&(S_1, S_1) + 2(S_0, S_2) = 0\ , (S_1, S_2) + (S_0, S_3) = 0 \\
&(S_2, S_2) + 2(S_1, S_3) + 2(S_0, S_4) = 0 \dots
\end{aligned} \tag{11.1.11}$$

The reader who prefers at this moment a simple example may look ahead at the Yang-Mills system in section 3. The condition $(S_0, S_0) = 0$ is automatically satisfied, since S_0 is independent of ϕ^*_A. The condition $(S_0, S_1) = 0$ expresses the gauge invariance of the classical action, since $S_0 = S(\phi^*_A = 0)$ and $S_1 = \phi^*_A \delta_{\text{BRST}} \phi^A + \cdots$. Hence $(S_0, S_1) = \partial S_0 / \partial \phi^I \delta_{BRST} \phi^I = 0$ which is equivalent to gauge invariance of S_0. The next relation is more interesting: $(S_1, S_1) = -2(S_0, S_2)$ states that (S_1, S_1) in general is only zero if one satisfies the **classical** field equations $\partial S_0 / \partial \phi^A = 0$. The bracket of S_1 with itself, $(S_1, S_1) = 2(\partial S_1 / \partial \phi^A) \left(\frac{\partial}{\partial \phi^*_A} S_1 \right)$ yields the BRST commutator. Indeed, $\frac{\partial}{\partial \phi^*_A} S_1$ are the BRST variations, and $\partial S_1 / \partial \phi^A\ \delta \phi^A$ states that we must vary once more the fields ϕ^A in the BRST transformation laws. (Once more, because S_1 itself contains already the BRST variations of ϕ^A). For ordinary gauge theories with a closed gauge algebra, the BRST transformations are nilpotent, as we discussed in chapter II, and hence for such theories $S_2 = 0$. But there exist also theories where two BRST transformations of ϕ^I yield another BRST transformation plus a term proportional to the field equations. If this is the case the theory is said to have an open gauge algebra. The best-known example is supergravity [2], but also simpler systems with this property exist. [3]

For gauge theories with open gauge algebras, the general form of the BRST algebra on the classical fields reads

$$\begin{aligned}&\delta_{BRST}(C^\alpha\Lambda_1)\delta_{BRST}(C^\alpha\Lambda_2)\phi \equiv \Lambda_1\Delta^{IJ}\left(\frac{\partial}{\partial\phi^J}S_0\right)\Lambda_2\\ &= ((\phi^I, S_1), S_1)\Lambda_1\Lambda_2 = -\frac{1}{2}((S_1, S_1), \phi^I)\Lambda_1\Lambda_2 = ((S_0, S_2), \phi^I)\Lambda_1\Lambda_2\end{aligned} \tag{11.1.12}$$

The first line defines Δ^{IJ} and the second line relates it to S_2. The Δ^{IJ} are super-antisymmetric because $\delta\phi^I = \Delta^{IJ}\frac{\partial}{\partial\phi^J}S_0$ is again a symmetry of the action. Such symmetries are trivial, and called equation of motion symmetries since for such $\delta\phi^I$ all terms in $\delta S_0 = (\partial S_0/\partial\phi^I)(\delta\phi^I)$ cancel. From (11.1.12) we obtain for S_2 the following result

$$S_2 = \frac{1}{2}\phi^*_J\phi^*_I\Delta^{IJ}. \tag{11.1.13}$$

One sometimes calls S_2 the "nonclosure term" in S.

It may be noted that the condition $(S_1, S_1) + 2(S_0, S_2) = 0$ has further homogeneous solutions for S_2; for example, any term of the form $S_2 = C^*_\alpha C^*_\beta f^{\alpha\beta}(\phi^A)$ is a solution of $(S_0, S_2) = 0$ since S_0 only depends on the classical fields but not on the ghosts C^α. The coefficients of these extra terms in S_2 are fixed at higher levels in antifields. In string theory models with infinitely many antifields are known. Borrowing Hamiltonian language, one may say that an action has rank n if $S_p = 0$ for $p \geq n+1$. So, for example, Yang-Mills theory has rank one, but supergravity has rank two.

Another complication one encounters in ordinary Lagrangian BRST quantization, and which is again nicely dealt with in the BV approach, are the "ghosts-for-ghosts". They are needed when the ghost-action obtained by Faddeev-Popov quantization, is itself gauge-invariant. As an example, consider an antisymmetric tensor field $A_{\mu\nu} =$

$-A_{\nu\mu}$. The Maxwell-like action $\mathcal{L} = -\frac{1}{12}F^2_{\mu\nu\rho}$ (where $F_{\mu\nu\rho} = \partial_\mu A_{\nu\rho} + \partial_\nu A_{\rho\mu} + \partial_\rho A_{\mu\nu}$) has a gauge invariance $\delta A_{\mu\nu} = \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu$, and fixing the gauge by $\partial_\mu A^{\mu\nu} = 0$ naive application of the Faddeev-Popov prescription would yield the ghost action $B_\nu \partial_\mu [\partial^\mu C^\nu - \partial^\nu C^\mu]$ which has clearly the gauge invariance $\delta C^\nu = \partial^\nu \Lambda_1$. Hence, one expects that one should once more apply the Faddeev-Popov trick, and add a new gauge choice $\partial_\nu C^\nu = 0$ and ghosts-for-ghosts $\bar{D}$ and D with action $\bar{D}\partial_\nu \partial^\nu D$. However, also the antighosts have a gauge invariance as is clear after partial integration, namely $\delta B_\nu = \partial_\nu \Lambda_2$, and fixing this gauge invariance by $\partial^\nu B_\nu$, it in turn leads to ghosts-for-ghosts $\bar{E}$ and E, with action $\bar{E}\partial_\nu \partial^\nu E$. [2] In fact, some antighosts-for-ghosts, are equal to some ghosts-for-antighosts, and there is no simple way of justifying this repeated FP quantization from a path-integral approach (although it can and has been done [5]). In this example one needs 3, not 4, ghosts and antighosts [6, 7], and we shall see how the BV formalism produces the correct (because of unitarity) result is a clear unambiguous way. When one has an infinite series of ghosts-for-ghosts-for-ghosts ..., it is in practice advisable to make a double expansion of S into antifields and into ghost generations.

The action we have been discussing so far is only part of the total action. We shall call this part the minimal action: $S^{\min}(\phi^A, \phi^*_A)$. It depends on ϕ^A and ϕ^*_A (with ϕ^A equal to ϕ^I, C^α and ghosts-for-ghosts if present). We must now introduce the antighosts (and extra ghosts if they exist).

Since in Faddeev-Popov quantization one contracts the BRST variation of the gauge function F^α with an antighost B_α, while S_1 contains the BRST variation, one expects that ϕ^*_A will be expressed in terms of B_α. This is done in two steps: first one adds **nonminimal terms** to the action

$$S^{\text{nonmin}} = \pi_\alpha B^{*\alpha} + \cdots \tag{11.1.14}$$

where $B^{*\alpha}$ are the antighost antifields and π_α the BRST auxiliary fields of chapter I (which are also the conjugate momenta of the Lagrange multipliers in the Hamiltonian formalism). Possible extra terms in S^{nonmin} denoted by $\cdots$ are restricted by requiring that

$$S = S^{\text{min}} + S^{\text{nonmin}} \tag{11.1.15}$$

still is nilpotent

$$(S, S) = 0 \tag{11.1.16}$$

The term in (11.1.14) satisfies this requirement because S^{min} contains neither B_α nor $\pi^{*\alpha}$. The second step is to eliminate all antifields by a canonical transformation. Before we can explain this canonical transformation, we need to discuss a remarkable property of the action S.

The action $S = S^{\text{min}} + S^{\text{nonmin}}$ is gauge invariant! Hence we must fix gauges before we can call it the quantum action. To prove the gauge invariance we consider the bracket

$$(S, S) = \partial S/\partial\phi^A \, \frac{\partial}{\partial\phi^*_A} S - \partial S/\partial\phi^*_A \, \frac{\partial}{\partial\phi^A} \, S \tag{11.1.17}$$

and introduce a variable z^a which stands for both ϕ^A and ϕ^*_A. Hence if there are N fields ϕ^A, the index a of z^a runs from 1 to $2N$. We also introduce a $2N \times 2N$ matrix Ω^{ab} in the space (ϕ^A, ϕ^*_A)

$$\Omega^{ab} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \tag{11.1.18}$$

This is all very similar to Hamiltonian dynamics. We can then write $(S, S) = 0$ as

$$(S, S) = \partial S/\partial z^a \; \Omega^{ab} \, \frac{\partial}{\partial z^b} S = 0 \tag{11.1.19}$$

This result can be interpreted as stating that there are equations of motion symmetries $\delta z^a = \Omega^{ab} \frac{\partial}{\partial z^b} S$. We get a more interesting result by differentiating once again w.r.t. z^c

$$\partial S/\partial z^a \; R^a{}_c = 0 \tag{11.1.20}$$

where

$$R^a{}_c = \Omega^{ab} \frac{\partial}{\partial z^b} \, \partial S/\partial z^c \tag{11.1.21}$$

Differentiating once more w.r.t. z^d, one obtains the following equation at those "points" z_0^a in field-antifield space where $\partial S/\partial z^a = 0$

$$\left(\frac{\partial}{\partial z^d}\,\partial S/\partial z^a\right)\,R^a{}_c = 0 \text{ at } z_0 \tag{11.1.22}$$

Contraction with Ω^{ed} yields then

$$R^e{}_a R^a{}_c(z_0) = 0 \text{ when } \partial S/\partial z^a\,(z_0) = 0 \tag{11.1.23}$$

Hence the matrix R is nilpotent when the field equations hold, so that its rank is at most N.[2] We shall first consider the case that the rank of R at z_0 is precisely N. These are the so-called irreducible theories; the reducible theories will be discussed later. We now claim that in that case the action S has N local gauge invariances. Indeed, from

$$\partial S/\partial z^a\;R^a{}_c = 0 \tag{11.1.24}$$

we see that $\delta z^a = R^a{}_c\lambda^c$ are $2N$ gauge invariances with arbitrary local parameters λ^c, but since $R^2 = 0$, only N of them are linearly independent at z_0. (The set $\delta z^a = R^a{}_c\lambda^c$ is linearly dependent at z_0 since it vanishes for $\lambda^c = R^c{}_d\eta^d$ with arbitrary η^d.) We have thus proven that S is gauge invariant. In order to discuss gauge-fixing, we first discuss canonical transformations. We follow here the discussion of canonical transformations, generating functionals etc. in the antifield formalism of ref. [9].

The condition $(S, S) = 0$ is invariant under canonical transformations. Canonical transformations are transformations

$$\begin{aligned}\phi^A &\to \phi'^A(\phi^B, \phi^*_B)\\ \phi^*_A &\to \phi^{*\prime}_A(\phi^B, \phi^*_B)\end{aligned} \tag{11.1.25}$$

such that the ϕ'^A and $\phi^{*\prime}_A$ satisfy the same brackets as ϕ^A, ϕ^*_A. It follows that also

$$S'(\phi^A, \phi^*_A) \equiv S(\phi'^A, \phi^{*\prime}_A) \tag{11.1.26}$$

[2] One particular class of solutions of the field equations for S is obtained by setting all antifields and ghosts equal to zero and taking a solution of the classical field equations. These solutions are given by $\partial S_0/\partial\phi^i = 0$ and play a role in the definition of reducible gauge theories.

satisfies $(S', S') = 0$. It can be shown that canonical transformations are always generated by a generator $\Psi = -\left(\frac{\partial}{\partial\phi^A}\psi\right)\frac{\partial}{\partial\phi^*_A} + \left(\frac{\partial}{\partial\phi^*_A}\psi\right)\frac{\partial}{\partial\phi^A}$ where ψ is a fermionic function of ϕ^A and ϕ^*_A called **the gauge fermion**.

$$\phi'^A = e^{-\Psi}\phi^A e^{\Psi} \equiv \phi^A - \left(\psi, \phi^A\right) + \frac{1}{2}\left(\psi, \left(\psi, \phi^A\right)\right) + \dots\ , \text{ idem for } \phi^*{}_A \tag{11.1.27}$$

Note that $-\Psi\phi^A + \phi^A\Psi$ is equal to $-\Psi$ acting on ϕ^A, which we may denote by $-(\Psi\phi^A)$, and this is equal to the BV bracket $-(\psi, \phi^A)$ as one exactly checks. Moreover, $-(\psi, \phi^A)$ is equal to (ϕ^A, ψ), and this shows that the expression for ψ is correct. We can now state how to eliminate the antifields and obtain the final quantum action. The gauge-fixed quantum action is obtained by projecting S' onto the surface where all transformed antifields vanish

$$S_{qu}(\phi) = S(\phi', \phi^{*\prime})_{|\phi'_*=0} \tag{11.1.28}$$

Since the gauge-fixed action is independent of antifields, it breaks the gauge invariance $\delta z^a = R^a{}_c\lambda^c$. If ψ only depends on fields (which is the usual case in applications) but not on antifields, we have

$$\begin{aligned} \phi'^A &= \phi^A \\ (\phi^*_A)' &= \phi^*_A - \partial\psi/\partial\phi^A \end{aligned} \tag{11.1.29}$$

(Since ψ is fermionic, it does not matter whether we use left or right derivatives.) It follows that the quantum action then is obtained by projecting $S(\phi^A, \phi^*_A)$ on the hypersurface $\sum$ defined by $\phi^*_A = \partial\psi/\partial\phi^A$

$$S_{qu}(\phi^A) = S(\phi^A, \phi^*_A = \partial\psi/\partial\phi^A) \tag{11.1.30}$$

Hence: the classical action is obtained from $S(\phi, \phi^*)$ by putting all $\phi^*_A = 0$, while the quantum action is obtained from $e^{-\psi}Se^{\psi}$ by putting all $\phi^{*\prime}_A = 0$. In other words: **projection onto the surface $\phi^*_A = 0$ before "rotating with ψ" gives the classical action, after "rotating with ψ" the quantum action**.

The restriction that ψ be ϕ^* independent can easily be removed. For general ψ one uses (11.1.27) and (11.1.28). If $\psi = \psi_0 + \psi_1 + \psi_2 + \dots$ contains terms ψ_0 independent of antifields, terms ψ_1 linear in antifields, terms ψ_2 quadratic in antifields etc., then

$$\begin{aligned} \phi'^A &= \phi^A - (\psi_1, \phi^A) - (\psi_2, \phi^A) + \dots + \frac{1}{2}(\psi_1, (\psi_1, \phi^A)) + \dots ; \\ & (\phi^*_A)' = \phi^*_A - (\psi_0, \phi^*_A) + \dots \end{aligned} \tag{11.1.31}$$

For the quantum action this means that one must make the replacements

$$\begin{aligned} \phi^A &\rightarrow \phi^A + \partial\psi_1/\partial\phi^*_{A|\Sigma} + \frac{1}{2}(\partial\psi_1/\partial\phi^*_B)\,(\frac{\partial}{\partial\phi^B}\partial\psi_1/\partial\phi^*_A)_{|\Sigma} + \dots \\ & \phi^*_A \rightarrow \partial\psi_0/\partial\phi^A + \dots \end{aligned} \tag{11.1.32}$$

A choice for ψ which is often used is

$$\psi = B_\alpha F^\alpha + \frac{1}{2}\lambda B_\alpha \pi_\beta \gamma^{\beta\alpha} \quad , \lambda \text{ a constant} \tag{11.1.33}$$

where B_α are antighosts and $F^\alpha = F^\alpha(\phi^A)$ are gauge conditions. If the $\gamma^{\beta\alpha}$ are field-independent constants, then (11.1.29) yields (recall that $\partial\psi/\partial\phi^A = \frac{\partial}{\partial\phi^A}\psi$ because ψ is anticommuting)

$$\begin{aligned} B^{*\alpha} &= F^\alpha + \frac{1}{2}\lambda\pi_\beta\gamma^{\beta\alpha} \\ \pi^{*\alpha} &= \frac{1}{2}\lambda B_\beta \gamma^{\alpha\beta} \\ \phi^*_A &= B_\alpha\, \delta F^\alpha/\delta\phi^A \end{aligned} \tag{11.1.34}$$

In that case the action takes on a familiar form

$$S_{qu} = S_{cl} + B_\alpha \partial F^\alpha/\partial\phi^A\ (\delta_{BRST}\phi^A)/\Lambda + \dots + \pi_\alpha F^\alpha + \frac{1}{2}\lambda\pi_\alpha\pi_\beta\gamma^{\beta\alpha} \tag{11.1.35}$$

The second term is clearly the Faddeev-Popov action corresponding to F^α and the dots indicate terms coming from S_2 and higher S_n (if present). The last two terms come from the nonminimal action in (11.1.14). The unweighted gauges (delta functions in the path integral) correspond to $\lambda = 0$.

We shall now show that the quantum action in (11.1.28) is BRST invariant. We define BRST transformations on all fields ϕ^I, B_α and π_α by

$$\boldsymbol{\delta_{\mathbf{BRST}}\ \mathbf{field}\ = (\mathbf{field}, S)_{|\Sigma}\Lambda} \tag{11.1.36}$$

Hence, if $S = S_0 + S_1 + S_2 + \ldots + S^{\text{nonmin}}$, one has

$$\delta_{BRST}\phi^A = \frac{\partial}{\partial\phi^{*A}}(S_1 + S_2 + S_3 \ldots + S^{\text{nonmin}})_{|\Sigma}\Lambda \tag{11.1.37}$$

In general one finds modifications of the naive BRST rules whenever $S_2, S_3 \ldots$ are nonvanishing. At this point one sees the usefulness of the antifield formalism. From now on, we denote all fields (gauge fields and other classical fields ϕ^I, ghosts C^α, ghosts-for ghosts, extra ghosts, antighosts B_α and auxiliary fields π_α) by ϕ^A.

To prove the BRST invariance of the quantum action, assuming that ψ does not depend on antifields, we must use the chain rule for $S(\phi^A, \phi^*_A = \partial\psi/\partial\phi^A)$. We get from the chain rule

$$\delta_{BRST}S_{qu} = \left(\partial S/\partial\phi^A\ \frac{\partial}{\partial\phi^*_A}S\right)_{|\Sigma}\Lambda + \partial S/\partial\phi^*_{A|\Sigma}\ \delta_{BRST}\ \partial\psi/\partial\phi^A \tag{11.1.38}$$

We claim that the second term vanishes! To see this we first evaluate it before projecting onto Σ. Using that ψ does not depend on antifields we obtain

$$(\partial S/\partial\phi^*_A)\ \left[\partial(\partial\psi/\partial\phi^A)/\partial\phi^B\right]\left(\frac{\partial}{\partial\phi^*_B}S\right) \tag{11.1.39}$$

Since ψ is anticommuting, $\partial\psi/\partial\phi^A$ equals $\frac{\partial}{\partial\phi^A}\psi$, so we can write the term under consideration as

$$(\partial S/\partial\phi^*_A)\ \left[\partial\left(\frac{\partial}{\partial\phi^A}\psi\right)/\partial\phi^B\right]\ \left(\frac{\partial}{\partial\phi^*_B}S\right) \tag{11.1.40}$$

It is easy to understand why this term vanishes: the symmetry under interchange of ϕ^*_A and ϕ^*_B is opposite to the symmetry under interchange of ϕ^A and ϕ^B. Hence, interchanging both ϕ^A with ϕ^B, and ϕ^*_A with ϕ^*_B, the term equals minus itself, and hence it vanishes.[3] Projecting onto $\sum$ keeps this term vanishing, of course. If ψ does depend on antifields, there is a second term in (11.1.39), but there are also further terms in $\phi'^A = \phi^A + \ldots$.

The first term in (11.1.38) is equal to $\frac{1}{2}(S,S)_{|\Sigma}\Lambda$. Hence, the quantum action is BRST invariant if and only if $(S,S)_{|\Sigma} = 0$. Since we wish this to be true for all possible gauge choices ψ, it seems likely that this implies $(S,S) = 0$. This is, of course, the basic equation of the antifield formalism, see (11.1.16).

2 BRST transformations and nilpotency

The BRST transformations rules of the classical fields and ghosts are given by the minimal action

$$\delta_{BRST}\phi^A = \left(\frac{\partial}{\partial\phi^*_A} S^{\min}(\phi,\phi^*)\right)_{|\Sigma} \Lambda \tag{11.2.1}$$

if S^{nonmin} does not depend on the corresponding antifields. If $S^{\min} = S_0 + S_1$, these are by definition the rules of chapter I, but if there are terms S_2 and higher, these rules are modified. The antifields B_α transform into π_α according to the definition(11.1.36) and (11.1.14) as

$$\delta_{BRST}B_\alpha = (-)^\alpha \pi_\alpha \Lambda = \Lambda\ \pi_\alpha \tag{11.2.2}$$

since $S^{\text{nonmin}} = \pi_\alpha B^{*\alpha}$. As always, the fields π_α are invariant; in the BV formalism this comes about because no antifield $\pi^{*\alpha}$ is introduced in the action, so that $\frac{\partial}{\partial\pi^{*\alpha}}S = 0$.

The reason one makes these definitions is, of course, that the quantum action is invariant. This we proved in the previous section. We next study the algebra, and

[3] At the risk of being too detailed, we show this explicitly: moving $\frac{\partial}{\partial\phi_B{}^*}S$ to the far left gives a factor $(-)^{(B+1)B} = 1$, and conversion to $\partial S/\partial\phi_B{}^*$ yields a factor $(-)^{B+1}$. Moving $\partial S/\partial\phi_A{}^*$ to the right yields a factor $(-)^{(A+1)(A+B+1)}$ and conversion to $\frac{\partial}{\partial\phi_A{}^*}S$ yields $(-)^{A+1}$. Altogether we find a sign factor $-(-)^{AB}$ under interchange of $\phi_A{}^*$ and $\phi_B{}^*$. For interchange of ϕ_A and ϕ_B we only get a sign $(-)^{AB}$ because ψ is fermionic. This proves the assertion.

consider two consecutive BRST variations

$$\begin{aligned} \delta_2\delta_1\phi^A &= \delta_2\left(\left(\frac{\partial}{\partial\phi^*_A}S\right)_{|\Sigma}\Lambda_1\right) \\ &= \delta_2\left(\left(\phi^A, S\right)_{|\Sigma}\Lambda_1\right) \end{aligned} \tag{11.2.3}$$

To evaluate $\delta_2(\phi^A, S)_{|\Sigma}$ we derive a lemma.

Lemma: Any function $F(\phi,\phi^*)_{|\Sigma}$ transforms under BRST transformations as follows

$$\delta_{BRST}(F(\phi,\phi^*)_{|\Sigma}) = (F,S)_{|\Sigma}\Lambda + \partial F/\partial\phi^*_{A|\Sigma}\left(\frac{d}{d\phi^A}S_{qu}\Lambda\right) \tag{11.2.4}$$

Hence, F goes into the bracket (F,S) plus a term which is proportional to the field equation of the full quantum action.

The proof of this lemma follows from the chain rule

$$\begin{aligned} &\delta_{BRST}(F(\phi,\phi^*)_{|\Sigma}) = (\partial F/\partial\phi^A)_{|\Sigma}\left(\frac{\partial}{\partial\phi^*_A}S\right)|_\Sigma\,\Lambda + \\ &(\partial F/\partial\phi^*_A)_{|\Sigma}[\partial(\partial\psi/\partial\phi^A)/\partial\phi^B]\left(\frac{\partial}{\partial\phi^*_B}S\right)_{|\Sigma}\Lambda \end{aligned} \tag{11.2.5}$$

Completing the first term to the full antibracket, we can extract a factor $\partial F/\partial\phi^*_A$ from the remainder

$$\begin{aligned} &(F,S)_{|\Sigma}\Lambda + (\,\partial F/\partial\phi^*_A)\,|_\Sigma\left[\frac{\partial}{\partial\phi^A}S + \right. \\ &\left.((\partial(\partial\psi/\partial\phi^A)/\partial\phi^B)\left(\frac{\partial}{\partial\phi^*_B}S\right)\right]_\Sigma\Lambda \end{aligned} \tag{11.2.6}$$

Because $\partial(\partial\psi/\partial\phi^A)/\partial\phi^B$ is equal to $\frac{\partial}{\partial\phi^A}(\partial\psi/\partial\phi^B)$ the terms within square brackets are just the full ϕ^A quantum field equation.

We return to the BRST commutator and find, applying the lemma to $F = (\phi^A, S)$, the following result

$$\begin{aligned} &\delta_2\delta_1\phi^A = ((\phi^A,S),S)_{|\Sigma}\Lambda_2\Lambda_1 \\ &+\left(\frac{\partial^2}{\partial\phi^*_A}S/\partial\phi^*_B\right)_{|\Sigma}\left(\frac{d}{d\phi^B}S_{qu}\right)\Lambda_2\Lambda_1 \end{aligned} \tag{11.2.7}$$

From the Jacobi identities we have

$$((\phi^A, S), S) = \frac{1}{2}((S, S), \phi^A) = 0 \tag{11.2.8}$$

so that

$$\delta_2\delta_1\phi^A = \left(\frac{\partial^2}{\partial\phi^*{}_A}\ S/\partial\phi^*_B\right)_{|\Sigma} \left(\frac{d}{d\phi^B}\ S_{qu}\right)\Lambda_2\Lambda_1 \tag{11.2.9}$$

Clearly, if S_n with $n \geq 2$ are vanishing, the BRST algebra is "closed" (nilpotent). Conversely if $S_2 \neq 0$ (or $S_n \neq 0$ with $n > 2$) then the BRST algebra may be "open" (it contains field equations of the **quantum** action). Usually closed classical gauge algebras lead to closed BRST algebras, but when the ghosts themselves become gauge fields, there appear ghosts-for-ghosts and this leads in general to nonclosure of the BRST algebra.

In the usual BRST formalism, the quantum action contains sources K_A for the fields ϕ^A. These sources were used to prove renormalizability of Yang-Mills theory by BRST techniques. Yang-Mills theory has a closed gauge algebra, but there are other interesting field theories without closed gauge algebra (and where one can prove that no simple set of auxiliary fields exist which close the gauge algebra). An example is $N = 4$ rigidly supersymmetric Yang-Mills theory. In this case one has only a rigid symmetry, and consequently, the ghosts are constants ("zero-modes"), but the BRST formalism applies equally well. In order to prove the renormalizability of such theories, one may use the BV formalism with sources K_A which are constants.

In Yang-Mills theory, the action contains the ghost-action

$$B_\alpha F^\alpha_{,I} R^I_\alpha C^\alpha \tag{11.2.10}$$

as well as the source terms $K_I R^I_\alpha$ and $L_\alpha C^\alpha$. This suggests to consider an extended quantum action

$$\hat{S}_{qu} = S(\phi^A, \phi^*_A = \partial\psi/\partial\phi^A + K_A) \equiv \hat{S}_{qu}(\phi, K) \tag{11.2.11}$$

We consider only gauge fermions ψ which are independent of antifields. Note that we now introduce external sources K_A for all fields, not only for the gauge fields and ghosts as in chapter II. As before we denote the set of all fields by ϕ^A. We shall now study this action in more detail.

We introduce now a hypersurface Σ' defined by

$$\phi^*_A = \partial\psi/\partial\phi^A + K_A \tag{11.2.12}$$

and define BRST transformations of all fields and sources by

$$\delta\text{ (object) } = (\text{object}, S) \mid_{\Sigma'} \Lambda \tag{11.2.13}$$

For the fields this yields

$$\begin{aligned} \delta\phi^A &= (\phi^A, S) \mid_{\Sigma'} \Lambda = \left\{ \frac{\partial}{\partial\phi^*_A} S(\phi, \phi^*) \mid_{\phi^*_B = \partial\psi/\partial\phi^B + K_B} \right\} \Lambda \\ &= \frac{\partial}{\partial K_A} \hat{S}(\phi, K)\Lambda \end{aligned} \tag{11.2.14}$$

Hence, for gauge theories with open gauge algebras, $\delta\phi^A$ contains extra terms with K_B.We shall now show that the extended quantum action $\hat{S}_{qu}(\phi, K)$ is BRST invariant. The proof with K sources is very similar to the proof without K sources. We define the K_A to be $BRST$ inert, which was also our approach in chapter II. Then one obtains

$$\begin{aligned} &\delta_{BRST}\hat{S}_{qu}(\phi, K) = \partial\hat{S}_{qu}(\phi, K)/\partial\phi^A\ \delta_{BRST}\phi^A \\ &= \partial S(\phi^B, \partial\psi/\partial\phi^B + K_B)/\partial\phi^A\ \delta_{BRST}\phi^A \\ &= (\partial S(\phi, \phi^*)/\partial\phi^A) \left(\frac{\partial}{\partial\phi^*_A} S \right) \mid_{\Sigma'} \Lambda + \\ &(\partial S(\phi, \phi^*)/\partial\phi^*_B) \mid_{\Sigma'} (\partial\partial\psi/\partial\phi^B/\partial\phi^A)\delta_{BRST}\phi^A \end{aligned} \tag{11.2.15}$$

The last term vanishes again due to symmetry. (As before, this term even vanishes before projecting unto $\sum'$). Thus

$$\delta_{BSRT}\hat{S}_{qu}(\phi, K) = \frac{1}{2}(S, S) \mid_{\Sigma'} \Lambda \tag{11.2.16}$$

which certainly vanishes as long as $(S, S) = 0$. Hence, the quantum action with K-sources is $BRST$ invariant under rules which keep K inert.

The action $\hat{S}_{qu}(\phi, K)$ has a dual BRST symmetry, denoted by δ'_{BRST}, under which ϕ is kept fixed but now K is varied. Varying K_A we obtain

$$\begin{aligned}&\delta'_{BRST}\hat{S}_{qu}(\phi, K) = \left\{\partial\hat{S}_{qu}(\phi, K)/\partial K_A\right\}\delta_{BRST}K_A\\ &= (\partial S/\partial\phi^*_A)\mid_{\Sigma'}\delta_{BRST}K_A\end{aligned} \tag{11.2.17}$$

To find out how K_A should transform, we record once more the variation of $\hat{S}_{qu}(\phi, K)$ if one varies only ϕ

$$\delta_{BRST}\hat{S}_{qu}(\phi, K) = (\partial S/\partial\phi^A)\mid_{\Sigma'}\left(\frac{\partial}{\partial\phi^*_A}S\right)\mid_{\Sigma'}\Lambda \tag{11.2.18}$$

Clearly, up to an arbitrary constant

$$\delta_{BRST}K_A = -\left(\frac{\partial}{\partial\phi^A}S\right)\mid_{\Sigma'}\Lambda \tag{11.2.19}$$

We can, in fact, forget altogether about antifields and define an antibracket for the variables ϕ^A, K_A

$$\begin{aligned}&(\phi^A, K_B) = \delta^A{}_B\ ;\ (\phi^A, \phi^B) = (K_A, K_B) = 0\\ &(F, G) \equiv \partial F/\partial\phi^A\frac{\partial}{\partial K_A}G - \partial F/\partial K_A\frac{\partial}{\partial\phi^A}G\end{aligned} \tag{11.2.20}$$

Then $\hat{S}(\phi, K)$ is invariant under

$$\delta\phi^A = (\phi^A, \hat{S})\Lambda = \left(\frac{\partial}{\partial K^A}\hat{S}\right)\Lambda \tag{11.2.21}$$

but also under

$$\delta K_A = (K_A, \hat{S})\Lambda = -\left(\frac{\partial}{\partial\phi^A}\hat{S}\right)\Lambda \tag{11.2.22}$$

(Now the constant in δK_A is fixed). We can thus define an extended BRST transformation Δ which acts both on ϕ^A (as $(\phi^A, \hat{S})\Lambda$) and on K_A (as $(K_A, \hat{S})\Lambda$) and which leaves $\hat{S}(\phi, K)$ invariant

$$\Delta\hat{S}(\phi, K) = (\hat{S}, \hat{S}) = (S, S)|_{\Sigma'} = 0 \tag{11.2.23}$$

This extended BRST transformation Δ is always nilpotent. To prove this we evaluate $\Delta_2\Delta_1\phi^A$ and $\Delta_2\Delta_1 K_A$. We begin with the former

$$\begin{aligned}
&\Delta_2(\Delta_1\phi^A) = \Delta_2\left[\frac{\partial}{\partial K_A}\hat{S}(\phi, K)\right]\Lambda_1 = \\
&\Delta_2(\phi^A, \hat{S})\Lambda_1 = ((\phi^A, \hat{S}\Lambda_2), \hat{S}\Lambda_1) = \\
&\frac{1}{2}(\phi^A, (\hat{S}, \hat{S}\Lambda_2))\Lambda_1 = 0
\end{aligned} \tag{11.2.24}$$

since $(\hat{S}, \hat{S}) = (S, S)_{|\Sigma'}$ and $\Delta_2\hat{S} = 0$ according to (11.2.23). The result for K_A is the same. The situation is similar to the case of theories with an open gauge algebra, where one can add BRST auxiliary fields and define nilpotent BRST transformations. Here the field-equation terms present in the BRST algebra for theories with an open algebra, are eliminated by introducing K-sources and defining an extended BRST transformation.

The invariance of $\hat{S}(\phi, K)$ under $\delta_{BRST}\phi^A$

$$\left(\partial\hat{S}/\partial\phi^A\right)\left(\frac{\partial}{\partial K_A}\hat{S}\right) = 0 \tag{11.2.25}$$

can be used to derive a Ward identity for the path integral

$$\exp[\frac{i}{\hbar}W(J, K)] = \int D\phi^A \exp\frac{i}{\hbar}[\hat{S} + J_A\phi^A] \tag{11.2.26}$$

namely

$$J_A\frac{\partial}{\partial K_A}W(J, K) = 0 \tag{11.2.27}$$

We assume here that the Jacobian $\frac{\delta}{\delta K_A}\hat{S}\frac{\overleftarrow{\partial}}{\partial\phi^A}$ vanishes; it is naively $\partial(\delta_{\mathrm{BRST}}\phi^A)/\partial\phi^A$, but it should, of course, be regularized. This is the same Ward identity as in chapter II, but now it also holds for theories with an open gauge algebra. Making the Legendre transformation

$$\Gamma(\phi, K) = W(J, K) - J_A\phi^A \tag{11.2.28}$$

we obtain the Ward identity for the effective action

$$\partial\Gamma/\partial\phi^A\ \frac{\partial}{\partial K_A}\Gamma = \frac{1}{2}(\Gamma, \Gamma) = 0 \tag{11.2.29}$$

If there are anomalies, it is modified into

$$(\Gamma, \Gamma) = \Delta.\Gamma \tag{11.2.30}$$

where Δ is a local operator. This Ward identity can be used to start the proof of renormalizability of theories with open algebras, similar to the analysis we carried out in chapter II.

We can now give a more abstract but also more powerful characterization of BRST symmetry. Let the action $S = S^{\rm min} + S^{\rm nonmin}$ satisfy $(S, S) = 0$. The quantum action is then obtained by a similarity transformation, namely write $\phi^A = e^\psi \phi^A_{\rm new} e^{-\psi}$ and $\phi^*_A = e^\psi \phi^*_{A,\rm new} e^{-\psi}$ and similarly for the nonminimal fields π_α, B_α and $B^{*\alpha}$, but do not afterwards set the new antifields equal to zero! Rather, interpreting $\phi^*_{A,\rm new}$ as the BRST sources K_A, we obtain the quantum action with gauge fixing and with BRST sources. This is the "extended quantum action", used for renormalization and it may contain terms with more than one K_A. It depends on $\phi^I_{\rm new}, C^\alpha_{\rm new}, K_A, \pi^{\rm new}_{\alpha,} B^{\rm new}_\alpha$ and K^α. (K^α is the source for the variations of the antighost $B^{\rm new}_\alpha$). The BRST transformation rules for all these fields and sources in the final gauge-fixed quantum action follow from

$$\delta\left(e^\psi \phi^*_{I,\rm new} e^{-\psi}\right) = \left(\frac{\partial}{\partial \phi^A} S\right)\left(\phi^* = e^\psi \phi^*_{\rm new} e^{-\psi}\right) \tag{11.2.31}$$

and idem for $C^*_\alpha, \phi^I, C^\alpha, K_A, \pi_\alpha, B_\alpha, K^\alpha, B^{*\alpha}$ by expanding left- and right-hand side.

For example for Yang-Mills theory with $\psi = \int b_\alpha F^\alpha$ and $F^\alpha = \partial^\mu A_\mu{}^a$ we get

$$\begin{aligned}\delta A^{*\mu}{}_a = \delta\left(K^\mu_a - \partial^\mu b_a\right) &= \frac{\partial}{\partial A_\mu{}^a} S = \frac{\partial}{\partial A_\mu{}^a} S^{(0)} + A^{*\mu}_b f^b{}_{ac} c^c \\ &= \frac{\partial}{\partial A_\mu{}^a} S_{cl} + (K_b{}^\mu - \partial^\mu b_b) f^b{}_{ac} c^c \end{aligned} \tag{11.2.32}$$

This can be decomposed into

$$\delta K_a{}^\mu = \frac{\partial}{\partial A_\mu{}^a} S_{qu} + \partial^\mu \pi_a, \delta b_a = \left(\frac{\partial}{\partial b^{*a}} S\right) = \pi_a. \tag{11.2.33}$$

which is indeed the correct result. (The term $\delta K_a{}^\mu = \partial^\mu \pi_a$ is needed to cancel $\delta\partial^\mu b_a$.) It is even simpler to use the canonical invariance of the BV bracket under similarity transformations, and to obtain

$$\begin{aligned}
\delta\phi^A_{\text{new}} &= \frac{\partial}{\partial K_A} S_{qu}, \delta K_A = -S_{qu}/\partial\phi^A_{\text{new}}, \phi^A = \{\phi^i, c^\alpha\} \\
\delta B^{\text{new}}_\alpha &= \frac{\partial}{\partial K^\alpha} S_{qu}, \delta\pi_\alpha = 0, \delta K^\alpha = -\partial S_{qu}/\partial B_{\alpha,\text{new}}
\end{aligned} \tag{11.2.34}$$

(Now for example the term $\delta K_a{}^\mu = \partial^\mu \pi_a$ in Yang-Mills theory comes from differentiating the gauge fixing term $\pi_a \partial \cdot A^a$ with respect to $A_\mu{}^a$.)

The extended BRST transformations are nothing else but the action of the BRST charge Q of chapter II on the various fields and sources. One might ask why we have gone through all the trouble of introducing antifields, if in the end we regain what we had before. The answer is that the results of chapter II deal only with the simplest gauge theory, Yang-Mills theory. For more complicated systems, the antifields formalism gives a systematic derivation of the action and BRST transformation laws (although other methods can also be used).

The set of "extended BRST transformations" is nilpotent, as we have seen. It can be decomposed into sectors with different antighost numbers which are each nilpotent and which commute with each other:

$$\delta_{\text{BRST}} = \beta + \gamma, \beta^2 = \gamma^2 = \beta\gamma + \gamma\beta = 0 \tag{11.2.35}$$

The set β has antighost number -1, while γ has antighost number 0, where the antighost number of K_A is as expected ($K_a{}^\mu$ has $+1, L_a$ has $+2$) and the classical fields have antighost number zero, but also the ghosts have antighost number zero and the antighosts have antighost number 1. For example, for Yang-Mills theory

$$\begin{aligned}
\beta A_\mu{}^a &= \beta C^a = \beta\varphi^i = 0, \beta K_a{}^\mu = \frac{\delta S_{cl}}{\delta A_\mu{}^a}, \beta L_a = D_\mu K_a{}^\mu - K_{\text{matter}} - \text{term} \\
\gamma A_\mu{}^a &= D_\mu c^a, \gamma c^a = \frac{1}{2} f^a{}_{bc} c^c \Lambda c^b, \gamma K_a{}^\mu =
\end{aligned} \tag{11.2.36}$$

This decomposition is used in the study of BRST cohomology.

3 Examples of irreducible theories

3.1 Pure Yang-Mills theory

As a first example of the BV formalism, we consider Yang-Mills theory. The classical action is given by

$$S_0 = S_{cl}(\phi^I) = \int -\frac{1}{4}(G_{\mu\nu}{}^a)^2 d^4x \tag{11.3.37}$$

The terms linear in antifields contain the BRST variations of ϕ^A, where ϕ^A denote the classical fields ϕ^I (namely $A_\mu{}^a$) and the ghosts C^α (namely C^a). One has

$$\begin{aligned} S_1 &= \phi^*_A(\delta_{BRST}\phi^A)/\Lambda = \int (A^{*\mu}{}_a D_\mu C^a \\ &+ C^*_a \frac{1}{2} f^a{}_{bc} C^c C^b) d^4x \end{aligned} \tag{11.3.38}$$

The action $S^{\min} = S_0 + S_1$ satisfies $(S_0 + S_1, S_0 + S_1) = 0$, and hence this is the complete minimal action; there are no $S_2, S_3 \ldots$ terms. The nonminimal action is given by $\pi_\alpha B^{*\alpha}$ in general, where $B^{*\alpha}$ are the antifield-antighosts, one for each local gauge invariance. In our case one thus gets

$$S^{\text{nonmin}} = \int (\pi_a B^{*a}) d^4x \tag{11.3.39}$$

Finally we need to choose the gauge fermion ψ. We take

$$\psi = \int (B_a F^a + \beta B_a \pi_b \delta^{ba}) d^4x \tag{11.3.40}$$

where β is a constant. From the general rule

$$\phi^*_A = \partial\psi/\partial\phi^A \tag{11.3.41}$$

we find

$$\begin{aligned} &A^{*\mu}{}_a(x) = \partial \int B_b(y) \partial F^b(y) d^4y / \partial A^a_\mu(x), C^*_a = \partial\psi/\partial C^a = 0 \\ &B^{*a} = \partial\psi/\partial B_a = F^a + \beta\pi_b\delta^{ba}. \end{aligned} \tag{11.3.42}$$

Thus the quantum action becomes

$$S_{qu} = S_{cl} + \int \left[B_a \left(\partial F^a / \partial A^b{}_\mu \right) D_\mu C^b + \pi_a F^a + \beta \pi_a \pi_b \delta^{ba} \right] d^4x \tag{11.3.43}$$

This is the standard quantum action; in particular we recognize the Faddeev-Popov ghost action, and the gauge fixing terms. Instead of δ^{ba} we could have used any metric. If it would have been field-dependent, it would have contributed extra terms to $\phi^*_A = \partial\psi/\partial\phi^A$, and thus extra terms to the action. As gauge choice F^a one may use anything one likes, for example $F^a = \partial^\mu A_\mu{}^a$, but also nonlinear gauges like

$$F^a = \partial^\mu A_\mu{}^a + \gamma A_\mu{}^a A^\mu{}_a \tag{11.3.44}$$

One could even allow ghost fields in F^a as in section 4 of chapter I. They would then lead to a nonvanishing result for $C^*_a = \partial\psi/\partial C^a$.

3.2 The point particle

As a second example, we consider the bosonic point particle with action

$$S = \int \left(p_\mu \dot{x}^\mu - \frac{1}{2} e p_\mu p_\nu \eta^{\mu\nu} \right) dt \ , \ \mu = 0, d-1. \tag{11.3.45}$$

This is the Hamiltonian form of the action $\int \frac{1}{2} e^{-1} \dot{x}^2 dt$. (In one dimension the covariant expression $\frac{1}{2}\sqrt{\det(-g)} g^{\alpha\beta} \partial_\alpha x \partial_\beta x$ reduces to $\frac{1}{2}[\det(-g)]^{-1/2} \dot{x}\dot{x}$ and $\det(-g) = e^2$). It has a local gauge invariance, namely general coordinate transformations in one dimension. One easily checks that they are given by

$$\begin{aligned} \delta x^\mu &= \dot{x}^\mu \xi, \ \delta p_\mu = \xi \dot{p}_\mu \ , \\ \delta e &= \frac{d}{dt}(e\xi) \end{aligned} \tag{11.3.46}$$

(One can view p_μ as a contravariant vector density $p^\alpha{}_\mu$ where α takes on only one value, and $\dot{x}^\mu = \partial x^\mu / \partial y^\alpha$ with $y^\alpha = \tau$. Then the term $p\dot{x}$ is a scalar density, and thus invariant. The transformation rule $\delta p^\alpha = \xi^\beta \partial_\beta p^\alpha - (\partial_\beta \xi^\alpha) p^\beta + (\partial_\beta \xi^\beta) p^\beta$ for a vector density reduces then to $\delta p = \xi \dot{p}$. And if the p^2 term is written as $p^\alpha p^\beta g_{\alpha\beta}/\sqrt{-g}$ we identify e with the usual density $\sqrt{-g}$, in agreement with $\delta e = \frac{d}{dt}(\xi e)$.

In point-particle circles, one usually simplifies these local symmetry laws by adding a separate equation of motion symmetry for x and p which removes time derivatives in the transformation rules (as should be the case for a Hamiltonian system)

$$\begin{aligned} \delta \text{ (new) } x^\mu &= \delta \text{ (old) } x^\mu - (\delta S_{cl}/\delta p_\mu)\xi = p^\mu e \xi \\ \delta \text{ (new) } p_\mu &= \delta \text{ (old) } p_\mu + (\delta S_{cl}/\delta x^\mu)\xi = 0 \end{aligned} \tag{11.3.47}$$

As a last simplification, one redefines $e\xi = \eta$, which is allowed as ξ is arbitrary anyhow. Then (11.3.45) is invariant under

$$\delta p^\mu = 0,\ \delta x^\mu = p^\mu \eta,\ \delta e = \dot{\eta} \tag{11.3.48}$$

We now construct the corresponding quantum action. The terms linear in antifields become

$$S_1 = \int (x^*_\mu p^\mu c + e^* \dot{c}) dt \tag{11.3.49}$$

where c is the ghost corresponding to the local gauge parameter η. There is no term with c^* since general coordinate transformations in one dimension commute, so that the structure constants of the local gauge algebra vanish: $f^\alpha{}_{\beta\gamma} = 0$. One may verify this also directly: $[\delta(\eta_1), \delta(\eta_2)]p_\mu = 0$, idem for x^μ and e. Since there are no pairs "field plus corresponding antifield" in (S_1), one obtains $(S_1, S_1) = 0$, and again S^{min} $= S_0 + S_1$. As nonminimal term we take

$$S^{\text{nonmin}} = \int (\pi b^*) dt \tag{11.3.50}$$

and with the gauge fermion

$$\psi = \int b(e-1) dt \tag{11.3.51}$$

we get

$$b^* = (e-1),\ x^*_\mu = 0, e^* = b \tag{11.3.52}$$

hence

$$S_{qu} = \int \left(p\dot{x} - \frac{1}{2}ep^2 + b\dot{c} + \pi(e-1) \right) dt \tag{11.3.53}$$

Let us check in this example that the Hessian $R_{ab} = \partial/\partial z^a \partial S^{\min}/\partial z^b$ with $z^a = \{\phi^A, \phi^*{}_A\}$ has maximal rank. This means that this system is irreducible. There are now $N = 2d+2$ fields ϕ^A, namely x^μ, p_μ, e and c. Hence the $(4d+4)\times(4d+4)$ matrix R should have rank $2d+2$.

We find for R

$$R = \begin{array}{ccccccccc} & x & p & e & c^* & x^* & p^* & e^* & c \\ x & 0 & -\frac{d}{dt} & 0 & 0 & 0 & 0 & 0 & 0 \\ p & \frac{d}{dt} & -g & -p & 0 & -c & 0 & 0 & x^* \\ e & 0 & -p & 0 & 0 & 0 & 0 & 0 & 0 \\ c^* & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ x^* & 0 & c & 0 & 0 & 0 & 0 & 0 & p \\ p^* & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ e^* & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{d}{dt} \\ c & 0 & -x^* & 0 & 0 & -p & 0 & \frac{d}{dt} & 0 \end{array} \tag{11.3.54}$$

We have put the anticommuting fields and the commuting fields together, such that R is a supermatrix. Clearly R is super-symmetric. The rank of N is the rank N_+ of the bose-bose-part R_{bb}, plus the rank N_- of the fermi-fermi part R_{ff}. (The superdeterminant is nonvanishing if both $\det R_{bb}$ and $\det R_{ff}$ are nonvanishing). We can put several rows and columns of R equal to zero, by taking suitable linear combinations. Then it is clear that in R_{bb} only $\begin{pmatrix} 0 & -d/dt \\ d/dt & 0 \end{pmatrix}$ times a $d \times d$ unit matrix remains, hence $N_+ = 2d$ (if $\mu = 0, \ldots d-1$). Similarly, in R_{ff} only $\begin{pmatrix} 0 & d/dt \\ d/dt & 0 \end{pmatrix}$ remains, hence $N_- = 2$. Therefore, the rank of R is indeed $2d+2$.

We may also check that $R_{ab}\Omega^{bc}R_{cd} = 0$, see (11.1.23), where $\Omega^{ab} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, up to field equations $\partial S^{\min}/\partial z^a$. One finds, for example, $\frac{d}{dt}c - c\frac{d}{dt}$ in the $(2,2)$ entry, which is equal to $\dot{c}$, which is indeed proportional to the e^* field equation.

4 Reducible gauge theories and ghosts for ghosts

When the Faddeev-Popov (FP) ghost-action itself is gauge-invariant, one must add further gauge-fixing terms for it, which in turn leads to ghosts-for ghosts. [4] If these ghosts-for-ghosts are gauge fields, one must add gauge fixing terms, etc. In principle this can be done by path-integral methods, [5] but it is easier to construct instead a BRST invariant quantum action, [6, 7] which contains a (FP) ghost-action and a (FP) ghosts-for-ghosts action. The further terms in the quantum action, as well as the BRST laws follow then from the requirement of BRST symmetry of the action.

It frequently (but not always as we shall show in an example) happens that the antighosts in the ghost action have also a gauge symmetry. Then one might expect that one would also need a FP action for the ghosts and antighosts which follow from the antighost gauge invariance. This leads to further ghosts in the system which are called “extra ghosts”. Historically the need for antighosts became clear when the real antisymmetric tensor field $A_{\mu\nu}$ with Maxwell-like action $(\partial_{[\mu}A_{\nu\rho]})^2$ was quantized. (The model was coupled to external gravity but that is not essential for the arguments, so we set the external gravitational fields to zero). The gauge invariance $\delta A_{\mu\nu} = \partial_\mu\Lambda_\nu - \partial_\nu\Lambda_\mu$ was fixed by the Lorentz-type gauge fixing term $(\partial^\mu A_{\mu\nu})^2 = 0$ and this led to the usual FP ghost action $\mathcal{L}(FP \text{ ghost}) = b^\nu\partial^\mu(\partial_\mu C_\nu - \partial_\nu C_\mu)$. Clearly this action has both a ghost gauge invariance $\delta C_\mu = \partial_\mu\Lambda_1$ and an antighost gauge invariance $\delta b^\nu = \partial^\nu\Lambda_2$, and naively (and incorrectly) one would expect that one should add two second-generation FP actions

$$\mathcal{L}\text{ (second ghosts) } = \tilde{b}\Box\tilde{c} + b\Box c \tag{11.4.55}$$

where b and c are the antighost and ghost for the ghost gauge invariance $\delta C_\nu = \partial_\nu\Lambda_1$, and $\tilde{b}$ and $\tilde{c}$ the corresponding fields for $\delta b^\nu = \partial^\nu\Lambda_2$. This led to a puzzle: by a duality transformation $A_{\mu\nu}$ is equivalent to a scalar field φ, but the trace anomalies for the system with $A_{\mu\nu}$ and ghosts turned out to be different from the trace anomaly for a scalar field. [11] The resolution was that there were extra ghosts. Due to these extra ghosts the trace anomalies became equal. [10] We shall discuss the quantization of antisymmetric tensor gauge fields in the examples in the next section.

From a formal point of view, gauge theories are defined to be reducible if the transformation laws of the classical fields, the ghosts, and the ghosts-for-ghosts etc. contain zero modes **when the classical equations of motion hold**. Let the classical transformation rules be given by

$$\delta\phi^i = R^i{}_\alpha \xi^\alpha \tag{11.4.56}$$

where ξ^α are the local gauge parameters. If there exist vectors $Z_1^\alpha{}_{\alpha_1}$ (labeled by α_1) such that

$$R^i{}_\alpha Z_1^\alpha{}_{\alpha_1} = 0 \text{ when } \frac{\partial}{\partial\phi^i} S_0 = 0 \tag{11.4.57}$$

one has by definition a first-rank reducible gauge theory. If $Z^\alpha{}_{\alpha_1}$ has itself further zero modes Z_2 (labeled by α_2)

$$Z_1^\alpha{}_{\alpha^1} Z_2^{\alpha_1}{}_{\alpha_2} = 0 \text{ when } \frac{\partial}{\partial\phi^i} S_0 = 0 \tag{11.4.58}$$

then one has a second rank reducible gauge theory, etc.

In the example of the antisymmetric tensor field $A_{\mu\nu}$ one has

$$\begin{aligned} \delta A_{\mu\nu} &= -\partial_\mu \Lambda_\nu + \partial_\nu \Lambda_\mu = (\delta^\rho_\mu \partial_\nu - \delta^\rho_\nu \partial_\mu)\Lambda_\rho \\ R^i{}_\alpha &= \delta^\rho_\mu \partial_\nu - \delta^\rho_\nu \partial_\mu, Z_1^\alpha{}_{\alpha_1} = \partial_\rho \end{aligned} \tag{11.4.59}$$

Since (11.4.57) is satisfied (we do not even need the classical field equation in this case) we have a first rank reducible system.

Another example is the symmetric traceless spin 5/2 fermion field $\psi_{\mu\nu} = \psi_{\nu\mu}$ with gauge invariance

$$\delta\psi_{\mu\nu} = \gamma_\mu \epsilon_\nu + \gamma_\nu \epsilon_\mu - \frac{1}{2}\eta_{\mu\nu}\gamma\cdot\epsilon \tag{11.4.60}$$

For this system $R^i{}_\alpha = R_{\mu\nu}{}^\rho = \gamma_\mu \delta^\rho_\nu + \gamma_\nu \delta^\rho_\mu - \frac{1}{2}\eta_{\mu\nu}\gamma^\rho$ and there are again zero modes $Z_{1,\rho} = \gamma_\rho$. Since $Z_{1,\rho} = \gamma_\rho$ itself has no further zero modes, this is also a first-rank reducible system.

As an example of an infinity reducible system consider the local κ symmetry of the superstring. The classical action has the symmetry transformation $\delta\theta = \not{p}\kappa$ where θ are fermionic coordinates and κ local anticommuting gauge parameters. In this case $R = \not{p}$, and there are zero modes $Z_1 = \not{p}$. Indeed $RZ = \not{p}\not{p} = 0$ if the classical field equation $p^2 = 0$ is satisfied. Furthermore $Z_1 = \not{p}$ has the zero mode $Z_2 = \not{p}$, because $Z_1 Z_2 = p^2 = 0$ on classical shell etc. Thus this system has reducibility rank infinity.

The meaning of first-rank reducibility is that the classical gauge transformation become linearly dependent on classical shell. Namely, $R^i{}_\alpha Z_1^\alpha{}_{\alpha_1}$ is a sum of local symmetries (sum over α) which vanishes when $\frac{\partial}{\partial\phi^i} S_0 = 0$. One could try to eliminate the redundant symmetries, but this would in general lead to nonlocality and violation of relativistic covariance. Rather, one opts to work with a set of local symmetries which is redundant on the classical shell and removes the redundant symmetries by ghosts-for-ghosts. This procedure is thus similar to the usual FP quantization but there one works with too many fields (longitudinal and timelike components of gauge fields) which one then removes by introducing ghosts and antighosts.

In addition to the concept of reducibility which leads to ghosts-for-ghosts there is another concept which plays a role in the most general gauge theories, and that is the gauge symmetry for antighosts. In the example with $A_{\mu\nu}$, $\mathcal{L}$ (ghosts) $= b^\nu(\partial_\mu C_\nu - \partial_\nu C_\mu)$ has the local gauge invariance $\delta b^\nu = \partial^\nu \Lambda$ and this leads to extra ghosts. In general one considers the gauge fermion $\psi =$ "b" "F" + more, where "b" denotes all possible kinds of antighosts, and "F" the corresponding gauge fixing terms. If ψ (which is always an integral over spacetime) has a local gauge invariance of the form δ"b", then there are extra ghosts. We shall give an example of systems with ghost-for-ghosts which have an extra ghost, and an example of a system with ghosts-for-ghosts without an extra ghost.

5 Examples of reducible gauge theories

The standard example of a reducible gauge theory is the Maxwell theory of an antisymmetric tensor field $A_{\mu\nu}$ coupled to gravity. [1, 4] We first discuss this system and then consider a more complicated system: the coupling of Yang-Mills fields A^a_μ to several antisymmetric tensor fields $B^a_{\mu\nu}$ in the adjoint representation of the Yang-Mills group.

5.1 Antisymmetric tensor gauge fields

As a first example of a reducible system we consider the two-index antisymmetric tensor field $A_{\mu\nu}$. Its classical action is a generalization of the Maxwell action,

$$\mathcal{L} = -\frac{1}{12}(F_{\mu\nu\rho})^2 \text{ with } F_{\mu\nu\rho} = \partial_\nu A_{\nu\rho} + \partial_\nu A_{\rho\mu} + \partial_\rho A_{\mu\nu} \tag{11.5.61}$$

We are in Minkowski space, and with the factor $\frac{1}{12}$ the kinetic term of, say A_{12}, has the standard normalization for a real scalar field, $\mathcal{L} = \frac{1}{2}(\partial_0 A_{12})^2$. Cross terms in $(F_{\mu\nu\rho})^2$ will be removed by the gauge fixing terms. The field $A_{\mu\nu}$ has the obvious gauge transformation

$$\delta A_{\mu\nu} = \partial_\mu \lambda_\nu - \partial_\nu \lambda_\mu \tag{11.5.62}$$

and if this was all, one would add a gauge fixing term, for example,

$$\mathcal{L}_{\text{fix}} = -\frac{1}{4}(\partial^\mu A_{\mu\nu})^2 \tag{11.5.63}$$

The action is then diagonal

$$\mathcal{L} + \mathcal{L}_{\text{fix}} = -\frac{1}{4}(\partial_\mu A^2_{\nu\rho} \tag{11.5.64}$$

and the gauge fixing term would lead to a corresponding ghost action according to the Faddeev-Popov procedure of varying $\partial^\mu A_{\mu\nu}$ and replacing λ_μ by ghosts C_μ

$$\begin{aligned}\mathcal{L}_{\text{ghost}} &= \bar{C}^\nu \partial^\mu(\partial_\mu C_\nu - \partial_\nu C_\mu) \\ &= -\frac{1}{2}(\partial^\mu \bar{C}^\nu - \partial^\nu \bar{C}^\mu)(\partial_\mu C_\nu - \partial_\nu C_\mu) \end{aligned} \tag{11.5.65}$$

However, this ghost action has the form of a complex Maxwell action, and thus it has two new gauge invariances,

$$\delta C_\mu = \partial_\mu \lambda, \qquad \delta \bar{C}_\mu = \partial_\mu \bar{\lambda} \tag{11.5.66}$$

where λ and $\bar{\lambda}$ are both real and independent from each other. This suggests that there are ghosts-for-ghosts [16]. If this is the case, one might think that one should do the covariant quantization procedure once more, replacing λ by a ghost-for-ghost C, and $\bar{\lambda}$ by a "ghost-for-antighost" $\bar{C}$, and then adding a suitable gauge fixing term and finally (the most complicated step as we shall see) a corresponding ghost action. A natural choice for the gauge fixing term would seem to be

$$\mathcal{L}'_{\text{fix}} = -(\partial^\mu \bar{C}_\mu)(\partial^\nu C_\nu) \tag{11.5.67}$$

because then $\mathcal{L}_{\text{ghost}} + \mathcal{L}_{\text{fix}} = -(\partial^\mu \bar{C}^\nu)(\partial_\mu C_\nu)$ is diagonal, with propagator $(-i/k^2)\eta^\nu{}_{\nu'}$. Such a gauge fixing term would be obtained in a path integral approach by fixing the two new local gauge symmetries with delta functions $\delta(\partial^\mu \bar{C}_\mu - \alpha)\delta(\partial^\nu C_\nu - \beta)$ and then integrating over α and β with $\int d\alpha d\beta e^{-\frac{i}{\hbar}\alpha\beta}$. If one would apply the Faddeev-Popov procedure separately to $\partial^\mu \bar{C}_\mu$ and to $\partial^\mu C_\mu$, one would find two new antighosts and two new ghosts

$$\mathcal{L}'_{\text{ghost}} = -\bar{E}\Box E - \bar{C}\Box C \quad \text{(incorrect)} \tag{11.5.68}$$

It is easy to see that this result is incorrect. A real $A_{\mu\nu} = -A_{\nu\mu}$ should represent one degree of freedom[4] whereas we find instead two degrees of freedom

$$6(A_{\mu\nu}) - 8(C_\mu, \bar{C}_\nu) + 4(\bar{E}, E, \bar{C}, C) = 2 \tag{11.5.69}$$

[4]Consider the "parent action" $\mathcal{L} = \frac{1}{6}\epsilon^{\mu\nu\rho\sigma} F_{\mu\nu\rho}\partial_\sigma \varphi - \frac{1}{12} F_{\mu\nu\rho} F^{\mu\nu\rho}$. Eliminating $F_{\mu\nu\rho}$ by using its

Apparently we only need three "ghosts-for-ghosts", instead of the four we got.

One can spot where the problem lies by introducing auxiliary fields which multiply the gauge fixing terms and indicating how these terms should be related to corresponding ghost action according to the BRST rules. For Maxwell theory this is well known

$$\begin{array}{ll} d(\partial^\mu A_\mu) & \qquad \delta_B A_\mu = \partial_\mu C \Lambda \\ \uparrow \;\; \downarrow & \\ \bar{C}(\Box C) & \qquad \delta_B \bar{C} = \Lambda d \end{array} \tag{11.5.70}$$

For the system with $A_{\mu\nu}$ one finds in this way [17].

$$\begin{array}{lll} d^\mu(\partial^\nu A_{\nu\mu}) + & \bar{d}(\partial^\mu C_\mu) & + (\partial^\mu d_\mu) C' \\ \uparrow \quad \downarrow & \uparrow \quad \downarrow & \quad \uparrow \quad \downarrow \\ \bar{C}^\mu(\Box \eta_{\mu\nu} - \partial_\mu \partial_\nu) C^\nu + & \bar{C} \Box C & + \; (\partial^\mu \bar{C}_\mu) d \end{array} \tag{11.5.71}$$

The first, the second and the last term fix the gauges as we discussed before. Clearly, we get only 3 new ghosts, namely $C, \bar{C}$ and C', because we already know that $\bar{C}_\mu$ transforms into d_μ and not into a new ghost field. We can exhibit these relations into two triangles [1], one for the "ghosts" (by which we mean ghosts, ghosts-for-ghosts, antighosts, ghosts for antighosts, antighosts for ghosts etc.), and another for the auxiliary fields.

$$\begin{array}{ccccccccc} & & A_{\mu\nu}(0) & & & & & & \\ & \swarrow & & \searrow & & & & d_\mu(0) & \\ & \bar{C}_\mu(-1) & & & C_\mu(1) & & & \cdot^{\cdot^{\cdot}} \; \cdot_{\cdot_{\cdot}} & \\ \swarrow & & \swarrow & & & \searrow & d(1) & & \bar{d}(-1) \\ C'(0) & & \bar{C}(-2) & & & C(2) & & & \end{array} \tag{11.5.72}$$

We have indicated the ghost number of each field in parentheses. Along the diagonal on the right one finds the minimal fields $A_{\mu\nu}, C_\mu, C$ which transform into each other as indicated by the arrows pointing in the lower-right direction: $\delta_B A_{\mu\nu} = \partial_\mu C_\nu - \partial_\nu C_\mu$ and $\partial_B C_\mu = \partial_\mu C$. The nonminimal ghosts $(\bar{C}_\mu, C', \bar{C})$ transform under BRST

algebraic field equation $F^{\mu\nu\rho} - \epsilon^{\mu\nu\rho\sigma}\partial_\sigma\varphi = 0$ yields $\mathcal{L} = -\frac{1}{2}(\partial_\sigma\varphi)^2$ (because $\epsilon^{\mu\nu\rho\sigma}\epsilon_{\mu\nu\rho\sigma'} = -6\delta^\sigma_{\sigma'}$). On the other hand, if we eliminate φ instead of $F_{\mu\nu\rho}$, the φ field equation yields $\epsilon^{\mu\nu\rho\sigma}\partial_\sigma F_{\mu\nu\rho} = 0$, whose solution is $F_{\mu\nu\rho} = \partial_\mu A_{\nu\rho} + \partial_\nu A_{\rho\mu} + \partial_\rho A_{\mu\nu}$. Substituting this result back into the action yields the Maxwell action for $A_{\mu\nu}$. Hence, $A_{\mu\nu}$ is equivalent in $3+1$ dimensions to a scalar field φ, and should correspond to one degree of freedom.

transformations into the corresponding auxiliary fields $(d_\mu, d, \bar{d})$. Of course C_μ and $\bar{C}_\mu$ are anticommuting, but $C, \bar{C}$ and C' are commuting, and $d, \bar{d}$ anticommuting. The arrows pointing in the lower-left direction indicate that $\bar{C}$ plays a role to fix the gauge symmetry of C_μ, while $\bar{C}_\mu$ should fix the symmetry of $A_{\mu\nu}$, and C' should fix the symmetry of $\bar{C}_\mu$. Namely, the BRST variations of $\bar{C}^\mu \partial^\nu A_{\nu\mu}$ and $\bar{C}\partial^\mu C_\mu$ and $(\partial^\mu \bar{C}_\mu)C'$ yield the terms in (11.5.71). Because $\bar{C}_\mu$ transforms into d^μ (antighosts always transform into BRST auxiliary fields in the BRST formalism) and not as $\delta\bar{C}_\mu = \partial_\mu \xi$, there is no action $-\bar{E}\Box E$ as in (11.5.68).

Having come this far, we now switch to the antifield formalism to fix the details. We begin with the minimal action

$$\mathcal{L}^{\min} = \mathcal{L}_{cl} + A^{*\mu\nu}(\partial_\mu C_\nu - \partial_\nu C_\mu) + C^{*\mu}\partial_\mu C \tag{11.5.73}$$

It clearly satisfies $(S^{\min}, S^{\min}) = 0$. As nonminimal action we take

$$\mathcal{L}^{\text{nonmin}} = \bar{C}^*_\mu d^\mu + \bar{d}\bar{C}^* + C^{'*} d \tag{11.5.74}$$

None of the fields in $\mathcal{L}^{\text{nonmin}}$ appear in $\mathcal{L}^{\min}$, hence $(S^{\min}, S^{\text{nonmin}}) = 0$ and also $(S^{\text{nonmin}}, S^{\text{nonmin}}) = 0$ as easily checks. Thus $(S, S) = 0$ where $S = S^{\min} + S^{\text{nonmin}}$.

As gauge fixing fermion we take

$$\psi = \int (\bar{C}^\mu \partial^\nu A_{\nu\mu} + \bar{C}\partial^\nu C_\nu + (\partial^\nu \bar{C}_\nu)C')d^4x \tag{11.5.75}$$

One finds then from $\phi^*_A = \partial\psi/\partial\phi^A$ the following results. For the minimal antifields

$$A^{*\mu\nu} = -\partial^{[\mu}\bar{C}^{\nu]}, \quad C^{*\mu} = -\partial^\mu \bar{C} \tag{11.5.76}$$

leads to

$$\mathcal{L}^{\min} = \mathcal{L}_{cl} + \mathcal{L}_{\text{ghost}} - (\partial^\mu \bar{C})(\partial_\mu C) \tag{11.5.77}$$

For the nonminimal antifields one obtains in a similar way

$$\bar{C}^*_\mu = \partial^\nu A_{\nu\mu} - \partial_\mu C'; \quad \bar{C}^* = \partial^\nu C_\nu; \quad C^{'*} = \partial^\nu \bar{C}_\nu \tag{11.5.78}$$

Substituting these results into $\mathcal{L}^{\text{nonmin}}$ yields

$$\mathcal{L}^{\text{nonmin}} = d^\mu(\partial^\nu A_{\nu\mu} - \partial_\mu C') + \bar{d}\partial^\nu C_\nu + (\partial^\nu \bar{C}_\nu)d \tag{11.5.79}$$

This is just the result we expected from our preliminary analysis, except for the term $-d^\mu \partial_\mu C'$ which contains C', "the third ghost".

To obtain standard diagonal kinetic terms for the ghosts we add extra terms to ψ which lead to squares of auxiliary fields in the action (similar to the term $\frac{1}{2\xi}(d)^2$ in Yang-Mills theory)

$$\psi_{\text{extra}} = \int [\alpha d_\mu \bar{C}^\mu + \beta d\bar{C} + \gamma C'\bar{d}] d^4x \tag{11.5.80}$$

Note that all terms in ψ_{extra} have ghost number -1. Then

$$\bar{C}_\mu^* \text{ (extra) } = \alpha d_\mu; \bar{C}^* \text{ (extra)} = \beta d; C'^* \text{ (extra)} = \gamma \bar{d} \tag{11.5.81}$$

and substituting these terms into $\mathcal{L}^{\text{nonmin}}$ yields

$$\mathcal{L}(\text{aux. fields}) = \alpha d_\mu d^\mu + \bar{d}\beta d + \gamma \bar{d} d \tag{11.5.82}$$

Eliminating these auxiliary fields yields the final action

$$\begin{aligned}\mathcal{L} = \; & - \frac{1}{12}(F_{\mu\nu\rho})^2 - \frac{1}{2}(\partial^\mu \bar{C}^\nu - \partial^\nu \bar{C}^\mu)(\partial_\mu C_\nu - \partial_\nu C_\mu) - (\partial^\mu \bar{C})\partial_\mu C \\ & - \frac{1}{4\alpha}\left\{(\partial^\nu A_{\nu\mu})^2 + (\partial_\mu C')^2\right\} - \frac{1}{\beta+\gamma}(\partial^\nu \bar{C}_\nu)(\partial^\mu C_\mu)\end{aligned} \tag{11.5.83}$$

We have obtained 3, not 4, propagating ghosts-for-ghosts, and the count of degrees of freedom is now correct.

So far we discussed free fields, so it is desirable to test these results for interacting theories. We consider two applications, both involving for once the coupling to gravity: one application to renormalizability (more precisely: the structure of the one-loop divergences), and another application to unitarity (the cutting rules extended to antisymmetric tensor fields).

5.2 Yang Mills fields coupled to antisymmetric tensors

A more complicated reducible system is obtained by coupling Yang-Mills fields to an antisymmetric tensor gauge field in the adjoint representation in Minkowski space.

The classical action reads[5]

$$S_{cl} = \left[\frac{1}{2}\tilde{B}^a_{\mu\nu}F^{\mu\nu}_a - \frac{1}{2}A^a_\mu A^\mu_a\right] d^4x \tag{11.5.84}$$

where $\tilde{B}_{\mu\nu}{}^a = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}B^{\rho\sigma,a}$ while $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^a{}_{bc}A^b_\mu A^c_\nu$. This action is invariant under the gauge transformation

$$\delta B^a_{\mu\nu} = D_{[\mu}\xi^a_{\nu]}\ ,\ \delta A^a_\mu = 0 \tag{11.5.85}$$

(use the Bianchi identities $D^\mu\tilde{F}_{\mu\nu} = 0$). Eliminating A^a_μ by iterating its algebraic field equation $A^a_\nu = -D^\mu\tilde{B}^a_{\mu\nu}$, one finds $\mathcal{L} = \frac{1}{2}(D^\mu\tilde{B}_{\mu\nu})^2 + \cdots$ which is equal to $\mathcal{L} = -\frac{1}{4}(D_\mu B_{\nu\rho})(D^\mu B^{\nu\rho} + D^\nu B^{\rho\mu} + D^\rho B^{\mu\nu})+$ interactions. This is the action for antisymmetric tensor gauge fields, coupled to Yang-Mills gauge fields, with the standard normalization. The interactions contain an infinite series in terms of $\partial_\mu B_{\nu\rho}$, so the model is not renormalizable.

Naive application of the Faddeev-Popov prescription would yield as ghost action corresponding to the gauge $D^\mu B_{\mu\nu} = 0$

$$\mathcal{L}^{\text{ghost}}_{FP} = B^\nu{}_a(D^\mu D_{[\mu}C^a_{\nu]} \tag{11.5.86}$$

In the abelian case this ghost action has again two gauge invariances

$$\delta B_\mu = \partial_\mu\lambda_1 \text{ and } \delta C_\mu = \partial_\mu\lambda_2 \tag{11.5.87}$$

but in the nonabelian case the gauge invariances $\delta B_\mu = D_\mu\lambda_1$ and $\delta C_\mu = D_\mu\lambda_2$ only hold on the classical shell (where $F^a_{\mu\nu} = 0$. The other classical field equation is $D^\mu\tilde{B}^a_{\mu\nu} + A^a_\nu = 0$). In fact, when the classical field equations are satisfied, the gauge

[5] In the literature one usually begins with $\mathcal{L} = B^a_{\mu\nu}F^{\mu\nu}_a$, but the choice $\tilde{B}^a_{\mu\nu}F^{\mu\nu}_a$ simplifies the formulas.

transformations themselves become clearly dependent (they have "zero modes"). This defines reducible theories. Formally these zero modes Z^α_β are defined by $R^I_\alpha Z^\alpha_\beta = 0$ when $\partial S_{cl}/\partial\phi^I = 0$. In our case, $R^I_\alpha = R_{\mu\nu}{}^\rho = (D_{[\mu})\delta^\rho_{\nu]}$ according to (11.5.85) and $Z^\alpha_\beta = Z_\rho$ corresponds to D_ρ.

For later use we first count the number of degrees of freedom. Acting with D^ν on $D^\mu\tilde{B}_{\mu\nu} + A_\nu = 0$ yields with $F_{\mu\nu}(A) = 0$ the equation $D^\mu A_\mu = \partial^\mu A_\mu = 0$. From $F_{\mu\nu}(A) = 0$ we find that A_μ is pure gauge, $A_\mu = g^{-1}\partial_\mu g$ and the field equation becomes $\partial^\mu(g^{-1}\partial_\mu g) = 0$. This is the field equation of a nonlinear σ model and describes one scalar field φ^a (for example, $g = \exp\varphi^a T_a$). The same result is obtained for the abelian theory if one counts the number of linearly **independent** gauge transformations. Three components of ξ_μ can fix the gauge $\partial^\mu B_{\mu\nu} = 0$. But then there are still two solutions of the homogeneous equation $\partial^\mu(\partial_\mu\xi_\nu - \partial_\nu\xi_\mu) = 0$. This leaves again one degree of freedom. (If one first solves A^a_μ from $D^\mu\tilde{B}_{\mu\nu} + A_\nu = 0$ and substitutes the result into $F_{\mu\nu}(A) = 0$, one finds a complicated interacting field theory for $B^a_{\mu\nu}$.) However for the nonabelian case the counting is more complicated because one gets different answers on-shell and off-shell. We now construct the correct quantum action.

We begin with the minimal action containing the fields $\phi^A = \{B^a_{\mu\nu}, A^a_\nu, C_\mu{}^a, C^a\}$ and their antifields. (The ghost-for-ghosts C^a have ghost number +2 and are commuting). The terms of $S_1^{\min}$ follow from (11.5.85) and (11.5.86)

$$S^{\min} = S_{cl} + \int [B^{*\mu\nu}_a D_\mu C^a_\nu + C^{*\mu}_a D_\mu C^a + \text{“more”}]\, d^4x \tag{11.5.88}$$

where "more" needs to be determined and may contain terms with C^*_a and with more than one antifield. We used that the structure constants of the classical gauge algebra vanish, and added the ghost-for-ghost C^a. Requiring $(S,S) = 0$, and using $(S_1, S_1) = f^{abc}B^{*\mu\nu}_a F_{\mu\nu,b}C_c$, one finds easily that only one further term is needed

$$S^{\min}_2 = \text{“more”} = -\frac{1}{2}B^{*\mu\nu}_a B^{*\rho\sigma}_b \epsilon_{\mu\nu\rho\sigma} f^{abc} C_c \tag{11.5.89}$$

To obtain this result, we used $\epsilon_{\mu\nu\rho\sigma}\epsilon^{\mu\nu\alpha\beta} = -2\left(\delta^\alpha_\rho\delta^\beta_\sigma - \delta^\beta_\rho\delta^\alpha_\sigma\right)$.

The nonminimal terms to be added are of the form πB^*

$$S^{\text{nonmin}} = \pi_a^\mu b_\mu^{*a} + \pi_a b^{*a} + \pi_a^0 d^{*a} \tag{11.5.90}$$

The b_a^μ and b_a are the expected antighosts for the ghosts $C_\mu{}^a$ and C^a, and π_μ^a and π_a are the corresponding auxilary fields. However **one extra** field appears: the extra-ghost d^a. In general the sequence of ghosts/antighosts (and corresponding auxilary fields) is given by

$$\begin{array}{ccccccc} & & & \phi^I & & & \\ & & \bar{C}_{0\alpha_0} & & C_0^{\alpha_0} & & \\ & C_1^{1\alpha_1} & & \bar{C}_{1\alpha_1} & & C_1^{\alpha_1} & \\ C_{2\sigma_2}^{11} & & C_2^{1\alpha_2} & & \bar{C}_{2\alpha_2} & & C_2^{\alpha_2} \end{array} \tag{11.5.91}$$

The fields $\phi^I, C_0^{\alpha_0}, C_1^{\alpha_1}, C_2^{\alpha_2}$ etc. appear in S^{min} and are the ghosts, ghosts-for-ghosts, ghosts-for-ghosts-for-ghosts etc. S^{nonmin} contains all fields below the upper diagonal. The line $\bar{C}_{0\alpha_0}, \bar{C}_{1\alpha_1}, \bar{C}_{2\alpha_2}$ gives the antighosts, antighosts for the ghosts-for-ghosts etc. The rest of the fields are the extra ghosts. We have previously denoted $\bar{C}_{0\alpha_0}$ by b_μ^a, and $C^{1\alpha_1}$ by b^a, and the extra ghost $\bar{C}_{1\alpha_1}$ by d^a. The field b_a^μ has ghost number -1, and b_a has ghost number -2, while d_a has ghost number zero. The need for the extra ghost d was already clear at the free field level. The term $\pi_a b^{*a}$ and a term $b_a D^\mu C_\mu^a$ in the gauge fermion lead to the gauge fixing term $\pi_a D^\mu C_\mu^a$, but the term $\pi_a^0 d^{*a}$ and a term $d_a D^\mu b_\mu^a$ in the gauge fermion will lead to the gauge fixing term $\pi_a^0 D^\mu b_\mu^a$ which fixes the antighost gauge invariance.

Motivated by these remarks, we can now write down a suitable gauge-fermion $\psi = \int (\psi_1 + \psi_2) d^4x$. We put in ψ_1 the gauge fixing terms of the form $B_\alpha F^\alpha$, and in ψ_2 terms of the form $B_\alpha \pi_\beta \gamma^{\beta\alpha}$ which lead to weighted gauges

$$\begin{aligned} \psi_1 &= b_a^\nu (D^\mu B_{\mu\nu}) + b_a (D^\mu C_\mu^a) + d^a (D_\mu b_a^\mu) \\ \psi_2 &= \lambda_1 b_\mu^a \pi_a^\mu \lambda_2 + \lambda_2 b^a \pi_a^0 + \lambda_3 d^a \pi_a \end{aligned} \tag{11.5.92}$$

The first two terms in ψ_1 fix the classical gauge invariance and the gauge invariance of the ghosts. Were it not for the last term, ψ would be gauge invariant under the antighost gauge invariance $\delta b_a^\nu = D^\nu \Lambda$ on the classical shell where $F_{\mu\nu} = 0$. The last term in ψ_1 also fixes this gauge invariance. The terms in ψ_2 are unique (up to rescalings of the π fields) if one requires that they have ghost number -1. (Note that π^μ and b_μ^* have ghost number zero, so the ghost number of b^* is $+1$ and of π is -1. Similarly, d^* has ghost number -1 and π^0 has then ghost number $+1$.) The antifields are then eliminated as follows

$$B_a^{*\mu\nu} = -D^{[\mu} b^{\nu]a} \tag{11.5.93}$$

$$C^*{}_a{}^\mu = -D^\mu b_a \tag{11.5.94}$$

$$b^*{}_\mu{}^a = D^\rho B_{\rho\mu} - D_\mu d^a + \lambda_1 \pi_\mu^a \tag{11.5.95}$$

$$b^{*a} = D^\mu C_\mu{}^a + \lambda_2 \pi_0^a \tag{11.5.96}$$

$$d^{*a} = D_\mu b_a^\mu + \lambda_3 \pi^a \tag{11.5.97}$$

Substituting these results into S, one finds the following terms in the quantum action

(i) The terms in $S_1^{\rm min}$ yield typical Faddeev-Popov actions of the form

$$-(D_\mu b_\nu^a)(D^{[\mu} C_a^{\nu]}) - (D^\mu b_a)(D_\mu C^a) \tag{11.5.98}$$

(ii) The term in $S_2^{\rm min}$ yields a peculiar anti-ghost interaction

$$-\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}(D_\mu b_\nu^a)(D_\rho b_\sigma^b)C^c f_{abc} \tag{11.5.99}$$

which we will discuss further below.

(iii) The nonminimal terms due to ψ_2 consist of π^2-type terms

$$\lambda_1 \pi_a^\mu \pi_\mu{}^a + \lambda_2 \pi_a \pi_0^a + \lambda_3 \pi_a^0 \pi^a \tag{11.5.100}$$

They are, of course, separately invariant, and without ψ_2 we would have found an unweighted gauge (delta-functions in the path-integral).

(iv) Finally, the most interesting terms come from ψ_1. First of all, there are the terms of the general form $\pi_\alpha F^\alpha$ where F^α are the gauge condition for the original gauge invariance, and the on-shell gauge invariance of the rank-one ghosts and antighosts

$$\pi^\mu_a(D^\rho B^a_{\rho\mu}) + \pi_a(D^\mu C^a_\mu) + \pi^0_a(D_\mu b^\mu_a) \tag{11.5.101}$$

(v) Then there is another term which looks like a gauge-fixing term and involves the extra ghosts

$$-\pi^\mu_a D_\mu d^a$$

If we eliminate the π-fields by using their algebraic field equation, we get Lorentz-type gauge-fixing terms for the classical action and for the $b^a_\mu c^a_\mu$ ghost action, and further a kinetic term for the extra ghost

$$(D_\mu d^a)(D^\mu d_a) + 2(D^\rho B^a_{\rho\mu})(D^\mu d_a)$$

One may check that the final quantum action is invariant under the following BRST rules

$$\delta B^a_{\mu\nu} = D_{[\mu}C^a_{\nu]} + (D_\mu b^b_\nu)C^c f^{abc}$$

$$\delta A^a_\mu = 0, \delta b^a_\mu = \pi^a_\mu, \delta\pi^a_\mu = 0, \delta\pi^a_0 = 0, \delta C^a = 0$$

$$\delta C^a_\mu = D_\mu C^a, \delta d^a = c^a, \delta b^a = \pi^a, \delta\pi^a = 0 \tag{11.5.102}$$

Furthermore $\delta^2 = 0$ except on $B^a_{\mu\nu}$, where it is proportional to the B-field equation. Thus the BRST algebra is open, and this explains the terms quadratic in $B^{*\mu\nu}$ in the minimal action (but note that the classical gauge algebra is closed!).

An interesting aspect of this model is that the tri-ghost coupling $(\partial_\mu b^a_\nu)$ $(\partial_\rho b^b_\sigma)$ $C^c\epsilon^{\mu\nu\rho\sigma}f^{abc}$ violates ghost number. Hence one can drop it without changing the set of Feynman diagrams. We then obtain a non-BRST invariant action (though it can be made BRST invariant) which, we claim, is unitary.

Let us now investigate whether this model is unitary. [14] For simplicity we use ordinary derivatives in ψ_1 instead of covariant derivatives. The final action after eliminating auxiliary fields reads

$$\begin{aligned}
\mathcal{L} &= \frac{1}{4}\tilde{B}^a_{\mu\nu}F_a^{\mu\nu} + \frac{1}{4}\left(A^a_\mu\right)^2 - \left(\partial^\mu B^a_{\mu\nu} + \partial_\nu d^a\right)^2 \\
&- \left(\partial^\mu B_a^\nu\right)\left(D_\mu C^a_\nu - D_\nu C^a_\mu\right) - \left(\partial^\mu B^a_\mu\right)\left(\partial^\nu C^a_\nu\right) - \left(\partial^\mu B^a\right)\left(\partial_\mu C^a\right) \\
&+\alpha\ \epsilon^{\mu\nu\rho\sigma}\left(D_\mu B^a_\nu\right)\left(D_\rho B^b_\sigma\right)C^c f_{abc}
\end{aligned} \tag{11.5.103}$$

We recall that the classical action (the first two terms) after elimination of A^a_μ reads $\mathcal{L} = -\frac{1}{2}(D_\mu B_{\nu\rho})^2 - \frac{1}{2}(D_\mu B_{\nu\rho})(D^\nu B^{\rho\mu} + D^\rho B^{\mu\nu})$, so that the third term in $\mathcal{L}$ cancels at the linearized level the terms with divergences of $B_{\mu\nu}$. The same holds for the first-generation ghosts. This leads to the standard Feynmanlike gauge where the kinetic terms contain only a Dalembertian. The extra ghost d^a does not couple; if we had used covariant derivatives in the gauge fixing terms, there would have been a coupling $-2(D^\mu B_{\mu\nu})(D^\nu d) \sim -F_{\mu\nu}\cdot(B_{\mu\nu}xd)$ and a kinetic term $-(D_\nu d)^2$.

The transformation rules which leaves this action invariant read

$$\begin{aligned}
&\delta B^a_{\mu\nu} = D_{[\mu}C^a_{\nu]} + gf^{abc}\left(D_\mu B^b_\nu\right)C^c\Lambda; \delta A^a_\mu = 0 \\
&\delta C^a_\mu = D_\mu C^c\Lambda; \delta C^a = 0; \delta B^a_\mu = -2\left(\partial^\rho B_{\rho\mu} + \partial_\mu d\right)\Lambda \\
&\delta d^a = 2\partial_\mu C^{a\mu}\Lambda; \delta B^a = \partial^\mu B_\mu\Lambda
\end{aligned} \tag{11.5.104}$$

$$\begin{matrix}\epsilon\epsilon & \epsilon\epsilon \\ \epsilon\epsilon & \epsilon\epsilon\end{matrix} \quad + \left(\quad \begin{matrix}\epsilon\epsilon & \epsilon\epsilon \\ \epsilon k & \epsilon\bar{k}\end{matrix} \quad + \text{ other graphs}\right)(A)$$

$$- \quad \begin{matrix}\epsilon\epsilon & \epsilon\epsilon \\ k\ \vec{k} & k\bar{k}\end{matrix} \quad - \quad \begin{matrix}k\bar{k} & k\bar{k} \\ \epsilon\epsilon & \epsilon\epsilon\end{matrix} \qquad (B)$$

$$+ \left(\quad \begin{matrix}\epsilon k & \epsilon\bar{k} \\ \epsilon k & \epsilon\bar{k}\end{matrix} \quad + \quad \begin{matrix}\epsilon k & \epsilon\bar{k} \\ \epsilon\bar{k} & \epsilon k\end{matrix} \quad + k \leftrightarrow \bar{k}\right)(C)$$

$$+ \left(\quad \begin{matrix}\epsilon k & \epsilon\bar{k} \\ k\bar{k} & k\bar{k}\end{matrix} \quad + \text{ 3 other graphs}\right) \qquad (D)$$

$$+ \qquad \begin{array}{cc} k\bar{k} & k\bar{k} \\ k\bar{k} & k\bar{k} \end{array} \qquad (E)$$

$$(11.5.105)$$

From $\delta\langle B_\mu\rangle \sim \langle\partial^\rho B_{\rho\mu}\rangle = 0$ We see that the unphysical cuts in (A), and all graphs in (B) cancel. From $\delta\langle B_\mu \partial^\sigma B_{\sigma\nu}\rangle = \langle\partial^\rho B_{\rho\mu}\partial^s B_{\sigma\nu}\rangle + \langle B_\mu \partial^\sigma(\partial_\sigma C_\nu - \partial_\nu C_\sigma)\rangle = 0$ we see, using that $\partial^\sigma(\partial_\sigma C_\nu - \partial_\nu C_\sigma)$ is a linearized field equation, that all graphs in (D) and (E) cancel, as well as the first graphs in (C). All these cancellations follow from the Ward identity that single or double divergences of a graph cancel provided all other lines are physical. We already derived in chapter IV the same by eliminating A_μ from its own nonpropagating field equation, we obtain a rather simple system, with gauge fields $B_{\mu\nu}$, ghosts B_μ and C_ν, as a ghosts-for-ghosts B and C. The propagators are

$$\begin{aligned} \langle B_{\mu\nu} B_{\rho\sigma} &= \frac{-i}{k^2}\left(\eta_{\mu\rho}\eta_{\nu\sigma} - \eta_{\mu\sigma}\eta_{\nu\rho}\right) \\ \langle C_\mu B_\nu\rangle &= \frac{-i}{k^2}\left(\eta_{\mu\nu}\right); \langle CB\rangle = -i/k^2 \end{aligned} \qquad (11.5.106)$$

Decomposing all $\eta_{\mu\nu}$ into $\epsilon^1_\mu\epsilon^1_\nu + \delta^2_\mu\epsilon^2_\nu + (k_\mu\bar{k}_\nu + \bar{k}_\mu k_\nu)/k\cdot\bar{k}$ we find for the numerator of the $B_{\mu\nu}$ propagator

$$\begin{aligned} &\eta_{\mu\rho}\eta_{\nu\sigma} - \rho\leftrightarrow\sigma = \left(\epsilon^1_\mu\epsilon^2_\nu - \epsilon^2_\mu\epsilon^1_\nu\right)\left(\epsilon^1_\rho\epsilon^2_\sigma - \epsilon^2_\rho\epsilon^1_\sigma\right) \\ &+\left\{\left(\epsilon^I_\mu k_\nu - \epsilon^I_\nu k_\mu\right)\left(\epsilon^I_\rho\bar{k}_s - \epsilon^I_\sigma\bar{k}_\rho\right)/\left(k\cdot\bar{k}\right) + k\leftrightarrow\bar{k}\right\} \\ &-\left(k_\mu\bar{k}_\nu - \bar{k}_\mu k_\nu\right)\left(k_\rho\bar{k}_\sigma - \bar{k}_\rho k_s\right)\left(k\cdot\bar{k}\right)^{-2} \end{aligned} \qquad (11.5.107)$$

The $\epsilon\epsilon$ term propagates clearly the physical mode, so that the other terms must cancel against ghost contributions. Considering first two-particle cuts through two $B_{\mu\nu}$ lines, we find the following set of cut graphs result for Yang-Mills theory.

We are left with the following graphs

$$\begin{array}{cccc} k,\mu\nu & & & \rho\sigma \\ & \epsilon_{[\mu}k_{\nu]} & \epsilon_{[\rho}k_{\sigma]} & \\ & \epsilon'_{[\alpha}\bar{k}'_{\beta]} & \epsilon'_{[\gamma}\bar{k}'_{\delta]} & \\ k',\alpha\beta & & & \gamma\delta \end{array} \qquad (11.5.108)$$

We clearly need a Ward identity which expresses these graphs into graphs with ghosts, in which one term is of thee form $\langle \partial^\nu B_{\mu\nu} B_{\alpha\beta} \rangle$. This Ward identity is

$$\begin{aligned} &\delta \langle B_\mu B_{\alpha\beta} \rangle = 0 = \langle \partial^\nu B_{\nu\mu} B_{\alpha\beta} \rangle \\ &+ \langle B_\mu \left(\partial_\alpha C_\beta - \partial_\beta C_\partial \right) \rangle \end{aligned} \tag{11.5.109}$$

Graphically it reads

$$\begin{array}{llll} \mu\nu & & & \\ & k_\nu & & \\ & & = & \qquad k'_\alpha \\ & & & \\ \alpha\beta & & \beta & \end{array} \tag{11.5.110}$$

Contracting k'_α with $(\epsilon'_\alpha \bar{k}'_\beta - \epsilon'_\beta \bar{k}'_\alpha)/k' \cdot \bar{k}'$, one is on-shell left with a factor $-\epsilon'_\beta$. Thus the above graphs are equal to

$$\begin{pmatrix} \mu & & \epsilon_\mu \\ & & \\ \beta & & \epsilon'_\beta \end{pmatrix} \qquad \begin{pmatrix} \epsilon_\rho \;\; \rho & & \\ & & \\ \epsilon_\gamma \;\; \gamma & & \end{pmatrix} \tag{11.5.111}$$

These graphs are part of cut ghost graphs. Adding also the ghost-for-ghost graphs one finds

$$\begin{array}{lcl} \epsilon\epsilon + k\bar{k} + \bar{k}k & & \\ \epsilon\epsilon + k\bar{k} + \bar{k}k & \qquad + \qquad & \end{array} \tag{11.5.112}$$

As well as graphs in which the arrows go clockwise. The leading terms with $\dot{\epsilon}\epsilon$ at both cut lines clearly cancel the graphs we found in (P). **The rest cancels as in ordinary Yang-Mills theory**, the only difference being that the gauge field is now a ghost, and the usual ghost has become a ghost-for-ghost. Thus, the theory with self-coupled antisymmetric tensors is unitary. It is not renormalizable (there are double-derivative interactions in the classical action), but similar models play a role in string theory and in low-energy phenomenology.

As a final comment we note that one could eliminate A^a_μ either from the classical action, or from the minimal action, or from the full quantum action. Elimination from the minimal action leads to terms quadratic in various antifields (because A_μ^a appears at most quadratically). Elimination of A_μ^a from the classical action and then applying the antifield formalism should lead to the same result.

5.3 Ghosts-for-ghosts without extra ghosts

As a last example we consider as a model which leads to ghosts-for-ghosts but without extra ghosts. Consider the following classical action in Euclidean space

$$\mathcal{L}_{cl} = Tr(F_{\mu\nu}^+ - G_{\mu\nu})(F^{+\mu\nu} - G^{\mu\nu}) \tag{11.5.113}$$

where $TrT_aT_b = -\frac{1}{2}\delta_{ab}$, and $F_{\alpha\beta}^+$ is the self dual part of the gauge field curvature, but $G_{\alpha\beta}$ is a self dual auxiliary field

$$F_{\mu\nu}^+ = \frac{1}{2}\left(F_{\mu\nu} + \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}\right) \tag{11.5.114}$$

Integrating in the path integral over $G_{\mu\nu}$ the action vanishes, and this model is an example of a topological field theory.[6] There are two classical gauge invariances: the usual Yang-Mills symmetry and a peculiar local gauge symmetry with parameter ϵ_μ

$$\begin{aligned} \delta A_\mu &= D_\mu\Lambda + \epsilon_\mu \\ \delta G_{\mu\nu} &= 2D_\mu^+\epsilon_\nu + [G_{\mu\nu}, \Lambda] \end{aligned} \tag{11.5.115}$$

The notation $D_\mu^+\epsilon_\nu$ denotes the self dual part of $D_\mu\epsilon_\nu$, namely $\frac{1}{2}\left(D_{[\mu}\epsilon_{\nu]} + \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}D^\rho\epsilon^\sigma\right)$. The invariance of the classical action is clear from the observation that $F_{\mu\nu}^+$ transforms into $2D_\mu^+\epsilon_\nu + [F_{\mu\nu}^+, \Lambda]$. If one writes the classical action as

$$\begin{aligned} \mathcal{L}_{cl} &= -\frac{1}{4}(F_{\mu\nu}^a)^2 - \frac{1}{4}(F^{\mu\nu,a})({}^*F_{\mu\nu}{}^a) \\ &- (G^{\mu\nu,a})\left[\frac{1}{2}G_{\mu\nu}^a - F_{\mu\nu}^{+a}\right] \end{aligned} \tag{11.5.116}$$

[6]Topological field theories have even less degrees of freedom than a point particle. Whereas a point particle with $x(t)$ and $p(t)$ has an infinite number of states (created by $a^\dagger \sim x + ip$), topological field theories usually have only a finite number of physical states (cohomology classes: polynomials in the fields and derivatives thereof which are BRST closed but not BRST exact).

It is clear that the action is an extension of the θ-term in QCD which vanishes on-shell. The symmetry $\delta A_\mu = \epsilon_\mu$ can be used to gauge away all of A_μ, but we shall apply BRST quantization to this system instead. [19]

The classical gauge invariances have zero modes, hence there are ghosts-for-ghosts. Indeed, the combination $\Lambda = \epsilon$ and $\epsilon_\mu = -D_\mu \epsilon$ leaves δA_μ and $\delta G_{\mu\nu}$ invariant when the classical field equation $-F^+_{\mu\nu} + G_{\mu\nu} = 0$ holds. Denoting the ghosts for Λ and ϵ_μ by c and ψ_μ, respectively, and the ghost-for-ghost for ψ_μ by ϕ, we obtain most of the terms of $S^{\min}$. Requiring $(S^{\min}, S^{\min}) = 0$ fixes the remaining terms

$$\begin{aligned}
&\mathcal{L}^{\min} = \mathcal{L}_{cl} + A^{\mu *}(D_\mu c + \psi_\mu) \\
&+G^{\mu\nu *}(D^+_\mu \psi_\nu + [G_{\mu\nu}, c]) + \psi^{\mu *}(D_\mu \psi + \{\psi_\mu, c\}) \\
&+c^* \left(\frac{1}{2}\{c, c\} + \phi\right) + \left[\phi^*[\phi, c] + \frac{1}{2}\{G^*_{\mu\nu}, G^{\mu\nu *}\}\phi\right]
\end{aligned} \tag{11.5.117}$$

The terms $[\psi_\mu, c]$ and cc in ψ^*_μ and c^* are the usual terms $\frac{1}{2} f_{\alpha\beta\gamma} c^\gamma \gamma^\beta$ which follow from the classical gauge algebra. Two transformations with ϵ^1_μ and ϵ^2_ν commute, while the Yang-Mills algebra yields the cc term. The gauge commutator $[\delta(\epsilon_\mu), \delta(\Lambda)] = \delta([\epsilon_\mu, \Lambda])$ produces the term $[\psi_\mu, c]$. The terms $\psi^{*\mu} D_\mu \phi$ and $\phi^*[c, \phi]$ correspond to the zero modes of the classical gauge transformations ($\delta c = \epsilon$ and $\delta \psi_\mu = D_\mu G$, respectively).

The terms in square brackets are needed to obtain $(S^{\min}, S^{\min}) = 0$. Whereas the term $\phi^*[\phi, c]$ might have been expected from rigid Yang-Mills symmetry, the term with two antifields $G^*_{\mu\nu}$ is less obvious. It is needed to cancel the terms in (S_1, S_1) which are due to varying the fields in $G^{\mu\nu *}(D^+_\mu \psi_\nu + [G_{\mu\nu}, c])$. The variation of ψ_ν into the ghost-for-ghost, $\delta \psi_\nu = D_\nu \phi$ yields a term $G^*[F^+, \phi]$, while the variation of c into the ghost-for-ghost, $\delta c = \phi$ yields a term $G^*[G, \phi]$. Together they yield $(S_1, S_1) = G^*[G - F^+, \phi]$ which is cancelled by $(S_0, S_2) = \partial S_0 / \partial G \frac{\partial}{\partial G^*} S_2 = (F^+ - G)[G^*, \phi]$. Hence, although the classical gauge algebra was closed, the ghosts-for-ghosts require terms with two antifields.

As nonminimal terms we choose $S^{\text{nonmin}} = \pi^{\mu\nu} b^*_{\mu\nu} + \pi b^* + \pi_1 \lambda^*$. With this choice $S = S^{\text{min}} + S^{\text{nonmin}}$ still satisfies $(S,S) = 0$. As gauge fermion we take

$$\psi = b^{\mu\nu} G_{\mu\nu} + b \partial^\mu A_\mu + \lambda D^\mu \psi_\nu \tag{11.5.118}$$

Thus $G_{\mu\nu}$ fixes the local ϵ_μ symmetry, $D^\mu A_\mu = \partial^\mu A_\mu$ fixes as usual the Yang-Mills symmetry, while $D^\mu \psi_\mu$ fixes the ghost-for-ghost symmetry $\psi_\mu = -D_\mu \epsilon$ with $\Lambda = \epsilon$. An important point is now whether there are extra ghosts. If there are local gauge transformations $\delta b^{\mu\nu}, \delta b$ and $\delta\lambda$ such that $\delta\psi = 0$, there are extra ghosts. There are none, hence we have here a reducible system without extra ghosts

6 Gauge-choice independence and master equation

We consider the path-integral in Minkowski space

$$Z_\psi = \int D\phi^A \exp \frac{i}{\hbar} [W_\psi + J(\phi)] \tag{11.6.1}$$

where $W_\psi = W(\phi, \phi^* = \partial\psi/\partial\phi)$ and W equals S plus possibly terms of order $\hbar$ and higher. For $J(\phi)$ we take $J_A \phi^A$. We want to prove that for vanishing J_A, Z_ψ is actually independent of ψ.

Under a small variation $\psi \to \psi + \delta\psi$, we find

$$Z_{\psi+\delta\psi} - Z_\psi = \int D\phi^A \exp \frac{i}{\hbar} [W_\psi + J(\phi)] \left(\frac{i}{\hbar} \partial W / \partial \phi^*_A \frac{\partial}{\partial \phi^A} \delta\psi \right)_{|\Sigma} \tag{11.6.2}$$

Assuming that $\delta\psi$ is independent of antifields, this can be written in terms of the antibracket

$$\delta_\psi Z = \langle -\frac{i}{\hbar} (W, \delta\psi)_{|\Sigma} \rangle \tag{11.6.3}$$

In the Hamiltonian case, one finds a result which looks similar,namely $\delta_\psi Z = \langle -\frac{i}{\hbar} \{Q_H, \delta\psi\} \rangle$. There one shows that this expression vanishes, by making a change of integration variables which is an "almost BRST transformation" obtained by replacing the BRST constant Λ by $\delta\psi$. The action was invariant, but the measure gave a contribution, namely the deviation of the Jacobian from unity was $\{Q, \delta\psi\}$. Since the sum of these two terms vanishes, it followed that $\delta_\psi Z = 0$.

In the BV case, we do not integrate over the full space ϕ^A, ϕ^*_A, but only over the ϕ^A. Nevertheless, we shall use the same "almost BRST transformation" to show that $\delta_\psi Z = 0$. The almost BRST transformation is now defined by

$$\begin{aligned} \delta\phi^A &= \left(\frac{\partial}{\partial\phi^*_A} W(\phi,\phi^*)\right)_{|\Sigma} \Lambda(\phi) \\ \Lambda(\phi) &= \frac{i}{\hbar}\delta\psi \end{aligned} \tag{11.6.4}$$

Let us again make a change of integration variables $\phi \to \phi'^A = \phi^A + \delta\phi^A$. If we do this everywhere, Z_ψ does not change (the Shakespeare theorem: "what is in a name?"). Hence

$$0 = \langle \delta(Jac)\rangle + \langle \delta \frac{i}{\hbar}(W + J(\phi))\rangle \tag{11.6.5}$$

For the Jacobian Jac we get

$$\begin{aligned} \delta(Jac) = \partial\delta\phi^A/\partial\phi^A(-)^A = \left[\left(\frac{\partial}{\partial\phi^*_A} W(\phi,\phi^*)\right)_{|\Sigma} / \frac{\partial}{\partial\phi^A}\right] \Lambda \\ + \left(\frac{\partial}{\partial\phi^*_A} W\right)_{|\Sigma} \left(\partial\Lambda/\partial\phi^A(-)^A\right) \end{aligned} \tag{11.6.6}$$

If we differentiate $\left(\frac{\partial}{\partial\phi^*_A} W(\phi,\phi^*)\right)_{|\Sigma}$ w.r.t. ϕ^A, we get two contributions: one from ϕ in W and one from ϕ^* in W. The latter vanishes, since it reads

$$\left(\frac{\partial^2}{\partial\phi^*_A} W/\partial\phi^*_B\right)_{|\Sigma} \left(\frac{\partial^2}{\partial\phi^B}\psi/\partial\phi^A\right) \tag{11.6.7}$$

which vanishes as the symmetry in ϕ^*_A, ϕ^*_B is opposite to that of ϕ^A, ϕ^B.

For the variation of the exponent we get

$$\begin{aligned} \frac{i}{\hbar}\Bigg[\partial W/\partial\phi^A \left(\frac{\partial}{\partial\phi^*_A} W\Lambda\right) + \partial W/\partial\phi^*_A[\partial(\partial\psi/\partial\phi^A)/\partial\phi^B]\left(\frac{\partial}{\partial\phi^*_B} W\right)\Lambda \\ + \partial J/\partial\phi^A \ \frac{\partial}{\partial\phi^*_A} W\Lambda\Bigg]_{|\Sigma} \end{aligned} \tag{11.6.8}$$

The second term vanishes from symmetry considerations as before. Hence, the variation of the exponent yields

$$\frac{i}{\hbar}\left[\frac{1}{2}(W,W)\Lambda + (J,W)\Lambda\right]_{|\Sigma} \tag{11.6.9}$$

We have thus obtained the identity

$$\begin{aligned} 0 &= [(\Delta W)\,\Lambda - (W,\Lambda)]_{\text{reg}} + \langle\frac{i}{2\hbar}\,(W,W)\,\Lambda + \frac{i}{\hbar}\,(J,W)\,\Lambda\rangle \\ \Delta W &\equiv \left(\frac{\partial}{\partial\phi_A^*}\partial W/\partial\phi_A\right)_{|\Sigma} \end{aligned} \tag{11.6.10}$$

The symbol <> denotes the path-integral average, and “reg” indicates that the Jacobian has as always to be regularized.

If we now take $\Lambda = \frac{i}{\hbar}\delta\psi$, and recall $\delta_\psi Z = \langle -\frac{i}{\hbar}(W,\delta\psi)_{|\Sigma}\rangle$, one obtains

$$\delta_\psi Z = \langle\left\{\Delta W + \frac{i}{2\hbar}(W,W) + \frac{i}{\hbar}(J,W)\right\}_{|\Sigma}\Lambda\rangle \tag{11.6.11}$$

Hence, Z_ψ is ψ-independent for vanishing J, provided $W(\phi,\phi^*)$ satisfies the following *master equation*

$$\langle\left(\Delta W + \frac{i}{2\hbar}(W,W)\right)_{|\Sigma}\rangle = 0. \tag{11.6.12}$$

Going back to $e^{\frac{i}{\hbar}W}$, we see that this can be written as

$$\langle\left(\Delta e^{\frac{i}{\hbar}W}(\phi,\phi^*)\right)_{|\Sigma}\rangle = 0 \tag{11.6.13}$$

Suppose now that W satisfies the stronger condition $\Delta W + \frac{i}{2\hbar}(W,W) = 0$. Expanding $W = S + \hbar M_1 + \hbar^2 M_2 + \cdots$ one obtains then

$$\begin{aligned} (S,S) &= 0 \\ (S,M_1) &= i\Delta S \quad \textit{etc.} \end{aligned} \tag{11.6.14}$$

Hence, the master equation contains the condition $(S,S)=0$, at order $\hbar = 0$, but the terms at higher orders in $\hbar$ can be viewed as the quantum extension of $(S,S)=0$.

7 From Hamiltonian-BRST to BV-BRST

In chapter II we showed that we can obtain the BRST invariant action of the Lagrangian formalism from the BRST-invariant action of the Hamiltonian formalism by integrating out the momenta. In the BV formalism one eliminates the antifields to obtain the quantum action. This suggests that there is a relation between these approaches. In fact, in the Hamiltonian formalism there are variables (the canonical momenta) which are absent in the BV formalism, while in the BV formalism there are variables (the antifields) which are absent in the Hamiltonian formalism.

This suggests to extend the BV action in a Hamiltonian direction, namely to enlarge the set of fields and antifields by including momenta, and then to proceed as before. Indeed, we shall be able to deduce from the Hamiltonian formalism the BV action in Hamiltonian form, *with* conjugate momenta *and* their antifields. We shall start with the minimal set of canonical variables p^i, q_i, C^α and $P(C)_\alpha$. The Lagrange multipliers λ^α will appear as the antifields of the momenta conjugate to the ghosts, $\lambda^\alpha = P(C)^{*\alpha}$. (The ghost numbers agree: C^α has ghost number $+1, P(C)_\alpha$ therefore -1, and $P(C)^*\alpha$ thus zero).

In the Hamiltonian formalism, we have as ingredients the BRST operator

$$Q_H = \pi_\alpha P(B)^\alpha + Q^1_H(p^i, q_i, C^\alpha)P(C)_\alpha, \tag{11.7.1}$$

and the BRST invariant Hamiltonian H_{BRST}

$$\begin{aligned} H_{BRST} &= H_0(p^i, q^i) + C^\alpha V_\alpha{}^\beta P(C)_\beta + \dots \\ &= H(p^i, q_i, C^\alpha, P(C)_\alpha) \end{aligned} \tag{11.7.2}$$

They satisfy the Poisson (or Dirac) brackets

$$\left\{Q^1_H, Q^1_H\right\} = 0, \quad \left\{Q^1_H, H\right\} = 0 \tag{11.7.3}$$

It is clear that the varabiales q_i, p^i, C^α and $P(C)_\alpha$ play a different role from $B_\alpha, P(B)^\alpha$, and since the minimal BV action depends only on the former, this is further evidence that the Hamiltonian and BV approach are related The set $z^a = \{p^i, q_i, P(C)_\alpha, C^\alpha\}$ satisfies the Poisson brackets

$$\left\{z^a, z^b\right\} = \Omega^{ab} \tag{11.7.4}$$

and the BRST transformations of z^a are given by

$$\delta_{BRST} z^a = \left\{z^a, Q_H^1 \Lambda\right\} \tag{11.7.5}$$

We can at once write down a BV action S_H depending on z^a and z_a^*, which reproduces these transformations and which is BRST invariant. It reads

$$S_H = \frac{1}{2}\Omega_{ba}\dot{z}^b z^a - H - \left\{\psi, Q_H^1\right\} + z_a^* \left\{z^a, Q_H^1\right\} \tag{11.7.6}$$

(The requirement of BRST invariance of the first term fixes the definition of the matrix Ω_{ab} as follows. From

$$\Omega_{ba}\dot{z}^b\{z^a, Q_H^1\} = \Omega_{ba}\dot{z}^b\{z^a, z^c\}\frac{\partial}{\partial z^c}Q_H^1 = \Omega_{ba}\Omega^{ac}\dot{z}^b\frac{\partial}{\partial z^c}Q_H^1 \tag{11.7.7}$$

we see that Ωba must be proportional to the inverse of Ω^{ac}. We put

$$\Omega_{ba}\Omega^{ac} = \delta_b{}^c \tag{11.7.8}$$

in which case one obtains $\frac{d}{dt}Q_H^1$. The overall sign in the action is fixed by requiring that for ordinary bosonic variables one obtains $p^i\dot{q}_i$. (Recall that $\{q_i, p^j\} = \delta_i{}^j$ in our conventions). Since the BV bracket satisfies $(z^a, z_b^*) = \delta^a{}_b$, we indeed obtain

$$\delta_{BRST} z^a = (z^a, S_H)\Lambda = \{z^a, Q_H^1\}\Lambda \tag{11.7.9}$$

Since Q_H^1 is nilpotent, S_H generates *always* nilpotent BRST transformations, like in Hamiltonian formalism but unlike non-Hamiltonian BV formalism. This suggests that also S_H is nilpotent. To prove this we note the following

$$\begin{aligned}(S_H, S_H) &= \partial S_H/\partial z^a \quad \partial/\partial z^*{}_a S_H = \\ &= \partial S_H/\partial z^a \left\{z^a, Q_H\right\} = \left\{S_H, Q_H\right\} = 0\end{aligned} \tag{11.7.10}$$

which vanishes since each term in S_H is separately BRST invariant. (The last term is BRST invariant because $\{z_a^*, Q_H^1\} = 0$ since $Q_H^1(z^a)$ only has a nonvanishing bracket with z^b, not with z_b^*. We can rewrite the last term of S_H as

$$\begin{aligned}
z_a^* \left\{ z^a, Q_H^1 \right\} &= -\frac{1}{2}\Omega^{ab}(z_a^* z_b^*, Q_H^1) \\
\text{since } \left(z_b^*, Q_H^2 \right) &= -\frac{\partial}{\partial z^b} Q_H^1 (\text{which equals } \Omega_{bc}\{z^c, Q_H\}) \\
\text{because } \{z^c, Q\} &= \left\{ z^c, z^d \right\} \frac{\partial}{\partial z^d} Q = \Omega^{cd} \frac{\partial}{\partial z^d} Q \\
\text{while} \quad \Omega_{bc}\Omega^{cd} &= \delta_b{}^d.
\end{aligned} \tag{11.7.11}$$

Hence, we have found the following BV action in Hamiltonian form

$$S_H = \frac{1}{2}\Omega_{ba}\dot{z}^b z^a - H - \left\{ \psi, Q_H^1 \right\} - \frac{1}{2}\Omega^{ab}\left(z_a^* z_b^*, Q_H^1 \right) \tag{11.7.12}$$

Note that S_H is nilpotent and contains at most one antifield, for any choice of ψ. We can view S_H as a gauge-fixed BV action, and by putting all antifields equal to zero, we should get the quantum action. Clearly, it coincides then with the quantum action of the Hamiltonian formalism. More interesting is to first declare some fields to be new antifields and the corresponding antifields to be minus the new fields - and then to put all antifields equal to zero. As we shall show the quantum action then takes on the familiar BV form. (The replacement $\phi^* = \psi$ and $\phi = -\psi^*$ preserves the BV bracket, and is thus a canonical transformation).

We shall now illustrate these ideas with some examples. We begin with the simplest case: gauge theories whose Hamiltonian bracket algebra has rank 1. The minimal BRST operator is then

$$Q_H^1 = C^\alpha \varphi_\alpha + \frac{1}{2} C^\alpha C^\beta f_{\beta\alpha}{}^\gamma P(C)_\gamma \tag{11.7.13}$$

Hence

$$\begin{aligned}
S_H \;=\; & p_i \dot{q}^i + \dot{C}^\alpha P(C)_\alpha - H + \\
& q^*\{q, Q_H^1\} + p^*\{p, Q_H^1\} + C^*\{C, Q_H^1\} + P(C)^*\{P(C), Q_H^1\}
\end{aligned}$$

$$\begin{aligned} = \; & \Big[p_i\dot{q}^i - H + q^*\{q, C^\alpha\varphi_\alpha\} + p^*\{p, C^\alpha\varphi_\alpha\} \\ & + C^*\left(\frac{1}{2}C^\alpha C^\beta f_{f\alpha}{}^\gamma\right)\,] + \dot{C}P(C) \\ & + P(C)^*\left(\varphi_\alpha + f_{\alpha\beta}{}^\gamma P(C)_\gamma C^\beta\right) \end{aligned} \tag{11.7.14}$$

Let us now make the canonical transformation

$$P(C)^{*\alpha} = \lambda^\alpha \quad , \quad P(C) = -\lambda^*_\alpha \tag{11.7.15}$$

Then

$$\begin{aligned} S_H \;=\; & \Big[p_i\dot{q}^i - H + \lambda^\alpha\varphi_\alpha + \phi^*_A\delta_{BRST}\phi^A\Big] \\ & +(\lambda^*_\alpha)\Big[\dot{C}^\alpha - \lambda^\beta C^\gamma f_{\gamma\beta}\alpha\Big] \end{aligned} \tag{11.7.16}$$

Clearly, we obtain now the action in Hamiltonian form, *with* Lagrange multipliers λ^α

$$S_0 = p_i\dot{q}^i - H + \lambda^\alpha\varphi_\alpha \tag{11.7.17}$$

together with S_1 which gives the BRST laws of q, p, C *and* λ. The latter reads

$$\delta\lambda^\alpha = \dot{C}^\alpha + f^\alpha{}_{b\gamma}\lambda^\beta C^\gamma \tag{11.7.18}$$

which is the transformation law for the time-components of the gauge fields.

For example, for a point particle we get

$$S_H = \dot{x}p + \dot{c}b - \frac{1}{2}(p^2 + m^2) + \lambda(p^2 + \dot{x}^* cp) \tag{11.7.19}$$

and defining

$$g = 1 + \lambda \tag{11.7.20}$$

we arrive at the usual result

$$S_H = \dot{x}p - \frac{1}{2}g(p^2 + m^2) + g^*\dot{c} + \dot{x}cp \tag{11.7.21}$$

8 Anomalies

If $\Gamma * \Gamma = \Delta$ then Δ is the consistent anomaly. It may depend on antifields (=BRST sources). For example, in a supersymmetric theory with auxiliary fields H^r, the BRST invariant action is the usual one, with only term linear in antifields. The anomaly satisfies the Wess-Zumino consistency condition $\Gamma * \Delta = 0$, and to lowest order in $\hbar$ the anomaly (denoted by Δ_1) is local and satisfies $\Gamma_0 * \Delta_1 = 0$ (where Γ_0 includes terms depending on H^r and $H^*{}_r$). One can then set the antifields for the auxiliary fields H^*_r equal to zero, and solve the auxiliary fields from the rest of the action: $H^r = H^r(\phi^A, \phi^*{}_A)$. If initially Δ_1 depended on H^r, then after elimination of H^r (and $H^*{}_r$ by setting $H^*{}_r = 0$) the object Δ_1 may start to depend on $\phi^*{}_A$. One can show that this new Δ_1 satisfies $\Gamma'_0 * \Delta_1 = 0$ where $\Gamma'_0 *$ is now the bracket obtained from $\Gamma_0(\phi, \phi^*, H(\phi, \phi^*), H^* = 0)$ in the usual way. This mechanism is similar to the generation of 4-ghost couplings after eliminating auxiliary fields from the quantum action in supergravity.

In supersymmetric theories this phenomenon may occur. In $N = 1$ susy YM theory, the loop calculations to obtain Δ_1 have not been performed but one finds that the susy extension of the abelian chiral anomaly $\Delta_1 = d_{IJK} \int d^4x \epsilon^{\mu\nu\rho\sigma} C^I F_{\mu\nu}{}^J F_{\rho\sigma}{}^K d^4x$ does not depend on auxiliary fields. However, for objects with dimension 3 instead of 5, they exist. For example, if C^I are ghosts for abelian gauge fields and D^I are auxiliary fields one finds a candidate for an anomaly

$$\Delta(\dim 3, \text{ghost } \#1) = \sum_{I,J} k_{I,J} \int d^4x (C^I D^J + \ldots) \tag{11.8.1}$$

where $k_{IJ} = -k_{JI}$, so that one needs at least two abelian gauge fields. One can also write down candidates for Δ_1 with dimension 5 and at least two abelian gauge fields which depend in a nontrivial way on antifields (nontrivial means: not as a BRST exact term).

F. Brandt, *Phys. Lett B* 1993.

G. Barnick and M. Henneaux, 1993. (Renorm. of gauge in v. ops. and anomalies in YM theory.)

Brandt has given explicit examples of Δ's depending on gauge invariant Noether currents in theories with at least two abelian gauge fields. We have also seen that such dependences can arise upon the **elimination** of auxiliary fields when the symmetry algebra is open. We now show, conversely, that upon the **addition** of auxiliary fields as one passes from Lagrange to Hamilton formulation, antifield dependence can arise.

We begin with the extended Fradkin-Vilkovisky action using (270) and (282)

$$\begin{aligned} S &= \int dt\{p^i\dot{q}_i + \pi(\lambda)^\alpha\dot{\lambda}_\alpha + \dot{c}^\alpha\pi(c)_\alpha + \dot{\pi}(b)^\alpha b_\alpha - (H_0 + c^\alpha V_\alpha{}^\beta\pi(c)_\beta) + \left\{\Psi, Q_H^{BRST}\right]_{PB} \\ &+ K_{p^i}\left\{p^i, Q_H^{BRST}\right]_{PB} + K_{q^i}\left\{q^i, Q_H^{BRST}\right]_{PB} + K_{\lambda^\alpha}\pi(b)^\alpha \\ &+ K_{c^\alpha}\left(-\frac{1}{2}c\Lambda c\right)^\alpha + K_{\pi(c)^\alpha}(-\psi^\alpha - (\pi(c)\Lambda c)^\alpha) + K_{b^\alpha}(-\pi(\lambda)^\alpha)\}, \end{aligned} \tag{11.8.2}$$

where the K's are sources for all non-inert BRST variations of fields. The BRST charge of (281) is given by $Q_H^{BRST} = c^\alpha\psi_\alpha + \pi(b)^\alpha\pi(\lambda)_\alpha + \frac{1}{2}(c\Lambda c)^\alpha\pi(c)_\alpha$ and generates canonical BRST transformations via the graded Poisson bracket $\{\ ,\ ]_{PB}$. The ψ_α are first class constraints with algebra $\{\psi_\alpha, \psi_\beta]_{PB} = f_{\alpha\beta}{}^\gamma\psi_\gamma, \{\psi_\alpha, H_0]_{PB} = V_\alpha{}^\beta\psi_\beta$. We denote $f_{\beta\gamma}{}^\alpha X^\alpha Y^\beta = (X\Lambda Y)^\alpha$ and repeated indices indicate also $\int d^3x$. As gauge fixing fermion we take (284) $\Psi = b_\alpha\chi^\alpha - \pi(c)_\alpha\lambda^\alpha$. Observe now that

$$\begin{aligned} \Gamma^{(0)}\star(\lambda^\alpha - K^\alpha_{\pi(c)}) &= (\delta S/\delta K_{\lambda\beta}\frac{\delta}{\delta\lambda^\beta} - \delta S/\delta\lambda^\beta\frac{\delta}{\delta K_{\lambda^\beta}} \\ &+ \delta S/\delta K_{\pi(c)^\beta}\frac{\delta}{\delta\pi(c)^\beta} - \delta S/\delta\pi(c)^\beta\frac{\delta}{\delta K_\pi(c)^\beta})(\lambda^\alpha - K^\alpha_{\pi(c)}) \\ &= \dot{c}^\alpha + V^\alpha{}_\beta c^\beta + ((\Lambda - K_{\pi(c)})\Lambda c)^\alpha. \end{aligned} \tag{11.8.3}$$

But this is exactly the BRST transformation of λ^α in the Lagrange formulation, see (144), only with λ replaced by $\lambda - K_{\pi(c)}$. Hence, since p_i, q^i and c^α have the same transformations in both Hamilton and Lagrange formulation (p_i albeit as a composite object in the latter), any consistent anomaly in the Lagrange formulation can be converted to a consistent phase space anomaly via the replacement $\lambda \to \lambda - K_{\pi(c)}$.

We now give two examples:

I) The consistent Adler-Bardeen-Bell-Jackiw (ABBJ) anomaly for Yang Mills fields coupled to chiral fermions.

It is easy to show that $\Delta = tr \int c(dAdA + \frac{1}{2}d(AAA))$ is a consistent anomaly where $A = A_\mu dx^\mu = \frac{i}{2}\tau^a A^a_\mu dx^\mu, c = \frac{i}{2}\tau^a c^a$ (we take the fundamental representation of $SU(2)$ for simplicity). Using that $(0 = trcFF; F = dA + AA)$ we may write

$$\begin{aligned} -\Delta &= tr \int c(AAAA + \frac{1}{2}dAAA) \\ &= \frac{1}{4}\int d^4x\{(\vec{E}^b + \epsilon^{bcd}A^c_0\vec{A}^d)\cdot(\vec{A}^b \times \vec{A}^a) + (\vec{A}^a A^b_0 + \vec{A}^b A^a_0)\cdot((\vec{\nabla}\times\vec{A}^b) \\ &+ \frac{1}{4}\epsilon^{bcd}(\vec{A}^c \times \vec{A}^d))\Big\} c^a \end{aligned} \tag{11.8.4}$$

Hence

$$\begin{aligned} \Delta &= -\frac{1}{4}\int d^4x \Big\{(\vec{E}^b + \epsilon^{bcd}(A^c_0 - K^c_{\pi(c)})\vec{A}^d)\cdot(\vec{A}^b \times \vec{A}^a) \\ &+ (\vec{A}^a(A^b_0 - K^b_{\pi(c)}) + \vec{A}^b(A^a_0 - K^a_{\pi(c)}))\cdot((\vec{\nabla}\times\vec{A}^b) + \frac{1}{4}\epsilon^{bcd}(\vec{A}^c \times \vec{A}^d))\Big\} c^a \end{aligned} \tag{11.8.5}$$

is the consistent ABBJ anomaly in phase space.

II) Virasoro gravity.

We take the action

$$S = \frac{1}{\pi}\int d^2z\left(-\frac{1}{2}\bar{\partial}\varphi^i\partial\varphi^i + \frac{1}{2}h\partial\varphi^i\partial\varphi^i\right) \tag{11.8.6}$$

which is invariant under $\delta\varphi^i = \epsilon\partial\varphi^i, \delta h = \bar{\partial}\epsilon + \epsilon\partial h - \partial\epsilon h$. Stelle et al.[?] found it advantageous to consider a derivative gauge fixing $\bar{\partial}h = 0$. The quantum action is then

$$S = \frac{1}{\pi}\int d^2z\left(-\frac{1}{2}\bar{\partial}\varphi^i\partial\varphi^i + \frac{1}{2}h\partial\varphi^i\partial\varphi^i + Q(b\bar{\partial}h)\right). \tag{11.8.7}$$

where $Q\varphi^i = c\varphi^i, Qh = \bar{\partial}c + c\partial h - \partial ch, Qc = c\partial c, Qb = \pi_{h,}Q\pi_h = 0$. And the usual consistent Virasoro anomaly is given by

$$\Delta = \int d^2zc\partial^3h.$$

However upon casting the action (11.8.7) into "1st order" (Hamiltonian) form

$$S = \frac{1}{\pi}\int d^2z \left(-\frac{1}{2}\bar{\partial}\varphi^i\partial\varphi^i + h\left(\frac{1}{2}\partial\varphi^i\partial\varphi^i + 2\pi_c\partial c + \partial\iota_c c\right) + \pi_h\bar{\partial}h - \pi_b\bar{\partial}b - \pi_c\bar{\partial}_c - \pi_b\pi_c\right)$$

Stelle et al. found a non-trivial antifield dependence in the Virasoro anomaly

$$\Delta = \int d^2zc\partial^3(h - K_{\pi_c})$$

whose origin is made clear by the preceding discussion.

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Chapter 12

The Yang-Baxter equation and the algebraic Bethe ansatz

The Yang-Baxter (YB) equation [1, 2] has been used in many areas: to obtain exact S matrices in $1+1$ dimensions [1, 3], to compute partition functions of classical models in statistical mechancs on two-dimensional lattices [2], and to study link and knot invariants in 3 dimensions. [4] It also appears in quantum groups [5] and in noncommutative geometry [6]. In this chapter we use a concrete physical model, the spin 1/2 Heisenberg chain, to study the YB equation. It also allows us to introduce quantum groups.

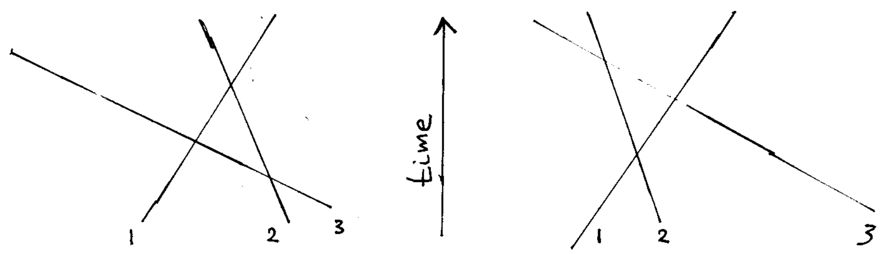
1 The Yang-Baxter equation

The YB equation is often introduced in an abstract manner, but it may help understanding its physical meaning if one begins instead with a concrete model in which the YB equation arises naturally. We shall mainly be interested in the application to the 1-dimensional Heisenberg chain of spin 1/2 particles [7], but in this introduction we begin with another application: the scattering of particles in $1+1$ dimensions.

Consider particles with indices α ($\alpha = 1, n$) with a 2-body S-matrix $S_{\alpha\beta}^{\alpha'\beta'}(\lambda - \mu)$

where λ and μ are the rapidities ($E = m\cosh\mu$ and $p = m\sinh\mu$, so $E^2 - p^2 = m^2$. Since $(p_1 - p_2)^2 = -m_1^2 - m_2^2 + 2m_1 m_2 \cosh(\mu_1 - \mu_2)$, the S matrix only depends on the difference of the rapidities). The S-matrix is an $n^2 \times n^2$ matrix, and for spin 1/2 particles $\alpha = 1, 2$ can be interpreted as their helicity. For spinless particles without internal indices, the S matrix is just a phase, $|S(\lambda_1 - \lambda_2)| = 1$, but in the more general case it is a unitary matrix.

Consider now the scattering of particles 1,2 and 3 by means of 2-body S matrices. The particles keep their momenta and energies due to energy momentum conservation in $1+1$ dimensions. As the figure indicates, assuming that time runs up, first



(12.1.1)

particles 2 and 3 may scatter, and then particles 1 and 3 and finally particles 1 and 2. Or, when particle 3 comes later, one has the situation that first particles 1 and 2 scatter, then particles 1 and 3, and finally particles 2 and 3. Requiring that the final 3-body S-matrix be the same leads to the equation

$$\begin{aligned} S^{(3)}_{123} &= S_{12}(\lambda_1 - \lambda_2) S_{13}(\lambda_1 - \lambda_3) S_{23}(\lambda_2 - \lambda_3) \\ &= S_{23}(\lambda_2 - \lambda_3) S_{13}(\lambda_1 - \lambda_3) S_{12}(\lambda_1 - \lambda_2) \end{aligned} \tag{12.1.2}$$

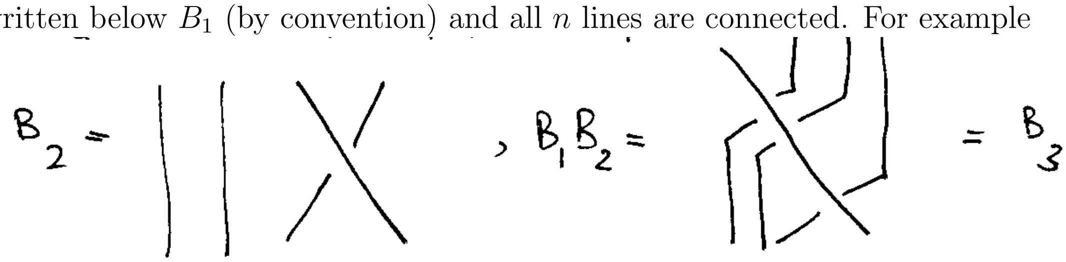
(The notation S_{12} stand for $S_{\alpha\beta}{}^{\alpha'\beta'}$ and indicates that one is dealing with the scattering of particles 1 and 2). Conventionally one writes R_{12} instead of S_{12}, and (12.1.2) can be written more explicitly as follows

$$\begin{aligned} & \left(R_{\gamma_1\gamma_2}{}^{\delta_1\delta_2} \delta_{\gamma_3}{}^{\delta_3}\right) \left(R_{\beta_1\beta_3}{}^{\gamma_1\gamma_3} \delta_{\beta_2}{}^{\gamma_2}\right) \left(R_{\alpha_2\alpha_3}{}^{\beta_2\beta_3} \delta_{\alpha_1}{}^{\beta_1}\right) \\ = & \left(R_{\gamma_2\gamma_3}{}^{\delta_2\delta_3} \delta_{\gamma_1}{}^{\delta_1}\right) \left(R_{\beta_1\beta_3}{}^{\gamma_1\gamma_3} \delta_{\beta_2}{}^{\gamma_2}\right) \left(R_{\alpha_1\alpha_2}{}^{\beta_1\beta_2} \delta_{\alpha_3}{}^{\beta_3}\right) \end{aligned} \tag{12.1.3}$$

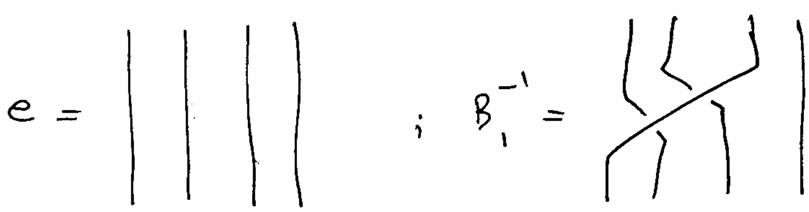
This is the YB equation. In this model the S matrix depends only on the difference of rapidities $\lambda - \mu$, but there are other models, for example the chiral Potts model and the Hubbard model, where R depends on λ and μ separately. Not all 3-body S matrices factorize in this way into 2-body S matrices, for example $\lambda\varphi^4$ theory in $1+1$ dimensions is a counter example. But sine-Gordon theory and Toda field theories have factorizable S matrices.[1] In field theories one also has further relations due to unitarity and crossing symmetry relating the matrix $S(\lambda-\mu)$ to $S(i\pi-\lambda+\mu)$.

Another way to introduce the Yang-Baxter equation is by means of braid group. A braid is a strand of n lines which may cross each other , and run from bottom to top. For example for $n = 4$

By definition the lines which cross from right to left pass above the lines which they cross. One defines a multiplication of two braids B_1 and B_2 by $B_1 B_2$, where B_2 is written below B_1 (by convention) and all n lines are connected. For example



There is a unit element (all n lines parallel) and for each braid B there is an inverse (the braid obtained by reflection along a horizontal line).



Note that in B_1^{-1} the left–movers cross the right–movers by passing below them

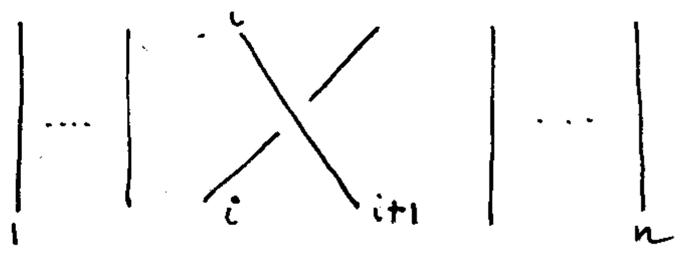
[1] In theories with factorizable 2-body S-matrices, no particle production can appear. One may check that in sine-Gordon theory there is no particle production, but only elastic scattering such as $a+b \to c+d$ scattering. Graphs with loops cancel tree graphs in the sine-Gordon model because one can absorb $\hbar$ into the coupling constant β after rescaling the scalar field φ [8].

if in B_1 they pass above, and vice-versa. So the braid group must contain both types of crossing. First adding a braid B_1 on top of B_2 and then a braid B_3 to the bottom of B_2, or first B_3 to the bottom of B_2 and then B_1 to the top of B_2 yields the same result because these operations do not interfere with each other (what happens at the bottom is unrelated to what happens at the top). Thus the multiplication is associative:

$$(B_1 B_2) B_3 = B_1 (B_2 B_3) \tag{12.1.4}$$

We have shown that braids with overhead crossings and underneath crossings form a group: the braid group.

Consider now special elements of the braid group, denoted by $s_1,\ s_2,\ \dots s_{n-1}$, where s_i is the braid which interchanges line i and $i+1$



It is clear that s_i and s_j commute if $|i-j| \geq 2$ (because they are then out of each other way), but s_i and s_{i+1} get tangled up. They satisfy, however, a simple equation

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \tag{12.1.5}$$

This is actually the Yang-Baxter equation again, which is clear if one rewrites s_i as $S_{\alpha_1\alpha_2}^{\ \ \beta_1\beta_2}$ with $S_{\alpha_1\alpha_2}^{\ \ \beta_1\beta_2} = \delta_{\alpha_1}^{\beta_2}\delta_{\alpha_2}^{\beta_1}$. Denoting $S_{\alpha_1\alpha_2}^{\ \ \beta_1\beta_2}$ by $\check{R}_{12}$, we can write (12.1.5) as follows

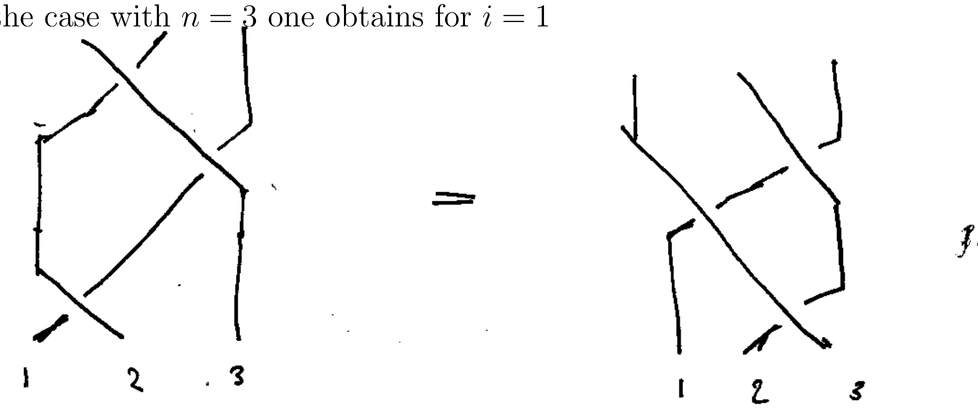
$$\check{R}_{12}\check{R}_{23}\check{R}_{12} = \check{R}_{23}\check{R}_{12}\check{R}_{23} \tag{12.1.6}$$

This is also called the YB equation, even though it appears to be different from the equation we obtained before which reads

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \tag{12.1.7}$$

These two equations are actually equivalent as can be seen by definig $P_{12}R_{12} = \check{R}_{12}$ where P_{12} is the exchange operator of particles (or lines) 1 and 2, defined by $P_{12} = \delta^{\beta_2}_{\alpha_1}\delta^{\beta_1}_{\alpha_2}\delta^{\beta_3}_{\alpha_3}$. Multiplication of the left-hand side of the R equation by $P_{12}P_{12}$, and multiplication of the right-hand side by $P_{23}P_{23}$ converts the R equation into the $\check{R}$ equation.[2]

For the case with $n = 3$ one obtains for $i = 1$



These braids are equal (note the equal sign in $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$) in the following sense: one can go from one to the other by stretching and shrinking. So they are topologically equivalent, and topologically equaivalent braids are identified (they correspond to the same abstract element of the braid group).

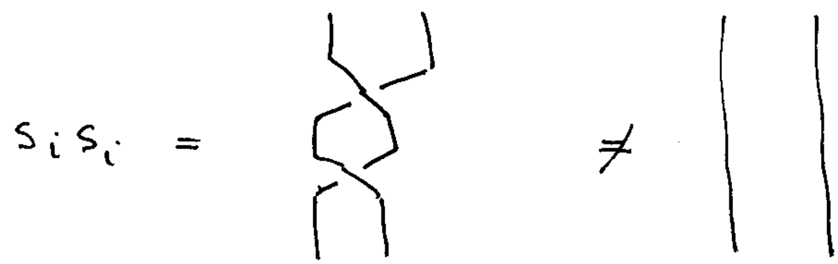
The definitions of the Yang-Baxter equation by means of 2-body scattering amplitudes and by means of braids are actually equivalent if one defines that the S-matrix corresponds to braids where left-moving lines cross over right-moving lines. The inverse of the S-matrix (S^{-1}) corresponds to the opposite case.

[2]The details are as follows: $P_{12}P_{12}R_{12}R_{13}R_{23} = (P_{12}R_{12})P_{12}R_{13}R_{23} = \check{R}_{12}P_{12}R_{13}R_{23}$. Since we must convert $P_{12}R_{13}$ into $\check{R}_{23}$ we insert $P_{23}P_{23}$ and obtain $\check{R}_{12}P_{23}(P_{23}P_{12}R_{13})R_{23} = \check{R}_{12}P_{23}(R_{23}P_{23}P_{12})R_{23} = \check{R}_{12}\check{R}_{23}(P_{23}P_{12}R_{23})$. We used that pulling P_{ab} to the right (or left) of any expression interchanges a and b in that expression. Finally we must convert $P_{23}P_{12}R_{23}$ into $\check{R}_{12}$. We therefore move P_{12} to the left of P_{23} and obtain $P_{23}P_{12}R_{23} = P_{12}P_{13}R_{23} = P_{12}R_{12}P_{13} = \check{R}_{12}P_{13}$. So we got the left-hand side of the $\check{R}$ equation, but with an extra factor P_{13} on the right-hand side. Repeating these steps for the right-hand side of the R equation which we multiply with $P_{23}P_{23}$, we find the right-hand side of the $\check{R}$ equation, again multiplied by the same factor P_{13}. Finally we remove these P_{13} by multiplying with P_{13}.

The braid group is clearly generated by the elementary braids s_i, which satisfy

$$s_i s_j = s_j s_i \quad \text{for } |i-j| \geq 2; \quad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \tag{12.1.8}$$

These are the same relations as the relations which define the symmetric group S_n of n objects (the group of permutation). However, there is a difference: in S_n one has the relation $s_i s_i = 1$, but in the braid group interchanging two objects twice need not yield the same situation as before



The YB equation is clearly invariant under the similarity transformation

$$S' = (M \otimes M) S (M \otimes M)^{-1} \tag{12.1.9}$$

However, this invariance group is not very large ($M \otimes M$ has n^2 independent parameter's, but S has n^4 parameters). There are more general similarity transformations of S satisfying (12.1.2) by matrices $\mathcal{M}$ which are not of the form $M \otimes M$; these matrices arise in the theory of quantum groups, but the complete number of such matrices (and their meaning) is not known. At first sight it may seem impossible that the YB can have any solutions. After all, the YB equation for ${S^{(3)}}_{\alpha\beta\gamma}{}^{\alpha'\beta'\gamma'}$ consists of n^6 equations for the n^4 parameters of S. Nevertheless, there are solutions. For example, for $n = 2$ (leading to 4×4 matrices for S) there are 12 solutions. For other values of n, the exact number of solutions is not known.

The YB equation was first introduced by C.N. Yang [1] who considered particles on a line, scattering by means of potentials with delta function singularities

$$V = \sum_{1 \leq i < j \leq N} 2c\delta(x_i - x_j) \tag{12.1.10}$$

Later R.J. Baxter considered two-dimensional lattices, where the four sides at each vertex carry Boltzmann weights $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ and introduced a matrix

$$R_{\alpha_1\alpha_2\alpha_3\alpha_4}(\lambda - \mu) \tag{12.1.11}$$

The parameter $\lambda - \mu$ depended on $e^{-E/kT}$ and characterized the energy and temperature.

One can also write down the expression for N-body S matrices (still in 1+1 dimensions) instead of 3-body S matrices in terms of products of $\frac{1}{2}N(N-1)$ 2-body S matrices. It is clear by moving one particle "up in time" that one does not get new information from the factorization of $S^{(N)}$ beyond what one got from the factorization of $S^{(3)}$, because when a 3-line vertex is created one only needs the factorization of $S^{(3)}$: factorization is a local concept.

One can also generalize the YB equation to higher dimensions. In $2+1$ dimensions one may consider lines instead of points, which move in a plane with different velocities. Two lines always cross at some point (if they are not parallel), but an "event" occurs then when 3 lines cross at the same point. In order to introduce the notion of factorizability one must consider 4 lines, and then the condition that the "4 line S-matrix" factorizes leads to "tetrahedron equations" which are a generalization of the YB equation. [8]

The range of the parameters λ and μ depends on the model. In models with trigonometric functions $\sin(\lambda - \mu)$, the range of $\lambda - \mu$ is clearly from 0 to 2π, but other models may require all complex λ and μ.

One can also consider one or more boundaries on the infinite line. One obtains then topologically nontrivial spaces (the half-line or line segments), and the requirement that $S^{(3)}$ factorizes leads then to another generalization of the YB equation. We shall discuss this in more detail in the case of the Heisenberg chain, but for particle scattering we mention that one introduces an $n \times n$ reflection matrix $K(\lambda)$ to describe reflection at the wall, which depends on the rapidity λ of the particle that is reflected at the boundary

$$K_\alpha{}^{\alpha'}(\lambda) \tag{12.1.12}$$

The reflection YB equation was introduced by Thierry Mieg, and used by Zhamolodchikov (who called it the boundary YB equation) for conformal field theory on a half-plane [12].

The Yang solution is of the form

$$S = \frac{\lambda - \mu - icP}{\lambda - \mu + ic} \tag{12.1.13}$$

where P is the permutation parameter which interchanges the two particles which scatter, $Pv \otimes w = w \otimes v$. Clearly, S is unitary. For example, if $\alpha, \beta = 1, 2$, there are four states 11,12,21,22 and S takes the form

$$S(\lambda - \mu) = \frac{1}{\lambda - \mu - ic} \begin{pmatrix} \lambda - \mu - ic & 0 & 0 & 0 \\ 0 & \lambda - \mu & -ic & 0 \\ 0 & -ic & \lambda - \mu & 0 \\ 0 & 0 & 0 & \lambda - \mu - ic \end{pmatrix} \tag{12.1.14}$$

Then $S^\dagger(\lambda - \mu) = S(\mu - \lambda)$. Omitting an overall scale factor one may consider the following solution for the R-matrix

$$R = (\lambda - \mu)I + \eta P \tag{12.1.15}$$

where η is the same for all 3 factors , while $\lambda - \mu$ are different. One class of solutions one may study are the "constant solutions", namely solutions which do not depend on the spectral parameters (rapidities) $\lambda - \mu$. There are two obvious solutions. First the unit matrix I is a solution of the YB equation. Also ηP is a solution. If one generalizes I to a diagonal matrix

$$R_{\alpha_1\beta_1}{}^{\alpha_2\beta_2} = d_{\alpha_1}{}^{\alpha_2}\delta_{\alpha_1}{}^{\alpha_2}\delta_{\beta_1}{}^{\beta_2} \tag{12.1.16}$$

the YB equation is still satisfied. However, if one tries to generalize the permutation matrix P to a P-term with an arbitrary coefficient

$$R_{\alpha_1}{}^{\beta_2} = P_{\alpha_1\alpha_2}\delta^{\beta_2}_{\alpha_1}\delta^{\beta_1}_{\alpha_2} \tag{12.1.17}$$

then this is only a solution if $P_{\alpha_1\alpha_2}$ is independent of α_1 and α_2.

We can of course remove the coefficient in front of the unit matrix by an overall rescaling. It is interesting to study the YB equation for small η. We expand $R(\lambda,\mu)$ as follows

$$R(\lambda,\mu) = I + \eta r(\lambda,\mu) + \mathcal{O}(\eta^2) \tag{12.1.18}$$

The terms without η and linear in η satisfy the YB equation trivially, but the terms quadratic in η lead to an interesting equation

$$\begin{aligned} &r_{12}(\lambda-\mu)r_{13}(\lambda-\nu) + r_{12}(\lambda-\mu)r_{23}(\mu-\nu) \\ &+r_{13}(\lambda-\nu)r_{23}(\mu-\nu) = \text{ same with } r_{23} \Leftrightarrow r_{12} \end{aligned} \tag{12.1.19}$$

Moving the terms on the right-hand side to the left yields

$$[\tau_{12},\tau_{13}] + [\tau_{12},\tau_{23}] + [\tau_{13},\tau_{23}] = 0 \tag{12.1.20}$$

This equation is sometimes called the classical YB equation because the expansion in terms of η is a kind of semiclassical expansion, and the first nontrivial terms are the terms with η^2. This equation yields a Lie-algebra structure if one views the operators τ_{12} as being composed of generators A_j and B_j acting in 3-spaces. For example, $r_{12} = \sum_j A_j \otimes B_j \otimes I$ and $r_{13} = \sum_j A_j \otimes I \otimes B_j$. Later we will discuss the Heisenberg spin 1/2 chain where operators $\sum_{j=1}^3 \sigma^j_{(k)} \otimes \sigma^j_{(a)}$ appear. More generally, for the spin S chain one encounters operators of the form $\sum_{j=1}^3 S^j_{(k)} \otimes \sigma^j_{(a)}$. The operators S act then in the first space, and $\sigma_{(a)}$ acts in the a-th space where $a = 1, 2$.

We now turn to an application to the spin 1/2 Heisenberg chain [27]. It consists of L sites with at each site a spin 1/2 particle whose spin can point up $|\uparrow\rangle$ or down

$|\downarrow\rangle$. The Hamiltonian is given by

$$H = J \sum_{j=1}^{L} \sum_{\alpha=1}^{3} \sigma_j^\alpha \sigma_{j+1}^\alpha \tag{12.1.21}$$

where σ_j^α are the 3 Pauli matrices at site j, and we impose the periodic boundary conditions $\sigma_{L+1} = \sigma_1$.[3] This is thus a model with only nearest-neighbor interactions. Heisenberg introduced this model in 1928 [7] to describe ferromagnetism ($J < 0$). (Antiferromagnetism corresponds to ($J > 0$)). In 1931 Bethe found the complete energy spectrum (2^L states) by making an ansatz for the x-dependence of the eigenfunctions (the coordinate Bethe ansatz [11]), so this covered both ferromagnetism and ferromagnetism. The ground state (lowest-energy state) is however very diferent in each case. An algebraic version of this ansatz, in terms of 2×2 operators which are x-independent, which will also appear in the discussion below of the solutions of the YB equation, is called the algebraic Bethe ansatz (ABA). We will show that in terms of Pauli matrices one can define an R-matrix and a T-matrix, such that

$$R(u,v)(T(u) \otimes I)(I \otimes T(v)) = (I \otimes T(v))(T(u) \otimes I)R(u,v) \tag{12.1.22}$$

where u and v are arbitrary complex numbers, and $R(u,v)$ is a matrix of numbers but the entries of the matrix $T(u)$ are operators. By multiplication by $(R(u,v))^{-1}$ from the right and taking the trace, one finds then that the operators $t(u) \equiv trT(u)$ and $t(v) \equiv trT(v)$ commute. (The operator $trT(u)$ is called the transfer matrix while $T(u)$ itself is called the monodromy matrix). Hence the set of operators $t(u)$ for all complex u form a set of commuting operators, and the Hamiltonian will be related to the derivative of $t(u)$. To diagonalize the operators $t(u)$ (and thus in particular to find the energy spectrum) the ansatz is made that the operator $B(u)$ in $T(u) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is a kind of creation operator. Repeated application of $B(u_1) \cdots B(u_N)$ to the Fock vacuum $|\uparrow\rangle_1 \otimes \cdots |\uparrow\rangle_N$ will lead to eigenstates of $t(u) = trT(u)$ provided the complex numbers $u_1, \cdots, u_N$ satisfy complicated coupled polynomial equations. The solution

[3]More precisely, the states $|\alpha\rangle_{L+1}$ with $\alpha = \pm 1/2$ at site $j = L+1$ and the states $|\beta\rangle_1$ with $\beta = \pm 1/2$ at site $j = 1$ are equal ($\alpha = \beta$).

of these equations is only partially known. In the thermodynamic limit $L \to \infty, N \to \infty, L/N$ = constant one obtains integral equations describing the physical ground state and genuine particles which lie above the physical ground state.

The model in (12.1.21) is sometimes called the XXX model. In the XYZ model one allows different coupling constants J_x, J_y and J_z for the terms $\sigma_j^x \sigma_{j+1}^x, \sigma_j^y \sigma_{j+1}^y$ and $\sigma_j^z \sigma_{j+1}^z$, and if $J_x = J_y \neq J_z$ one has by definition the XXZ model. The YB equation and its solutions for the XXZ model are closely related to those of the XXX model, but for the XYZ model the situation is much more complicated. For the XXX model there is no energy gap in the spectrum in the thermodynamic limit, but for the XYZ model there is an energy gap, while for the XXZ model there is only an energy gap if the two coupling constants are sufficiently different. We discuss below mainly the XXX model.

2 The spin $1/2$ Heisenberg chain

It is useful to rewrite the Hamiltonian of the XXX model in terms of the permutation operator P_{ij} which is defined by

$$P_{ij} = \frac{1}{2} + \frac{1}{2}\vec{\sigma}_i \cdot \vec{\sigma}_j = \frac{1}{2} + \frac{1}{2}\sigma_i^z \sigma_j^z + \sigma_i^+ \sigma_j^- + \sigma_i^- \sigma_j^+ \tag{12.2.1}$$

with $\sigma^{\pm} = \frac{1}{2}(\sigma^1 \pm i\sigma^2)$ and $i \neq j$. This operator acts on the 4 states $|\ \alpha\rangle_i \otimes |\ \beta\rangle_j$ where $\alpha, \beta = \pm 1/2$ and $|\ \frac{1}{2}\rangle_i = |\uparrow\rangle_i$ and $|\ -1/2\rangle_j = |\downarrow\rangle_j$. One has for example

$$P_{ij}\ |\uparrow\rangle_i \otimes |\uparrow\rangle_j = |\uparrow\rangle_i \otimes |\uparrow\rangle_j; P_{ij}\ |\uparrow\rangle_i \otimes |\downarrow\rangle_j = |\downarrow\rangle_i \otimes |\uparrow\rangle_j \tag{12.2.2}$$

So in general $P_{ij} \mid \alpha\rangle_1 \otimes \cdots \mid \beta\rangle_i \otimes \cdots \mid \gamma\rangle_j \otimes \cdots \mid \delta\rangle_L$ is equal to $\mid \alpha\rangle_1 \otimes \cdots \mid \gamma\rangle_i \otimes \cdots \mid \beta\rangle_j \otimes \cdots \mid \delta\rangle_L$. Note that $P_{ij} = P_{ji}$ and $P_{ik}P_{ki} = I$. (We do not define P_{ij} for $i = j$. If $i = j$ one would obtain from (12.2.1) $P_{ii} = 2$ and $P_{ik}P_{ki} = 4$ for $i = k$. We therefore always take $i \neq j$ in P_{ij}). Another relation which will be used frequently is $P_{ik}P_{kj} = P_{kj}P_{ij}$. One can check this relation by acting on a general state: then on the left j goes to k, and k goes to i (so j goes to i), k goes to j, and i goes to k. The same happens on the right. It is faster to note that if one pulls P_{ik} from the left to the right past a product of P_{mn} operators, then the net result is that i and k are interchanged in each of the P_{mn}. The relation $P_{kj}P_{ij} = P_{ik}P_{kj}$ is just a special case. The Hamiltonian can be written in terms of P_{ij} as follows

$$H = J\sum_{j=1}^{L}(2P_{j\,j+1} - 1) \tag{12.2.3}$$

We shall now first construct other operators depending on P_{ij} and then later come back to H.

It is useful to denote the 4 states

$$|\uparrow\rangle_i\otimes\, |\uparrow\rangle_j, |\uparrow\rangle_i\otimes\, |\downarrow\rangle_j, \langle\downarrow|_i \otimes |\uparrow\rangle_j, |\downarrow\rangle_i\otimes\, |\downarrow\rangle_j \tag{12.2.4}$$

by $\mid I\rangle$ where $I = 1, 2, 3, 4$. Then P_{ij} becomes an operator $\hat{P}$ whose action on $\mid I\rangle$ is given in terms of a 4×4 matrix P_{IJ}

$$\hat{P} \mid I\rangle = P_{IJ} \mid J\rangle \text{ with } P_{IJ} = \begin{pmatrix} 1 & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & 1 \end{pmatrix} \tag{12.2.5}$$

Consider now the operator

$$R_{ij}(u, v) \equiv u - v + P_{ij} \tag{12.2.6}$$

where u and v are complex numbers. It satisfies the YB equation

$$\begin{aligned} &R_{12}(u, v)R_{13}(u, w)R_{23}(v, w) \\ &\qquad = R_{23}(v, w)R_{13}(u, w)R_{12}(u, v) \end{aligned} \tag{12.2.7}$$

To prove this note that the terms without any P and with one P match. The terms with three P's also match. They read

$$P_{12}P_{13}P_{23} = P_{23}P_{13}P_{12} \tag{12.2.8}$$

and they agree because moving P_{12} past the two other P's interchanges the indices 1 and 2. Alternatively one can check this relation by acting on states: the operator $P_{12}P_{13}P_{23}$ acts on $\mid\alpha\rangle_1 \mid\beta\rangle_2 \mid\gamma\rangle_3$ as follows: P_{23} yields $\mid\alpha\rangle_1 \mid\gamma\rangle_2 \mid\beta\rangle_3$, then P_{13} yields $\mid\beta\rangle_1 \mid\gamma\rangle_2 \mid\alpha\rangle_3$ and finally P_{12} yields $\mid\gamma\rangle_1 \mid\beta\rangle_2 \mid\alpha\rangle_3$. On the other hand $P_{23}P_{13}P_{12}$ yields successively $\mid\beta\rangle_1 \mid\alpha\rangle_2 \mid\gamma\rangle_3 \to\mid\gamma\rangle_1 \mid\alpha\rangle_2 \mid\beta\rangle_3 \to\mid\gamma\rangle_1 \mid\beta\rangle_2 \mid\alpha\rangle_3$ which is the same result. Finally, the terms with two P's also match.[4]

Next we define the Yang-Baxter algebra in this model. We divide the YB equation by $u-v$. Setting $w=1/2$ and replacing u and v by iu and iv, and denoting 1 by a, 2 by b and 3 by j, we obtain after multiplication by P_{ab} from the left[5]

$$\begin{aligned}&\left(\frac{i}{v-u}+P_{ab}\right)\left(u-\frac{i}{2}\sigma_a^\alpha\sigma_j^\alpha\otimes I_b^2\right)\left(v-\frac{i}{2}I_a^2\otimes\sigma_b^\beta\sigma_j^\beta\right)\\&\quad=\left(v-\frac{i}{2}\sigma_a^\alpha\sigma_j^\alpha\otimes I_b^2\right)\left(u-\frac{i}{2}I_a^2\otimes\sigma_b^\beta\sigma_j^\beta\right)\left(\frac{i}{v-u}+P_{ab}\right)\end{aligned}\tag{12.2.9}$$

We view this equation as an operator which acts simultaneously in 3 spaces: two auxiliary spaces a and b, and a quantum space j. The matrices I_a^2 and I_b^2 are the unit operators in these auxiliary spaces (in our case 2×2 matrices). The direct product $\otimes$ is a tensor product between the two auxiliary spaces but the entries of the 4×4 matrix acting in this tensored space are still quantum operators (depending on σ_j).

Introducing the notation[6]

$$\begin{aligned}&\frac{i}{v-u}+P=\check{R}(u,v)\\&u-\frac{i}{2}\sigma^\alpha\sigma_j^\alpha=L_j(u)\end{aligned}\tag{12.2.10}$$

we find the Yang-Baxter algebra (YBA)

$$\check{R}_{12}(u-v)(L_j(u)\otimes I)(I\otimes L_j(v))=(L_j(v)\otimes I)(I\otimes L_j(u))\check{R}_{12}(u-v)\tag{12.2.11}$$

[4]For the terms with two P's one finds for all terms proportional to u

$$P_{13}P_{23}+P_{12}P_{23}=P_{23}P_{12}+P_{23}P_{13}$$

Since $P_{13}P_{23}=P_{23}P_{12}$ and $P_{12}P_{23}=P_{23}P_{13}$ this relation holds.

[5]For example $P_{12}R_{23}(v,w)=P_{12}(v-w+P_{23})=(v-w)P_{12}+P_{12}P_{23}=(v-w+P_{13})P_{12}=R_{13}(v,w)P_{12}$ because $P_{12}P_{23}=P_{13}P_{12}$. Similarly $P_{12}P_{13}=P_{23}P_{12}$ and $P_{12}P_{13}=P_{23}P_{12}$.

[6]One sometimes defines $\tilde{R}(u-v)=PR(u-v)$.

An alternative form of the YBA in terms of R_{12} is obtained by multiplication with P_{ab} from the left.

$$R_{12}(u-v)(L_j(u)\otimes I)(I\otimes L_j(v)) = (I\otimes L_j(v))(L_j(u)\otimes I)R_{12}(u-v) \quad (12.2.12)$$

This can be symbolically written as

$$R_{12}(u-v)L_j^1(u)L_j^2(v) = L_j^2(v)L_j^1(u)R_{12}(u-v) \qquad (12.2.13)$$

For ordinary matrices L one can write this as $R_{12}L_j(u)\otimes L_j(v) = L_j(u)\otimes L_j(v)R_{12}(u-v)$ but but clearly not for matrices L with operator-valued entries. The name YB algebra is used to indicate that the entries of $L_j(u)$ are generators of an algebra with structure constants $R(u,v)$.

We can view the $L_j(u)$ as a representation of the YBA (whose abstract definition will be given later) with R matrix $\check{R}(u,v)$. We shall call $L_j(u)$ the L matrix. Because the matrix $\check{R}(u,v)$ is acting in the direct product of the two auxiliary spaces a,b, it can be represented by a 4×4 matrix. The matrix $L_j(u)$ acts in one of the auxiliary spaces and can be written as a 2×2 matrix whose entries are operators which act in the quantum space j.

$$\check{R}(u,v) = \begin{pmatrix} f(v,u) & & & \\ & g(v,u) & 1 & \\ & 1 & g(v,u) & \\ & & & f(v,u) \end{pmatrix} \quad \begin{array}{l} f(v,u) = 1 + \frac{i}{v-u} \\ \\ g(v,u) = \frac{i}{v-u} \end{array}$$

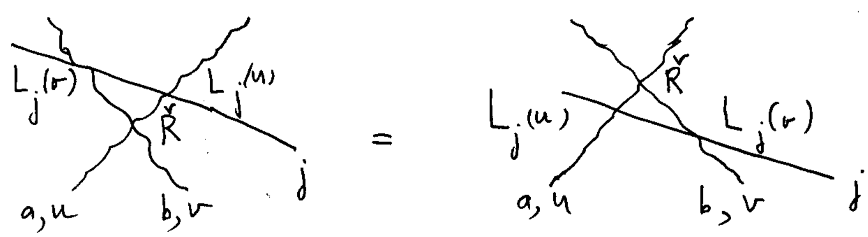
$$L_j(u) = \begin{pmatrix} u - \frac{i}{2}\sigma_j^z & -i\sigma_j^- \\ -i\sigma_j^+ & u + \frac{i}{2}\sigma_j^z \end{pmatrix}; \quad \sigma_j^\pm = \frac{1}{2}(\sigma_j^1 \pm +i\sigma_j^2) \qquad (12.2.14)$$

One can also write out the $\check{R}LL$ equation explicitly with indices; the result is (12.4.6) with T replaced by L.

It is useful for suggesting generalizations (such as the boundary Yang-Baxter equation) to write the YBA as a kind of Feynman diagram

$$(12.2.15)$$

The auxiliary particles a and b are denoted by wigly lines. The matrix $\check{R}(u,v)$ denotes an interaction (scattering) of these auxiliary particles and the operators $L_j(u)$ denote interactions between an auxiliary particle and the particle at site j, while the solid line $\big|$ denotes the particles at site j. From this picture one may view the YBA as stating that the scattering of two auxiliary particles with each other ($\check{R}$) and with a j-particle (L_j) is independent of the order in which these interactions take place (first ab interactions or first aj and bj interactions). The YB equation itself has a similar pictorial interpretation as we already discussed.



3 Quantum groups

The $\check{R}LL$ relation in (12.2.13) is a good starting point to introduce quantum groups [4]. We write this equation as

$$\check{R}^{ij}{}_{kl} T^k{}_r T^l{}_s = T^i{}_k T^j{}_l \check{R}^{kl}{}_{rs} \tag{12.3.1}$$

For simplicity we take all indices i, j, k, l, r, s in the range (1,2). This leads to $Sl_q(2)$ as we shall see. Extension to $Sl_q(n)$ is obtained by letting the indices range from 1 to n.

The group $Gl(2, R)$ is given by real nonsingular 2×2 matrices $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. If $\det T = 1$ one obtains the group $Sl(2, R)$, and $Gl(2, C)$ and $Sl(2, C)$ results if the entries a, b, c, d are complex numbers. We now consider abstract operators a, b, c, d which satisfy (12.3.1). For simplicity we take a particular $\check{R}$ matrix

$$\check{R} = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}; \quad \lambda \equiv q - \frac{1}{q} \tag{12.3.2}$$

where q is any (complex) number. The matrix $\check{R}$ satisfies the $\check{R}$ equation as one may verify. The 16 equations in (12.3.1) lead to 6 relations between the operators a, b, c and d

$$\begin{array}{ll} ab = qba & bc = cb \\ ac = qca & bd = qdb \\ ad = da + \lambda bc & cd = qdc \end{array} \tag{12.3.3}$$

For $q = 1$ these are the commutation relations for arbitrary numbers, but for $q \neq 1$ one obtains a deformation of the usual commutation relations.

One can define a determinant of T such that it commutes with a, b, c and d

$$\det T = ab - qbc \tag{12.3.4}$$

One can then define an inverse of T by

$$T^{-1} = \frac{1}{\det T} \begin{pmatrix} d & -\frac{1}{q}b \\ -qc & a \end{pmatrix} \tag{12.3.5}$$

The ordinary matrix product of two matrices $T_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$ and $T_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$ which each satisfy the six relations in (12.3.3) with the same q is a matrix T_{12} which again satisfies (12.3.3) with the same q. It is a good exercise to check this explicitly. Of course, ordinary matrix multiplication is associative, whether or not the entries are noncommutating operators. The unit element $E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ also satisfies the six conditions. The set of 2×2 matrices T with $\det T = 1$ forms then by definition the quantum group $Sl_q(2)$. This name suggests it is a group, but it is actually not a group, because T^{-1} satisfies the six conditions with q replaced by q^{-1}. Thus quantum groups are not groups; rather, they form an algebra obtained by formal power series in a, b, c, d modulo the six relations and modulo the relation $ab - qbc = 1$.

The six conditions followed from the YB equation, and they do not lead to further conditions on a, b, c, d. This is nontrivial as the following example shows. Assume $yx = yx + x^2 + y^2$, and try to order the monomial y^2x into such a way that x stands to the left of all y. The first step yields $y^2x = y(xy + x^2 + y^2)x$ but if we order $y(x^2) = (yx)x = (xy + x^2 + y^2)x$, the terms y^2x cancel, and we find another constraint among x and y, namely $xyx + (xyx + x^3) + y^3 = 0$ which can be written in ordered form as $x^3 + y^3 + x^2y + xy^2 = 0$.

If one modifies the six relations a bit one gets such extra constraints. For example, if one sets $bd = q'db$ with $q' \neq q$, and tries to rearrange the product abd into the form dba by first moving a past b, and then compares with what one gets if one first moves b past d, one finds

$$\begin{aligned} & (ab)d = qbad = qb(da + \lambda bc) = qq'dba + \lambda qb^2c \\ = \; & a(bd) = q'adb = q'(da + \lambda bc)b = qq'dba + \lambda q'b^2c \end{aligned} \qquad (12.3.6)$$

Hence one gets the extra constraint $b^2c = 0$ in this case. The statement that there are no further conditions in the algebra of polynomials in a, b, c, d is known in mathematics as the Poincaré-Birkhoff-Witt theorem.

The matrix T acts on a two dimensional linear vector space written as $x = \begin{pmatrix} x^1 \\ x^2 \end{pmatrix}$. Thus

$$Tx = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \end{pmatrix} = \begin{pmatrix} ax^1 + bx^2 \\ cx^1 + dx^2 \end{pmatrix} \equiv \begin{pmatrix} \Delta(x^1) \\ \Delta(x^2) \end{pmatrix} \qquad (12.3.7)$$

One may view the a, b, c, d as operators in one space, and x^1 and x^2 as operators acting in another space. Then a, b, c and d commute with x^1 and x^2, and to stress this point one sometimes writes $a \otimes x^1$ instead of ax^1. For notational simplicity we shall not introduce these symbols $\otimes$. Let us impose the following conditions as x^1 and x^2

$$x^1x^2 = qx^2x^1 \qquad (12.3.8)$$

Then it is straightforward to show that also $\Delta(x^1)$ and $\Delta(x^2)$ satisfy this property

$$\Delta(x^1)\Delta(x^2) = q\Delta(x^2)\Delta(x^1) \qquad (12.3.9)$$

as a consequence of the six relations between a, b, c and d, and the relation between x^1 and x^2.

It is natural to extend these considerations to fermions and to introduce a vector $\xi = \begin{pmatrix} \xi^1 \\ \xi^2 \end{pmatrix}$ which satisfies some deformed Grassman relations. One consistent deformation is as follows

$$\xi^1\xi^2 = -\frac{1}{q}\xi^2\xi^1, \quad \xi^1\xi^1 = 0, \quad \xi^2\xi^2 = 0 \tag{12.3.10}$$

It is consistent in the sense that also $\Delta(\xi^1)$ and $\Delta(\xi^2)$ satisfy these relations.

How can we find in general such constant deformations of (anti)commutation relations for coordinates? Consider two vectors x and y, and define

$$xy = \frac{1}{q}\check{R}yx \tag{12.3.11}$$

More explicitly this reads

$$\begin{pmatrix} x^1y^1 \\ x^1y^2 \\ x^2y^1 \\ x^2y^2 \end{pmatrix} = \frac{1}{q}\begin{pmatrix} q & & & \\ & \lambda & 1 & \\ & 1 & 0 & \\ & & & q \end{pmatrix}\begin{pmatrix} y^1x^1 \\ y^1x^2 \\ y^2x^1 \\ y^2x^2 \end{pmatrix} \tag{12.3.12}$$

Direct evaluation yields

$$\begin{array}{ll} x^1y^1 = y^1x^1 & x^2y^1 = \frac{1}{q}y^1x^2 \\ x^1y^2 = \frac{1}{q}y^2x^1 + \frac{1}{q}\lambda\ y^1x^2 & x^2y^2 = y^2x^2 \end{array} \tag{12.3.13}$$

When $y = x$ this reduces to $x^1x^2 = qx^2x^1$. For the anticommuting coordinates one may show that

$$\xi\eta = -q\check{R}\eta\xi \tag{12.3.14}$$

leads to

$$\begin{array}{lcl} \xi^1\eta^1 = -q^2\eta^1\xi^1 & ; & \xi^2\eta^1 = -q\eta^1\xi^2 \\ \xi^1\eta^2 = -q\eta^2\xi^1 - \lambda q\eta^1\xi^2 & & \xi^2\eta^2 = -q^2\eta^2\xi^2 \end{array} \tag{12.3.15}$$

For $\eta = \xi$, one finds again $\xi^1\xi^1 = \xi^2\xi^2 = 0$ and $\xi^1\xi^2 = -\frac{1}{q}\xi^2\xi^1$. Again $\Delta(x)$ and $\Delta(y)$ satisfy $\Delta(x)\Delta(y) = \frac{1}{q}\check{R}\Delta(y)\Delta(x)$, and $\Delta(\xi)$ and $\Delta(\eta)$ satisfy $\Delta(\xi)\Delta(\eta) = -q\check{R}\Delta(\eta)\Delta(\xi)$.

Let us now take 3 vectors x, y and z. We impose that all coordinates have the same deformed commutation relations

$$xy = \frac{1}{q}\check{R}yx; \quad yz = \frac{1}{q}\check{R}zy; \quad xz = \frac{1}{q}\check{R}zx \tag{12.3.16}$$

Consider now xyz (8 combinations). We want to order this into zyx, but this can be done in two ways: first interchange y and z (and then x with z), or first interchange x with y (and then x with z). If we require that both ways yield the same answer, one finds the YB equation for $\check{R}$

$$\check{R}_{12}\check{R}_{23}\check{R}_{31} = \check{R}_{23}\check{R}_{12}\check{R}_{23} \tag{12.3.17}$$

4 Transfer matrices

Let us now return to the XXX model. The YBA is the starting point for the process of diagonalizing the Hamiltonian (and other commuting operators). Up to now, the YBA acted only in one of the L quantum spaces, namely in quantum space j

$$\check{R}_{12}(u,v)L_j^1(u)L_j^2(v) = L_j^1(v)L_j^2(u)\check{R}_{12}(u,v) \tag{12.4.1}$$

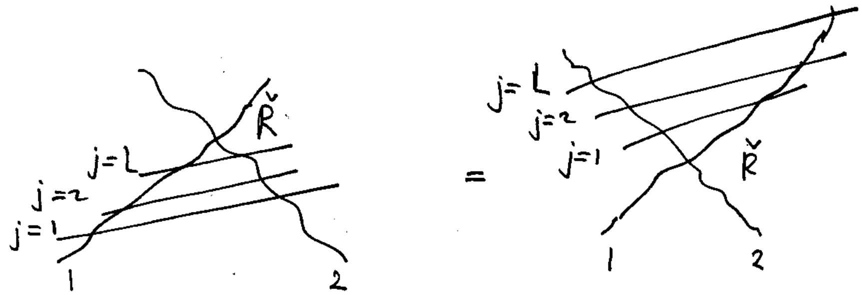
We would expect to need also operators which act in more than only one quantum space. This is the motivation for introducing the monodromy matrix $T(u)$

$$T(u) = L_L(u)\cdots L_2(u)L_1(u) \tag{12.4.2}$$

(All matrices in $T(u)$ act in the same auxiliary space hence there are no direct product signs $\otimes$ in this equation.) The four entries in $T(u)$ depend on all $3L$ quantum operators $(\sigma_1^\alpha, \sigma_2^\alpha, \ldots \sigma_L^\alpha.)$ We can start with the YBA for $j = 1$ and multiply it with $L_2^1(v)L_2^2(u)$ from the left. Due to the YBA, one can pull $\check{R}$ through $L_2^1(v)L_2^2(u)$. Using that $L_j^1(u)$ and $L_k^2(v)$ commute for $j \neq k$, and repeating this process L times, the result can be written in terms of $T_1(u)$ and $T_2(v)$. The relation

$$\check{R}_{12}(u,v)T^1(u)T^2(v) = T^1(v)T^2(u)\check{R}_{12}(u,v) \tag{12.4.3}$$

where now we have operators which act simultaneously in all L quantum spaces and in the two auxiliary spaces. One may depict this relations graphically as follows



Clearly, one can generalize this equation and consider operators $T(u)$ acting in some quantum Hilbert space (the generalization of $\mid \alpha_1\rangle_1 \otimes \cdots \mid \alpha_L\rangle_L$) as well as in an auxiliary space. There are always two auxiliary spaces whose tensor product is denoted by $\otimes$. If the auxiliary space has n states, T is an $n \times n$ matrix and $\check{R}$ is an $n^2 \times n^2$ matrix. More generally, T is an operator in the auxiliary space which is also an operator in the quantum space, and $\check{R}$ is an operator in the direct product of two auxiliary spaces. This is the formulation used for quantum groups.

We introduced above $L_j(u)$ or $T(u)$ as special cases of $R(u)$ or products of $R(u)$ for the spin 1/2 chain. One can also introduce abstractly operators $T(u)$ and R-matrices such that (12.4.3) holds. One can then consider three spaces in which T operators act as $T(u) \otimes T(v) \otimes T(w)$. One can use the $\check{R}TT = TT\check{R}$ equation to change this to $T(w) \otimes T(v) \otimes T(u)$ in 2 different ways, and the YB equation is then a sufficient consistency condition. Namely, we may begin with $R_{23}R_{13}R_{12}T_1T_2T_3$ and rewrite this as $R_{23}R_{13}T_2T_1T_3R_{12} = R_{23}T_2R_{13}T_1T_3R_{12}$, and then rewrite $R_{13}T_1T_3$ as $T_3T_1R_{13}$. Or we may use the YB equation to first rewrite the three R's as $R_{12}R_{13}R_{23}$ and then proceed as before. The result is

$$\begin{aligned} T(u) \otimes T(v) \otimes T(w) \equiv T_1T_2T_3 &= R_{12}^{-1}R_{13}^{-1}R_{23}^{-1}T_3T_2T_1R_{23}R_{13}R_{12} \\ &= R_{23}^{-1}R_{13}^{-1}R_{12}^{-1}T_3T_2T_1R_{12}R_{13}R_{23} \end{aligned} \quad (12.4.4)$$

Clearly, if $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$ then both results agree.

We can expand $T(u)$ into a power series

$$T(u) = \sum_{n=0}^{L} t_n u^n \tag{12.4.5}$$

The operators t_n satisfy then generalized commutation relations which also depend on the matrix $\check{R}(u,v)$. Thus the matrix $\check{R}(u,v)$ can be viewed as a generalized set of structure constants. For our explicit spin 1/2 model, the set of all commutation relations can be deduced if one specifies 6 operators and their commutation relations, together with their Serre relations (nonlinear multiple-commutator relations between these 6 operators from which one can construct all other commutators). These 6 operators are the two 2×2 operators t_1 and t_2. (The operators t_1 and t_2 contain each 4 entries, but there are two central elements, obtained from the quantum determinant, which can be written in terms of the entries of t_1 and t_2. This reduces the 8 operators to 6 independent ones).

We shall now take a trace of the $\check{R}TT = TT\check{R}$ relation. Because there are various spaces in which $\check{R}$ and T act, it may be useful to first write this relation with indices

$$\check{R}(u,v)^{\alpha\beta}{}_{\gamma\delta} T^{\gamma}{}_{\epsilon}(u) T^{\delta}{}_{\zeta}(v) = T^{\alpha}{}_{\gamma}(v) T^{\beta}{}_{\delta}(u) \check{R}(u,v)^{\gamma\delta}{}_{\epsilon\zeta} \tag{12.4.6}$$

where $\alpha, \beta \ldots$ can be $+1/2$ or $-1/2$. We contract by $\check{R}^{-1}$ from the right (multiply with $\{\check{R}(u,v)^{-1}\}^{\epsilon\zeta}{}_{\eta\theta}$ and sum over ϵ and ζ). Then we take the trace in the two auxiliary spaces (sum over $\alpha = \eta$ and $\beta = \theta$). Using the cyclicity of the trace (because the entries of $\check{R}$ are just ordinary complex numbers in our model we may interchange $\check{R}$ and $\check{R}^{-1}$) one obtains

$$\begin{aligned} trT_1(u)T_2(u) &= T^{\alpha}{}_{\alpha}(u) T^{\beta}{}_{\beta}(v) \\ &= TrT_2(v)T_1(u) = T^{\beta}{}_{\beta}(v) T^{\alpha}{}_{\alpha}(u) \end{aligned} \tag{12.4.7}$$

We have thus obtained a set of commuting operators for the XXX model.[7] Among them are the Hamiltonian H and the momentum (shift) operator $\hat{\Pi}$.

[7] For the XXX model the functions $f(u,v)$ and $g(u,v)$ parametrizing $\check{R}(u,v)$ were given by $g/f = \frac{i}{u-v+i}$ and $1/f = \frac{u-v}{u-v+i}$. For the XXZ model the R matrix is given by $R(u) = \sum_{a=0}^{3} w_a(u)\sigma^a \otimes \sigma^a$ with

We now construct the shift operator $\hat{\Pi}$, following [12]. This construction is a good exercise to become familiar with the techniques used in the studies of the YB equation, but it will not be used afterwards. To construct the shift operator $\hat{\Pi}$, we consider $L_j(-i/2)$. Recalling the definition of $L_j(u)$ we find

$$L_j(-i/2) = -\frac{i}{2} - \frac{i}{2}\sigma_0^\alpha \sigma_j^\alpha = -iP_{0j} \tag{12.4.8}$$

where P_{0j} acts both in the auxiliary space (denoted by 0) and the quantum space j. Then

$$\begin{aligned} T\left(\frac{-i}{2}\right) &= (-i)^L P_{0L} \dots P_{02} P_{01} \\ &= (-i)^L P_{01} P_{1L} \dots P_{13} P_{12} \equiv (-i)^L P_{01} \hat{U} \end{aligned} \tag{12.4.9}$$

where $\hat{U}$ is the shift operator $P_{1L} \dots P_{12}$ which shifts $1 \to 2, 2 \to 3 \dots, L \to 1$. This is a relation containing operators which act in one auxiliary space (P_{01} acts as a 2×2 matrix) and the L quantum spaces. Taking the trace in the auxiliary space yields

$$trT\left(\frac{-i}{2}\right) = (-i)^L (trP_{01})\hat{U} = (-i)^L \hat{U} \tag{12.4.10}$$

(Use $P_{01} = \frac{1}{2} + \frac{1}{2}\vec{\sigma}_0 \cdot \vec{\sigma}_j$ and $tr\vec{\sigma}_0 = 0$ while $tr\frac{1}{2} = 1$). Since $\frac{\partial}{\partial u} L_j(u) \equiv L'_j(u) = I$ we can evaluate the derivative of $T(u)$ at $u = -i/2$ as follows

$$T'\left(u = \frac{-i}{2}\right) = (-i)^{L-1}(P_{0,L-1} \dots P_{01} + P_{0L}P_{0,L-2}..P_{01} + .. + P_{0L}..P_{03}P_{01} + P_{0L}..P_{02}) \tag{12.4.11}$$

We insert $P_{0L}^2 = 1$ on the left-hand side of the first term, and $P_{0,L-1}^2$ between P_{0L} and $P_{0,L-2}$ in the second term, etc. This yields then[8]

$$T'\left(u = -\frac{i}{2}\right) = (-i)^{L-1}(P_{0L} \dots P_{01} P_{L-1,L} + P_{0L} \dots P_{01} P_{L-2,L-1}$$

$\sigma^0 = I, w_0 = \sinh(u + \frac{1}{2}\eta)\cosh(\frac{1}{2}\eta), w_3 = \sinh(\frac{1}{2}\eta)\cosh(u + \frac{1}{2}\eta)$ and $w_1 = w_2 = \sinh(\frac{1}{2}\eta)\cosh(\frac{1}{2}\eta)$. To recover the R-matrix of the XXX model $R = u + P$, rescale u to $u\eta$, take the limit $\eta \to 0$ and divide by η. The $\check{R}$ matrix has then $g/f = \sinh(\eta)/\sinh -i\eta(u - v + i)$ and $1/f = \sinh -i\eta(u - v)/\sinh -i\eta(u - v + i)$. The parameter $q = e^{i\eta}$ is often called the deformation parameter. For the XYZ model one finds new off-diagonal 2×2 submatrices in the $\check{R}$ matrix which complicate the algebra enormously.

[8] Use $P_{0j}(P_{0j} \cdots P_{01}) = (P_{0j} \cdots P_{01})P_{j-1,j}$ in the first step.

$$
\begin{aligned}
&+\ldots+P_{0L}\ldots P_{01}P_{12}+P_{0L}\ldots P_{01}P_{01}) \\
&=(-i)^{L-1}P_{01}\hat{U}(P_{L-1,L}+\ldots P_{12}+P_{01}) \\
&=(-i)^{L-1}P_{01}\hat{U}(P_{12}+\ldots+P_{L-1,L})+(-i)^{L-1}P_{02}\hat{U}P_{L1} \qquad (12.4.12)
\end{aligned}
$$

We used in the last step that $P_{01}\hat{U}P_{01}=P_{02}\hat{U}P_{L1}$ which follows from $1\to 0\to 1, L\to 0, 0\to 2$ and $j\to j+1$ for $j=2,\ldots,L-1$.

Taking the trace in the auxiliary space leads to

$$
trT'\left(u=\frac{-i}{2}\right)=(-i)^{L-1}\hat{U}\left(\sum_{j=1}^{L}P_{j,j+1}\right) \qquad (12.4.13)
$$

where we used the periodic boundary condition $\sigma_{L+1}=\sigma_1$. Substituting $trT(u=-i/2)=(-i)^L\hat{U}$ we get

$$
trT'(u=-i/2)=itrT(u=-i/2)\left(\sum_{j=1}^{L}P_{j,j+1}\right) \qquad (12.4.14)
$$

Hence

$$
\begin{aligned}
\sum_{j=1}^{L}P_{j,j+1} &= \frac{1}{2}\frac{1}{J}(H+LJ)=\frac{-itrT'}{trT}(u=-i/2) \\
&= -i\frac{\partial}{\partial u}\ln trT(u=-i/2) \qquad (12.4.15)
\end{aligned}
$$

Clearly it is useful to introduce

$$
\tau(u)=-i\ln trT(u) \qquad (12.4.16)
$$

The operators $\tau(u)$ still commute for different u

$$
[\tau(u),\tau(v)]=0 \qquad (12.4.17)
$$

and the expansion of $\tau(u)$ about $u=-i/2$ yields

$$
\begin{aligned}
\tau(u) &= -i\ln tr(-i)^L\hat{U}+(u+i/2)\left(\frac{\hat{H}}{2J}+\frac{1}{2}L\right)+\ldots \\
&= -\frac{\pi L}{2}+\hat{\Pi}+(u+i/2)\left(\frac{\hat{H}}{2J}+\frac{1}{2}L\right)+\mathcal{O}(u+i/2)^2 \qquad (12.4.18)
\end{aligned}
$$

Here

$$\hat{\Pi} = -i \ln \hat{U}, (\hat{U} = e^{i\hat{\Pi}}) \tag{12.4.19}$$

is the generator of the shift operator, which can be written in terms of $\hat{U}$ as follows [5]

$$\hat{\Pi} = \frac{2\pi}{L} \sum_{n=1}^{L-1} \left(\frac{1}{2} + \frac{\hat{U}^n}{e^{-i\frac{n\pi}{L}} - 1} \right) \tag{12.4.20}$$

5 The algebraic Bethe ansatz

In our model $T(u)$ acts in the auxiliary space as a 2×2 matrix. Hence we can define

$$T(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}, trT(u) = A(u) + D(u) \tag{12.5.1}$$

The matrices $T(u) \otimes T(v)$ are then 4×4 matrices which act as follows on the states $|\uparrow\rangle \otimes |\uparrow\rangle, |\uparrow\rangle \otimes |\downarrow\rangle, |\downarrow\rangle \otimes |\uparrow\rangle$ and $|\downarrow\rangle \otimes |\downarrow\rangle$ in the direct product of the two auxiliary spaces

$$T(u) \otimes T(v) = \left(\begin{array}{cc|cc} AA' & AB' & BA' & BB' \\ AC' & AD' & BC' & BD' \\ \hline CA' & CB' & DA' & DB' \\ CC' & CD' & DC' & DD' \end{array} \right) \tag{12.5.2}$$

where $A = A(u)$ and $A' = A(v)$, etc. We also recall the form of $\check{R}(u, v)$

$$\check{R}(u, v) = \begin{pmatrix} f(v, u) & 0 & 0 & 0 \\ 0 & g(v, u) & 1 & 0 \\ 0 & 1 & g(v, u) & 0 \\ 0 & 0 & 0 & f(v, u) \end{pmatrix} \tag{12.5.3}$$

where $f(v, u) = 1 + \frac{i}{v-u}$ and $g(v, u) = \frac{i}{v-u}$.

The matrix equation $\check{R}_{12}(u, v)T^1(u)T^2(v) = T^1(v)T^2(u)\check{R}_{12}(u, v)$ leads to 16 commutation relations between the operators A, B, C and D. We shall only need three of them to diagonalize the transfer matrix; we determine them in turn.

(I) 1st row, 4th column:

$$f(v, u)BB' = B'Bf(v, u) \tag{12.5.4}$$

This implies that the operators $B(u)$ commute with each other for different u

$$[B(u), B(v)] = 0 \tag{12.5.5}$$

(II) 2nd row, 4th column:

$$g(v,u)BD' + DB' = B'Df(v,u) \tag{12.5.6}$$

We rewrite this as a rule how to pull D through B

$$D(u)B(v) = f(v,u)B(v)D(u) - g(v,u)B(u)D(v) \tag{12.5.7}$$

(III) 1st row, 3rd column

$$f(v,u)BA' = A'B + B'Ag(v,u) \tag{12.5.8}$$

This yields the rule how to pull $A(u)$ past $B(v)$. Interchanging u and v we get

$$A(u)B(v) = f(u,v)B(v)A(u) - g(u,v)B(u)A(v) \tag{12.5.9}$$

Summarizing

$$\begin{aligned} [B(u), B(v)] = 0; \qquad & A(u)B(v) = f(u,v)B(v)A(u) - g(u,v)B(u)A(v) \\ & D(u)B(v) = f(v,u)B(v)D(u) - g(v,u)B(u)D(v) \end{aligned} \tag{12.5.10}$$

We shall now consider a series of states obtained by acting with $T(u)$ on the Fock vacuum $|\uparrow\rangle_L \otimes \ldots |\uparrow\rangle_1$. We write this vacuum as $\binom{1}{0}^{\otimes L} \equiv | \Omega\rangle$. It follows that $\sigma_j^+|\Omega >= 0$. To evaluate $T(u) \mid \Omega\rangle$ we begin by working out $L_1(u) \mid \Omega\rangle$. One finds

$$L_1(u) \mid \Omega\rangle = \begin{pmatrix} u - \frac{i}{2}\sigma_1^z & -i\sigma_1^- \\ -i\sigma_1^+ & u + \frac{i}{2}\sigma_1^z \end{pmatrix} \mid \Omega\rangle = \begin{pmatrix} u - i/2 & *_1 \\ 0 & u + \frac{i}{2} \end{pmatrix} \mid \Omega\rangle \tag{12.5.11}$$

where $*_1$ denotes an expression which we do not need. Similarly

$$\begin{aligned} L_2(u)L_1(u) \mid \Omega\rangle &= \begin{pmatrix} (u - i/2)(u - i/2\sigma_2^z) & *_2 \\ (u - i/2)(-i\sigma_2^+) & (u + \frac{i}{2})(u + \frac{i}{2}\sigma_2^z) - i\sigma_2^+ *_1 \end{pmatrix} \mid \Omega\rangle \\ &= \begin{pmatrix} (u - i/2)^2 & *_2 \\ 0 & (u + \frac{i}{2})^2 \end{pmatrix} \mid \Omega\rangle \end{aligned} \tag{12.5.12}$$

Continuing this till we reach $L_L(u)$ we obtain

$$T(u) \mid \Omega\rangle = \begin{pmatrix} (u-i/2)^L & *_L \\ 0 & (u+i/2)^L \end{pmatrix} \mid \Omega\rangle \tag{12.5.13}$$

Thus the entries $A(u)$ and $D(u)$ are diagonal on $\mid \Omega\rangle$ with eigenvalues $a(u)$ and $d(u)$, respectively, and $C(u)$ vanishes on $\mid \Omega\rangle$. We also recall that the $B(u)$ are commuting

$$\begin{aligned} &A(u) \mid \Omega\rangle = a(u) \mid \Omega\rangle \ , \ a(u) = (u-i/2)^L \\ &D(u) \mid \Omega\rangle = d(u) \mid \Omega\rangle \ , \ d(u) = (u+i/2)^L \\ &C(u) \mid \Omega\rangle = 0 \ ; \ [B(u), B(v)] = 0 \end{aligned} \tag{12.5.14}$$

The algebraic Bethe Ansatz is then the assumption that the $C(u)$ are annihilation operators on $\mid \Omega\rangle$ and $B(u)$ are creation operators, and to consider the following set of states

$$\mid \vec{u}\rangle = B(u_1)\dots B(u_N) \mid \Omega\rangle \tag{12.5.15}$$

where u_j are still to be determined, and N runs from 0 up to $[L/2]$. (For fixed N there will be several solutions for $\vec{u}$).

$$\textbf{Theorem} : A(u) \mid \vec{u}\rangle = a(u) \prod_{j=1}^{N} f(u,u_j) \mid \vec{u}\rangle + \sum_{n=1}^{N} \Lambda_n B(u) \prod_{j=1, j\neq n}^{N} B(u_j) \mid 0\rangle \tag{12.5.16}$$

We used here the A, B commutation relation. A term with $B(u)$ can only be produced when $A(u)B(u_j)$ is replaced by $B(u)A(u_j)$. Then $B(u)$ will appear in the second term in (12.5.16), and can be pulled to the left because $B(u)$ commutes with all $B(u_j)$. Eventually $A(u)$ or $A(u_j)$ reaches $\mid \Omega\rangle$ and then becomes $a(u)$ or $a(u_j)$. We shall calculate the Λ_n later.

Similarly

$$D(u) \mid \vec{u}\rangle = d(u) \prod_{j=1}^{N} f(u_j,u) \mid \vec{u}\rangle + \sum_{n=1}^{N} \tilde{\Lambda}_n B(u) \prod_{j=1, j\neq n}^{N} B(u_j) \mid \Omega\rangle \tag{12.5.17}$$

We also calculate the $\tilde{\Lambda}_n$ below.

Our strategy is now to consider $A(u) \mid \vec{u}\rangle + D(u) \mid \vec{u}\rangle = trT(u) \mid \vec{u}\rangle$, and to choose the $u_1, \ldots, u_N$ such that all Λ_n and $\tilde{\Lambda}_n$ vanish. If this is achieved, we have an eigenstate of $trT(u)$ (and thus also of the Hamiltonian and other commuting operators).

To determine Λ_n and $\tilde{\Lambda}_n$, let us begin with Λ_1. Then we must collect all terms without $B(u_1)$.

$$
\begin{aligned}
&A(u)B(u_1)\ldots B(u_N) \mid \Omega\rangle \\
&= \{f(u,u_1)B(u_1)A(u) - g(u,u_1)B(u)A(u_1)\}B(u_2)\ldots B(u_N) \mid \Omega\rangle \\
&= -g(u,u_1)B(u)A(u_1)B(u_2)\ldots B(u_N) \mid \Omega\rangle + \text{ terms with } B(u_1) \\
&= -g(u,u_1)a(u_1)B(u)\left(\prod_{j=2}^{N} f(u_1,u_j)\right)B(u_2)\ldots B(u_N) \mid \Omega\rangle + \text{ terms with } B(u_1)
\end{aligned} \tag{12.5.18}
$$

Hence

$$
\Lambda_1 = -a(u_1)g(u,u_1)\prod_{j=2}^{N} f(u_1,u_j) \tag{12.5.19}
$$

For Λ_n one uses $A(u_l)B(u_k) = f(u_l,u_k)B(u_k)A(u_l) - g(u_l,u_k)B(u_l)A(u_k)$. There are then several terms with a factor $B(u)$, and one can write them as one terms by using the relation involving fg which follow from the $RTT = TTR$ equation. (One may check this for $n = 2$). The result is

$$
\Lambda_n = -a(u_n)g(u,u_n)\prod_{j=1,j\neq n}^{N} f(u_n,u_j) \tag{12.5.20}
$$

An easy proof which avoids cumbersome combinatorics is to use that the initial state is symmetric in the $u_1, \ldots, u_N$, and thus the result for λ_n should be related by symmetry to the result for λ_1 which we already found rather easily.

In a similar manner one finds

$$
\tilde{\Lambda}_n = -d(u_n)g(u_n,u)\prod_{j=1,j\neq n}^{N} f(u_j,u_n) \tag{12.5.21}
$$

We have thus obtained the following result: $|\,\vec{u}\rangle = B(u_1)\dots B(u_N)\binom{1}{0}^{\otimes L}$ is an eigenstate of $trT(u)$ with eigenvalue

$$\lambda(u) = a(u)\prod_{j=1}^{N} f(u,u_j) + d(u)\prod_{j=1}^{N} f(u_j,u) \tag{12.5.22}$$

provided $\Lambda_n + \tilde{\Lambda}_n = 0$ for all $n = 1,\dots N$.

The states $|\,\vec{u}\rangle$ are the highest-weight states of the spin operator $S^z = \frac{1}{2}\sum_{j=1}^{N}\sigma_j^z$ because

$$S^z\,|\,\vec{u}\rangle = \frac{1}{2}(L-2N)\,|\,\vec{u}\rangle;\, S^+\,|\,\vec{u}\rangle = 0 \tag{12.5.23}$$

Acting with $S^- = \frac{1}{2}\sum_{j=1}^{N}\sigma_j^-$ on $|\,\vec{u}\rangle$ gives the rest of this multiplet.[9]

As we already noted, $|\,\vec{u}\rangle$ becomes an eigenstate of $T(\vec{u})$ if $\Lambda_n + \tilde{\Lambda}_n = 0$ for all $n = 1,\dots N$. This implies that the following set of conditions

$$\frac{a(u_n)}{d(u_n)}\frac{g(u,u_n)}{g(u_n,u)} + \prod_{j=1,j\neq n}^{N}\frac{f(u_j,u_n)}{f(u_n,u_j)} = 0 \tag{12.5.24}$$

Substitute now the special values $f(u,v) = 1 + \frac{i}{u-v}$ and $g(u,v) = \frac{i}{u-v}$ for our spin 1/2 chain. One obtains then the following set of algebraic equations

$$\left(\frac{u_n - i/2}{u_n + i/2}\right)^L = \prod_{j=1,j\neq n}^{N}\left(\frac{u_n - u_j - i}{u_n - u_j + i}\right);\, n = 1,\dots N \tag{12.5.25}$$

[9] Use $[T(u),\sigma_0^\alpha + 2S^\alpha] = 0$ where $S^\alpha = \frac{1}{2}\sum_{j=1}^{L}\sigma_j^\alpha$. This follows from $[L_j(u),\sigma_0^\alpha + \sigma_j^\alpha] = 0$ recalling that $L_j(u) = \begin{pmatrix} u - \frac{i}{2}\sigma_j^z & -i\sigma_j^- \\ -i\sigma_j^+ & u + \frac{i}{2}\sigma_j^z \end{pmatrix}$ can be written as $u - \frac{i}{2}\vec{\sigma}_0\cdot\vec{\sigma}_j$.
From this relation one finds for $\alpha = z$

$$\left[\begin{pmatrix} A & B \\ C & D \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right] = 2[S^z, T(u)] = 2\begin{pmatrix} [S^z,A] & [S^z,B] \\ [S^2,C] & [S^z,D] \end{pmatrix}$$

The commutator yields

$$\begin{pmatrix} A & -B \\ C & -D \end{pmatrix} - \begin{pmatrix} A & B \\ -C & -D \end{pmatrix} = \begin{pmatrix} 0 & -2B \\ 2C & 0 \end{pmatrix};\text{ hence}$$

$[S^z,B] = -B$. It follows that $S^z\,|\,\vec{u}\rangle = -N + \frac{1}{2}L$. Similarly for S^+ but then one must consider the eigenstates for which the u_j are special. One finds $[S^+,B] = A-D$. One must then pull $A-D$ through the $B(u_j)$. (The algebra is similar to the calculation of eigenstates of $T(u)$. One finds that $[S^+,B]\,|\,\vec{u}\rangle = 0$ leads again to the Bethe equations.)

For given N, these are N complex equations for the N complex variables $u_1, \dots u_N$. They are sometimes called the Bethe equations (BE). Let us denote the number of solutions by $n(L, N)$.

6 Solutions of the Bethe equations

To determine whether one obtains indeed all 2^L eigenstates, one must determine $n(L, N)$. There is a hypothesis, the string hypothesis, according to which one finds indeed 2^L solutions. This hypothesis classifies solutions, for example for $N = 2$ there are real solutions (all u_j real) and complex solutions. Under complex conjugation a solution $\{u_1, \dots u_N\}$ always goes over into itself because for example $u_2^* = u_1$ etc.

For $N = 1$ one has $\left(\frac{u-i/2}{u+i/2}\right)^L = 1$. Defining $\frac{u-i/2}{u+i/2} = e^{ip}$ this leads to $p = \frac{2\pi n}{L}$. The case $n = 0$ is excluded since it implies that $\frac{u-i/2}{u+i/2} = 1$. Hence there are $L-1$ solutions

$$n(L, N = 1) = L - 1 \tag{12.6.1}$$

which together with the state with $S_z = \frac{1}{2}L - 1$ in the multiplet which starts from $\mid \Omega\rangle$ as highest state, lead to L states with one spin flipped. The p are the momenta of a free particle with periodic boundary condition.

The total number of states with 2 spins flipped is $\binom{L}{2}$. Since $S_z = \frac{1}{2}L - N$ shows that $\mid \vec{u}\rangle$ has N spins flipped, one expects that for $N = 2$ one should find $n(L, N = 2) = \binom{L}{2} - n(L, N = 0) - n(L, N = 1) = \frac{1}{2}L(L-3)$. This is indeed the case as we now show.

Defining $X_1 = \frac{2u_1+i}{2u_1-i}$ and $X_2 = \frac{2u_2+i}{2u_2-i}$ the ABE can be written as

$$X_1^L = -\frac{X_1X_2 - 2X_1 + 1}{X_1X_2 - 2X_2 + 1}; X_2^L = -\frac{X_1X_2 - 2X_2 + 1}{X_1X_2 - 2X_1 + 1} \tag{12.6.2}$$

Clearly $(X_1X_2)^L = 1$, hence $X_1X_2 = \omega = \exp 2\pi i n/L$. Substitution of $X_2 = \omega/X_1$ leads to

$$X_1^L = -X_1\frac{\omega - 2X_1 + 1}{\omega X_1 - 2\omega + X_1} \Longrightarrow (\omega + 1)X_1^L - 2\omega X_1^{L-1} - 2X_1 + (\omega + 1) = 0$$

(12.6.3)

We continue with odd L. Introducing $\sqrt{\omega} = \exp in\pi/L$ for odd n, and $\sqrt{\omega} = -\exp in\pi/L$ for even n, we have $(\sqrt{\omega})^N = -1$ in both cases. Introducing $X_1 = \sqrt{\omega}\hat{X}_1$ we arrive at

$$-\sqrt{\omega}\left[\left(\sqrt{\omega}+\frac{1}{\sqrt{\omega}}\right)\hat{X}_1^L - 2\hat{X}_1^{L-1} + 2\hat{X}_1 - \left(\sqrt{\omega}+\frac{1}{\sqrt{\omega}}\right)\right] = 0 \qquad (12.6.4)$$

Since $\sqrt{\omega}+\frac{1}{\sqrt{\omega}} = 2\cos\frac{n\pi}{L}$ for odd n, and $\sqrt{\omega}+\frac{1}{\sqrt{\omega}} = -2\cos n\pi/L$ for even n, while for real u_1 we can always write $\hat{X}_1 = \exp i\alpha$, we obtain for real solutions

$$\left(\pm\cos\frac{n\pi}{L}\right)\sin\frac{L\alpha}{2} = \sin\frac{(L-2)\alpha}{2} \text{ for } \begin{cases} n = 1, 3, \ldots L-2 \\ n = 0, 2, 4, \ldots L-1 \end{cases} \qquad (12.6.5)$$

Complex solutions will be discussed later.

We study this further for $L = 5$. We can solve the equation $\pm\cos\frac{n\pi}{L} = \frac{\sin\frac{1}{2}(L-2)\alpha}{\sin\frac{1}{2}L\alpha}$ graphically. For $L = 5$ we get $f(\alpha) = \frac{\sin(3/2)\alpha}{\sin(5/2)\alpha}$ which we plot. We expect 5 real and/or complex solutions. The real solutions are

$n = 0 : -1 = f(\alpha) : 1$ root $: X = e^{i\pi} = -1, u_1 = 0.$

$n = 1 : \cos\frac{\pi}{5} = 0.809 = (1+\sqrt{5})/4 = f(\alpha) : 4$ roots at $\alpha = \frac{\pi}{5}, \arccos\left[\frac{1}{4}\left(\sqrt{5}-5\right)\right],$ $2\pi - \arccos\left[\frac{1}{4}\left(\sqrt{5}-5\right)\right], 2\pi - \frac{\pi}{5}$ and $X = e^{i\pi/5}e^{i\alpha}$

$n = 2 : -\cos\frac{2\pi}{5} = -0.309 = (1-\sqrt{5})/4 = f(\alpha) : 2$ roots at $\alpha = \frac{3\pi}{5}$ and $2\pi = \frac{3\pi}{5}$ and $X = e^{2i\pi/5}e^{i\alpha}$

$n = 3 : \cos\frac{3\pi}{5} = -0.309 = f(\alpha) : 2$ roots $: X = e^{3i\pi/5}e^{i\alpha}$

$n = 4 : -\cos\frac{4\pi}{5} = 0.809 = f(\alpha) : 4$ roots $: X = e^{4\pi i/5}e^{i\alpha}$

However, not all roots are allowed or give independent solutions. We must exclude the following cases[10]

$$X_1^L = -1, X_1 = 1, X_1 = \omega, X_1 \text{ and } \omega/X_1 \text{ give same solution} \qquad (12.6.6)$$

Then the root at $n = 0$ is lost, and only roots at $\alpha < \pi$ remain because if X_1 corresponds to α then ω/X_1 corresponds to $-\alpha = 2\pi - \alpha \bmod 2\pi$.

For L **even** (the most interesting case for antiferromagnets because then the ground state can have spin zero) we define $\sqrt{\omega} = \exp i\pi/L$. If n is odd, $(\sqrt{\omega})^N = -1$ and from here we can use the same steps as for the case L odd, yielding

$$\cos\frac{n\pi}{L} = \frac{\sin\frac{(L-2)\alpha}{2}}{\sin\frac{L\alpha}{2}} \text{ for } L \text{ even and } n \text{ odd} \qquad (12.6.7)$$

The solutions of this equation can again be read off from a plot. For L even and $\sqrt{\omega} = \exp in\pi/L$ but n even we have $(\sqrt{\omega})^N = +1$. The sign of the first two terms in (B) is now flipped, and one arrives at

$$\cos\frac{n\pi}{L} = \frac{\cos\frac{1}{2}(L-2)\alpha}{\cos\frac{1}{2}L\alpha} \text{ for } L \text{ even and } n \text{ even} \qquad (12.6.8)$$

We consider the case $L = 4$. We expect $\frac{1}{2}L(L-3) = 2$ roots. We plot for $L = 4$ both curves

$$\begin{array}{ll}
\cos\frac{n\pi}{L} = \frac{\sin\frac{1}{2}(L-2)\alpha}{\sin\frac{1}{2}L\alpha} = \frac{1}{2\cos\alpha} & \cos\frac{n\pi}{L} = \frac{\cos\frac{1}{2}(L-2)\alpha}{\cos\frac{1}{2}L\alpha} = \frac{\cos\alpha}{\cos 2\alpha} \\
n = 1 : \cos\frac{\pi}{4} = \frac{1}{2}\sqrt{2} = \frac{1}{2\cos\alpha} \Longrightarrow \alpha = \frac{\pi}{4}, \frac{7\pi}{4} & n = 0 : 1 = f(\alpha) : \alpha = 0, \frac{2\pi}{3} \\
n = 3 : \cos\frac{3\pi}{4} = -\frac{1}{2}\sqrt{2} = \frac{1}{2\cos\alpha} \Longrightarrow \alpha = \frac{3\pi}{4}, \frac{5\pi}{4} & n = 2 : 0 = f(\alpha) : \alpha = \frac{\pi}{2}
\end{array}$$

The root at $\alpha = \pi/4$ yields $X = e^{i\pi/4}e^{i\pi/4} = i$ while the root at $\alpha = \frac{3\pi}{4}$ yields $X = e^{3i\pi/4}e^{i3\pi/4} = -i$. The root at $\alpha = 0$ yields $X = 1$, the root at $\alpha = 2\pi/3$ yields $X = e^{2\pi i/3}$ while the root at $\alpha = \pi/2$ yields $X = e^{2i\pi/4}e^{i\pi/2} = -1$. Thus we seem to

[10] If $(X_1)^L = -1$ one gets $X_1 = X_2$, hence $u_1 = u_2$, but then the state $B(u_1)B(u_2) \mid \Omega\rangle$ vanishes. Next $X_1 = 1$ violates $X_1 = \frac{2u_1+i}{2u_1-i}$. Similarly $X_1 = \omega$ implies $X_2 = 1$ which violates $X_2 = \frac{2u_2+i}{2u_2-i}$. Finally ω/X_1 yields u_2 if X_1 yields u_1 and $B(u_1)B(u_2) \mid \Omega\rangle$ is the same state as $B(u_2)B(u_1) \mid \Omega\rangle$.

get five roots: $X = i, X = -i, X = 1, X = \frac{-1+i}{\sqrt{2}}$ and $X = -1$. However, all roots are excluded except

$$X_1 = \frac{-1+i}{\sqrt{2}} \tag{12.6.9}$$

Hence we expect one complex root for $L = 4$.

To find the complex roots of the algebraic Bethe equations in the two-particle sector ($N = 2$), we use the fact that for complex solutions $u_2 = (u_1)^*$. If $u_2 = u_1^*$ the energy E becomes real, as it should, but we do not give a formal proof of the relation $u_2 = u_1^*$ and refer instead to the literature. Setting $u_1 = \frac{1}{2}(x + iy)$ we find

$$\left(\frac{u_1 + i/2}{u_1 - i/2}\right)^L = \left(\frac{x + i(y+1)}{x + i(y-1)}\right)^L = \frac{(u_1 - u_2) + i}{(u_1 - u_2) - i} = \frac{y+1}{y-1} \tag{12.6.10}$$

Dividing by the complex conjugate

$$\left(\frac{x - i(y+1)}{x - i(y-1)}\right)^L = \frac{y+1}{y-1} \tag{12.6.11}$$

we obtain

$$\left[\frac{x + i(y+1)}{x + i(y-1)}\right]^L \left[\frac{x - i(y-1)}{x - i(y+1)}\right]^L = 1 \tag{12.6.12}$$

Hence

$$\frac{x + i(y+1)}{x + i(y-1)} \frac{x - i(y-1)}{x - i(y+1)} = \omega \ , \omega = e^{2\pi i n/L}, n = 0, 1, \ldots L-1 \tag{12.6.13}$$

This leads to a quadratic complex equation for x and y

$$[x + i(y+1)][x - i(y-1)] = \omega[x + i(y-1)][x - i(y+1)] \tag{12.6.14}$$

whose real and imaginary parts are

$$\begin{aligned} x^2 + y^2 - 1 &= (\cos 2\pi n/L)(x^2 + y^2 - 1) + (\sin 2\pi n/L)(2x) \\ 2x &= (\cos 2\pi n/L)(-2x) + \left(\sin \frac{2n\pi}{L}\right)(x^2 + y^2 - 1) \end{aligned} \tag{12.6.15}$$

Rearranging terms we find

$$\begin{aligned}(1-\cos 2\pi n/L) &= (\sin 2\pi n/L)\frac{2x}{x^2+y^2-1}\\ (1+\cos 2\pi n/L)\frac{2x}{x^2+y^2-1} &= \sin 2\pi n/L \Longrightarrow tg\frac{\pi n}{L} = \frac{2x}{x^2+y^2-1}\end{aligned} \tag{12.6.16}$$

These curves are circles

$$\left(x-\frac{1}{tg\ n\pi/L}\right)^2 + y^2 = \frac{1}{(\sin n\pi/L)^2} \tag{12.6.17}$$

Instead of taking the ratio of (12.6.10) and (12.6.11) we can also take the product

$$\left[\frac{x^2+(y+1)^2}{x^2+(y-1)^2}\right]^L = \frac{(y+1)^2}{(y-1)^2} \tag{12.6.18}$$

This yields a curve in the $x-y$ plane which contains the whole x-axis ($y=0$) and further a deformed ellipse which intersects the x-axis at $x=\pm\sqrt{L-1}$, and a deformed hyperbole which touches the ellips at $x=0, y=\pm 1$ and whose asymptotes are $y=\pm\sqrt{L-1}x$.

Consider the case $L=4$. Then we need points which lie both on this curve and on the circles

$$\left(x-\frac{1}{tg\ n\pi/4}\right)^2 + y^2 = \frac{1}{\sin^2 n\pi/4} \text{ for } n=0,1,2,3 \tag{12.6.19}$$

More explicitly, these circles are given by

$$\begin{aligned}&n=0: x=0\ (y-\ \text{axis})\ ; n=2: x^2+y^2=1\\ &n=1: (x-1)^2+y^2=2\ ; n=3: (x+1)^2+y^2=2\end{aligned} \tag{12.6.20}$$

Hence the solutions are:

$$n=0: x=0, y=\pm 1,\ \text{and } x=0, y=0$$

$$\begin{aligned}
&n = 1 : x = 0, y = \pm 1 \text{ and } x = 1 \pm \sqrt{2}, y = 0 \text{ and } x = x_0, y = \pm y_0 \\
&n = 2 : x = 0, y \pm 1 \text{ and } y = 0, x = \pm 1 \\
&n = 3 : x = 0, y = \pm 1 \text{ and } x = -1 \pm \sqrt{2}, y = 0 \text{ and } x = -x_0, y = \pm y_0
\end{aligned} \tag{12.6.21}$$

There is indeed only one bound state ($x = \pm x_0, y = \pm y_0$).

From the eigenvalues $\lambda(u)$ in (12.4.9) we can find expressions for the shift operator $\hat{\prod}$ and Hamiltonian $\hat{H}$. Since $\lambda(u) = trT(u) = a(u)\prod_{j=1}^N f(u, u_j) + d(u)\prod_{j=1}^N f(u_j, u)$ and $\tau(u) = -i \ln trT(u) = -\frac{\pi L}{2} + \hat{\prod} + (u + i/2)\left(\frac{\hat{H}}{2J} + \frac{1}{2}L\right) + \ldots$ we find

$$\begin{aligned}
&(-i)^L \exp i\hat{\prod} = \tau(u = -i/2) = (-i)^L \prod_{j=1}^N f(-i/2, u_j) \\
&= (-i)^L \prod_{j=1}^N \left(1 + \frac{i}{-i/2 - u_j}\right) = (-i)^L \prod_{j=1}^N \left(\frac{i/2 - u_j}{-i/2 - u_j}\right) \\
&= (-i) \prod_{j=1}^N \frac{u_j - i/2}{u_j + i/2} = (-i)^L e^{iP}
\end{aligned} \tag{12.6.22}$$

Recalling that $\ln \frac{u_j - i/2}{u_j + i/2} = ip_j$ we find for the eigenvalue of $\hat{\prod}$ on the states $|\ \vec{u}\rangle$ in (12.4.1) the following result

$$P = \sum_{j=1}^N p_j \text{ mod } 2\pi \tag{12.6.23}$$

For the Hamiltonian similar results hold. We have $\frac{\partial}{\partial u}\tau(u) = \frac{\hat{H}}{2J} + \frac{1}{2}L$ where

$$-i\frac{\lambda'(u = -i/2)}{\lambda(u = -i/2)} = -i\left[\frac{a'(-i/2)}{a(-i/2)} + \sum_{j=1}^N \frac{\partial_u f'(u = -i/2, u_j)}{f(u = -i/2, u_j)}\right] \tag{12.6.24}$$

Recalling $a(u) = (u - \frac{i}{2})^L$ and $f(u, v) = 1 + \frac{i}{u-v}$ we get

$$\begin{aligned}
&\frac{\partial}{\partial u}\tau(u) = -i\left[\frac{L(-i)^{L-1}}{(-i)^L} + \sum_{j=1}^N \frac{-i}{(-i/2 - u_j)^2}\left(\frac{-i/2 - u_j}{i/2 - u_j}\right)\right] \\
&= L - \sum_{j=1}^N \frac{1}{u_j^2 + 1/4}
\end{aligned} \tag{12.6.25}$$

Hence

$$\frac{E}{2J} = \frac{1}{2}L - \sum_{j=1}^{N} \frac{1}{u_j^2 + 1/4} \ . \tag{12.6.26}$$

This can be rewritten as follows. From

$$\begin{aligned} 2\cos p_j &= e^{ip_j} + e^{-ip_j} = \frac{u_j - i/2}{u_j + i/2} + \frac{u_j + i/2}{u_j - i/2} \\ &= \frac{2u_j^2 - 1/2}{u_j^2 + 1/4} = 2 - \frac{1}{u_j^2 + 1/4} \end{aligned} \tag{12.6.27}$$

we find

$$\frac{E}{2J} = \frac{1}{2}L + \sum_{j=1}^{N} 2\cos p_j - 2N \tag{12.6.28}$$

For $N = 0$ we find $E = JL$ which agrees with the expectation value of $\hat{H} = J \sum \vec{\sigma}_j \cdot \vec{\sigma}_{j+1}$ on the state $| \, \Omega\rangle = \binom{1}{0}^{\otimes L}$.

7 The boundary Yang-Baxter equation

The Yang-Baxter equation and the corresponding Yang-Baxter algebra describe the Heisenberg spin 1/2 chain (and many other models) with periodic boundary conditions. It is natural to investigate other boundary conditions. There exists an extension of the YB equation which takes more general boundary conditions into account, called the boundary Yang-Baxter equation (BYB), also called the reflection equation [12]. We now discuss this equation. One of the reasons this equation is interesting is that for certain matrices K one recovers in the continuum limit certain conformal field theories defined on a half-plane. Hence, the BYB equation can be viewed as a regularization method of these two-dimensional conformal field theories. (Another interesting result is that one can recover certain **super** conformal field theories in this way. Does this suggest a solution to the well-known problem of describing fermions and supersymmetry on the lattice?)

The basic idea is to introduce a boundary matrix $K(u)$ which acts only in the auxiliary space but not in the quantum spaces labeled by $j = 1, \dots L$. This matrix must satisfy the condition

$$[R(u-v), K(u) \otimes K(v)] = 0 \tag{12.7.1}$$

As we shall see, the matrix K will be associated with the matrix T, and can be interpreted as a change of the periodic boundary conditions. Thus the relation $[R, KK] = 0$ can be viewed as a symmetry property of R. We shall consider K matrices which are independent of u and v.

If one multiplies the RTT equation by $K \otimes K$ from the left one obtains the following result

$$\begin{aligned} K_1(u)K_2(v)R_{12}(u-v)T_1(u)T_2(v) &= R_{12}(u-v)K_1(u)K_2(v)T_1(u)T_2(v) \\ &= K_1(u)K_2(v)T_2(v)T_1(u)R_{12}(u-v) \end{aligned} \tag{12.7.2}$$

Multiplying by R_{12}^{-1} from the left (or from the right) and taking the trace as before, we now arrive at commuting transfer matrices involving the boundary operator

$$[trK(u)T(u), trK(v)T(u)] = 0 \tag{12.7.3}$$

The operator $trK(u)T(u)$ generalizes the transfer matrix $trT(u)$ which we derived before for periodic boundary conditions. In the spin 1/2 model, the choice $K = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ leads to antiperiodic boundary conditions (it requires that if $|\alpha\rangle_N = |\uparrow\rangle$ then $|\alpha\rangle_1 = |\downarrow\rangle$, and if $|\alpha\rangle_N = |\downarrow\rangle$ then $|\alpha\rangle_1 = |\uparrow\rangle$). For the spin 1/2 model, $K \otimes K$ becomes in matrix form $\begin{pmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{pmatrix}$, and one may check that it indeed commutes with $\check{R} = \frac{i}{v-u} + P$. Other choices of K correspond to other boundary conditions. This is thus an obvious extension of the closed chain. In fact, under a similarity transformation $R \to R' = (M \otimes M)R(M \otimes M)^{-1}$, the matrix R' satisfies again the

YBE. However, the only R matrix which is invariant under all constant similarity transformations $(M \otimes M)R(M \otimes M)^{-1} = R$ is the Yang matrix $R(u) = uI + iP$.

The $KKRTT$ equation can also be written as $R\tilde{T}\tilde{T} = \tilde{T}\tilde{T}R$ where now $\tilde{T} = KT$, since K is not an operator (only a numerical matrix). Going through the same steps as for the case of periodic boundary conditions, one finds for $K = \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{pmatrix}$ that only the right-hand side of (12.5.25) gets multiplied by $e^{2i\alpha}$. Taking the product over all $n = 1, \ldots, N$ one finds that all factors cancel leaving only $e^{2i\alpha N}$. The left-hand side yields $\exp iL \sum_{n=1}^{N} p(\lambda_n)$. The meaning of α follows from $P = 2\alpha(N/L)$.

The matrix $K = \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{pmatrix}$ corresponds to the condition $\psi(x+L) = e^{2i\alpha}\psi(x)$ in the coordinate Bethe ansatz. (One associates the vacuum of the corresponding quantum mechanical system with all spins up, and nonvanishing $\psi(x)$ with some spins down. Clearly $K = \begin{pmatrix} 1 & 0 \\ 0 & e^{-2i\alpha} \end{pmatrix}$ keeps the vacuum invariant, and leads to the "twisted-boundary condition" on $\psi(x)$. If a magnetic flux goes through the loop formed by the closed chain, the wave function of an electrically charged particle acquires a phase factor $e^{-2i\alpha}$). The other matrix $K = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ can also be generalized to $\begin{pmatrix} 0 & e^{i\alpha} \\ e^{-i\alpha} & 0 \end{pmatrix}$ and corresponds to another type of antiperiodic (or twisted) boundary conditions, namely on the spin states rather than the wave function.

We begin the discussion of the open chain by using a physical picture of particles with 2-body interactions in a finite (1-dimensional) box. The operators K are viewed as walls on which the auxiliary particles reflect. Thus K is an operator which maps from the one auxiliary space to the second. The $\check{R}$-matrix could be interpreted as a 4×4 matrix in the spin 1/2 model; similarly, the K-matrix can be interpreted as a 2×2 matrix mappingfrom the states $|\alpha\rangle_a$ in the bottom row to the states $|\beta\rangle_b$ in the top row.

Recalling that the essence of the YB and YBA equations was that the order in which processes occurred was irrelevant, we can at once write down a picture which suggests how to write down the BYB equation (time is again running upwards)

$$(12.7.4)$$

$$K_2(v)\check{R}_{12}(u+v)K_1(u)\check{R}_{12}(u-v) = \check{R}_{12}(u-v)K_1(u)\check{R}_{12}(u+v)K_2(v) \quad (12.7.5)$$

One may view u as the rapidity of particle 1 and v as the rapidity of particle 2; then the reflected particles have opposite rapidities but the orientation at the second R_{12} vertex is different

$$u \;\; R_{12}(u,v); u \;\; R_{12}(u,-v) \quad (12.7.6)$$

The R_{ij} are the R-matrices defined before, they act in the tensor product $|\alpha\rangle_a \otimes |\beta\rangle_b$ of the two auxiliary spaces (two horizontal wigly lines as before). The operator K acts on the particle (spin) at the first (or last) site. There are thus 2 operations to close (connect) the two wigly lines: tracing (which gave us before the transfer matrix) or adding a boundary K.

It is also possible to introduce two boundaries, one on the left and one as the right, with boundary matrices K_l and K_r, and to construct the analogue of the RTT relations. The transfer matrices become double-row transfer matrices which commute with themselves for different u. We refer the reader to the literature [12]. We introduced before the monodromy matrix $T(u) = L_L(u)\dots L_1(u)$. We generalize this now to Sklyanin's double-row monodromy matrices (one with K_l and another with K_r at the end)

$$\begin{aligned}\tilde{T}^r(u) &= L_M(u)\dots L_1(u)K^r(u)L_1^{-1}(-u)\dots L_M^{-1}(-u)\\ \tilde{T}^{l,t}(u) &= L^t_{M+1}(u)\dots L^t_L(u)K^{l,t}(u)L_L^{t,-1}(-u)\dots L^{t,-1}_{M+1}(-u) \quad (12.7.7)\end{aligned}$$

(the spaces $j = 1,\dots M$ in T_r are different from those in T_l). Pictorially T_l and T_r can be written as follows

$$(12.7.8)$$

As the notation indicates, the matrices T_l and T_r satisfy a generalized RTT equation which leads again to commuting transfer matrices after taking a suitable trace. To derive this equation, we start from the $RKRK$ equation, multiply again by L^2L^1 and use the RLL equation to move the L^2L^1 operators to the other side of the equation. As we now show, this procedure yields

$$\begin{aligned}&\check{R}(u,v)(\tilde{T}_r(u)\otimes I)\check{R}(u,-v)(I\otimes\tilde{T}_r(v)) = \\ &(I\otimes\tilde{T}_r(v))\check{R}(u,-v)(\tilde{T}_r(u)\otimes I)\check{R}(u,v)\end{aligned} \tag{12.7.9}$$

For $\tilde{T}_r = K_r$ we recover $\check{R}_{12}(u-v)K_1\check{R}_{12}(u+v)K_2 = K_2\check{R}_{12}(u+v)K_1\check{R}_{12}(u-v)$. It is useful to suppress the tensor notation and use subscripts 1 and 2 to denote the first and second auxiliary space, respectively.

We should then prove the following relation

$$\begin{aligned}&\check{R}_{12}(u-v)\tilde{T}_1^r(u)R_{12}(u+v)\tilde{T}_2^r(v) \\ &\quad = \tilde{T}_2^r(v)R_{12}(u+v)\tilde{T}_1^r(u)R_{12}(u-v)\end{aligned} \tag{12.7.10}$$

with $T^r(u)$ given in (12.7.7). We indicate the steps in the proof. In the first step[11] we use $\check{R}_{12}(u-v) = \check{R}_{21}(u-v)$

$$\check{R}_{12}(u-v)\tilde{T}_1^r(u)R_{12}(u+v)\tilde{T}_2^r(v) =$$

$$\check{R}_{12}(u-v)T_1(u)K_1^r(u)T_1^{-1}(-u)\check{R}_{12}(u+v)T_2(v)K_2^r(v)T_2^{-1}(-v)$$

$$T_2(v)\check{R}_{12}(u+v)T_1^{-1}(-u)$$

$$T_2(v)K_1^r(u) \qquad K_2^r(v)T_1^{-1}(-u)$$

$$T_2(v)T_1(u)\check{R}_{12}(u-v)$$

$$K_2^r(v)\check{R}_{12}(u+v)K_1^r(u)\check{R}_{12}(u-v)$$

$$K_2^r(v)T_1(u) \qquad T_2^{-1}(-v)T_1^{-1}(-u)\check{R}_{12}(u-v)$$

[11]The property $R_{12}(u) = R_{21}(u)$ reads explicitly $R_{\alpha\beta}{}^{\gamma\delta}(u) = R_{\beta\alpha}{}^{\delta\gamma}(u)$. Substitute this into the $RTT = TTR$ relation, replace u by $-u$, and then interchange $1 \leftrightarrow 2$ and $u \leftrightarrow v$. The result is $T_1^{-1}(-u)R_{12}(u+v)T_2(v) = T_2(v)R_{12}(u+v)T_1^{-1}(-u)$.

$$T_2^{-1}(-v)K_1^r(u)$$

$$T_2^{-1}(-v)\check{R}_{12}(u+v)T_1(u) \qquad \text{q.e.d.} \tag{12.7.11}$$

Similarly, $\tilde{T}^l(u)$ satisfies

$$\begin{aligned}\check{R}_{12}(-u+v)\tilde{T}_1^{l,t}(u)\check{R}_{12}(-u-v-2\eta)\tilde{T}_2^{l,t}(v) = \\ \tilde{T}_2^{l,t}(v)R_{12}(-u-v-2\eta)\tilde{T}_1^{l,t}(u)R_{12}(-u+v)\end{aligned} \tag{12.7.12}$$

To prove this, we need further properties of the $\check{R}$ matrix

$$\begin{aligned}&\check{R}_{\alpha\beta}{}^{\gamma\delta}(u) = \check{R}_{\gamma\delta}{}^{\alpha\beta}(u); (R_{12}^{t_1}(u) = R_{12}^{t_2}(u)) \\ &R_{12}(u)R_{12}(-u) = \rho(u)I \text{ (with } \rho(u) \text{ a function)} \\ &R_{12}^{t_1}(u)R_{12}^{t_1}(-u-2\eta) = \tilde{\rho}(u)I; (R_{\gamma\beta}{}^{\alpha\delta}(u)R_{\epsilon\delta}{}^{\gamma\zeta}(u-2\eta) = \tilde{\rho}(u))\end{aligned} \tag{12.7.13}$$

The transfer matrix (more precisely, the double-row transfer matrix) is defined by

$$T(u) = tr\tilde{T}_l(u)\tilde{T}_r(u) = \tag{12.7.14}$$

and it commutes again with itself for different u

$$[T(u), T(v)] = 0 \tag{12.7.15}$$

To prove that the transfer matrices commute for different u, we start from $tr\ \tau(u)\tau(v)$ and use the crossing relation and the unitarity relation to insert four R matrices into the trace. Then we use the $R\tilde{T}\tilde{T} = \tilde{T}\tilde{T}R$ relation to invert the order of the operators which depend on u and v, and finally we run the derivation backwards to remove the four R-matrices. The details are as follows

$$\begin{aligned}tr\ \tau(u)\tau(v) &= (tr_1\tilde{T}_1^l(u)\tilde{T}_1^r(u))tr_2(\tilde{T}_2^l(v)\tilde{T}_2^r(v)) \\ &= (tr_1\tilde{T}^{l,t_1}(u)\tilde{T}_1^{r,t_1})tr_2(\tilde{T}_2^l(v)\tilde{T}_2^r(v) \\ &= tr_{12}\tilde{T}_1^{l,t_1}(u)\tilde{T}_1^{r,t_1}(u)\tilde{T}_2^l(v)\tilde{T}_2^r(v) \\ &= tr_{12}\tilde{T}_1^{l,t_1}(u)\tilde{T}_2^l(v)\tilde{T}_1^{r,t_1}(u)\tilde{T}_2^r(v)\end{aligned} \tag{12.7.16}$$

We used that $\tilde{T}_1^r(u)$ and $\tilde{T}_2^l(v)$ commute because they act in the first M and last $M-N$ quantum spaces, respectively. Next we insert a factor proportional to unity

$$R_{12}^{t_1}(u+v)R_{12}^{t_1}(-u-v-2\eta) = R_{12}^{t_2}(-u-v-2\eta)R_{12}^{t_1}(u+v) = \tilde{\rho}(u)I \quad (12.7.17)$$

in the middle and write

$$\tilde{T}_1^{l,t_1}(u)\tilde{T}_2^l(v)R_{12}^{t_2}(-u-v-2\eta) = (\tilde{T}_1^{l,t_1}(u)R_{12}(-u-v-2\eta)\tilde{T}_2^{l,t_2}(v))^{t_2} \quad (12.7.18)$$

Similarly we write

$$R_{12}^{t_1}(u+v)\tilde{T}_1^{r,t_1}(u)\tilde{T}_2^r(v) = (\tilde{T}_1^r(u)R_{12}(u+v)\tilde{T}_2^r(v))^{t_1} \quad (12.7.19)$$

Finally we use the identity $tr_{12}A^{t_2}B^{t_1} = trA^{t_{12}}B$ where $t_{12} = t_2t_1$ and arrive at

$$tr_{12}(\tilde{T}_1^{l,t_1}R_{12}(-u-v-2\eta)\tilde{T}_2^{l,t_2})^{t_{12}}\tilde{T}_1^r(u)R_{12}(u+v)T_2^r(v) \quad (12.7.20)$$

Next we use the unitarity relation

$$R_{12}(-u+v)R_{12}(u-v) = \rho(-u+v)I \quad (12.7.21)$$

to insert two more R matrices in the middle, and use $A^{t_{12}}R_{12}(-u+v) = (R_{12}(-u+v)A)^{t_{12}}$ to move $R_{12}(-u+v)$ to the far left. We then have the $R\tilde{T}R\tilde{T}$ matrices in the correct position to apply the $R\tilde{T}R\tilde{T}$ identity once on the left and once on the right. This yields

$$\begin{aligned} tr_{12}\ &\{\tilde{T}_2^{l,t_2}R_{12}(-u-v-2\eta)\tilde{T}_1^{l,t_1}T_{12}(-u+v)\}^{t_{12}} \\ &\{\tilde{T}_2^r(v)R_{12}(u+v)T_1^r(u)R_{12}(u-v)\} \end{aligned} \quad (12.7.22)$$

From here on we repeat the whole chain of arguments in reverse order, and obtain $tr\tau(v)\tau(u)$. Hence $\tau(u)$ and $\tau(v)$ indeed commute.

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Chapter 13

The Gribov problem

The basic reason for fixing the gauge in path integrals is to make the integral convergent by restricting the sum over paths to precisely one representative from each gauge orbit. More precisely, one replaces the summation over all ${A_\mu}^a$ by a summation over representatives (which yields a finite result) times an integration over the gauge group. The latter is an infinite constant which is dropped. (At each point in spacetime the volume of the compact gauge group is a finite constant, but the product of these constants over all spacetime points is an infinite, field-independent constant.) The result of the gauge fixing procedure is an extra factor $\Delta_F(A_\mu)\delta(\partial^\mu {A_\mu}^a - f^a)$ in the measure at each point in spacetime, or a similar expression for other gauges such as the Coulomb gauge $\partial^i {A_i}^a - f^a = 0$. However, the equation $\partial^\mu {A_\mu}^a = f^a$ or $\partial^i {A_i}^a = f^a$ has in general more than one solution. Thus, the commonly used gauge fixing procedure does not fix the gauge for such useful gauges as $\partial^\mu A_\mu = f$ or $\partial^i A_i = f$! This is the Gribov problem.

There is at present no complete solution of the Gribov problem. In perturbation theory there are no Gribov copies,[1] and if one only uses gauges without Gribov copies (such as ${A_3}^a = 0$ or, more generally, $n \cdot A = 0$) for nonperturbative aspects,

[1]This statement is incomplete. As we shall discuss, it is plausible that there are no **nearby** Gribov copies if one requires that fields fall off to zero at infinity, but for a system in a box with length L and periodic boundary conditions there are nearby Gribov copies.

there is not even a Gribov problem. For practical calculations at the perturbative level one should then use relativistic gauges with $\partial^\mu A_\mu$ because $n \cdot A$ gauges are difficult to compute with: the propagators have multiple singularities and lead to nonlocal counter terms, and only a few one-loop calculations have been performed. For practical calculations at the nonperturbative level one could use lattice gauge theory. However, lattice gauge theory has its own unsolved problems so one would like to be able to perform nonperturbative calculations in continuum field theory using relativistic gauges $\partial^\mu A_\mu - f = 0$, and for this problem no solution had yet been found.

There is, actually, a second problem with the usual Faddeev-Popov quantization method. Inserting unity into the path integral yields $| \det \partial^\mu D_\mu |$ for $\Delta_F(A_\mu)$, but integrating the ghost action $\mathcal{L}$ (ghost) $= b\partial^\mu D_\mu c$ over the antighosts b and ghosts c leads to $\det \partial^\mu D_\mu$ without absolute value sign. If one uses BRST quantization, one needs the action with $\mathcal{L}$ (ghost), so **BRST quantization seems inconsistent with Faddeev-Popov quantization**. Since BRST transformations are infinitesimal (in the sense that exponentiating $\delta c = c^2\Lambda$ leads to $c' = \exp c^2\Lambda$ but expanding the exponent only $1 + c^2\Lambda$ remains as $\Lambda^2 = 0$), they seem not to know about Gribov copies. Yet many people believe that BRST symmetry is the correct symmetry at the quantum level. This seems to suggest that one should use $\det \partial^\mu D_\mu$ instead of $| \det \partial^\mu D_\mu |$ in the path integral. There is a way that both results $| \det \partial^\mu D_\mu |$ and $\det \partial^\mu D_\mu$ are correct, namely if Gribov copies would cancel in pairs with opposite signs: then using $| \det \partial^\mu D_\mu |$ and restricting the sum to one Gribov copy would still be equivalent to using $\det \partial^\mu D_\mu$ without any restriction on the range of integration over ${A_\mu}^a$ in the path integral. We discuss this in more detail below. [2]

It is clear that the problem of Gribov copies is intimately linked to properties

[2] In lattice gauge theories one need not fix the gauge, and hence here no Gribov problem exists: one just integrates over all the links. It has been claimed that if one nevertheless adds the operator $\det \partial^\mu D_\mu$ to the measure (without absolute value signs!) Gribov copies cancel in pairs and one obtains the same results. It is not yet clear whether similar results hold in continuum field theory.

of the operators $\partial^\mu D_\mu$ or $\partial^i D_i$. The study of the operator $\partial^\mu D_\mu$ is hindered by the fact that when $\partial^\mu A_\mu$ does not vanish, for example, when one uses weighted gauges $\partial^\mu A_\mu - f = 0$, the operator $\partial^\mu D_\mu$ is not hermitian, and arguments to the extent that "the operator $\partial^\mu D_\mu$ is sufficiently large" need a more precise mathematical definition. For a self-adjoint operator one may define the size of an operator by the size of its eigenvalues.

For abelian gauge fields in Euclidean space, no Gribov problem exists since the condition for Gribov copies reads $\Delta^2\lambda = 0$ where λ is the gauge parameter. This equation has no solutions other than constant λ, and for constant λ there are no Gribov copies since $A_\mu + \partial_\mu\lambda = A_\mu$. For nonabelian ${A_\mu}^a(x)$ which are small everywhere in spacetime, no Gribov problem exists as we shall show, hence as far as perturbation theory is concerned, no inconsistencies arise. But for large ${A_\mu}^a$, i.e., at the nonperturbative level, the gauge fixing procedure breaks down, and this necessitates revision and modification of the path integral for quantum gauge field theory.

When there are multiple Gribov copies, the problem arises what to do. Should one only pick one copy, and if so, which copy? If one goes through the usual argument of inserting unity in the path integral in the form of $\Delta \int \delta[\partial\cdot A^\Omega - f]d\Omega$ where $d\Omega$ is the gauge-invariant Haar measure, one finds that $\Delta^{-1} = \sum_k \Delta_F^{-1}(A^{(k)})$ where the sum runs over all Gribov copies on a given gauge orbit. This Δ is still gauge invariant because the Haar measure $d\Omega$ is gauge invariant, but the Faddeev-Popov determininant in the path integral would be replaced by $[\sum_k \Delta_F^{-1}(A^{(k)})]^{-1}$, and this expression can no longer be exponentiated by using ghosts. Hence, though formally correct, this solution is useless for practical applications. We discuss this further in a separate section below.

Another proposal is to add a factor $[1 + n(A)]^{-1}$ to the path integral measure where $n(A)$ is the number of copies for a given field configuration ${A_\mu}^a(x)$.[3] That is

[3]In string theory a similar problem arises: the gauge $h_{\alpha\beta} = h^{(0)}_{\alpha\beta}$ does not fix the gauge symmetries which correspond to conformal Killing vectors. One divides then by the volume of the group of conformal Killing vectors which makes sense since the rest of the path integral does not change under these

not a practical solution since one does not have available an explicit formula for $n(A)$. It is also not clear that this procedure is correct. Actually, what physical principle decides which alternative is correct, unitarity perhaps?

It has been suggested [10] to restrict the path integral to those field configurations for which $\Delta_F(A) \geq 0$, in the belief that inside this "Faddeev-Popov horizon" no Gribov copies exist. This is false: as we shall show, in the gauge $\partial^\mu A_\mu = f$ there exist Gribov copies for which $\Delta_F(A) > 0$ and which are connected to the origin $A_\mu = 0$ by a straight line λA along which $\Delta_F(\lambda A)$ always remains positive.

Some people have proposed to restrict the path integral over ${A_\mu}^a$ for each gauge orbit to that Gribov copy for which

$$||{A_\mu}^U||^2 \equiv \int ({A_\mu}^{a,U})^\dagger A^{\mu,a,U} d^4x \tag{13.0.1}$$

is minimal where ${A_\mu}^U = U^{-1}(\partial_\mu + A_\mu)U$. There may be gauge orbits for which the absolute minimum is attained by two or more fields ${A_\mu}^U$. Presumably the set of gauge orbits whose absolute minimum is degenerate has measure zero in the total space of gauge orbits, in which case the ambiguity in choosing the Gribov copy in these cases is irrelevant. However, there is no simple formula which one can stick inside the path integral such that it selects the minima of $||{A_\mu}^U||$, hence from a practical point of view, also this is not a solution.

One may summarize the problem of Gribov copies by the following figure

(13.0.2)

Γ: hyperplane with $\partial \cdot A = 0$

Ω: Gribov region with $\partial \cdot A = 0$ and $-\partial D \geq 0$

Λ: Fundamental modular region with $||A|| \leq ||A^\Omega||$.

$A^I \epsilon \Omega, A^{II} \epsilon \Lambda$ and A^{III} outside Ω are Gribov copies which all lie on the same gauge

symmetries. In gauge field theory different Gribov copies have the same $\partial \cdot A$ but different A, so it is not clear that dividing by $1 + n(A)$ is the correct procedure.

orbit A_1^u. The region Ω is the set of all local and absolute minima, while Λ denotes the set of all absolute minima.

We shall first follow the original article by Gribov, and work through a toy model which is so simple that it allows explicit solutions: SU(2) gauge theory with time-independent spherically symmetric gauge fields in the Coulomb gauge. Then we shall consider the relativistic gauges $\partial^\mu A_\mu{}^a = f^a$ and relate the occurrence of Gribov copies to a variational problem for the functional integral in (20.0.1). In the next section we reanalyze in full generality the Faddeev-Popov method in the presence of Gribov copies. We also consider the temporal and other $n \cdot A = 0$ gauges and show that for them no Gribov problem arises. We shall end with a toy model (constant fields) which can be exactly solved and where we see that one must take all Gribov copies into account. This toy model contains constant fields. A slightly less trivial toy model with quantum mechanical point particles (fields depending only on time) is given in [8].

1 Gribov copies in the Coulomb gauge

Following Gribov's original article [1] we consider SU(2) gauge theory with time-independent spherically-symmetric gauge fields

$$A_j(\vec{x}) = f_1(r)\frac{\partial}{\partial x^j}\left(\frac{i\vec{x}\cdot\vec{\sigma}}{r}\right) + f_2(r)\left(\frac{i\vec{x}\cdot\vec{\sigma}}{r}\right)\frac{\partial}{\partial x^j}\left(\frac{i\vec{x}\cdot\vec{\sigma}}{r}\right) + f_3(r)\left(\frac{i\vec{x}\cdot\vec{\sigma}}{r}\right)\frac{x_j}{r} \tag{13.1.1}$$

By spherically symmetry we mean invariance under simultaneous space rotations which act on $\vec{x}$ and gauge rotations which act on $\vec{\sigma}$. In the term with $f_2(r)$ only terms with one Pauli matrix remain, thus $A_j(\vec{x})$ is indeed Lie-algebra valued.[4] The fields $A_0{}^a$ play no role in the following discussion. Of course, in realistic integrals one cannot restrict one's attention to time-independent spherically-symmetric gauge fields

[4]Use that in $\vec{x}\cdot\vec{\sigma}(\sigma^j/r - x^j\vec{x}\cdot\vec{\sigma}/r^2)$ the contribution without σ matrices vanishes.

with $A_0 = 0$, but since already for this special class of fields we can demonstrate the occurrence of Gribov copies, this proves that Gribov copies also exist in the realistic case. The set of spherically symmetric Gribov copies has measure zero in the space of all gauge field configurations, hence this example does not really prove that there is a problem, but it clarifies the mathematical issues involved.

We consider the gauge orbit of $A_j(\vec{x})$ due to spherically symmetric elements of SU(2)

$$\begin{aligned} \tilde{A}_j(\vec{x}) &= U^{-1}(\partial_j + A_j)U \\ U &= \exp \frac{1}{2}\alpha(r)\left(\frac{i\vec{x}\cdot\vec{\sigma}}{r}\right) = \cos\frac{\alpha}{2} + \left(\frac{i\vec{x}\cdot\vec{\sigma}}{r}\right)\sin\frac{\alpha}{2} \end{aligned} \tag{13.1.2}$$

Since $\tilde{A}_j$ is again spherically symmetric, it can be parametrized by functions $\tilde{f}_j$ as in (20.0.2). It is a matter of straightforward algebra to find the functions $\tilde{f}_j$

$$\begin{aligned} \tilde{f}_1 &= f_1\cos\alpha + \left(f_2 + \frac{1}{2}\right)\sin\alpha \\ \tilde{f}_2 &= -\frac{1}{2} - f_1\sin\alpha + \left(f_2 + \frac{1}{2}\right)\cos\alpha \\ \tilde{f}_3 &= f_3 + \frac{1}{2}\frac{d}{dr}\alpha(r) \end{aligned} \tag{13.1.3}$$

A special role will be played by the constant $-1/2$ in $\tilde{f}_2$; it comes of course from $U^{-1}\partial_j U = (\cos\frac{\alpha}{2} - \frac{i\vec{x}\cdot\vec{\sigma}}{r}\sin\frac{\alpha}{2})(\frac{i\sigma_j}{r} - \frac{i\vec{x}\cdot\vec{\sigma}x_j}{r^3})\sin\frac{\alpha}{2}$ by using $-\sin^2\frac{\alpha}{2} = -\frac{1}{2} + \frac{1}{2}\cos\alpha$.

We consider the Coulomb gauge, and ask for which A_j there exist Gribov copies. Hence we must solve the equation

$$\partial^j A_j = \partial^j \tilde{A}_j \tag{13.1.4}$$

Note that we do not require $\partial^j A_j = 0$ because in the path integral we need $\delta(\partial^i A_i - f)$, not $\delta(\partial^j A_j)$. It is straightforward to evaluate $\partial^j A_j$

$$\partial^j A_j = \left(-\frac{2}{r^2}f_1 + \frac{2}{r}f_3 + f_3'\right)\left(\frac{i\vec{x}\cdot\vec{\sigma}}{r}\right) \tag{13.1.5}$$

Hence, there are Gribov copies when

$$-\frac{2}{r^2}\left[f_1\cos\alpha+\left(f_2+\frac{1}{2}\right)\sin\alpha\right]+\frac{2}{r}\left[f_3+\frac{1}{2}\frac{d}{dr}\alpha\right]$$
$$+\left[\frac{d}{dr}f_3+\frac{1}{2}\frac{d^2}{dr^2}\alpha\right]=-\frac{2}{r^2}f_1+\frac{2}{r}f_3+\frac{d}{dr}f_3 \tag{13.1.6}$$

The terms with f_3 cancel, and rearranging the rest of the terms we find

$$\alpha''+\frac{2}{r}\alpha'-\frac{4}{r^2}f_1(\cos\alpha-1)-\frac{4}{r^2}\left(f_2+\frac{1}{2}\right)\sin\alpha=0 \tag{13.1.7}$$

where $\alpha'=\frac{d}{d\tau}\alpha$. To remove the factors with r^{-1} and r^{-2}, we introduce the "time" variable $\tau=\ln r$, and obtain

$$\ddot{\alpha}+\dot{\alpha}+4f_1(1-\cos\alpha)-(4f_2+2)\sin\alpha=0 \tag{13.1.8}$$

where $\dot{\alpha}=\frac{d}{d\tau}\alpha(\tau)$.

This is the equation of a pendulum, with a damping force $\dot{\alpha}$, and a constant "gravitational" force $(4f_2(r)+2)$ pointing downwards, and with two forces of strength $4f_1$, one of which acting horizontally towards the right, and the other being orthogonal to the pendulum and acting anticlockwise.

> The condition that the original gauge field $A_j{}^a(\vec{x})$ be regular at the origin requires that f_1, f_2 and f_3 vanish at $r=0$ $(\tau=-\infty)$. Moreover, we require that $rA_j{}^a(\vec{x})$ tends to zero for large r. This corresponds to forces f_j which tend to zero for large times.

Requiring U to be regular at the origin $r=0$, we find the initial condition $\alpha(\tau=-\infty)=\pm 2n\pi$. Since $U=\cos\frac{\alpha}{2}+i\frac{\vec{x}\cdot\vec{\sigma}}{r}\sin\frac{\alpha}{2}$, these initial conditions all specify the unit element.

We then get the following picture: at $\tau=-\infty$, the pendulum points upwards, but as time τ goes on, forces start acting on the pendulum and it starts to move away from its metastable equilibrium. Since the forces tend to zero for $\tau\to\infty(rA_j{}^a(\vec{x})$ tends to zero for large $r)$, the pendulum will in general come to rest at it stable equilibrium

$\alpha = \pi$, but then $\tilde{f}_2 \to -\frac{1}{2} + \frac{1}{2}\cos\alpha = -1$, while $\tilde{f}_1$ and $\tilde{f}_3$ vanish, hence $\tilde{A}_j(\vec{x}) \sim \epsilon_{jkl}\frac{x^k}{r^2} i\sigma^l$. These solutions are not square integrable. In path integrals one does not require the fields to be square integrable, but we shall rule these configurations out as Gribov copies by the boundary condition that $rA_j(\vec{x})$ tends to zero as r tends to infinity.

For generic but small f_j (fields A_j used in perturbation theory), the pendulum will always end up at $\alpha = \pi$. Since $\tilde{A}_j$ with $\tilde{f}_2 = -1$ are not small, these solutions can be ignored in perturbation theory. This is generally true: Gribov copies are only a problem for nonperturbative physics.

As a special case suppose one starts with the vacuum $A_i{}^a = 0$. Then we obtain as condition for the occurrence of Gribov copies of the vacuum

$$\ddot{\alpha} + \dot{\alpha} - 2\sin\alpha = 0 \tag{13.1.9}$$

which is a pendulum in a constant gravitational field with friction. It is clear that either $\alpha(\tau)$ stays at its unstable position $\alpha(\tau) = 0$ at all times, or it falls (clockwise or anticlockwise) and comes to rest at $\alpha(\tau = +\infty) = \pi$ without having come back to $\alpha = 0$. Since we have ruled out such $\tilde{A}_j$, we conclude that there are no Gribov copies of the vacuum.

Converseley, by choosing the pure gauge $A_j = U^{-1}\partial_j U$ with $U = \exp\frac{1}{2}\beta(r)(\frac{i\vec{x}\cdot\vec{\sigma}}{r})$, we find forces $f_1 = \frac{1}{2}\sin\beta$, $f_2 = -\frac{1}{2} + \frac{1}{2}\cos\beta$ and $f_3 = \frac{1}{2}\dot{\beta}$. Regularity requires that β vanishes for small and large r. The equation for Gribov copies becomes

$$\ddot{\alpha} + \dot{\alpha} + 2\sin\beta - 2\sin(\alpha + \beta) = 0 \tag{13.1.10}$$

For given $\beta(r)$ there are two solutions for $\alpha(r)$ which end up on top as $r \to 0$: one moves clockwise back in time and the other anticlockwise. But taking these solutions and moving forward in time, the pendulum should again end up on top and only for very special forces will this happen. Thus the equation $\partial^i A_i^\Omega - f = 0$ with A pure gauge has only Gribov copies for very special f.

The Gribov copies correspond to the case where the forces f_j are carefully chosen such that at $\tau = +\infty$ the pendulum ends up in its metastable position ($\alpha = 0$). If the original field A_j is square integrable, the f_j tend fast to zero for large times and then we find the approximate equation $\ddot{\alpha} + \dot{\alpha} - 2\alpha = 0$, whose solution is $\alpha(\tau) \sim e^{-2\tau} = \frac{1}{r^2}$. Thus then also these Gribov copies are square integrable. The other solution $\alpha(\tau) \sim e^{\tau}$ does not fall off at large times.

As a particular case, consider forces such that $\alpha(r)$ remains always small (so that the forces f_j keep propping the pendulum up against the pull of gravity). Then one finds the approximate equation

$$\ddot{\alpha} + \dot{\alpha} - (4f_2 + 2)\alpha = 0 \tag{13.1.11}$$

Thus, for very special field configurations (appropriate f_j), near Gribov copies (small α) exist. For example, consider a gauge field A_j with only a f_2 component. It clearly satisfies $\partial^i A_j = 0$, see (20.0.6), and if a Gribov copy exists (if f_2 is such that the pendulum stays near 12 o'clock and ends up for $\tau \to +\infty$ at 12 o'clock), the relation between α and f_2 in (20.0.12) must hold. One may then check that also the following relation holds as a consequence of (20.0.12)

$$D^j(A)\partial_j \left(i\alpha \frac{\vec{x} \cdot \vec{\sigma}}{r} \right) = \partial^j D_j(A) \left(i\alpha \frac{\vec{x} \cdot \vec{\sigma}}{r} \right) = 0 \tag{13.1.12}$$

Hence, if a nearby Gribov copy exists for such A_j, the Faddeev-Popov determinant develops a zero eigenvalue. This means that the measure becomes singular at such fields A_j, and also that in the spectrum of the ghost fields zero-mass bound states appear. (We call them bound states because they are square integrable, so not part of the continuous spectrum, and we call them massless because massive fields satisfy $(\partial^\mu D_\mu + m^2)\varphi = 0$). We shall later show that also in other gauges such as $\partial^\mu A_\mu = 0$, nearby gauge copies exist whenever $\partial^\mu D_\mu(A)$ develops a zero eigenvalue.

2 The relativistic gauge $\partial^\mu A_\mu = 0$

By far the most used gauge is $\partial^\mu A_\mu = f$, so it is important to study the Gribov problem for this gauge. We ask: when is $\partial^\mu A_\mu = \partial^\mu {A_\mu}^U$ where ${A_\mu}^U = U^{-1}(\partial_\mu + A_\mu)U$? It is clear that the condition on U reads

$$\partial^\mu\{U^{-1}(\partial_\mu U + A_\mu U)\} - \partial^\mu A_\mu = \partial^\mu\{U^{-1}[D_\mu(A), U]\} = 0 \tag{13.2.1}$$

(For gauge fields A_μ satisfying $\partial^\mu A_\mu = 0$, this result can be rewritten a covariant conservation law of a current by multiplication by U from the left and U^{-1} from the right,

$$D_\mu(A) j^\mu = 0, \; j^\mu = \partial^\mu U U^{-1} \tag{13.2.2}$$

However, we continue with the more general case that $\partial^\mu A_\mu$ does not vanish). For an abelian theory with generator T and $U = \exp\varphi T$ this equation reduces to $\Box\varphi = 0$, and in Euclidean space there are no solutions for φ which vanish at infinity. Hence, for abelian gauge theories in Euclidean space there is no Gribov problem. For nonabelian theories we already demonstrated in the example of SU(2) in the Coulomb gauge that Gribov copies are present. In Minkowski spacetime it is difficult to even formulate the problem; for example, even in the abelian case the condition for Gribov copies in the Lorentz gauge is $\Box\lambda = 0$ of which there are infinitely many solutions.

For matrices U near the identity, $U \simeq 1 + \omega^a T_a$, (20.0.14) reduces to

$$\partial^\mu D_\mu(A)\omega = 0 \tag{13.2.3}$$

Hence, if a Gribov copy exists **near** fields ${A_\mu}^a$ it approximately solves the equation $\partial^\mu D_\mu(A)\omega = 0$ where $U = 1 + \omega$. Conversely, when the Faddeev-Popov determinant develops a zero eigenvalue ω, a nearby Gribov copy exists. For the vacuum $A_\mu = 0$ we can actually prove that there are no Gribov copies at all. The equation for Gribov copies is $\partial^\mu(U^{-1}\partial_\mu U) = 0$. This equation can then be written as actually proportional to the field equation of the following action[5]

$$\mathcal{L} = Tr(U^{-1}\partial^\mu U)(U^{-1}\partial_\mu U), U = e^\omega \tag{13.2.4}$$

It is actually easy to show that the action has no nontrivial solutions by using a kind of "Derrick theorem". Assume that $\omega_0(x)$ is a solution, then $S(\omega_0 + \delta\omega) = S(\omega_0) + \mathcal{O}(\delta\omega)^2$. Define a variation $\delta\omega(x)$ by $\omega_0(\lambda x) = \omega_0(x) + \delta\omega(x)$ for λ near 1. One finds then

$$S(\omega_0 + \delta\omega) = \int Tr(\tilde{U}^{-1}\partial_\mu \tilde{U})(\tilde{U}^{-1}\partial^\mu \tilde{U})d^4x \tag{13.2.5}$$

where $\tilde{U} = U(\omega_0(\lambda x))$. Now replace x by λx in the derivatives and in d^2x. Then $S(\omega_0 + \delta\omega) = \lambda^2 S(\omega_0)$, but since $\frac{\partial}{\partial\lambda}(\lambda^2 S(\omega_0))$ must vanish at $\lambda = 1$, the only solution is $S(\omega_0) = 0$, i.e., ${e_\alpha}^a\partial_\mu\omega^\alpha = 0$. Thus there are no Gribov copies for ${A_\mu}^a = 0$.

One may study Gribov copies far away from A_μ by using a variational principle. This has the advantage that one can demonstrate the existence of Gribov copies on plausible grounds without explicitly solving the differential equations. Consider the functional[6]

$$\begin{aligned} f_A(U) &= 2\int tr(A^U)^\dagger A^U d^4x = \int |{A_\mu}^{a,U}|^2 d^4x \equiv ||A^U||^2 \\ &= 2\int tr(\partial^\mu U^{-1}\partial_\mu U - 2(\partial^\mu U)U^{-1}A_\mu - A^\mu A_\mu) \end{aligned} \tag{13.2.6}$$

($trT_aT_b = -\frac{1}{2}\delta_{ab}$ and $T_a^\dagger = -T_a$). Since this functional is positive definite, it seems

[5] Introduce group vielbeins by $U^{-1}\partial_\mu U = T_a e_\alpha^a(\omega)\partial_\mu\omega^\alpha$. The field equation can then be written as $\partial^\mu[{e_\gamma}^a {e_\beta}^a \partial_\mu\omega^\beta] - (\partial_\gamma {e_\alpha}^a)\, {e_\beta}^a\partial_\mu\omega^\alpha\partial^\mu\omega^\beta = 0$. It reduces to

$$(\partial_\alpha {e_\gamma}^a - \partial_\gamma {e_\alpha}^a){e_\beta}^a\partial_\mu\omega^\alpha\partial^\mu\omega^\beta + {e_\gamma}^a\partial^\mu({e_\beta}^a\partial_\mu\omega^\beta) = 0$$

The first term vanishes according to the Cartan-Maurer equation $\partial_\alpha {e_\gamma}^a - \partial_\gamma e_\alpha^a = {f^a}_{bc}e_\alpha^b {e_\gamma}^c$. The second term is the Gribov equation.

[6] In Gribov's original article, the functional $\int(\partial^j U^{-1}\partial_j U - 2(\partial^j U)U^{-1}A_j)d^4x$ was considered. The functional in (20.0.17) is due to ref (–). The analysis of this functional in ref [5] is restricted to square integrable functions. Although this set of functions has measure zero in the set of functions over which one integrates in the path integral (Brownian motion), it is hoped that the features discovered in this analysis remain true for the more general set of functions. For an analysis which does not require L_2 functions, see [Zwanziger].

likely that it has an absolute minimum which correspond to some U_0.[7] This can actually be proven for D-dimensional gauge fields ${A_\mu}^a$ in L_2, and unitary matrices $U(x)$ in the gauge group satisfying

$$||U||^2 \equiv \int tr(U^\dagger - 1)(U-1)\frac{d^Dx}{x^2} + \int tr(\partial^\mu U^\dagger)(\partial_\mu U)d^Dx < \infty \tag{13.2.7}$$

These matrices form a subgroup, the Sobolev group G(S).[8] (To show that UV lies in G(S) if U and V are in G(S), use that $U^{-1}\partial_\mu U$ and $\partial_\mu VV^{-1}$ lie in L_2). If A_μ lies in L_2, also ${A_\mu}^U$ lies in L_2 for U satisfying (20.0.19), because the L_2 norm $||{A_\mu}^U|| = ||A_\mu + \partial_\mu UU^{-1}||$ satisfies the triangle inequality, and $U\partial_\mu U^{-1}$ lies in L_2. Conversely, if both A_μ and ${A_\mu}^U$ lie in L_2, then U lies in $G(S)$ because ${A_\mu}^U - UA_\mu U^{-1} = U^{-1}\partial_\mu U$ lies in L_2. Thus the group $G(S)$ is the natural gauge group for square-integrable gauge fields. It can be rigorously proven that $f_A(U)$ attains an absolute minimum for $A\epsilon L_2$ and $U\epsilon G(S)$.

If U_0 is an extremum of $f_A(U)$, the field ${A_\mu}^{U_0}$ is transversal and small fluctuations around this extremum yield the Faddeev-Popov operator. Indeed, setting $U = U_0 \exp\omega$ one obtains from (20.0.17) with A_μ replaced by $A_\mu^{U_0}$ and $U = e^\omega$

$$f_A(U) = f_A(U_0) - 4\int tr(\partial^\mu\omega){A_\mu}^{U_0}d^4x - 2\int tr[(\partial^\mu\omega)D_\mu(A^{U_0})\omega]d^4x + \mathcal{O}(\omega^3) \tag{13.2.9}$$

Integrating the first term in the integral by parts shows that ${A_\mu}^{U_0}$ is transversal. Integrating the second term by parts yields the Faddeev-Popov operator.

This demonstrates for $A_\mu\epsilon L_2$ that on each gauge orbit A^U there is one (or more) relative or absolute minima at which ${A_\mu}^{U_0}$ is transversal and the Faddeev-Popov

[7] Since (20.0.17) is linear in A, we can consider such large A that for some U it becomes negative. At the minimum of $f_A(U)$ on the gauge orbit of such large $A, \partial^\mu A_\mu = 0$ and the Faddeev-Popov operator becomes positive semi definite. Thus Gribov copies appear for large A_μ.

[8] For any smooth w, which is only nonvanishing on a compact doman in R^D, one may show

$$\int trw^\dagger w\frac{d^Dx}{x^2} \leq \frac{4}{(D-2)^2}\int tr(\partial^\mu w^\dagger)(\partial_\mu w)d^Dx \tag{13.2.8}$$

To prove (20.0.19), consider $\int |\partial_\mu(wr^{-\alpha})|^2 r^{2\alpha}d^Dx \geq 0$. The terms with $\partial_\mu w^\dagger w$ and $w^\dagger\partial_\mu w$ can be rewritten as terms with $w^\dagger w$ by using $\partial_\mu w^\dagger w + w^\dagger\partial_\mu w = \partial_\mu(w^\dagger w)$ and partially integrating. Then minimize w.r.t. the parameter operator is positive definite if the minimum is as isolated minimum. Thus for $A_\mu \epsilon L_2$ Gribov copies do exist.

A natural set C of A_μ to consider are those transversal A_μ for which the Faddeev-Popov operator is positive or zero[9]

$$C = \{A_\mu | A_\mu \epsilon L_2, \partial^\mu A_\mu = 0, \partial^\mu D_\mu(A) \geq 0\} \tag{13.2.10}$$

This domain is by definition the Gribov region. Its boundary is called the Gribov horizon; at the Gribov horizon the Faddeev-Popov operator $\partial^\mu D_\mu$ develops a zero eigenvalue. It follows from (20.0.20) that the Gribov region is the set of local (relative) minima of the norm functional.

The set of absolute minima on all gauge orbits is given by

$$B = \{A_\mu | A_\mu \epsilon L_2, f(A^U) \geq f(A) \text{ for all } U \epsilon \mathcal{G}(S)\} \tag{13.2.11}$$

We shall call this set the fundamental domain. Clearly the fundamental domain is contained in the Gribov region

$$B \subset C \tag{13.2.12}$$

Both B and C are convex: if A_μ and A'_μ lie in B, also ${A_\mu}^\alpha \equiv \alpha A_\mu + (1-\alpha)A'_\mu$ lies in B for $0 \leq \alpha \leq 1$. This follows from

$$||A^U||^2 - ||A||^2 = 2\int tr(\partial_\mu U^{-1}\partial_\mu U - 2\partial_\mu U U^{-1} A_\mu)d^4x \geq 0, \text{ idem for } A' \tag{13.2.13}$$

because $||{A_\mu}^{\alpha,U}||^2 - ||A^\alpha||^2$ is equal to $2\int tr(\partial^\mu U^{-1}\partial_\mu U - 2\partial_\mu U U^{-1}{A_\mu}^\alpha)d^4x$, which is a sum of $\alpha f_A(U)$ and $(1-\alpha)f_{A'}(U)$, and each functional $f_A(U)$ and $f_{A'}(U)$ is minimized by A and A', respectively, because A and A' lie in B by assumption. Also C is convex (because $\partial^\mu A_\mu$ and $\partial^\mu D_\mu(A) \geq 0$ are linear in A_μ). Note that the subspace $\partial^\mu A_\mu = 0$ contains both B and C. Since $A_\mu = 0$ lies in B, any field αA_μ with $0 \leq \alpha \leq 1$ lies in B if A_μ lies in B. (Idem for C).

[9]If $\partial^\mu A_\mu = 0$, the operator $\partial^\mu D_\mu$ is hermititan. With suitable boundary conditions it becomes even self-adjoint and has then a complete spectrum with real eigenvalues.

We now show that for general A_μ there exist Gribov copies inside the Faddeev-Popov horizon.[10] Suppose a field A_μ lies on the boundary of C. Then it satisfies $\partial^\mu A_\mu = 0$ and $\Delta_F(A) = 0$, hence $\partial^\mu D_\mu(A)\omega = 0$ for some ω. Hence the first two terms in the expansion of $f_A(e^\omega)$ vanish as we showed before, $f(e^\omega) = 2\int \partial^\mu \omega A_\mu + \int \partial^\mu \omega D_\mu(A)\omega + \dots$. The third term in the expansion

$$f_A(e^\omega) = \frac{1}{3}\int tr[A_\mu, [\omega, [\omega, \partial^\mu \omega]\,]+\dots \tag{13.2.14}$$

is odd in ω, hence one can lower the value of $f_A(U)$ at $U = 1$ by moving from $U = 1$ to a nearby group element $U = e^{\omega_0}$ with $\omega_0 = \lambda\omega$ and $|\lambda|$ sufficiently small. So $f_A(e^{\omega_0}) < f_A(I)$. Consider now the "point" $A^\epsilon \equiv (1-\epsilon)A$. It lies in C because C is convex. It lies actually **inside** C because the operator $M(A^\epsilon) = \partial^\mu D_\mu(A^\epsilon)$ is equal to $(1-\epsilon)M(A)+\epsilon M(A=0)$, and since $M(A=0) = \partial^\mu\partial_\mu$ is strictly positive while $M(A)$ is semidefinite positive, $M(A^\epsilon)$ is strictly positive. Thus the point A^ϵ is a relative minimum of the norm functional. The function $g(\epsilon) = f_{A^\epsilon}(e^{\omega_0}) - f_{A^\epsilon}(I)$ satisfies $g(0) < 0$, and thus, by continuity, there exists an $\epsilon = \epsilon_0$ for which $f_{A^{\epsilon_0}}(e^{\omega_0}) < f_{A^{\epsilon_0}}(I)$. This proves that there are points on the gauge orbit of A^ϵ with smaller norm than A^ϵ. Thus the absolute minimum on these gauge orbits does not coincide with the relative minimum at the Gribov horizon. Hence there exist Gribov copies inside the horizon.

Finally we note that if $A_\mu(x)$ lies in the Gribov region, also $\beta A_\mu(\beta x)$ lies in the Gribov region. Similarly, if $A_\mu(x)$ lies in the fundamental domain, also $\beta A_\mu(\beta x)$ lies in the fundamental domain. This follows easily from $\partial^\mu(\partial_\mu + A_\mu(\beta x) = \beta^2\partial^\mu D_\mu(A)$.

[10] On compact spaces, it can be shown that every gauge orbit passes inside the Gribov horizon [27]. In fact, in a box with periodic boundary conditions, there are Gribov copies of the vacuum due to $\exp 2\pi i T_a x^1/L$ (if T_a has integer entries after suitable scaling) at a distance $\frac{2\pi}{L}$, hence nearby Gribov copies exist. As discussed in the text, for ${A_\mu}^a$ which fall off at infinity fast enough, it seems plausible that no Gribov copies exist.

3 Inserting unity into the path integral

We now reconsider the Faddeev-Popov approach to path-integral quantization, paying close attention to the Gribov problem. One can use a phase-space approach, as in the original article of Faddeev and Popov, or a configuration-space path integral, by inserting unity into the latter. We follow here the latter approach.

Consider the ill-defined path integral $Z = \int dA_\mu{}^a \exp \frac{i}{\hbar} S(A)$ where $S(A)$ is the classical gauge-invariant action. Consider then the following identity

$$\int \delta(\partial^\mu A_\mu{}^U - f) d\mu = \sum_I |\det \frac{\partial}{\partial \omega^b} (\partial^\mu A_\mu{}^{U_I})^a\,|^{-1} \tag{13.3.1}$$

Here $d\mu = \det e_\alpha{}^a(\omega)(\Pi d\omega^\alpha)$ is the Haar measure for integration over a compact group, with $e_\alpha{}^a(\omega)$ the group vielbein and ω^α the group parameters.[11] It satisfies the property

$$\int d\mu h(U(\omega)) = \int d\mu h(U(\omega)V) = \int d\mu h(VU(\omega)) \tag{13.3.2}$$

for any function f on the group. As usual, $A_\mu{}^U = U^{-1}(\partial_\mu + A_\mu)U$ is the gauge transform of A_μ with U a group element, and f is an arbitrary function which vanishes sufficiently fast at infinity. The gauge orbit $\partial^\mu A_\mu{}^U$ is assumed to have N_A+1 solutions $A_\mu{}^{U_I}$ with $I = 0, \ldots, N$ which satisfy $\partial^\mu A_\mu{}^{U_I} = f$. Hence, the matrices U_I depend themselves both on A_μ and f, and we shall explicitly indicate this by using the notation $U_I^f(A)$. Then we have the following resolution of unity

$$\begin{aligned} I &= \Delta_F{}^f(A) \int \delta(\partial^\mu A_\mu{}^U - f) d\mu \\ \Delta_F{}^f(A)^{-1} &= \sum_{I=0}^{N} |\det \frac{\partial}{\partial \omega^b} \partial^\mu A_\mu{}^{U_I{}^f(A)}|^{-1} \end{aligned} \tag{13.3.3}$$

Since the Haar measure is group invariant, $\Delta_F{}^f(A)$ is gauge invariant for fixed but arbitrary f and A. Hence

$$\Delta_F{}^f(A) = \Delta_F{}^f(A^U) \tag{13.3.4}$$

[11]The group vielbein $e_\alpha{}^a(\omega)$ is defined by $g^{-1}dg = d\omega^\alpha e_\alpha{}^a T_a$ for $g \epsilon U$.

(Note that one must transform in (13.3.4) both the explicit A_μ and the A_μ implicitly contained in U_I). Insert now the resolution of unity in (13.3.3) into the path integral Z, and interchange the order of integration over $dA_\mu{}^a$ and $d\mu$. One obtains then

$$Z = \int d\mu \left[\int dA_\mu{}^a \Delta_F{}^f(A^U)\delta(\partial^\mu A_\mu{}^U - f) e^{\frac{i}{\hbar}S(A)} \right] \tag{13.3.5}$$

Using the gauge invariance of the classical action, we may certainly replace $S(A)$ by $S(A^U)$, and since the structure constants are traceless, the Jacobian for $dA_\mu \to dA_\mu{}^U$ is unity. [12] Denoting $A_\mu{}^U$ by B_μ, we then find

$$Z = \int d\mu \int dB_\mu{}^a \Delta_F{}^f(B)\delta(\partial^\mu B_\mu - f) e^{\frac{i}{\hbar}S(B)} \tag{13.3.6}$$

The integral has been factorized into an infinite group factor $\int d\mu$ which we drop, and a remainder which defines the path integral Z.

Let us now study the expression $\Delta_F{}^f(B)$ in more detail. If there are no Gribov copies, the solution of $\partial^\mu A_\mu{}^U \equiv \partial^\mu B_\mu = f$ is unique. For general f and B, the expression $\partial^\mu B_\mu{}^{U_0{}^f(B)}$ which appears in $\Delta_F{}^f$ is very complicated, but for $f = \partial^\mu B_\mu$ we obtain a simple relation

$$\partial^\mu B_\mu{}^{U_0{}^f(B)} \mid_{f=\partial\cdot B} = \partial^\mu B_\mu \tag{13.3.7}$$

The reason is that $B_\mu{}^{U_0{}^f(B)} \mid_{f=\partial\cdot B}$ is a point on the gauge orbit of B_μ, and the only point on the gauge orbit of B_μ which satisfies $\partial^\mu B_\mu{}^U = f$ is B_μ itself. This implies that $U_0{}^f(B) = I$ for $f = \partial^\mu B_\mu$. So we need only consider small gauge transformations. In that case the Faddeev-Popov determinant simplifies to

$$\det\left(\frac{\partial}{\partial\omega^b}\partial^\mu B_\mu{}^{U_0{}^f(B),a}\right) = \det(\partial^\mu D_\mu(B)^a{}_b) \tag{13.3.8}$$

which is indeed the usual result. However, when there are Gribov copies, $\partial^\mu B_\mu{}^{U_I{}^f(B)} = \partial^\mu B_\mu$ only tells us that for $f = \partial \cdot B$, one of the solutions U_I corresponds to $U_0 = I$,

[12] This often used argument is incomplete. One should regulate the Jacobian with a suitable regulator, and show that the regulated Jacobian is unity.

but the N other solutions ${U_I}^f(B)$ for $I = 1, \ldots, N$ are all very complicated nonlinear nonlocal expressions. In that case the Faddeev-Popov determinant becomes

$$\left[{\Delta_F}^{f=\partial\cdot B}(B)\right]^{-1} = \mid \det \partial^\mu D_\mu(B)^{-1} \mid + \sum_{I=1}^{N} \mid \det \frac{\partial}{\partial\omega^b} \partial^\mu {B_\mu}^{{U_I}^f(B),a} \mid^{-1}_{f=\partial\cdot B} \qquad (13.3.9)$$

and it is clear that one cannot exponentiate this expression by ghosts and antighosts because one finds the inverse of a sum of inverses of determinants. Furthermore, as we already discussed in the introduction, one finds the absolute value of the determinants, while the ghost action corresponds only to a determinant, not its absolute value. Since the various terms in ${\Delta_F}^{-1}$ are not equal to each other, the proposal to add a factor $1+n_A$ does not seem correct, and selecting one of the solutions does not seem correct either.

4 Gribov copies in a simple toy model

It may be useful to check these general ideas in a simple model. Consider the function $f(x, y) = f(x^2 + y^2)$. It plays the role of the classical gauge action and is clearly invariant under the gauge transformations

$$x_\theta = x\cos\theta - y\sin\theta; y_\theta = y\cos\theta + x\sin\theta \qquad (13.4.1)$$

The gauge group is $SO(2)$ and the gauge orbits are circles in the $x - y$ plane. We consider the integral

$$Z = \int dxdyf(x^2 + y^2) \qquad (13.4.2)$$

which we view as a path integral for constant fields x and y. We fix the gauge[13] by $y_\theta = y\cos\theta + x\sin\theta = 0$ and consider

$$\int \delta(y_\theta)d\theta = \Delta_F(x, y)^{-1} = \sum_I \frac{1}{\mid \frac{\partial}{\partial\theta} y_{\theta_I} \mid} = \sum_I \frac{1}{\mid -y\sin\theta_I(x, y) + x\cos\theta_I(x, y)\mid} \qquad (13.4.3)$$

[13]Unfortunately we cannot mimick the field theory case by fixing the gauge by $y_\theta = f$ because for nonvanishing f the straight line $y_\theta - f = 0$ in the x, y plane for fixed θ and f does not go through the origin so that there are circles ("gauge orbits") which the gauge cannot reach.

The integral $\int_0^{2\pi} d\theta$ is the Haar measure for the group $SO(2)$ (the group vielbein for an abelian group is unity). The functions $\theta_I(x,y)$ are the solutions of $y_\theta = 0$, hence $tg\theta = -y/x$. There are two solutions $\theta_1(x,y) = -arctg\, y/x$ and $\theta_2(x,y) = \pi - arctg\, y/x$. Hence there is one Gribov copy. The gauge invariance of $\Delta_F(x,y)$ can be checked explicitly. The first solution for θ yields (extracting a factor $|\cos\theta_I| = \frac{1}{\sqrt{1+tg^2\theta_I}}$ from the denominator)

$$\frac{\sqrt{1+(y/x)^2}}{|x - y(-y/x)|} = \frac{1}{\sqrt{x^2+y^2}} \tag{13.4.4}$$

which is clearly gauge-invariant. The Gribov copy yields the same contribution. (This is a peculiarity of this model, and not expected in the case of field theories.)[14] Note that $|-y\sin\theta + x\cos\theta|^{-1}$ for $\theta = 0$ yields the (inverse of) the usual Faddeev-Popov determinant which is clearly gauge-invariant, but for $\theta = \theta_1(x,y)$ or $\theta = \theta_2(x,y)$ it is gauge-invariant. With $\Delta_F(x,y) = \frac{r}{2}$ the integral can now be completed

$$\begin{aligned} Z &= \int dxdy \frac{\sqrt{x^2+y^2}}{2} \int d\theta \delta(y_\theta) f(x^2+y^2) \\ &= \int d\theta [\int dx'dy' \frac{\sqrt{x'^2+y'^2}}{2} \delta(y') f(x'^2+y'^2)] \\ &= (2\pi) \int_{-\infty}^{\infty} dx' \frac{|x'|}{2} f(x'^2) = (2\pi) \int_0^{\infty} drrf(r^2) \end{aligned} \tag{13.4.5}$$

which is the correct result. The variables x_θ and y_θ played the role of ${A_\mu}^U$, and x', y' correspond to B_μ in (13.3.6). The factor $\frac{1}{2}$ in (13.4.5) is due to the Gribov copy. Note that we need the Gribov copy to get the correct answer. This example is in certain respects too simple: fixing the gauge by fixing one of the integration variables ($y_\theta = 0$) corresponds more to axial gauges $A_3 = 0$ than to the relativistic or Coulomb gauge, and also both copies give here the same result. However, this example does clearly show that the Faddeev-Popov determinant is gauge-invariant only after expressing

[14]Note that if one did not take the absolute value of Δ_F, the Gribov copy would cancel the result of θ_1 because $\cos\theta_2 = -\cos\theta_1$. It has been claimed that this is a general result that also holds in field theory: summing over all Gribov copies without taking the absolute value of the Faddeev-Popov determinant yields zero due to an index theorem, see [7].

the group parameter θ into "gauge fields" x, y, and it also shows that ignoring Gribov copies is patently wrong.

5 No Gribov copies in perturbation theory or axial gauges

In this section we shall argue that in perturbation theory (small ${A_\mu}^a$ at all x) there are no Gribov copies in the gauge $\partial^\mu A_\mu = 0$ if one requires that ${A_\mu}^a$ tends to zero for large x. Furthermore, we shall discuss that on compact spaces there are always Gribov copies, but that the axial-gauge can only be defined on noncompact spaces and there it is without Gribov copies.

Suppose 4-dimensional Euclidean space has been compactified to S_4 (by stereographic projection, for example). Then $A_\mu(x, y, z, t)$ should tend to the same $A_\mu(\infty)$ as $x^2 + y^2 + z^2 + t^2 \to \infty$. One can then view A_μ as a connection on S_4 which is well-defined at the north pole. (This discussion makes even sense in Minkowski spacetime, although most of the analysis of compact surfaces deals with the Euclidean case).

Suppose one begins with a field configuration $A_\mu(x, y, z, t)$ in the axial gauge: $A_3(x, y, z, t) = 0$ for all x, y, z, t and $A_\mu(x, y, z, t)$ tends to $A_\mu(\infty)$ as x, y, z, t tend to infinity. Then A_μ is in the axial gauge on S_4. Consider now a gauge transformation which keeps A_μ in the axial gauge.

$$A_3^\Omega = U^{-1}(\partial_3 + A_3)U = 0 \tag{13.5.1}$$

One can solve this equation for U by a path-ordered integral.

$$U(x, y, z, t) = [P \exp - \int_{-\infty}^{z} A_3(x, y, z', t)dz']V(x, y, t) \tag{13.5.2}$$

where $V(x, y, t)$ is the integration constant. Clearly $(\partial_3 + A_3)U = 0$, and for $z \to -\infty$ the group elements U tend to V. Hence

$$A_\mu^\Omega(x, y, z = -\infty, t) = V^{-1}(\partial_\mu + A_\mu)V \tag{13.5.3}$$

In order that A_μ^Ω lies again in S_4, we must require that $V^{-1}(\partial_\mu + A_\mu)V$ be independent of x, y, x, t. Since we assumed that A_μ tends to zero at infinity this means that $V^{-1}\partial_\mu V$ must be independent of x, y, t. Hence $V(x, y, z) = V_0 = \text{constant}$.

However, for $z \to \infty$ the gauge matrix $U(x, y, z, t)$ tends to $F(x, y, t)V$ where $F(x, y, t) = P \exp \int_{-\infty}^{\infty} A_3(x, y, z', t)dz'$. Since A_μ was completely arbitrary on S_4, F does not vanish in general. It follows that $A_\mu(x, y, z, t)$ tends for $z \to \infty$ to $V^{-1}(\partial_\mu + A_\mu)V + V^{-1}(F^{-1}(\partial_\mu + A_\mu)F)V$. Hence $F^{-1}(\partial_\mu + A_\mu)F$ should tend to the same group element in all directions in order that A_μ^U still lies on S_4, but A_μ was arbitrary on S_4 and F only depends on A_3. Hence $F^{-1}(\partial_\mu + A_\mu)F$ will not be constant in general. Thus a gauge transformation on a connection on S_4 in the axial gauge which keeps the connection in the axial gauge necessarily moves that connection off S_4. There are no Gribov copies of the axial gauge on S_4 (or other compact spaces; the proof is the same).

The same reasoning shows that if one starts in R^4 with a connection $A_\mu(x, y, z, t)$ which vanishes at infinity, one cannot reach the gauge $A_3^U(x, y, x, t) = 0$ such that $A_\mu^U(x, y, z, t)$ still vanishes at infinity. Hence, the axial gauge is without Gribov copies if one imposes vanishing at infinity.

We now give an argument that in perturbation theory there are no Gribov copies in the gauge $\partial^\mu A_\mu = 0$. Consider the gauge orbit which contains in particular the point $A_\mu = 0$. It intersects the hyperplane of all A_μ satisfying $\partial^\mu A_\mu = 0$ at right angles. Namely a point on this gauge orbit near $A_\mu = 0$ is given by $A_\mu^g = \partial_\mu \omega$, and the inner product of this A_μ^g with A_μ in the hyperplane vanishes.

$$\begin{aligned} \langle A^g, A \text{ (hyper) } \rangle &= \langle \partial^\mu \omega, A \text{ (hyper) } \rangle \\ &= -\langle \omega, \partial^\mu A_\mu \text{ (hyper) } \rangle = 0 \end{aligned} \tag{13.5.4}$$

It seems then plausible (but this is not a rigorous mathematical fact) that nearby gauge orbits also start out almost orthogonal to the hyperplane, and need to go a long way before they bend over and intersect the hyperplane to produce a Gribov copy.

$$\tag{13.5.5}$$

One can draw an analogy with an ordinary integral with a saddle point approximation. Let the function to be integrated by given be $\exp[-\lambda f(x)]$ (corresponding to $\frac{-1}{g^2}S(A)$). Approximate $\int dx e^{-\lambda f(x)}$ around a solution x_0 as

$$e^{-\lambda f(x_0)} \int_{x_0-\epsilon}^{x_0+\epsilon} dx \exp -\lambda \left[\frac{1}{2}(x-x_0)^2 f''(x_0) + \frac{1}{3!}(x-x_0)^3 f'''(x_0) + \cdots\right] \tag{13.5.6}$$

The Gaussian approximation is only good when $\epsilon^2 \lambda f''(x_0) \gg 1$. Perturbation theory is allowed if $\epsilon << f''/f'''$, independently of λ. Then one may write the integral as

$$e^{-\lambda f(x_0)} \int_{-\sqrt{\lambda}\epsilon}^{\sqrt{\lambda}\epsilon} \exp\left[-\frac{1}{2}u^2 f''(x_0) + \frac{1}{3!}\frac{u^3}{\sqrt{\lambda}} f'''(x_0) + \cdots\right] \frac{du}{\sqrt{\lambda}} \tag{13.5.7}$$

and for $\lambda \to \infty$ this integral becomes the usual integral $\int_{-\infty}^{\infty}$ with good perturbative properties.

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Chapter 14

Supersymmetry

Supersymmetry (susy) has become part of modern quantum field theory. It is a symmetry between bosonic and fermionic fields, and a current theme of research in gauge field theory is to assume that many field theories in which susy is not manifest, have a susy at a deeper level. This susy can be explicitly, spontaneously or dynamically broken. It may be due to strings in QCD or be a minimally supersymmetric (but still field theoretic) extension of the Standard Model (the so-called MSSM model). Even if a given field theory is not susy by itself, one can often learn a great deal about it by extending it first to a susy field theory, just as in mathematics one may learn much about real functions by using complex function theory. If nature is supersymmetric, one should discover susy particles and this discovery will have monumental implications for particle physics. For mathematicians it has already become a potent tool to prove theorems, but if nature is not susy this latter application may well be its final legacy.

We begin with an introduction into rigid $N = 1$ susy, both in x-space and in superspace. This is used for the kink and the MSSM. Then we discuss models in x-space with rigid $N = 2$ susy which are used for monopoles, and models with rigid $N = 4$ susy which are used for instantons. We end with $N = 2$ superspace which we use to explain the origin of the BRST symmetry of topological field theory.

1 The Poincaré supersymmetry algebras.

The particles of supersymmetric field theories transform into each other under the symmetries of the theory. It is thus important to determine the multiplets of supersymmetry. The supersymmetry generators together with the Poincaré generators form the so-called super Poincaré algebras. We begin by deriving these algebras and determining their representations in terms of particles.

We consider extensions of the Poincaré algebra with N susy generators $Q^{\alpha i}$ with $i = 1, N$ and their hermitian conjugates[1] $(Q^{\alpha i})^\dagger$. This means that we begin with the Poincaré algebra $[J_{\mu\nu}, P_\rho] = \eta_{\nu\rho}P_\mu - \eta_{\mu\rho}P_\nu$ and $[P_\mu, P_\nu] = 0$ and $[J_{\mu\nu}, J_{\rho\sigma}] = \eta_{\nu\rho}J_{\mu\sigma} + 3$ terms, and add

$$[Q^{\alpha i}, P_\mu] = 0; [Q^{\alpha i}, J_{\mu\nu}] = \frac{1}{2}(\sigma_{\mu\nu})^\alpha{}_\beta Q^{\beta i} \tag{14.1.1}$$

The first relation states that the charges $Q^{\alpha i}$ are space and time independent (conserved), while the second relation states that $Q^{\alpha i}$ is a chiral spinor ($\alpha = 1, 2$) and with spin 1/2. The spin 1/2 Lorentz generators are defined by

$$(\sigma^{\mu\nu})^\alpha{}_\beta = \frac{1}{2}\left[(\sigma^\mu)^{\alpha\dot{\gamma}}(\bar{\sigma}^\nu)_{\dot{\gamma}\beta} - \mu \leftrightarrow \nu\right] \tag{14.1.2}$$

where $\sigma^\mu = (I, \vec{\sigma})$ and $\bar{\sigma}^\mu = (-I, \vec{\sigma})$ with $\vec{\sigma}$ the three Pauli matrices, see (14.5.1).

We fix the relation between $J_{\mu\nu}$ and angular momentum such that the $Q^{\alpha i}$ in (14.1.1) for $\alpha = 1$ raise the z component of angular momentum by 1/2. The generators $J_{\mu\nu}$ and P_μ are antihermitian and if $J_{12} \mid \lambda\rangle = -i\lambda \mid \lambda\rangle$ for a state $\mid \lambda\rangle$, the eigenvalue λ is the z component of angular momentum. Then $Q^{\alpha i} \mid \lambda\rangle$ for $\alpha = 1$ has eigenvalue $\lambda + \frac{1}{2}$ according to (14.1.1). We also require that if $P_\mu \mid q\rangle = -iq_\mu \mid q\rangle$ then q_0 is positive for physical states, so q_0 is the energy.

We will also encounter internal symmetry generators T_r and central charges Z^{ij}. We can always take combinations $T - T^\dagger$ and $i(T + T^\dagger)$ to achieve that the T_r are

[1] In our conventions, $(Q^{\alpha j})^\dagger = \bar{Q}_j^{\dot{\alpha}}$, see (14.5.15) and (14.5.16), but $Q_\alpha^j = Q^{j\beta}\epsilon_{\beta\alpha}$ and $(Q_\alpha^j)^\dagger = -\bar{Q}_{\dot{\alpha}j}$ because we take $\epsilon_{\alpha\beta} = -\epsilon_{\dot{\alpha}\dot{\beta}}$.

antihermitian. We could decompose the Z^{ij} into $Z_1^{ij} + iZ_2^{ij}$ but we shall instead make a field redefinition later such that Z^{ij} becomes real.

According to the Coleman-Mandula theorem [1], all internal symmetry generators must commute with the Poincaré generators. Hence

$$[T_r, P_\mu] = [T_r, J_{\mu\nu}] = [Z^{ij}, P_\mu] = [Z^{ij}, J_{\mu\nu}] = 0 \tag{14.1.3}$$

An important second part of the Coleman-Mandula theorem is that the internal symmetry algebra can only be a direct sum of a semisimple Lie algebra and a set of abelian generators which commute with the semisimple generators. This will be the reason below that the Z^{ij} are central charges.

The nontrivial part of the N-extended susy algebra reads

$$\begin{aligned} &[Q^{\alpha i}, T_r] = (t_r)^i{}_j Q^{\alpha j} \\ &\{Q^{\alpha i}, Q^{\beta j}\} = -\epsilon^{\alpha\beta} Z^{ij} \\ &\{Q^{\alpha i}, \bar{Q}_j^{\dot\beta}\} = 2i\delta_j^i \sigma^{\mu,\alpha\dot\beta} P_\mu; \bar{Q}_j^{\dot\beta} \equiv (Q^{\beta j})^\dagger \end{aligned} \tag{14.1.4}$$

In a positive definite Hilbert space, $\{Q^{\alpha i}, \bar{Q}_i^{\dot\alpha}\}$ is positive definite, and this explains why we defined $P_0 = -iq_0$ with q_0 the energy. Because $[Q^{\alpha i}, P_\mu] = 0$, there can be no (selfdual or anti-selfdual part of) Lorentz generators on the right-hand side of the $\{Q, Q\}$ anticommutator,[2] so the only possibility are "internal symmetry" generators Z^{ij}.

Taking the hermitian conjugates yields

$$\begin{aligned} &[\bar{Q}_i^{\dot\alpha}, T_r] = (t_r^*)_i{}^j \bar{Q}_j^{\dot\alpha} \text{ with } (t_r^*)_i{}^j \equiv (t_r{}^i{}_j)^* \\ &\{\bar{Q}_i^{\dot\alpha}, \bar{Q}_j^{\dot\beta}\} = \epsilon^{\dot\alpha\dot\beta} Z_{ij} \text{ with } Z_{ij} \equiv (Z^{ij})^\dagger \text{ and } \epsilon^{\dot\alpha\dot\beta} = -\epsilon^{\alpha\beta} \\ &\{Q^{\beta j}, \bar{Q}_i^{\dot\alpha}\} = 2i\delta_i^j \sigma^{\mu,\beta\dot\alpha} P_\mu \end{aligned} \tag{14.1.5}$$

[2]If we decompose $J_{\mu\nu}(\sigma^\mu)^{\alpha\dot\beta}(\sigma^\nu)^{\gamma\dot\delta} \equiv J^{\alpha\dot\beta,\gamma\dot\delta}$ into $\epsilon^{\alpha\gamma} J^{\dot\beta\dot\delta} + \epsilon^{\dot\beta\dot\delta} J^{\alpha\gamma}$, the $J^{\alpha\gamma}$ and $J^{\dot\beta\dot\delta}$ are symmetric and give the selfdual and antiselfdual parts of $J_{\mu\nu}$, respectively. One might then expect also a term with $J^{\alpha\beta}$ in $\{Q^{\alpha i}, Q^{\beta j}\}$, at least for the $N = 1$ case, but the Jacobi identity $[\{Q^{\alpha i}, Q^{\beta j}\}, P_\mu] = \{Q^{\alpha i}, [Q^{\beta j}, P_\mu]\} + \alpha i \leftrightarrow \beta j$ rules this out.

where we used that the matrices $\sigma^\mu = (I, \vec{\sigma})$ are hermitian, so $(\sigma^{\mu,\alpha\dot{\beta}})^* = \sigma^{\mu,\beta\dot{\alpha}}$. From the $(Q, \bar{Q}, T)$ Jacobi identities one learns that

$$\begin{aligned} [\{Q^{\alpha i}, \bar{Q}^{\dot{\beta}}_j\}, T_r] &= \{Q^{\alpha i}, [\bar{Q}^{\dot{\beta}}_j, T_r]\} + \{\bar{Q}^{\dot{\beta}}_j, [Q^{\alpha i}, T_r]\} \\ 0 &= (t_r^*)_j{}^i + (t_r)^i{}_j \end{aligned} \tag{14.1.6}$$

Hence the matrices t_r are antihermitian which is not surprising because we took the generators T_r as antihermitian. Thus the generators T_r generate $U(N)$, or a subgroup of $U(N)$. We now proceed to determine this subgroup.

By a unitary redefinition $Q^{\alpha i} \to U^i{}_j Q^{\alpha j}$ one can always cast the complex antisymmetric matrices Z^{ij} into canonical form $Z \to (UZU^T)$ with all $z_m > 0$

$$Z^{ij} = \begin{pmatrix} \ddots & & & & & & & \\ & \begin{pmatrix} 0 & -z_m \\ z_m & 0 \end{pmatrix} & \ddots & & & & & \\ & q_m \text{ times} & & \begin{pmatrix} 0 & -z_m \\ z_m & 0 \end{pmatrix} & & & & \\ & & & & 0 & & & \\ & & & & & 0 & & \\ & & & & q_0 \text{ times} & & \ddots & \\ & & & & & & & 0 \end{pmatrix} \tag{14.1.7}$$

This is a result in matrix theory [2]; in our case the Z^{ij} are operators, but the same steps which lead to (14.1.7) can be used when Z^{ij} are operators. Under this redefinition Ut_rU^{-1} remains antihermitian and the $\{Q, \bar{Q}\}$ anticommutator is unchanged. Since Z^{ij} is a $N \times N$ matrix, one has the relation

$$2q_1 + 2q_2 + \cdots + 2q_m + q_0 = N \tag{14.1.8}$$

The QQQ Jacobi identity leads to

$$[Q^{\alpha i}, \epsilon^{\beta\gamma} Z^{jk}] + [Q^{\beta j}, \epsilon^{\gamma\alpha} Z^{ki}] + [Q^{\gamma k}, \epsilon^{\alpha\beta} Z^{ij}] = 0 \tag{14.1.9}$$

and reveals that Q commutes with Z^{ij}. (Take special values for α, β, γ and use (14.1.7).) The QQZ Jacobi identity reveals then that the Z^{ij} commute with themselves. From the QQT Jacobi identity one learns that $[Z^{ij}, T_r]$ is proportional to

$(t_r)^i{}_{i'}Z^{i'j} + (t_r)^j{}_{j'}Z^{ij'}$. Hence the Z^{ij} form an ideal of the internal symmetry algebra. Since a semisimple Lie algebra has no ideal, it follows from the Coleman-Mandula theorem that the Z^{ij} must form an abelian subset which commutes with the T_r. Hence, the Z_{ij} commute with all other generators and with themselves: **the Z^{ij} are central charges**. This means that they take on the same value on all components of a representation, hence one can view them as constant complex $N \times N$ matrices.

The requirement that $[Z^{ij}, T_r] = 0$ vanishes implies with the QQT Jacobi identity that (in matrix notation)

$$t_r Z + Z t_r^T = 0 \tag{14.1.10}$$

Since Z in (14.1.7) is a symplectic metric (a real antisymmetric nondegenerate matrix), we find that in the sectors with central charges, the group G is reduced to a unitary-symplectic group $Usp(2q_m)$. Hence, theories with N-extended rigid susy have as internal symmetry group

$$G = U(q_0) \otimes Usp(2q_1) \otimes \ldots \otimes Usp(2q_m) \tag{14.1.11}$$

For the case $N = 2$ we find that $Z = \begin{pmatrix} 0 & -z \\ z & 0 \end{pmatrix}$ with $z \geq 0$, and the symmetry group is $G = U(2)$ if $z = 0$, or $G = Usp(2)$ if $z > 0$. The generators of $U(2)$ in the fundamental representation are $(-iI, -i\vec{\sigma})$. The generators of $Usp(2)$ are only $(-i\vec{\sigma})$, i.e. $Usp(2) = SU(2)$. So if a central charge is present, the $U(1)$ is lost.

For the case $N = 4$ there are several possibilities. If there are no central charges, $G = U(4)$. With one central charge z_1 the group is $G = U(2) \times Usp(2)$. With two equal central charges ($z_1 = z_2$) one has $G = Usp(4)$, and with two unequal central charges ($z_1 > 0, z_2 > 0, z_1 \neq z_2$) one has $G = Usp(2) \otimes Usp(2)$.

These symmetry groups (or subgroups of them) should be realized on the particles (the on-shell fields). Not every on-shell symmetry can be extended to an off-shell symmetry (example: dualities in supergravity theories) but we shall see that the internal symmetry group G is also realized on the off-shell fields. In fact, both the on-shell physical states and the off-shell fields of a susy action (with auxiliary fields) form a representation of the susy algebra.

2 Multiplets of states of extended susy

Particles form representations of the N-extended super Poincaré algebra. We shall only construct here representations of the super algebra and not of the corresponding group. Not even that: we only construct representations of a subalgebra of the full super Poincaré algebra, namely the "little algebra". By "boosting" one then gets the representation of the full algebra. The situation is similar to representations of the Poincaré algebra (see below). We consider only states with momentum $m^2 = -q^2 \geq 0$ (physical mass-shell conditions). The representations for $m^2 \neq 0$ are all the same and continuous in m, but at $m = 0$ there are other representations: there is a mass discontinuity at $m = 0$.

We consider separately three classes of irreducible representations (irreps):
(A) massless representations (which have no central charges as we shall see),
(B) massive representations without a central charge,
(C) massive representations with a central charge. When one or more central charges saturate the bound $\mid z \mid \leq m$ multiplet shortening occurs.

A. Massless irreps.

Consider a momentum eigenstate state $|q >$ with momentum $q_m = (q, 0, 0, q)$ along the positive z-axis. All generators which map this state into (a multiple of) itself span by definition the little algebra. These are

$$Q^{\alpha i}, P_m, T_r, Z^{ij}, J_{12}, J_{10} - J_{13} \equiv t_1; J_{20} - J_{23} \equiv t_2 \tag{14.2.1}$$

The last three generators need a short explanation. From $P_1(J_{12} \mid q\rangle) = P_2 \mid q\rangle + J_{12}P_1 \mid q\rangle = -iq_1 J_{12} \mid q\rangle$ it follows that $J_{12} \mid q\rangle$ has the same momentum P_1 as $\mid q\rangle$.

The same is true for P_2, P_3 and P_0. Hence $J_{12} \mid q\rangle$ is proportional to $\mid q\rangle$. Hence J_{12} is part of the little algebra. More generally, $P_\mu \frac{1}{2}\lambda^{\rho\sigma} J_{\rho\sigma} \mid q\rangle$ is equal to $-i\lambda_\mu{}^\sigma q_\sigma \mid q\rangle$ plus a term proportional to $q_\mu \left(\frac{1}{2}\lambda^{\rho\sigma} J_{\rho\sigma} \mid q\rangle\right)$. Hence when $\delta q_\mu = \lambda_\mu{}^\sigma q_\sigma = 0$ the corresponding generator lies in the little algebra. From $\delta q_\mu = \lambda_\mu{}^\sigma q_\sigma = 0$ we find 4 conditions (for $\mu = 0, 1, 2, 3$) on $\lambda_\mu{}^\sigma$

$$\lambda_0{}^3 = \lambda_1{}^0 + \lambda_1{}^3 = \lambda_2{}^0 + \lambda_2{}^3 = \lambda_3{}^0 = 0 \tag{14.2.2}$$

The remaining Lorentz parameters are λ^{12}, $\lambda^{10} - \lambda^{13}$ and $\lambda^{20} - \lambda^{23}$. Hence J_{12}, $J_{10} - J_{13}$ and $J_{20} - J_{23}$ form part of the little algebra. They generate the noncompact Euclidean group E_2 (the two translations and one rotation of the two-dimensional Euclidean plane).

For finite-dimensional unitary representations only the compact subgroup (J_{12}) can be nonvanishing. (Noncompact generators can generate infinite dimensional unitary representations but not finite-dimensional ones). Hence

$$\begin{aligned} t_1 \mid q, 0, 0, q\rangle &= t_2 \mid q, 0, 0, q\rangle = 0 \\ J_{12} \mid q, 0, 0, q\rangle &= -i\lambda \mid q, 0, 0, q\rangle \end{aligned} \tag{14.2.3}$$

Since later J_{12} will become part of the rotation group $SU(2)$, λ can only be half-integer or integer.

Thus, for the massless representations of the Poincaré algebra, the little group is generated by E_2, P_m, and the unitary irreps are determined by the states $\mid q, \lambda\rangle$ with $q^2 = 0$ and $\lambda = \frac{n}{2}$. We now extend these representations of the little algebra of the Poincaré algebra to the little algebra of the super Poincaré algebra. As we shall see and might expect, we need in general several irreducible representations of the bosonic little algebra to construct one irreducible representation of the little super algebra.

The susy generators satisfy the following relations when acting on the states $\mid q, \lambda\rangle$

$$\{Q^{\alpha i}, (Q^{\beta j})^\dagger\} \;\; = \;\; 2i\delta^i_j (\sigma^m)^{\alpha\dot{\beta}} P_m = 2i\delta^i_j (\sigma^0 + \sigma^3)^{\alpha\dot{\beta}} (-iq_0)$$

$$= \quad 4q_0\delta^i_j \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{14.2.4}$$

In a positive definite Hilbert space q_0 must be positive, and this agrees with our convention that q_0 is the energy.

So $\{Q^{2i}, (Q^{2i})^\dagger\} = 0$ hence $Q^{2i} = 0$ (in the positive definite Hilbert space of states). Then the anticommutator

$$\{Q^{\alpha i}, Q^{\beta j}\} = -\epsilon^{\alpha\beta} Z^{ij} \tag{14.2.5}$$

yields for $\alpha = 1$, $\beta = 2$ that $Z^{ij} = 0$. We conclude that **there are no central charges for massless representations.** The remaining relations read

$$\begin{aligned} &\{Q^{1i}, (Q^{1j})^*\} = 4q_0\delta^i_j \\ &\{Q^{1i}, Q^{1j}\} = \{(Q^{1i})^\dagger, (Q^{1j})^\dagger\} = 0 \end{aligned} \tag{14.2.6}$$

and show that the $Q^{1i}, (Q^{1i})^\dagger$ are fermionic creation and annihilation operators.

Thus the states are obtained by acting with $(Q^{1i})^\dagger$ on a vacuum $\mid \Omega\rangle$. The vacuum is annihilated by the annihilation operators, namely $Q^{\alpha i} \mid \Omega\rangle = 0$. We find then the following states.

$$\mid \Omega\rangle = \mid q, \lambda\rangle, (Q^{1i})^\dagger \mid \Omega\rangle, (Q^{1i})^\dagger (Q^{1j})^\dagger \mid \Omega\rangle \ldots \tag{14.2.7}$$

There are 2^N states if there is only one vacuum $\mid \Omega\rangle$ (i.e., if the vacuum is in the trivial representation of G). Then the $(Q^{1i})^\dagger \mid \Omega\rangle$ form the $\mathbf{N}$ of $U(N)$ and $(Q^{1i})^\dagger (Q^{1j})^\dagger \mid \Omega\rangle$ form the antisymmetric tensor representation, etc. However, the vacuum $\mid \Omega\rangle$ can be in a nontrivial representation R of the $U(N)$ group with generators T_r. Since the $(Q^{1i})^\dagger$ are in the fundamental representation $\mathbf{N}$ of $U(N)$, the states $(Q^{1i})^\dagger \mid \Omega\rangle$ etc. form tensor products of irreps of T_r.

Because it follows from (14.1.1) that $[(Q^{1i})^\dagger, J_{12}] = \frac{-i}{2}(Q^{1i})^\dagger$, the $(Q^{1i})^\dagger$ lower the z-component of angular momentum by 1/2. If the three-momentum is in the direction of the positive z-axis, then $(Q^{1i})^\dagger$ lowers the helicity. If we assume that the vacuum

$|\,\Omega\rangle = |\,q,\lambda\rangle$ has helicity λ, no states with helicity less than $\lambda - N$ can occur since we can at most have the product of N operators $(Q^{1i})^\dagger$.

Before going on, we must discuss a property of the field theories which will describe the particles in the representations of the super Poincaré algebra. Under the conditions of local Lagrangian field theory (string theory does not satisfy this condition and might lead to CPT violation), relativistic invariance and the spin-statistics connection, field theory with a hermitian action is CPT invariant. This means that for every particle with momentum $(\vec{q}, q_0)$, helicity λ in a representation R of the internal symmetry group, there must be a CPT conjugate of this particle, which is a particle with the same momentum $(\vec{q}, q_0)$, helicity $-\lambda$ and in the complex conjugate representation R^*. So, if $|\,A\rangle$ denotes all particles $|\,\vec{q}, q_0, \lambda, R\rangle$ as well as the set of their CPT conjugates $|\,\vec{q}, q_0, -\lambda, R^*\rangle$, and the CPT operation is denoted by Ω, then the fact that A is closed under CPT implies

$$\Omega \,|\, A\rangle = S^A{}_{A'} \,|\, A'\rangle \tag{14.2.8}$$

In order that CPT is an involution $(\Omega\Omega = I)$ we get a constraint on S

$$S^* S = I \tag{14.2.9}$$

We now quote a theorem of semisimple Lie algebras which we will apply to (14.2.9). To understand this theorem we recall that a representation R is called real if it can be made real by a similarity transformation. If R and its complex conjugate R^* are related by $R^* = SRS^{-1}$ but R cannot be made real, we call it a pseudoreal representation.

Theorem: If a unitary representation satisfies $R^* = SRS^{-1}$ one has the following group theoretical fact [3]

$$\begin{aligned} &R \text{ real } \Leftrightarrow S \text{ symmetric and } S^*S = +I \\ &R \text{ pseudoreal } \Leftrightarrow S \text{ antisymmetric and } S^*S = -I \end{aligned} \tag{14.2.10}$$

Consider now **all** spin 0 states in a multiplet with momentum $\vec{q}$. They form a (reducible or irreducible) representation R of the internal symmetry group. We denote the states of this set by $\mid A\rangle = \mid \lambda = 0, \vec{q}, q_0, R\rangle$. The action of CPT maps this set of states into $\mid \lambda = 0, \vec{q}, q_0, R^*\rangle$. Let us denote the states in this set by $\mid B\rangle$. Then $\Omega \mid A\rangle = S^A{}_B \mid B\rangle$. Under an internal symmetry transformation $\mid A\rangle \to R^A{}_{A'} \mid A'\rangle$, we have $\Omega \mid A\rangle \to \Omega(R^A{}_{A'} \mid A'\rangle) = (R^*)^A{}_{A'}\Omega \mid A'\rangle$, and if we combine this with $\Omega \mid A\rangle = S^A{}_B \mid B\rangle \to S^A{}_B R^B{}_{B'} \mid B'\rangle$, we find the relation $R^*S = SR$. Hence the matrix S in $\Omega \mid A\rangle = S^A{}_B \mid B\rangle$ satisfies $R^* = SRS^{-1}$. This shows that the representation R of the spin 0 states with momentum $\vec{q}$ cannot be complex but it can at best only be real or pseudoreal. However, in order that CPT is an involution, $\Omega\Omega = I$, one needs also

$$S^*S = I \tag{14.2.11}$$

and (14.2.10) shows that this is only possible for real representations, but not for pseudoreal representations.[3] We conclude that the spin 0 states with a given momentum can only be in a real representation of the internal symmetry group.

For example, if one has two spin 0 states in the **2** of $SU(2)$ or $U(2)$, we need to double the number of states. In the field theory, two spin 0 states correspond to two real scalar fields φ^i, and real fields can not transform as the **2** of $SU(2)$. One must then add another doublet of scalar fields.

One can still construct CPT self conjugate representations if both the vacuum $\mid \Omega\rangle$ and the spin 0 states $\mid A\rangle$ are in a pseudoreal representation. Then we still have $S^*S = I$ because of $(-1)(-1) = +1$.

Example: **massless $N = 1$ scalar (Wess-Zumino) multiplet.** For $N = 1$, the R-symmetry group is $G = U(1)$. For $\mid \Omega\rangle = \mid q, \lambda\rangle$ we find helicity $(\lambda, \lambda - 1/2)$ doublets,

[3] In the case of parity we shall see that $\pi^2 = -I$ on fermions instead of $\pi^2 = I$. On bosons $\pi^2 = I$ always, and since the $\lambda = 0$ states are bosons, we do not consider the possibility that $\Omega^2 = -I$ on the scalars.

in particular $(1,1/2),(1/2,0),(0,-1/2)$ and $(-1/2,-1)$ doublets. In order that all particles with their CPT conjugates appear we need the $(1,1/2)$ and $(-1/2,-1)$ doublets together. Also $(1/2,0)$ must come together with $(0,-1/2)$. The $U(1)$ phase of the two helicity zero (spin 0) states must be opposite in order that these two scalars form a real representations of $G=U(1)$, and also the helicity $1/2$ and $-1/2$ states have opposite phases. The $U(1)$ phase of the fermion differs from the $U(1)$ phase of the scalar by the $U(1)$ phase of the creation generator $(Q^{1j})^\dagger$ (with $j=1$). Hence a 4-component Majorana fermion transforms under $U(1)$ as $\delta(U(1))\psi = i\alpha\gamma_5\psi$. (Note that $i\alpha\gamma_5\psi$ is again a Majorana spinor). The scalars φ and φ^* transform as $\delta\varphi = i\beta\varphi$ and $\delta\varphi^* = -i\beta\varphi^*$. One can produce these results for ψ and φ in superspace by assuming that the chiral superfield[4] Φ has $U(1)$ weight $\varphi_\Phi = \beta$ and θ has $U(1)$ weight $\varphi_\theta = \beta - \alpha$. Since $\Phi = \varphi(y) + i\theta^\alpha\psi_\alpha(y) + i\theta^\alpha\theta_\alpha F(y)$ where y is $U(1)$ inert (because θ^α and $\bar\theta_{\dot\alpha}$ transform with an opposite phase), the transformation rule

$$\Phi'(y,\theta) = e^{i\varphi_\Phi}\phi(y, e^{i\varphi_\theta}\theta) \tag{14.2.12}$$

reproduces the transformation rules of the component fields in $\phi'(y,\theta) = \varphi'(y) + i\theta^\alpha\psi'_\alpha(y) + i\theta^\alpha\theta_\alpha F'(y)$.

Example: **the massless $N=1$ vector multiplet.** The vacuum with helicity $\lambda = 1$ leads to a helicity $(1,1/2)$ doublet. The CPT conjugate has helicity $(-1,-1/2)$, and the corresponding field theory contains a vector field A_μ and a Majorana spinor $(\lambda^\alpha, \bar\lambda_{\dot\alpha})$. The 4-component spinor transforms again under $U(1)$ and $\delta\lambda = i\alpha\gamma_5\lambda$, but the vector is inert under $U(1)$ because it is real. These results can also be obtained from superspace where the vector multiplet corresponds to a **real** superfield $V(x^\mu, \theta^\alpha, \bar\theta_{\dot\alpha})$. The vector field A_μ appears as $A_\mu\bar\theta^{\dot\alpha}\bar\sigma^\mu_{\dot\alpha\beta}\theta^\beta$, and since $\delta V = 0$ under $U(1)$, and the phases of θ^α and $\bar\theta_{\dot\alpha}$ cancel each other, A_μ is again $U(1)$ inert.

[4] As we shall discuss, a chiral superfield Φ is a function of x^μ, θ^α and $\bar\theta_{\dot\alpha}$ satisfying $\bar D_{\dot\alpha}\Phi = 0$ where $\bar D_{\dot\alpha} = \partial/\partial\bar\theta^{\dot\alpha} + i\bar\sigma^\mu_{\dot\alpha\beta}\theta^\beta\partial_\mu$. It can be written as a function of only θ^α and y^μ, where y^μ satisfies $\bar D_{\dot\alpha}y^\mu = 0$ and is given by $y^\mu = x^\mu + i\bar\theta^{\dot\alpha}\bar\sigma^\mu_{\dot\alpha\beta}\theta^\beta$.

Example: **the massless $N = 2$ hypermultiplet.** For $N = 2$ one has the R-symmetry group $G = U(2)$. Starting with a vacuum $\mid \Omega\rangle$ with $\lambda = 1/2$ in the trivial representation of G, we find 4 states

$$\mid \Omega\rangle, (Q^{1i})^\dagger \mid \Omega\rangle, (Q^{11})^\dagger (Q^{12})^\dagger \mid \Omega\rangle \tag{14.2.13}$$

with $\lambda = 1/2$ in **1** of $SU(2), \lambda = 0$ in **2**, and $\lambda = -1/2$ in **1**. Since the **2** of $SU(2)$ is pseudoreal, we must add a second multiplet. Thus the particle content corresponds to a complex Dirac spinor and two massless complex spin 0 fields, double the content of the Wess-Zumino model. This corresponds to two chiral superfields ϕ^i and ϕ_i in complex-conjugate representations of the Yang-Mills gauge group with gauge index i. (The same arguments which were used to prove that the scalars form a real representation of the R symmetry group can be used to show that they form a real representation of the internal Yang-Mills group). We shall only construct the action after we have discussed the massive model in order to be able to notice the differences with the massless case.

Example: **the massless $N = 2$ vector multiplet.** We begin with a vacuum $\mid \Omega\rangle$ with helicity $\lambda = 1$, in the trivial representation of $SU(2)$. By acting with $(Q^{1i})^\dagger$ we create a helicity 1/2 $SU(2)$-doublet, and $(Q^{11})^\dagger (Q^{12})^\dagger \mid \Omega\rangle$ yields a helicity 0 $SU(2)$-singlet. Together with the CPT conjugate multiplet with helcities $(-1, -1/2, 0)$ we obtain the following fields: a vector field V_μ, a Majorana spinor doublet $\lambda^i (i = 1, 2)$ and two scalar fields M and N which are inert under $SU(2)$. The $U(1)$ weight of the vector V_μ vanishes, but the λ^i_L has weight β, and $(M + iN)$ has weight 2β. (Of course, λ^i_R has weight $-\beta$ and $M - iN$ has weight -2β). In superspace this multiplet consists of an $N = 1$ superfield V and a chiral superfield S. The coupling of the $N = 2$ vector multiplet to an $N = 2$ hypermultiplet in terms of $N = 1$ superfields involves the trilinear chiral superfield coupling $\varphi^i S \varphi_i$.

Example: **the massless $N = 2$ linear multiplet.** In this case we begin with a vacuum in a nontrivial representation R of $U(2)$, namely $\mid \Omega\rangle =\mid q, \lambda = \frac{1}{2}, R = \mathbf{2}$ of

$U(2)\rangle$. By acting with $(Q^{1i})^\dagger$ we find the following states

$$\begin{aligned}
&\mid \Omega\rangle: \text{ helicity} + 1/2 \text{ doublet}\\
&(Q^{1j})^\dagger \mid \Omega\rangle: \text{ helicity 0 singlet and triplet}\\
&(Q^{11})^\dagger (Q^{12})^\dagger \mid \Omega\rangle: \text{ helicity } -1/2 \text{ doublet.}
\end{aligned} \tag{14.2.14}$$

Since the singlet and triplet of $SU(2)$ are real, there is no CPT-doubling. The total number of states is thus the same as for the hypermultiplet but they form different $U(2)$ multiplets. The free field theory for this multiplet is

$$\mathcal{L} = -\frac{1}{2}V_\mu^2 - \frac{1}{2}(\partial_\mu t^{ij})^2 - \frac{1}{2}\bar{\lambda}\not{D}\lambda \tag{14.2.15}$$

with complex $\lambda, t^{ij} = t^{ji}$ and $(t^{ij})^a st = t_{ij}$, while $\partial^\mu V_\mu = 0$.

Example: **the massless** $N = 4$ **model**. The helicity content of the particles is

$$1(\mathbf{1}), \frac{1}{2}(\mathbf{4}), 0(\mathbf{6}), -\frac{1}{2}(\mathbf{4}), -1(\mathbf{1}) \tag{14.2.16}$$

Since the **6** of $SU(4)$ is real there is no multiplet doubling. The $U(1)$ charge must change sign under CPT, thus the $U(1)$ charges are given by

$$U(1): e^{2i\alpha}, e^{i\alpha}, 1, e^{-i\alpha}, e^{-2i\alpha} \tag{14.2.17}$$

but since the vector is real, all fields are $U(1)$ inert.

B. Massive irreps without Z^{ij}.

Take $q_\mu = (m, 0, 0, 0)$. The generators of the little group are now $P_m, Q^{\alpha i}, (Q^{\alpha i})^\dagger, T_r$ and **all** J_{ij} with $i, j = 1, 3$. The susy algebra now yields

$$\begin{aligned}
&\{Q^{\alpha i}, (Q^{\beta j})^\dagger\} = 2i\delta^i_j(\sigma^{0\alpha\dot{\beta}})(-im) = 2m\delta^i_j\delta^\alpha_\beta\\
&\{Q^{\alpha i}, Q^{\beta j}\} = \{(Q^{\alpha i})^\dagger, (Q^{\beta j})^\dagger\} = 0 \text{ (because } Z^{ij} = 0)
\end{aligned} \tag{14.2.18}$$

The $(Q^{\alpha i})^\dagger$ transform under J_{ij} as spin $1/2$ (not helicity $1/2$), and the vacuum $\mid \Omega\rangle$ must be in a representation R of $U(N)$ with a given spin J (not helicity).

The vacuum $\mid \Omega\rangle$ is annihilated by all $Q^{\alpha i}$ for $\alpha = 1, 2; i = 1, \dots N$ and we get states by acting with $(Q^{\alpha i})^\dagger$ on $\mid \Omega\rangle$. There are 2^{2N} states which split into 2^{2N-1} bosonic states and 2^{2N-1} fermionic states.

Example: **massive multiplets of $N = 1$ susy.** The R-symmetry group is still $G = U(1)$. We drop the index i since $i = 1$. The states are $\mid \Omega\rangle, (Q^\alpha)^\dagger \mid \Omega\rangle$ and $(Q^1)^\dagger(Q^2)^\dagger \mid \Omega\rangle$. If the vacuum has spin J, these states form a (spin J, spin $J-1/2$, spin $J+1/2$, spin J) multiplet.

For $J = 0$ one finds a spin $(0, \frac{1}{2}, 0)$ multiplet: the WZ model with a mass term. There is no problem with CPT since $G = U(1)$: the spin 0 particles have opposite $U(1)$ charge (as in the massless case), while also the left-handed and right-handed parts of the spinor have opposite $U(1)$ charge. One can build these massive multiplets from massless multiplets by a Higgs mechanism. For example the massive spin $(1, 1/2, 1/2, 0)$ multiplet arises when a massless vector (coming from the $(1, \frac{1}{2})$ and $(-1, -\frac{1}{2})$ multiplets) eats one spin 0 from the WZ multiplet.

$$\left(\begin{array}{cccc} \text{spin } 1 & \text{spin } 1/2 & \text{spin } 1/2 & \text{spin } 0 \\ 1 & & & \\ & 1/2 & 1/2 & \\ 0 & & & 0 \\ & -1/2 & -1/2 & \\ -1 & & & \end{array}\right) \tag{14.2.19}$$

Example: **massive $N = 2$ multiplets with $U(2)$-singlet spin-zero vacuum.** The states with their $SU(2)$ and spin representations are as follows

$$\begin{array}{ccccc} & \mid \Omega\rangle & (Q^{\alpha i})^\dagger \mid \Omega\rangle & (Q^{\alpha i})^\dagger (Q^{\beta j})^\dagger \epsilon_{ij} \mid \Omega\rangle & (Q^{\alpha i})^\dagger (Q^{\beta j})^\dagger \epsilon_{\dot\alpha\dot\beta} \mid \Omega\rangle \\ SU(2) & \mathbf{1} & \mathbf{2} & \mathbf{1} & \mathbf{3} \\ \text{spin} & J=0 & J=1/2 & J=1 & J=0 \end{array}$$

$$\begin{array}{cc} (Q^{\alpha i})^\dagger (Q^{\beta j})^\dagger (Q^{\gamma k})^\dagger \mid \Omega\rangle & (Q^{11})^\dagger (Q^{12})^\dagger (Q^{21})^\dagger (Q^{22})^\dagger \mid \Omega\rangle \\ \ydiagram{2,1}\,\mathbf{2} & \ydiagram{2}\,\mathbf{1} \\ J = \frac{1}{2} & J = 0 \end{array} \tag{14.2.20}$$

This multiplet has one spin 1 singlet, two spin 1/2 doublets, and a triplet and two singlets of spin 0. We do not need multiplet doubling in this case because the **1** and

3 of $SU(2)$ are real. One can again build this massive $N=2$ multiplet from massless $N=2$ representations by a Higgs mechanism. The massless $N=2$ irreps which are needed have the following helicities: $(1,1/2,1/2,0),(-1,-1/2,-1/2,0)$ and one hypermultiplet.

$$\begin{array}{ccc} \text{spin } 1 & \text{spin } 1/2 & \text{spin } 0 \\ 1 & & \\ & \frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}\ \frac{1}{2} & 0\,0\,0\,0\,0 \\ 0 & & \\ & -\frac{1}{2}\ -\frac{1}{2}\ -\frac{1}{2}\ -\frac{1}{2} & \\ -1 & & \end{array} \tag{14.2.21}$$

C. Massive irreps with Z^{ij} and BPS bounds.

We begin with $q_m=(m,0,0,0)$ and $\{Q^{\alpha i},(Q^{\beta j})^\dagger\}=2i\delta^i_j\sigma^{m,\alpha\dot\beta}P_m=2m\delta^i_j\delta^\alpha_\beta$. Because Z^{ij} has the form given in (14.1.7), we decompose for even N the indices i and j into ma and nb where $a,b=1,2$ and $m,n=1,N/2$. Thus m and n now label the 2×2 blocks of Z^{ij}, and a and b label the entries inside a given block. One obtains then

$$\begin{aligned} &\{Q^{\alpha ma},(Q^{\beta nb})^\dagger\}=2m\delta^\alpha_\beta\delta^m_n\delta^a_b \\ &\{Q^{\alpha ma},Q^{\beta nb}\}=-\epsilon^{\alpha\beta}Z^{ij}=\epsilon^{\alpha\beta}\delta^{mn}\epsilon^{ab}z_m \text{ with } z_m\geq 0 \\ &\{(Q^{\alpha ma})^\dagger,(Q^{\beta nb})^\dagger\}=-\epsilon^{\dot\alpha\dot\beta}\delta^{mn}\epsilon^{ab}z_m \text{ (since } \epsilon^{\alpha\beta}=-\epsilon^{\dot\alpha\dot\beta}) \end{aligned} \tag{14.2.22}$$

To diagonalize this into a set of fermionic harmonic oscillators, we construct the following linear combinations[5]

$$S^{\alpha n}_j=\frac{1}{\sqrt{2}}[Q^{\alpha n1}+(-)^j(Q^{\beta n2}\epsilon_{\beta\alpha})^\dagger];j=1,2 \tag{14.2.23}$$

One may check that this indeed diagonalizes the anticommutation relations in (14.2.22)

$$\begin{aligned} &\{S^{\alpha m}_i,(S^{\beta n}_j)^\dagger\}=\delta^j_i\delta^\alpha_\beta\delta^m_n 2(m-(-)^j z_m) \\ &\{S,S\}=\{S^*,S^*\}=0 \end{aligned} \tag{14.2.24}$$

[5]One needs linear combinations of Q's and $Q^\dagger$'s with the same n. Furthermore, one can only combine $Q^{\alpha n1}$ for $\alpha=1$ with $(Q^{\beta n1})^\dagger$ for $\beta=2$ because they both raise helicity. This yields (14.2.23).

Positivity of the norm in Hilbert space requires $z_m \leq m$. When $z_m = m$, the corresponding operators with $j = 2(S_2^{\alpha m}$ and $(S_2^{\alpha m})^*)$ vanish. One gets then **multiplet shortening**. As we shall discuss, this leads to the BPS bound in field theories.

Example: **massive** $N = 2$ **multiplets.** (Recall that for $N = 1$ there are no Z^{ij}). $G = SU(2)$. If $z < m$, one has symmetry group $USp(2) = SU(2)$ without $U(1)$. The states are generated by all $(S_i^{\alpha n})^*$ with $i = 1, 2$ and $\alpha = 1, 2$ and $n = 1$. Hence we get the same number of states as when there are no central charges (in that case the states were generated by all $(Q^{\alpha j})^*$ with $j = 1, 2$). For an $SU(2)$ singlet vacuum, we find a vector which is a singlet, two spin 1/2 particles which are doublets and spin 0 in a triplet and two singlets. This is the same content as in (14.2.20) and corresponds to the coupling of a $N = 2$ vector multiplet to a hypermultiplet with a Higgs mechanism.

If $z = m$, only $(S_1^{\alpha n})^*$ (with $n = 1$) can act while $S_2^{\alpha n}$ (with $n = 1$) vanishes. This is the same number of charges as $(Q^{1j})^*$ for $j = 1, 2$. Hence, then we get the same number of states as for massless $N = 2$ irreps. In particular we get the hypermultiplet after CPT induced multiplet doubling. In the corresponding field theory the limit from $m \neq 0$ to $m = 0$ is continuous because for $m \neq 0$ we have maximal $z = m$.

For the vector multiplet we can get masses from a Higgs mechanism as we already mentioned. Again the limit $m \to 0$ is smooth in the action for the same reasons.

Example: **massive** $N = 4$ **multiplets.** The only possibility to get a massive multiplet with spins $J \leq 1$ is to have both central charges z_1 and z_2 saturate the bound

$$z_1 = z_2 = m \tag{14.2.25}$$

Then $S_2^{\alpha n}$ and $(S_2^{\alpha n})^\dagger$ vanish, and $(S_{(1)}^{\alpha n})^\dagger$ for $\alpha = 1, 2$ and $n = 1, 2$ generate the same number of states as the $(Q^{1j})^*$ for $j = 1, 4$ in the massless case. The symmetry group is $Usp(4)$. It is generated by those 4×4 matrices M which are generators t of $U(4)$

which also satisfy $tZ + Zt^T = 0$. Writing the t and Z as follows

$$U(4): \left(\begin{array}{c|c} & \sigma^\mu \\ \hline -\sigma^\mu & \end{array}\right), \left(\begin{array}{c|c} 0 & i\sigma^\mu \\ \hline i\sigma^\mu & 0 \end{array}\right), \left(\begin{array}{c|c} i\sigma^\mu & \\ \hline & 0 \end{array}\right), \left(\begin{array}{c|c} 0 & \\ \hline & i\sigma^\mu \end{array}\right)$$

$$Z = \left(\begin{array}{c|c} \Omega & 0 \\ \hline 0 & \pm\Omega \end{array}\right) \quad M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{14.2.26}$$

the condition $MZ + ZM^T = 0$ implies

$$A\Omega + \Omega\, A^T = 0, B\Omega \pm \Omega C^T = 0, D\Omega + \Omega\, D^T = 0 \tag{14.2.27}$$

This leaves the following generators of $USp(4)$ if we choose the minus sign in Z

$$\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}, \begin{pmatrix} i\vec{\sigma} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & i\vec{\sigma} \end{pmatrix} \tag{14.2.28}$$

These are the Dirac matrices $-\gamma^0, i\gamma^k$, and $\gamma^{\mu\nu}$ and Z is the charge conjugation matrix C^- which satisfies $C^-\gamma^\mu = \gamma^{\mu,T}C^-$. (The plus sign corresponds to $C^+ \equiv C^-\gamma^5$ which satisfies $C^+\gamma^\mu = \gamma^{\mu,T}C^+$).

3 Parity

Half of the spin 0 particles in susy multiplets are scalars and half are pseudoscalars. For example, in the Wess Zumino multiplet, A has positive parity and B negative parity. For a massive $N = 1$ vector multiplet, a massless vector field eats one of the spin 0 particles of an $N = 1$ scalar multiplet to become massive, and the remaining scalar has opposite parity to that of the massive vector. We now want to understand this from the susy algebra.

We consider a larger algebra which contains both the generators of the super Poincaré algebra and the parity operator Π. The parity operator acts on the momentum generators as $\Pi P_m \Pi^{-1} = (-\vec{P}, P_0)$. Further $\Pi J_{ij} \Pi^{-1} = J_{ij}$ but $\Pi J_{i0} \Pi^{-1} = -J_{i0}$, just like in QED for $F_{\mu\nu}$ and $\vec{E}$ and $\vec{B}$. For Q^α it is easiest to first use 4-component spinor notation.

$$\{Q^\alpha, \bar{Q}_\beta\} \sim (\gamma^m)^\alpha{}_\beta P_m \tag{14.3.1}$$

Clearly $\Pi Q^{\beta}\Pi^{-1} = \alpha(\gamma^4)^{\beta}{}_{\gamma}Q^{\gamma}$ with α a phase because then $\gamma_4\gamma^k\gamma_4 = -\gamma^k$ for $k = 1, 2, 3$ compensates the minus sign in $\Pi\vec{P}\Pi^{-1} = -\vec{P}$. To fix α we consider the Majorana condition $Q^T C = Q^{\dagger} i\gamma^0$. Under parity

$$Q^T \alpha \gamma_4^T C = Q^{\dagger}\alpha^* \gamma_4 i\gamma^0 \tag{14.3.2}$$

and using $\gamma_4^T C = -C\gamma_4$ we find $\alpha = \pm i$. We redefine $Q \to -Q$ if necessary such that $\alpha = i$. Then

$$\Pi Q^{\alpha i}\Pi^{-1} = i(\gamma^4)^{\alpha}{}_{\beta}Q^{\beta i} \tag{14.3.3}$$

(Note that Π^2 on fermions yields now -1, so it is not an involution. However, physical observables are even in the number of fermions, and thus for physical observables Π^2 is still $+1$. The -1 for Π^2 on fermions is similar to the -1 for rotations of fermions over 2π, and poses no problem for the same reasons).

Reverting to 2 component spinors using

$$Q^{\alpha i} = \begin{pmatrix} Q^{\alpha i} \\ \bar{Q}_{\dot{\alpha} i} \end{pmatrix}, \gamma^4 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \tag{14.3.4}$$

we obtain

$$\Pi Q^{\alpha i}\Pi^{-1} = i\bar{Q}_{\dot{\alpha} i}; \Pi\bar{Q}_{\dot{\alpha} i}\Pi^{-1} = iQ^{\alpha i} \tag{14.3.5}$$

Hence $(Q^{\alpha i})^{\dagger} = -\bar{Q}_i^{\dot{\alpha}} = -\epsilon^{\dot{\alpha}\dot{\beta}}\bar{Q}_{\dot{\beta} i}$ transforms as follows under the parity operator

$$(Q^{1i})^{\dagger} \to iQ^{2i}, (Q^{2i})^{\dagger} \to -iQ^{1i} \tag{14.3.6}$$

Consider now an $N = 1$ doublet whose vacuum $\mid \Omega\rangle$ has momentum $\vec{q}$ along the positive z axis and helicity $\lambda = 1/2$. Consider also the parity conjugated multiplet that starts with a vacuum $\mid \Omega'\rangle$ with momentum $-\vec{q}$ and $\lambda = -1/2$. Under parity a state with λ goes into a state with $-\lambda$, hence Π maps $\mid \Omega\rangle$ to $\mid \Omega'\rangle$. We fix the phase of $\mid \Omega'\rangle$ by defining $\Pi \mid \Omega\rangle =\mid \Omega'\rangle$. Since $Q^{\alpha i} \mid \Omega\rangle = 0$ for all α with $i = 1$, it follows from $\Pi Q^{\alpha i} \mid \Omega\rangle = 0 = \Pi Q^{\alpha i}\Pi^{-1}\Pi \mid \Omega\rangle$ that $(Q^{\alpha i})^{\dagger} \mid \Omega'\rangle = 0$ for all α with $i = 1$.

Furthermore, on the multiplet with $+\vec{q}$ the $Q^{2i}, (Q^{2i})^\dagger$ vanish, but on the multiplet with $-\vec{q}$ the $Q^{1i}, (Q^{1i})^\dagger$ vanish. (The reason is the following. The expression $\gamma^m P_m$ is proportional to $(\sigma^3 + \sigma^0)$ for $+\vec{q}$ but proportional to $(\sigma^3 - \sigma^0)$ for $-\vec{q}$.) The operator Q^{2i} with $i = 1$ lowers the z-component of angular momentum, so it raises the helicity of the states with $-\vec{q}$. We then find the following quartet of states

$$\begin{array}{ll} \mid \Omega\rangle \text{ with } \lambda = 1/2 & \\ (Q^{11})^* \mid \Omega\rangle \text{ with } \lambda = 0 & Q^{21} \mid \Omega'\rangle \text{ with } \lambda = 0 \\ & \mid \Omega'\rangle \text{ with } \lambda = -1/2 \end{array} \tag{14.3.7}$$

We are interested in the action of Π on the spin 0 states. Recalling that $\Pi \mid \Omega\rangle = \mid \Omega'\rangle$, it follows that

$$\Pi(Q^{11})^* \mid \Omega\rangle = \Pi(Q^{11})^* \Pi^{-1} \Pi \mid \Omega\rangle = iQ^{21} \mid \Omega'\rangle \tag{14.3.8}$$

Hence Π maps the two $\lambda = 0$ states into each other.

The sum of these states is invariant under Π while the difference is mapped into minus itself. Let the corresponding fields be A and B. To show that the field for the $\lambda = 0$ state belonging to the multiplet with $\mid \Omega\rangle$ is $A + iB$ instead of $A + B$, one must consider the properties of these states under the charge conjugation operator C. An analysis of the super Poincaré algebra extended with C along the same lines as used for Π should reveal that also under C the two $\lambda = 0$ states are mapped into each other. This then proves that the two $\lambda = 0$ states correspond to $A + iB$ and $A - iB$, respectively.

4 $N = 1$ susy field theories x-space

To make the SM supersymmetric, one needs the so-called N=1 version of supersymmetry since the $N \geq 2$ formulations do not allow chiral fermions.[6] Furthermore, we

[6] After it was found that neutrinos are massive, this argument was no longer valid. We continue, however, with $N = 1$.

need rigid supersymmetry (susy with constant spinorial parameters) instead of local supersymmetry (=supergravity) if one wants to exclude gravity with its nonrenormalizability problems. In order to obtain a renormalizable model, we can only allow particles with spins up to and including one. In this section we present the most general renormalizable model with unbroken N=1 rigid supersymmetry.

The action for the coupling of Yang-Mills multiplets to matter multiplets ("vector multiplets" to "scalar multiplets") is a sum of

(i) gauge-invariant kinetic terms for gauge bosons, gauginos (susy partners of the gauge bosons), complex scalars and chiral (left-handed, so complex) spinors as well as auxiliary fields D^a (real) and F^i (complex),

(ii) Yukawa terms which couple gauginos to chiral spinors and complex scalars, and a coupling of D^a to a particular combination of the complex scalars, namely $\phi_i^*(T_a)^i{}_j\phi^j$, where T_a are the generators of the Yang-Mills group.

(iii) terms depending on the superpotential $W = \lambda_i\phi^i + \frac{1}{2}m_{ij}\phi^i\phi^j + \frac{1}{3}\lambda_{ijk}\phi^i\phi^j\phi^k$, which yield masses and Yukawa couplings for the scalars and chiral spinors. Gauge invariance requires that λ_i, m_{ij} and λ_{ijk} are gauge invariant tensors. For nonabelian simple groups m_{ij} is then proportional to the Killing metric and λ_{ijk} is proportional to the d-symbol.

We now discuss these sectors separately, and at the end of this section we summarize the results. This whole section is in x-space. In the next section we discuss the corresponding $N = 1$ superspace.

The kinetic terms for the vector multiplet and the scalar multiplet together with the auxiliary fields are separately supersymmetric if the gauge coupling constant g is set to zero. When g is nonzero, one introduces minimal gauge couplings by means of the usual gauge-covariant derivatives:

$$\begin{aligned}
\mathcal{L}\,(\text{kin}) &= -\frac{1}{4}(G^a_{\mu\nu})^2 - \frac{1}{2}\bar{\chi}^a\gamma^\mu D_\mu\chi^a + \frac{1}{2}(D^a)^2 \\
&- (D_\mu\varphi^*_i)\,(D^\mu\varphi^i) - \bar{\psi}_{Li}\gamma^\mu D_\mu {\psi_L}^i + F^*_i F^i \\
G^a_{\mu\nu} &= \partial_\mu G^a_\nu - \partial_\nu G^a_\mu + g f^a{}_{bc} G^b_\mu G^c_\nu \\
D_\mu\chi^a &= \partial_\mu\chi^a + g f^a{}_{bc} G^b_\mu \chi^c\ ;\ \bar{\chi}^a = \chi^T C = \chi^\dagger i\gamma^0 \\
D_\mu\varphi^i &= \partial_\mu\varphi^i + g G^a_\mu (T_a)^i{}_j\varphi^j ; D_\mu\varphi^*_i = \partial_\mu\varphi^*_i - g G^a_\mu \varphi^*_j (T_a)^j{}_i \\
D_\mu\psi^i_L &= \partial_\mu\psi^i_L + g G^a_\mu (T_a)^i{}_j \psi^j_L, \psi_L = \frac{1}{2}(1+\gamma_5)\psi
\end{aligned} \tag{14.4.1}$$

We used that T_a are antihermitian: $((T_a)^i{}_j)^* = -(T_a)^j{}_i$. Since χ^a are Majorana spinors, they satisfy the reality condition that their Majorana conjugate $\chi^{a,T}C$ is equal to their Dirac conjugate $(\chi^a)^\dagger i\gamma^0$. The charge conjugation matrix C satisfies $C\gamma_\mu C^{-1} = -\gamma^T_\mu$. (In a real representation of $\gamma_1, \gamma_2, \gamma_3$ and γ_0, we can choose $C = i\gamma^0$. Then χ^a is, in fact, "really real". We shall, however, use the complex representation which corresponds to two-component spinor formalism.) In general the indices a in $G^a_{\mu\nu}$ etc. should be contracted with the Killing metric g_{ab} defined by $f_{ap}{}^q f_{bq}{}^p = -g_{ab}\chi$ with χ the dual Coxeter number. For $SU(N), \chi = N$ but we have chosen a normalization of the generators such that $g_{ab} = \delta_{ab}$. Then $trT_aT_b = -\frac{1}{2}\delta_{ab}$ for the fundamental representation $\mathbf{N}$ of $SU(N)$. The indices of χ^a are then raised and lowered by δ^{ab} and δ_{ab}, and so it makes no difference whether we write $\bar{\chi}^a$ or $\bar{\chi}_a$. We recall that $\bar{\psi}_{Li} \equiv (\psi^i_L)^\dagger i\gamma^0 = \bar{\psi}_i \frac{1}{2}(1-\gamma_5)$ with $\bar{\psi}_i = (\psi^i)^\dagger i\gamma^0$.

The gauge transformations with gauge parameter λ^a under which $\mathcal{L}$ (kin) and the other terms in the action are invariant, are given by

$$\begin{aligned}
\delta(\text{gauge})G^a_\mu &= D_\mu\lambda^a = \partial_\mu\lambda^a + g f^a{}_{bc} G^b_\mu \lambda^c \\
\delta(\text{gauge})\chi^a &= g f^a{}_{bc}\chi^b\lambda^c ; \delta(\text{gauge})D^a = g f^a{}_{bc} D^b \lambda^c \\
\delta(\text{gauge})\varphi^i &= -g\lambda^a (T_a)^i{}_j \varphi^j ; \delta(\text{gauge})\psi^i_L = -g\lambda^a (T_a)^i{}_j \psi^j_L \\
\delta(\text{gauge})F^i &= -g\lambda^a (T_a)^i{}_j F^j
\end{aligned} \tag{14.4.2}$$

The antihermitian matrices T_a satisfy $[T_a, T_b] = f_{ab}{}^c T_c$ and in the adjoint representation $(T_a)^b{}_c = f^b{}_{ac}$. Of course, G^a_μ, χ^a and D^a transform the same way for constant λ^a since they form a super multiplet. This fixes the covariant derivatives. The same holds for the scalar multiplet $(\varphi^i, \psi^i_L, F^i)$.

The supersymmetry transformations under which $\mathcal{L}$ (kin) is invariant, are given by

$$\begin{aligned}
\delta G^a_\mu &= i\bar{\epsilon}\gamma_\mu\gamma_5\chi^a \quad , \quad \bar{\epsilon} = \epsilon^T C = \epsilon^\dagger i\gamma^0 \\
\delta\chi^a &= \frac{i}{2}\gamma^\mu\gamma^\nu G_{\mu\nu}{}^a\gamma_5\epsilon + \alpha D^a\epsilon \\
\delta D^a &= \beta\bar{\epsilon}\not{D}\chi^a, \ (\alpha = \beta = 1, \text{ see below})
\end{aligned} \tag{14.4.3}$$

The factors of i are needed to ensure the reality of these transformation rules. (For example, $(\delta\chi^a)^\dagger i\gamma^0 = \delta\chi^{a,T}C$).

We now first consider the case $g=0$ (abelian case). The action is susy invariant if $\alpha = \beta$, and α and β must be real to keep $\delta\chi^a$ a Majorana spinor and δD^a real. The susy commutator on $G_\mu{}^a$ yields

$$[\delta(\epsilon_1), \delta(\epsilon_2)]G_\mu{}^a = (2\bar{\epsilon}_2\gamma^\sigma\epsilon_1)\partial_\sigma G_\mu{}^a - \partial_\mu[2\bar{\epsilon}_2\gamma^\sigma\epsilon_1 G^a_\sigma] \tag{14.4.4}$$

which is clearly a sum of a translation and an abelian gauge transformation. The same commutator on χ^a contains terms with $\bar{\epsilon}_2\gamma_\sigma\epsilon_1$ and $\bar{\epsilon}_2\gamma_\sigma\gamma_\tau\epsilon_1$. The latter cancel if $\alpha\beta = 1$, and for $\alpha\beta = 1$ one finds that all terms with $\not{\partial}\chi$ ("equation of motion terms") cancel, whereas one is left with the same translation as for G^a_μ. As a thorough check one may then verify that also $[\delta(\epsilon_1), \delta(\epsilon_2)]D^a$ becomes equal to $2\epsilon_2\gamma^\sigma\epsilon_1\partial_\sigma D^a$. We choose $\alpha = \beta = +1$, which amounts to a choice of sign for D^a.

For $g \neq 0$, we simply covariantize these transformation laws, and $G^a_{\mu\nu}$ is then the Yang-Mills curvature. As far as dimensions are concerned, a term $DA^\mu A_\mu$ would also be allowed, but it is ruled out by gauge invariance (and would also destroy susy). The cancellation of the terms in the variation of the action proceeds as before, with D_μ

instead of ∂_μ everywhere[7] (the Bianchi identity $D_{[\mu}G_{\rho\sigma]}=0$ is needed, as well as the well-known fact that $\delta G^a_{\mu\nu}=D_\mu \delta A_\nu{}^a - D_\nu \delta A_\mu{}^a$). However, there is one extra term, due to varying the connection in $-\frac{1}{2}\bar\chi \not{D}\chi$. This variation is given by

$$-\frac{1}{2}(\bar\chi^a\gamma^\mu \; g f^a{}_{bc}\chi^c) i\bar\epsilon\gamma_\mu\gamma_5\chi^b \tag{14.4.5}$$

and it vanishes since $(\bar\chi^a\gamma^\mu\chi^b)\gamma_\mu\chi^c$, when totally antisymmetrized in abc, is identically zero.

We could have removed the matrices γ_5 in δG^a_μ and $\delta\chi^a$ by redefining $i\gamma_5\epsilon=\tilde\epsilon$ or $i\gamma_5\chi'=\chi$ (these are also Majorana spinors). This would have introduced factors γ_5 in δD and in the term with D in $\delta\chi$.

$$\begin{aligned}
&\delta A^a_\mu = \bar\epsilon\gamma_\mu\chi^a \\
&\delta\chi^a = -\frac{1}{2}\gamma^\mu\gamma^\nu G^a_{\mu\nu}\epsilon + i\gamma_5\epsilon D^a \\
&\delta D^a = i\bar\epsilon\gamma_5 \not{D}\chi^a \; .
\end{aligned} \tag{14.4.6}$$

As far as invariance of the gauge action is concerned, we could even suppress the factors γ_5 in all terms. Then the susy commutator $[\delta(\epsilon_1),\delta(\epsilon_2)]$ would no longer be proportional to a translation but if one is only interested in invariance of the action, this would not be inadmissable. However, for the coupling to scalar multiplets one needs the transformation rules as given in (14.4.3), although replacing χ by $i\gamma_5\chi'$ or ϵ by $i\gamma_5\epsilon'$ everywhere is, of course, allowed. (See the footnote above (14.4.14))

In a similar manner we analyze the kinetic sector of the scalar multiplet. We begin with

$$\begin{aligned}
\delta\varphi^i &= \sqrt{2}\bar\epsilon_R\psi^i_L \; ; \; \delta\varphi^*_i = \sqrt{2}\bar\psi_{iL}\epsilon_R \\
\delta\psi^i{}_L &= \sqrt{2}\gamma^\mu D_\mu\varphi^i\epsilon_R + a\sqrt{2}F^i\epsilon_L \; ; \; \delta\bar\psi_{iL} = -\sqrt{2}\bar\epsilon_R\gamma^\mu D_\mu\varphi^*_i + a^*\sqrt{2}\bar\epsilon_L F^*_i
\end{aligned}$$

[7] The variations of the gauge action yields $(D^\mu G^a_{\mu\nu})(\delta A^{\nu a})(-\bar\chi^a)\gamma^\mu[\gamma^{\rho\sigma}(D_\mu F^a_{\rho\sigma})\frac{i}{2}\gamma_5\epsilon + D^a\epsilon]$. With $\gamma^\mu\gamma^{\rho\sigma}=\gamma^{\mu\rho\sigma}+\eta^{\mu\rho}\gamma^\sigma-\eta^{\mu\sigma}\gamma^\rho$ the term with $\gamma^{\mu\rho\sigma}\equiv\gamma^{[\mu}\gamma^{\rho\sigma]}$ cancels due to the Bianchi identity, while the terms with the field equation $D^\mu G^a_{\mu\nu}$ cancel separately. The variations with D^a cancel if one chooses δD^a appropriately.

$$\delta F^i = \sqrt{2} b \bar{\epsilon}_L \not{D} \psi_L{}^i \; ; \; \delta F_i^* = -\sqrt{2} b^* (D_\mu \bar{\psi}_{Li}) \gamma^\mu \epsilon; (a = b = 1, \text{ see below}) \tag{14.4.7}$$

The left-handed part of ϵ is needed in $\delta\psi^i \sim F^i\epsilon$ and δF^i, while the right-handed part occurs in $\delta\varphi^i$ and $\delta\psi_L^i \sim \not{D}\varphi^i\epsilon$. Again putting first $g = 0$, the variations of the kinetic terms yield

$$\begin{aligned} \frac{1}{\sqrt{2}}\delta\mathcal{L}(\text{kin}) &= -(\partial_\mu\varphi_i^*)\partial^\mu[\bar{\epsilon}\psi_L^i] - (\partial^\mu\varphi^i)\partial_\mu[\bar{\psi}_{iL}\epsilon] \\ &- \bar{\psi}_{iL}\gamma^\mu\partial_\mu[\gamma^\nu\partial_\nu\varphi^i\epsilon + aF^i\epsilon] - (-\bar{\epsilon}\gamma^\mu\partial_\mu\varphi_i^* + a^*\bar{\epsilon}F_i^*)\not{\partial}\psi_L^i \\ &+ bF_i^*\bar{\epsilon}\not{\partial}\psi_L^i + b^*F^i(-\partial_\mu\bar{\psi}_{Li}\gamma^\mu\epsilon) = 0 \text{ if } a = b^* \end{aligned} \tag{14.4.8}$$

The susy commutator on φ^i yields $[\delta(\epsilon_1), \delta(\epsilon_2)]\varphi^i = 2(\bar{\epsilon}_2\gamma^\sigma\epsilon_1)\partial_\sigma\varphi^i$ which shows that the overall normalization with factors $\sqrt{2}$ of the transformation laws for the scalar multiplet agrees with that of the vector multiplet. The susy commutator on ψ_L^i yields

$$\begin{aligned} [\delta(\epsilon_1), \delta(\epsilon_2)]\psi_L^i &= (\gamma^\mu\epsilon_{2R})\partial_\mu[\bar{\epsilon}_{1R}\psi_L^i] + a\epsilon_{2L}b\bar{\epsilon}_{1L}\;\not{\partial}\psi_L^i - (1 \leftrightarrow 2) \\ &= -\frac{1}{4}(\bar{\epsilon}_{1R}O\epsilon_{2R})[\gamma^\mu O\partial_\mu\psi_L^i] - \frac{1}{4}ab(\bar{\epsilon}_{1L}O\epsilon_{2L})O\not{\partial}\psi_L^i - 1 \leftrightarrow 2 \end{aligned} \tag{14.4.9}$$

The tensor terms with $O = \gamma^{\alpha\beta}$ cancel as $\bar{\epsilon}_{1R}\gamma^{\alpha\beta}\epsilon_{2R}$ and $\bar{\epsilon}_{1L}\gamma^{\alpha\beta}\epsilon_{2L}$ vanish, and only the vector and axial vector terms with $O = \gamma_\sigma$ and $O = i\gamma_\sigma\gamma_5$ remain. In fact, their contributions are equal and one finds the same translation as for the other fields provided $ab = 1$. Finally, as a check one may evaluate $[\delta(\epsilon_1), \delta(\epsilon_2)]$ on F^i, and one finds the correct translation for $ab = 1$. We choose $a = b = 1$ which amount to choosing the phase of F^i conveniently.

For $g \neq 0$ we replace ∂_μ by D_μ in action and transformation laws. This yields minimal gauge couplings of the form $G_a^\mu J_\mu^a$ where J_μ^a is quadratic in matter fields, but in order to obtain a supersymmetric action, one also needs couplings of the gaugino and auxiliary field D to similar currents bilinear in matter fields as we now discuss.

The Yukawa couplings. In susy, the minimal gauge couplings of the form $G_\mu\varphi_i^*\partial^\mu\varphi^j$ induce further couplings because G_μ^a, χ^a and D^a form a multiplet. These extra terms are Yukawa couplings of the form $\chi\varphi_i^*\psi^i$ and "D-terms" of the form $D\varphi^*\varphi$. Since these terms are the partners of the minimal gauge coupling terms, they are all proportional to g. For dimensional reasons they cannot contain derivatives. In fact, dimensional arguments allow only these terms. More precisely

$$\begin{aligned}\mathcal{L}(\text{Yuk}) + \mathcal{L}(D\text{−coupling}) &= i\sqrt{2}g\varphi_i^*\bar{\chi}^a(T_a)^i{}_j\psi_L^j + i\sqrt{2}g\bar{\psi}_{iL}(T^a)^i{}_j\chi^a\varphi^j \\ &- ig\varphi_i^*(T_a)^i{}_j\varphi^j D^a \end{aligned} \tag{14.4.10}$$

Since in our conventions T_a is antihermitian we need a factor i in front of the last term to make it real. To prove the invariance of these terms under supersymmetry, we should also take into account the minimal gauge coupling terms in the matter action. These were contained in the covariant derivatives and read separately

$$\begin{aligned}\mathcal{L}(\text{min. coupl. matter}) &= -g\partial^\mu\varphi_i^* G_\mu^a(T_a)^i{}_j\varphi^j \\ &+ g(\varphi_i^*)G_\mu{}^a(T_a)^i{}_j\partial_\mu\varphi^j + g^2\varphi_i^*(T_a G_\mu^a T_b G^{\mu b})^i{}_j\varphi^j \\ &- g\bar{\psi}_{Li}\gamma^\mu G_\mu{}^a(T_a)^i{}_j\psi^j \end{aligned} \tag{14.4.11}$$

but we should only use here the variations of G_μ^a because all other variations were already taken care of previously by working with covariant derivatives. It is easy to check that the variations of the φ's in $\mathcal{L}$ (D-coupling) cancel the variations in $\mathcal{L}$(Yuk) due to $\delta\chi \sim D$. The factor $\sqrt{2}$ in front of the Yukawa couplings is due to the $\sqrt{2}$ in $\delta\varphi^i = \sqrt{2}\bar{\epsilon}\psi_L^i$. To check to normalization of $\mathcal{L}$ (D-coupling) with respect to $\mathcal{L}$ (min. coupl. matter) we consider the variations of the form $g^2\varphi^*\varphi\bar{\epsilon}G_\mu\gamma^\mu\chi$. These cancel, too. One may check that all other variations cancel.

Since $F = 0$ and $F^* = 0$ are the field equations of F^* and F, respectively, and field equations transform into field equations, we must update the transformation laws for F and F^* such that they contain the field ψ and $\bar{\psi}$ field spectrum. One finds

$$\begin{aligned}\delta F^i &= \sqrt{2}\bar{\epsilon}_L(\not{D}\psi_L{}^i - i\sqrt{2}g(T_a)^i{}_j\chi^a\varphi^j) \\ \delta F_i^* &= -\sqrt{2}(D_\mu\bar{\psi}_{Li} + i\sqrt{2}g\varphi_j^*(T_a)^j{}_i) \end{aligned} \tag{14.4.12}$$

The superpotential. The last set of terms depends on an analytic function $W(\varphi)$ of φ, called the "superpotential". They are separately invariant, and read

$$\begin{aligned}\mathcal{L}(W) &= F^i \frac{\partial W}{\partial \varphi^i} + F_i^* \frac{\partial W^*}{\partial \varphi_i^*} - \frac{1}{2}\frac{\partial^2 W}{\partial \varphi^i \partial \varphi^j}(\psi_L^{iT} C \psi_L^j) \\ &- \frac{1}{2}\frac{\partial^2 W^*}{\partial \varphi_i^* \partial \varphi_j^*}\left(\psi_{iL}^\dagger C^{-1} \psi_{jL}^*\right) \end{aligned} \tag{14.4.13}$$

After eliminating F^i and F_i^* from their algebraic field equation, one obtains $-\frac{\partial W}{\partial \varphi_i^*}\frac{\partial W}{\partial \varphi^i}$ for minus the potential, which contains a mass term $-\varphi_i^*(mm^*)^i{}_j\varphi^j$ for the scalars, while the last two terms yield $-\frac{1}{2}m_{ij}\psi_L^{iT}C\psi_L^j + h.c.$ Note that $\psi_L^{iT}C$ is not the same as $\bar{\psi}_{iL} \equiv (\psi_L^i)^\dagger i\gamma^0$; for example, they transform differently under $SU(N)$. To prove the invariance of $\mathcal{L}(W)$, one may show that the variation of $\frac{\partial^2 W}{\partial \varphi^i \partial \varphi^j}$ produces terms proportional to $\frac{\partial^3 W}{\delta \varphi^i \partial \varphi^j \partial \varphi^k}$ with 3 fields ψ_L; the vanishing of these terms can be demonstrated by a Fierz rearrangement. All other variations cancel straightforwardly.

Summary. The action for Yang-Mills fields $G_\mu{}^a$ (where $a = 1, \ldots \dim G$ is the adjoint index) and gauginos χ^a (Majorana fermions) coupled to complex scalars φ^i and (complex) left-handed fermions ψ_L^i (in a representation $(T^a)^i{}_j$ of G) is given by

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}(G_{\mu\nu}^a)^2 - \frac{1}{2}\bar{\chi}^a \not{D}\chi^a + \frac{1}{2}(D^a)^2 \\ &- (D_\mu\varphi_i^*)(D^\mu\varphi^i) - \bar{\psi}_{iL}\not{D}\psi_L^i + F_i^* F^i \\ &+ F^i\frac{\partial W}{\partial \varphi^i} + F_i^*\frac{\partial W^*}{\partial \varphi_i^*} - \frac{1}{2}\frac{\partial^2 W}{\partial \varphi^i \partial \varphi^j}(\psi_L^{iT} C \psi_L^j) \\ &+ \frac{1}{2}\frac{\partial^2 W^*}{\partial \varphi_i^* \partial \varphi_j^*}\left(\bar{\psi}_{jL} C^{-1} (\bar{\psi}_{iL})^T\right) \\ &+ i\sqrt{2}g\left\{\varphi_i^* \bar{\chi}^a (T_a)^i{}_j \psi_L^j + \bar{\psi}_{iL}(T_a)^i{}_j \chi^a \varphi^j\right\} \\ &- ig\varphi_i^*(T_a)^i{}_j\varphi^j D^a(-\eta^a D_a \text{ for } U(1) \text{ groups}) \end{aligned} \tag{14.4.14}$$

Since D^a varies into a total derivative, we can add a term $-\eta_a D^a$ to the action, with η_a a constant and still keep susy. If D^a belongs to a $U(1)$ group (in which case we can omit the index a), it is gauge-invariant, and under susy $\delta D^a = \bar{\epsilon}\gamma^\mu\partial_\mu\chi^a$. Then the term $\eta_a D^a$ is both susy and gauge invariant.

The auxiliary fields D^a (real) and F^i (complex) yield the potential of the scalar fields. After elimination of F^i, F_i^* and D^a, it is given by[8]

$$V(\varphi^i, \varphi_j^*) = \left|\frac{\partial W}{\partial \varphi^i}\right|^2 + \frac{1}{2}(ig\varphi_i^*(T_a)^i{}_j\varphi^j)^2 \tag{14.4.15}$$

Note that both terms are positive semi-definite (T_a is antihermitian). We can rewrite this potential as

$$V(\varphi^i, \varphi_j^*) = F_i^* F^i + \frac{1}{2} D_a D^a \tag{14.4.16}$$

with F, F^* and D being solutions of their nonpropagating field equations:

$$F^i = -\frac{\partial W^*}{\partial \varphi_i}; F_i^* = -\frac{\partial W}{\partial \varphi^i}; D^a = ig(\varphi^* T_a \varphi) \tag{14.4.17}$$

5 $N = 1$ Susy field theories in superspace

We now review $N = 1$ Minkowski superspace. We begin with some properties of 2-component spinors. The Dirac matrices $\gamma^{m,A}{}_B$ with $m = 0,3$ and $A, B = 1,4$ and 4-component Dirac spinors ψ^A are given by

$$\gamma^m = \left(\begin{array}{c|c} 0 & -i\sigma^{m,\alpha\dot\beta} \\ \hline i\bar\sigma^m_{\dot\alpha\beta} & 0 \end{array}\right) \begin{array}{l} \sigma^m = (I, \vec\sigma) \\ \bar\sigma^m = (-I, \vec\sigma) \end{array} \; \psi^A = \begin{pmatrix} \chi^\alpha \\ \bar\zeta_{\dot\alpha} \end{pmatrix} \begin{array}{l} \alpha = 1,2 \\ \dot\alpha = 1,2 \end{array} \tag{14.5.1}$$

The matrix $\gamma_5 \equiv \gamma^1\gamma^2\gamma^3 i\gamma^0$ is diagonal, with $+I$ and $-I$ along the diagonal. The charge conjugation matrix C_{AB} satisfies $C\gamma^m C^{-1} = -\gamma^{m,T}$ and is given by

$$C = \begin{pmatrix} \epsilon_{\alpha\beta} & 0 \\ 0 & \epsilon^{\dot\alpha\dot\beta} \end{pmatrix} \text{ with } \epsilon_{\alpha\beta} = \epsilon^{\alpha\beta} = -\bar\epsilon_{\dot\alpha\dot\beta} = -\epsilon^{\dot\alpha\dot\beta} \tag{14.5.2}$$

We raise and lower two-component spinor indices with ϵ tensors with the "north-west to east-south rule"

$$\begin{aligned} \theta^\alpha &= \epsilon^{\alpha\beta}\theta_\beta, \theta_\beta = \theta^\alpha \epsilon_{\alpha\beta} \\ \bar\theta^{\dot\alpha} &= \epsilon^{\dot\alpha\dot\beta}\bar\theta_{\dot\beta}, \bar\theta_{\dot\alpha} = \bar\theta^{\dot\beta}\epsilon_{\dot\beta\dot\alpha} \end{aligned} \tag{14.5.3}$$

[8]The variation of the gauge fields G^a_μ in the scalar kinetic term (proportional to $\bar\epsilon\gamma_\mu\gamma_5\chi$) cancels half of the variation of the quarks in the Yukawa term (the terms with the γ_5 from ψ_L). The other half cancels the variation of D (proportional to $\bar\epsilon \not{D}\chi$) in the term with $D\varphi^*\varphi$. One can everywhere replace χ by $i\gamma_5\chi'$ (which is again a Majorana spinor). Then $\delta G^a_\mu \sim \bar\epsilon\gamma_\mu\chi'$ and $\delta D \sim \bar\epsilon\gamma_5\not{D}\chi'$, and the gaugino action does not change, but in the Yukawa term the χ'-dependent terms get a relative minus sign and the prefactor i is canceled.

(This proves that numerically $\epsilon^{\alpha\beta}=\epsilon_{\alpha\beta}$ and $\epsilon^{\dot{\alpha}\dot{\beta}}=\epsilon_{\dot{\alpha}\dot{\beta}}$ because $\epsilon^{\alpha\beta}\equiv\epsilon^{\alpha\gamma}\epsilon^{\beta\delta}\epsilon_{\gamma\delta}$ and similarly for $\epsilon^{\dot{\alpha}\dot{\beta}}$). The contractions of 4-component and 2-component spinors are then related as follows

$$\begin{aligned}\bar{\psi}\lambda &\equiv \psi^T C\lambda = \psi^A C_{AB}\lambda^B = \psi_B\lambda^B \\ &= \psi^\alpha\epsilon_{\alpha\beta}\lambda^\beta + \bar{\psi}_{\dot{\alpha}}\epsilon^{\dot{\alpha}\dot{\beta}}\bar{\lambda}_{\dot{\beta}} = \psi_\beta\lambda^\beta + \bar{\psi}_{\dot{\alpha}}\bar{\lambda}^{\dot{\alpha}} \end{aligned} \tag{14.5.4}$$

(We raise and lower four-component spinor indices with the charge conjugation matrix, using again the "north-west rule". Hence $\psi_B=\psi^A C_{AB}$).

Lowering the indices of $\sigma^{m,\alpha\dot{\beta}}$ with ϵ tensors leads to a matrix $\sigma^m_{\alpha\dot{\beta}}$ (namely $\sigma^m_{\alpha\dot{\beta}} = \sigma^{m,\gamma\dot{\delta}}\epsilon_{\gamma\alpha}\epsilon_{\dot{\delta}\dot{\beta}}$) which is related to $\bar{\sigma}^m_{\dot{\alpha}\beta}$ by complex conjugation (or transposition since the matrices σ^m are hermitian)

$$\bar{\sigma}^m_{\dot{\alpha}\beta} = (\sigma^m_{\alpha\dot{\beta}})^* = \sigma^m_{\beta\dot{\alpha}} \tag{14.5.5}$$

To check this, note that $\sigma^m_{\alpha\dot{\beta}} = -\epsilon_{\alpha\gamma}\sigma^{m,\gamma\dot{\delta}}\epsilon_{\dot{\delta}\dot{\beta}} = -(\sigma^2\sigma^m\sigma^2)_{\alpha\dot{\beta}}$. Similarly

$$\bar{\sigma}^{m,\dot{\alpha}\beta} = \sigma^{m,\beta\dot{\alpha}} = (\sigma^{m,\alpha\dot{\beta}})^* \tag{14.5.6}$$

A four-component Majorana spinor is a spinor whose charge conjugate $\psi^T C$ is equal to its Dirac conjugate $\psi^\dagger i\gamma^0$. Hence $\psi^T C=\psi^\dagger i\gamma^0$. Using the representation in (14.5.1) it has in two-component formalism the form

$$\psi = \begin{pmatrix} \psi^{\alpha i} \\ \bar{\psi}_{\dot{\alpha}i} \equiv (\psi^i_\alpha)^* \end{pmatrix}; \bar{\psi} = (\psi^i_\alpha, -\bar{\psi}^{\dot{\alpha}}_i) \tag{14.5.7}$$

Because 2-component spinor formalism is used in superspace we rewrite some of the results of the previous section from this point of view. The kinetic actions for the fermions can be written in two-component formalism as follows

$$\begin{aligned} -\frac{1}{2}\bar{\chi}^a\gamma^\mu D_\mu\chi^a &= \frac{i}{2}\bar{\chi}^a_{\dot{\alpha}}(\sigma^\mu)^{\alpha\dot{\alpha}}D_\mu\chi^a_\alpha + \frac{i}{2}\chi^a_\alpha(\sigma^\mu)^{\alpha\dot{\alpha}}D_\mu\bar{\chi}^a_{\dot{\alpha}} = i\bar{\chi}^a_{\dot{\alpha}}(\sigma^\mu)^{\alpha\dot{\alpha}}D_\mu\chi^a_\alpha \\ -\frac{1}{2}\bar{\psi}\not{D}\psi &= -\frac{1}{2}\bar{\psi}\not{D}\psi_L - \frac{1}{2}\bar{\psi}\not{D}\psi_R \\ &= \frac{i}{2}\bar{\psi}^{\dot{\alpha}}_i\bar{\sigma}^\mu_{\dot{\alpha}\alpha}D_\mu\psi^{i\alpha} + \frac{i}{2}\psi^i_\alpha(\sigma^\mu)^{\alpha\dot{\alpha}}D_\mu\bar{\psi}_{i\dot{\alpha}} \\ &= i\bar{\psi}_{\dot{\alpha}i}(\sigma^\mu)^{\alpha\dot{\alpha}}D_\mu\psi_{i\alpha} \end{aligned} \tag{14.5.8}$$

This agrees with (14.4.1). The terms with the susy extension of the minimal gauge couplings in $\mathcal{L}$ (kin) were given in (14.4.14)

$$\mathcal{L}\text{ (minimal) } = i\sqrt{2}g(\varphi^*\bar{\chi}\psi_L + \bar{\psi}_L\chi\varphi) - ig\varphi^* D\varphi \tag{14.5.9}$$

The last term, $-ig\varphi_i^* D^a (T_a)^i{}_j \varphi^j$, is hermitian. The sums of the two terms $i\sqrt{2g}(\varphi_i^*\bar{\chi}_a$ $(T_a)^i{}_j\psi_L^j + \bar{\psi}_{L,i}\chi^a(T_a)^i{}_j\varphi^j)$ in (14.5.9) is also hermitian because

$$\begin{aligned}
&(i\varphi_i^*\bar{\chi}_a(T_a)^i{}_j\psi_L^j)^\dagger = (i\varphi_i^*\chi_\alpha^a T_{aj}^i\psi^{j\alpha})^\dagger = -i(-\bar{\psi}_j^{\dot{\alpha}})(-T_a)^j{}_i\bar{\chi}_{\dot{\alpha}}^a\varphi^i \\
&= i\bar{\psi}_{L,i}\chi^a(T_a)^i{}_j\varphi^j = i(-\bar{\psi}_i^{\dot{\alpha}})\bar{\chi}_{\dot{\alpha}}^a(T_a)^i{}_j\varphi^j
\end{aligned} \tag{14.5.10}$$

Finally, the terms due to the superpotential $W = \frac{1}{2}m_{ij}\varphi^i\varphi^j + \frac{1}{3}\lambda_{ijk}\varphi^i\varphi^j\varphi^k$ in (14.4.13) read

$$\begin{aligned}
W &= F^i\frac{\partial W}{\partial\varphi^i} + F_i^*\frac{\partial W}{\partial\varphi_i^*} - \frac{1}{2}\frac{\partial^2 W}{\partial\varphi^i\partial\varphi^j}(\psi_L^{\alpha i T}\epsilon_{\alpha\beta}\psi_L^{j\beta}) \\
&+ \frac{1}{2}\frac{\partial^2 W}{\partial\varphi_i^*\partial\varphi_j^*}(\bar{\psi}_{Lj}C^{-1}\bar{\psi}_{iL}^T)
\end{aligned} \tag{14.5.11}$$

Using that $\bar{\psi}_{Lj}C^{-1}\bar{\psi}_{iL}^T = -\psi_{jL}^\dagger C^{-1}\psi_{jL}^*$, the mass terms are manifestly real. In 2-component formalism they are given by

$$\begin{aligned}
\mathcal{L}\text{ (mass) } &= -\frac{1}{2}m_{ij}\psi_\alpha^i\psi^{j\alpha} + \frac{1}{2}(m_{ij})^*(\psi_j^{\dot{\beta}})(-\epsilon_{\dot{\beta}\dot{\alpha}})(\psi_i^{\dot{\alpha}}) \\
&= -\frac{1}{2}m_{ij}\bar{\psi}_A^i\psi^{j,A}
\end{aligned} \tag{14.5.12}$$

After these preliminaries we are ready to discuss superspace. The coordinates of $N = 1$ Minkowski superspace are x^μ for $\mu = 0, 3$ and θ_α for $\alpha = 1, 2$. We define $\bar{\theta}_{\dot{\alpha}} = (\theta_\alpha)^*$. Hence $\binom{\theta^\alpha}{\bar{\theta}_{\dot{\alpha}}}$ is a Majorana spinor. Then $\theta^\alpha \equiv \epsilon^{\alpha\beta}\theta_\beta$ satisfies $(\theta^\alpha)^* = -\bar{\theta}^{\dot{\alpha}}$ (where $\bar{\theta}^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}}\bar{\theta}_{\dot{\beta}}$) because $\epsilon^{\alpha\beta} = -\epsilon^{\dot{\alpha}\dot{\beta}}$. The derivatives $\frac{\partial}{\partial\theta^\alpha}$ and $\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}}$ satisfy reality properties which follow from $\{\frac{\partial}{\partial\theta^\alpha}, \theta^\beta\} = \delta_\alpha{}^\beta$ and $\{\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}}, \bar{\theta}^{\dot{\beta}}\} = \delta_{\dot{\alpha}}{}^{\dot{\beta}}$. Using $(\theta^\beta)^\dagger = -\bar{\theta}^{\dot{\beta}}$ we find

$$\left(\frac{\partial}{\partial\theta^\alpha}\right)^\dagger = -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} \tag{14.5.13}$$

Similarly we find from $\left[\frac{\partial}{\partial x^\mu}, x^\nu\right] = \delta_\mu{}^\nu$ and $(x^\nu)^\dagger = x^\nu$ that $\left(\frac{\partial}{\partial x^\mu}\right)^\dagger = -\frac{\partial}{\partial x^\mu}$.

We define the covariant derivatives of rigid $N = 1$ superspace by

$$\begin{array}{ll} D_\alpha = \frac{\partial}{\partial \theta^\alpha} + i\sigma^m_{\alpha\dot\beta}\bar\theta^{\dot\beta}\frac{\partial}{\partial x^m} & \{D_\alpha, \bar D_{\dot\beta}\} = 2i\sigma^m_{\alpha\dot\beta}\partial_m \\ \bar D_{\dot\alpha} = \frac{\partial}{\partial \bar\theta^{\dot\alpha}} + i\theta^\beta\bar\sigma^m_{\dot\alpha\beta}\frac{\partial}{\partial x^m} & \{D_\alpha, D_\beta\} = \{\bar D_{\dot\alpha}, \bar D_{\dot\beta}\} = 0 \end{array} \tag{14.5.14}$$

It follows that

$$(D_\alpha)^\dagger = -\bar D_{\dot\alpha} \tag{14.5.15}$$

Hence for a real scalar $S, (D_\alpha S) \equiv [D_\alpha, S] = \chi_\alpha$ and $(\bar D_{\dot\alpha} S) \equiv [\bar D_{\dot\alpha}, S] = \bar\chi_{\dot\alpha}$ form a Majorana spinor.

These derivatives are called covariant derivatives because they anticommute with the supersymmetry generators Q^α and $\bar Q_{\dot\alpha}$ where

$$\begin{array}{c} Q_\alpha = \dfrac{\partial}{\partial \theta^\alpha} - i\bar\theta^{\dot\beta}\bar\sigma^m_{\dot\beta\alpha}\dfrac{\partial}{\partial x^m}; \bar Q_{\dot\alpha} = \dfrac{\partial}{\partial \bar\theta^{\dot\alpha}} - i\theta^\beta\sigma^m_{\beta\dot\alpha}\dfrac{\partial}{\partial x^m} \\ \{D_\alpha, Q_\beta\} = \{D_\alpha, \bar Q_{\dot\beta}\} = \{\bar D_{\dot\alpha}, Q_\beta\} = \{\bar D_{\dot\alpha}, \bar Q_{\dot\beta}\} = 0 \end{array} \tag{14.5.16}$$

Again $(Q_\alpha)^\dagger = -\bar Q_{\dot\alpha}$, and $\{Q_\alpha, \bar Q_{\dot\beta}\} = -2i\sigma^m_{\alpha\dot\beta}\partial_m$ (note the opposite sign as compared to $\{D_\alpha, \bar D_{\dot\beta}\}$).

The superfields we need transform as $\delta\Phi =(\epsilon_\alpha Q^\alpha + \bar\epsilon_{\dot\alpha}\bar Q^{\dot\alpha})\Phi$. So, they transform as scalar fields under supersymmetry: only the coordinates transform but these superfields have no supersymmetry indices. Because $\{D, Q\} = 0$, also $\bar D_{\dot\beta}\Phi$ transforms as a superfield

$$\delta\bar D_{\dot\beta}\Phi = \bar D_{\dot\beta}(\epsilon_\alpha Q^\alpha + \bar\epsilon_{\dot\alpha}\bar Q^{\dot\alpha})\Phi = (\epsilon_\alpha Q^\alpha + \bar\epsilon_{\dot\alpha}\bar Q^{\dot\alpha})\bar D_{\dot\beta}\Phi \tag{14.5.17}$$

6 The gauge action in $N = 1$ superspace

The superfields for an $N = 1$ vector multiplet are real e^V (so with $V = V^a T_a$ and $(V^a)^\dagger = -V^a$ and $T_a^\dagger = -T_a$. The superfield $V^a(x, \theta, \bar\theta)$ is unconstrained except for $(V^a)^\dagger = -V^a$). As we shall see, the covariant x-space spinors and curvatures are contained in the superfield $W_\alpha = \bar{D}^2 e^{-V} D_\alpha e^V$ where $\bar{D}^2 \equiv \bar{D}^{\dot\alpha}\bar{D}_{\dot\alpha}$. Many results simplify in the Wess-Zumino (WZ) gauge (see below). In WZ gauge the x-space gauge field is given by

$$2i(\partial_{\alpha\dot\beta} + A_{\alpha\dot\beta}) = \{\bar{D}_{\dot\beta}, e^{-V} D_\alpha e^V\}_| = \{\bar{D}_{\dot\beta}, \mathcal{D}_\alpha\}_| = 2i\mathcal{D}_{\alpha\dot\beta\,|} \tag{14.6.1}$$

where $\mathcal{D}_\alpha \equiv e^{-V} D_\alpha e^V$. The vertical bar indicates that one should set $\theta = \bar\theta = 0$ in the result. (This is the chiral representation. All our results are in the chiral representation[9]). Further, the gaugino and auxiliary field (denoted by D(aux) to avoid confusion with derivatives D_α) are given in WZ gauge by[10]

$$\begin{aligned} \lambda_\alpha &= W_{\alpha|} = \bar{D}^2 e^{-V} D_\alpha e^V{}_| = [\bar{D}^{\dot\beta}, \{\bar{D}_{\dot\beta}, \mathcal{D}_\alpha\}]_| \\ D\,(\text{aux}) &= e^{-V} D^\alpha e^V \bar{D}^2 e^{-V} D_\alpha e^V{}_| = \mathcal{D}^\alpha W_{\alpha|} \end{aligned} \tag{14.6.2}$$

Also $\mathcal{D}^\alpha W_\alpha$ is shorthand for $\{\mathcal{D}^\alpha, W_\alpha\}$; in fact, one needs an (anti)commutator in order that D(aux) be Lie-algebra valued. Thus all fields of x-space are defined in terms of (anti) commutators of $D_{\dot\alpha}$ and $\mathcal{D}_\alpha$.

The choice of e^V as gauge field is not obvious. It would seem more natural to begin with a supervector field $A_\Lambda(x,\theta,\bar\theta)$ which decomposes into $A_\mu(x,\theta,\bar\theta)$, $A_\alpha(x,\theta,\bar\theta)$ and $A_{\dot\alpha}(x,\theta,\bar\theta)$. To avoid higher-spin x-fields in the $\theta, \bar\theta$ expansions, one must impose constraints on the supercurvatures and by solving these constraints one arrives at e^V as gauge field. One of these constraints is $\{\bar{D}_{\dot\beta}, \mathcal{D}_\alpha\} = 2i\mathcal{D}_{\alpha\dot\beta}$ which we already encountered in (14.6.1).

[9] If one writes the operator $\mathcal{D}_\alpha = e^{-V} D_\alpha e^V$ as $D_\alpha + \Gamma_\alpha$ where $\Gamma_\alpha = e^{-V}(D_\alpha e^V)$ is a connection, then $2iA_{\alpha\dot\beta} = (D_{\dot\beta}\Gamma_\alpha)$ in the chiral representation. The gauge covariant derivative $\bar{\mathcal{D}}_{\dot\beta}$ is equal to $\bar{D}_{\dot\beta}$ in the chiral representation.

[10] In general the x-space components are given by $D\,(\text{aux}) = e^{V/2}(\mathcal{D}^\alpha W_\alpha) e_|^{-V/2}$ which is real as we explain in a footnote below (14.6.9), and $\lambda_\alpha = e^{V/2} W_\alpha e^{-V/2}{}_|$. For the proper definition of the gauge field in a general gauge we note that $e^{V/2}\{\bar{D}_{\dot\beta}, \mathcal{D}_\alpha\}e^{-V/2}$ is hermitian. Hence we define $2i(\partial_{\alpha\dot\beta} + A_{\alpha\dot\beta}) = e^{V/2}\{\bar{D}_{\dot\beta}, \mathcal{D}_\alpha\}e^{-V/2}{}_|$, from which one obtains $A_{\alpha\dot\beta}$. Thus all x-space components in a general gauge are obtained by making a similarity transformation with $e^{V/2}$ on (14.6.1) and (14.6.2). Since the action is invariant under similarity transformations, we shall continue to use the simpler formulas without $e^{V/2}$.

The Lagrangian for the gauge fields is given by $trW^\alpha W_\alpha$. (It is real by itself up to a imaginary total x-derivative as we shall show. The real parts of $W^\alpha W_\alpha$ and $\bar{W}^{\dot{\alpha}}\bar{W}_{\dot{\alpha}}$ agree and the imaginary terms are opposite). Gauge transformations with chiral parameter $\Lambda = \Lambda^a T_a$ are defined by

$$(e^V)' = e^{\bar{\Lambda}} e^V e^{-\Lambda}, \bar{\Lambda} = -\Lambda^\dagger = (\Lambda^a)^* T_a \text{ and } \bar{D}_{\dot{\alpha}}\Lambda = 0 \tag{14.6.3}$$

If Λ is chiral, $\bar{\Lambda}$ and $(\Lambda^a)^*$ are antichiral.[11] Clearly $(e^V)'$ is again real. One can use this gauge freedom to go to the Wess-Zumino gauge where the $\theta, \bar{\theta}$ independent term and the terms proportional to $\theta, \theta^2, \bar{\theta}$ and $\bar{\theta}^2$ in V vanish.

Since $\bar{D}^2(e^{-V} D_\alpha e^V)$ transforms into $e^\Lambda(\bar{D}^2 e^{-V} D_\alpha e^V)e^{-\Lambda}$ because $D_\alpha\bar{\Lambda} = 0$ and $\bar{D}_{\dot{\alpha}}\Lambda = 0$, the gauge action is gauge invariant.[12] Since the product of three $\bar{D}$ derivatives vanishes, W^α is chiral, and so is the Lagrangian. Since $\mathcal{L}$ depends only on y^μ and θ^α, the gauge action needs a $d^2\theta$ integral, not a $d^4\theta = d^2\theta d^2\bar{\theta}$ integral. Inside a $\int d^4x$ integral $d\theta^\alpha$ is equivalent to D_α. Normalizing the $d^2\theta$ integral by $d^2\theta = D^2$ we obtain for the gauge action

$$\begin{aligned} S\text{ (gauge)} &= \int d^4x d^2\theta trW^\alpha W_\alpha = \int d^4x D^2 trW^\alpha W_\alpha \\ &= \int d^4x tr\mathcal{D}^2(W^\alpha W_\alpha) = 2\int d^4x tr[W^\alpha(\mathcal{D}^2 W_\alpha) - (\mathcal{D}^\beta W^\alpha)(\mathcal{D}_\beta W_\alpha)] \end{aligned} \tag{14.6.4}$$

where we could replace D^2 inside the trace by $\mathcal{D}^2$ because $trW^\alpha W_\alpha$ is a scalar under gauge transformations. Note that by $\mathcal{D}^2(W^\alpha W_\alpha)$ we mean $\{\mathcal{D}^\beta, [\mathcal{D}_\beta, W^\alpha W_\alpha]\}$. Inside the trace all connections indeed cancel, $tr[D^\beta(\Gamma_\beta W^\alpha W_\alpha - W^\alpha W_\alpha \Gamma_\beta)] = 0$.

To recover the result $-\frac{1}{4}G_{\mu\nu}^2 - \frac{1}{2}\bar{\lambda}\not{D}\lambda + \frac{1}{2}D(\text{aux})^2$ of (14.4.1) we must work out $\mathcal{D}^{(\beta}W^{\alpha)}$ and $\mathcal{D}^2 W_\alpha$. The term with $\mathcal{D}^{[\beta}W^{\alpha]}\mathcal{D}_{[\beta}W_{\alpha]}$ clearly gives $\frac{1}{2}D(\text{aux})^2$, because

[11] A chiral field ϕ satisfies $\bar{D}_{\dot{\alpha}}\phi = 0$, and an antichiral field $\bar{\Lambda}$ satisfies $D_\alpha\bar{\Lambda} = 0$. One can solve the differential equation $\bar{D}_{\dot{\alpha}}\phi = 0$ by writing ϕ as a function of y^μ and θ^α where $\bar{D}_{\dot{\alpha}}y^\mu = 0$. The solution for y^μ is $y^\mu = x^\mu + i\theta^\alpha\sigma^\mu_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}$. Since $\bar{D}_{\dot{\alpha}}\theta^\alpha = 0, \phi(y^\mu, \theta^\alpha)$ is indeed chiral.

[12] When D_α in $\bar{D}^2 e^{-V} D_\alpha e^V e^{-\Lambda}$ acts on $e^{-\Lambda}$ the result $\bar{D}^2 D_\alpha e^{-\Lambda} = \bar{D}^{\dot{\alpha}}\{\bar{D}_{\dot{\alpha}}, D_\alpha\}e^{-\Lambda}$ vanishes since $\bar{D}^{\dot{\alpha}}$ commutes with $\{\bar{D}_{\dot{\alpha}}, D_\alpha\}$ and $\bar{D}^{\dot{\alpha}}e^{-\Lambda} = 0$. Hence one can extract the factor $e^{-\Lambda}$ on the right.

$\mathcal{D}^{[\beta}W^{\alpha]} = -\frac{1}{2}\epsilon^{\beta\alpha}\mathcal{D}^{\gamma}W_{\gamma}$ (use $\mathcal{D}^{\gamma}W_{\gamma} = \mathcal{D}^{\gamma}W^{\delta}\epsilon_{\delta\gamma}$). However, the other terms require a bit of work.

Derivation of $\mathcal{D}^{(\beta}W^{\alpha)}$.

$$\begin{aligned} &\mathcal{D}^{\beta}W^{\alpha} = \{\mathcal{D}^{\beta},[\bar{D}^{\dot\beta},\{\bar{D}_{\dot\beta},\mathcal{D}^{\alpha}\}] = \text{ with Jacobi for } \mathcal{D}^{\beta}, \bar{D}^{\dot\beta} \text{ and } \{\bar{D}_{\dot\beta},\mathcal{D}^{\alpha}\} \\ &= -[\{\bar{D}_{\dot\beta},\mathcal{D}^{\alpha}\},\{\mathcal{D}^{\beta},\bar{D}^{\dot\beta}\}] + \{\bar{D}^{\dot\beta},[\{\bar{D}_{\dot\beta},\mathcal{D}^{\alpha}\},\mathcal{D}^{\beta}]\} \end{aligned} \tag{14.6.5}$$

The first term contains the Yang-Mills curvature because it is the commutator of two covariant derivatives $\mathcal{D}_{\alpha\dot\beta} = \sigma^{\mu}_{\alpha\dot\beta}\mathcal{D}_{\mu}$. The second term can be cast into the same form as the first term by using the Jacobi identity once more and then using $\{\mathcal{D}^{\alpha},\mathcal{D}^{\beta}\} = 0$

$$[\{\bar{D}_{\dot\beta},\mathcal{D}^{\alpha}\},\mathcal{D}^{\beta}] = [\mathcal{D}^{\alpha},\{\bar{D}_{\dot\beta},\mathcal{D}^{\beta}\}] \tag{14.6.6}$$

The second term in (14.6.5) then becomes

$$\begin{aligned} &\{\bar{D}^{\dot\beta},[\mathcal{D}^{\alpha},\{\bar{D}_{\dot\beta},\mathcal{D}^{\beta}\}]\} = \text{ (again with Jacobi)} \\ &= [\{\bar{D}^{\dot\beta},\mathcal{D}^{\alpha}\},\{\bar{D}_{\dot\beta},\mathcal{D}^{\beta}\}] - \{\mathcal{D}^{\alpha},[\bar{D}^{\dot\beta},\{\bar{D}_{\dot\beta},\mathcal{D}^{\beta}\}]\} \end{aligned} \tag{14.6.7}$$

Note that the last term in (14.6.7) is equal to $-\mathcal{D}^{\alpha}W^{\beta}$. Bringing it to the left-hand side of (14.6.5) we arrive at

$$\{\mathcal{D}^{\alpha},[\bar{D}^{\dot\beta},\{\bar{D}_{\dot\beta},\mathcal{D}^{\beta}\}]\} + \alpha \leftrightarrow \beta = 2[\{\bar{D}^{\dot\beta},\mathcal{D}^{\alpha}\},\{\bar{D}_{\dot\beta},\mathcal{D}^{\beta}\}] \tag{14.6.8}$$

Or

$$\begin{aligned} &\mathcal{D}^{(\alpha}W^{\beta)}{}_{|} = (2i)^2[\partial^{\alpha\dot\beta} + A^{\alpha\dot\beta}, \partial^{\beta}{}_{\dot\beta} + A^{\beta}{}_{\dot\beta}] = (2i)^2\sigma^{\mu,\alpha\dot\beta}\sigma^{\nu\beta}{}_{\dot\beta}F_{\mu\nu} \\ &= (2i)^2(\sigma^{\mu\nu})^{\alpha\beta}F_{\mu\nu} = (2i)^2\frac{1}{2}(\sigma^{\mu\nu})^{\alpha\beta}(F_{\mu\nu} - \frac{i}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}) = (2i)^2(-2)F^{\alpha\beta} \end{aligned} \tag{14.6.9}$$

where[13] $F_{\alpha\dot\beta,\gamma\dot\delta} = \epsilon_{\dot\beta\dot\delta}F_{\alpha\gamma} + \epsilon_{\alpha\gamma}F_{\dot\beta\dot\delta}$. Hence $D^{(\beta}W^{\alpha)}$ yields a kind of complex selfdual part of $F_{\mu\nu}$. Namely $(F_{\mu\nu})^d \equiv F_{\mu\nu} - \frac{i}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$ satisfies $F^{dd}_{\mu\nu} = F_{\mu\nu}$. Of course, one

[13] We define $[\mathcal{D}_{\alpha\dot\beta},\mathcal{D}_{\gamma\dot\delta}] = F_{\alpha\dot\beta,\gamma\dot\delta}$ and $F_{\alpha\dot\beta,\gamma\dot\delta} = \sigma^{\mu}_{\alpha\dot\beta}\sigma^{\nu}_{\gamma\dot\delta}F_{\mu\nu}$. Further we used that $(\sigma^{\mu\nu})^{\alpha\beta} = (\sigma^{\mu\nu})^{\beta\alpha}$ and $\frac{1}{2}(\sigma^{\mu\nu})^{\alpha\beta}(\sigma_{\rho\sigma})_{\beta\alpha} = \delta^{\mu}_{\rho}\delta^{\nu}_{\sigma} - \delta^{\mu}_{\sigma}\delta^{\nu}_{\rho} + i\epsilon^{\mu\nu}{}_{\rho\sigma}$ with $\epsilon^{0123} = 1$.

cannot define a real selfdual part of $F_{\mu\nu}$ in Minkowski space. Note that superspace provides automatically an (imaginary) topological $F\tilde{F}$ term in the action!

Derivation of $\mathcal{D}^2 W^\gamma$.

We use the reality condition[14] $\mathcal{D}^\alpha W_\alpha = \bar{D}^{\dot\alpha}(e^{-V}\bar{W}_{\dot\alpha}e^V)$ and the identity $\mathcal{D}^\gamma\mathcal{D}^\beta W_\beta = -\frac{1}{2}\epsilon^{\gamma\beta}\mathcal{D}^2 W_\beta = -\frac{1}{2}\mathcal{D}^2 W^\gamma$. (The latter identity follows from the fact that $\mathcal{D}^\gamma$ and $\mathcal{D}^\beta$ anticommute). Using the reality condition we get

$$\mathcal{D}^\gamma\mathcal{D}^\beta W_\beta = \mathcal{D}^\gamma\bar{D}^{\dot\beta}(e^{-V}\bar{W}_{\dot\beta}e^V) = \{\mathcal{D}^\gamma, \bar{D}^{\dot\beta}\}e^{-V}\bar{W}_{\dot\beta}e^V = 2i(\partial^{\gamma\dot\beta} + A^{\gamma\dot\beta})\bar\lambda_{\dot\beta} \quad (14.6.10)$$

We used that $\mathcal{D}^\gamma(e^{-V}\bar{W}_{\dot\beta}e^V) = e^{-V}(D^\gamma\bar{W}_{\dot\beta})e^V = 0$. (Note that when we replaced D^γ by $\mathcal{D}^\gamma$, this derivative $\mathcal{D}^\gamma$ acts on W_α as an anticommutator). Hence $\mathcal{D}^2W^\gamma$ equals $-4i$ times the Yang-Mills covariant derivative of $\bar\lambda_{\dot\beta}$. This differs according to (14.5.8) from the Dirac action by the divergence of the axial current of the gaugino.

The action then becomes (after suitable rescalings)

$$\mathcal{L} = -\frac{1}{8}(F_{\mu\nu} - i{}^*F_{\mu\nu})^2 + \frac{1}{2}D(\text{aux})^2 - \frac{1}{2}\bar\lambda\not{D}\lambda - \frac{i}{2}\partial_\mu[\bar\lambda^{\dot\alpha}\bar\sigma^\mu_{\dot\alpha\beta}\lambda^\beta] \qquad (14.6.11)$$

Note that all fields are $\bar{D}$ or $\mathcal{D}$ projections of W_α and $e^{-V}\bar{W}_{\dot\beta}e^V$ (not of $\bar{W}_{\dot\beta}$). Note also that both imaginary terms are total derivatives.

The general susy Lagrangians transform into total x-space derivatives, indicating that susy is a spacetime symmetry. This can be readily checked for this case: $D^2W^\beta W_{\beta\,|}$ transforms into $\bar\epsilon^{\dot\alpha}\bar{D}_{\dot\alpha}D^2W^\beta W_{\beta\,|}$ (the term with $\epsilon^\alpha D_\alpha$ obviously cancels) and this is indeed a total x-derivative, namely $-8i\bar\epsilon^{\dot\alpha}\partial_{\dot\alpha\alpha}(D^\alpha W^\beta)W_{\beta\,|}$.

[14] This relation is an **identity**. It is difficult to prove, but note that it is gauge covariant. From $(D_\alpha)^\dagger = -\bar{D}_{\dot\alpha}$ and $(D^\alpha)^\dagger = \bar{D}^{\dot\alpha}$ one finds $\bar{W}_{\dot\alpha} \equiv (W_\alpha)^\dagger = -D^2e^V\bar{D}_{\dot\alpha}e^{-V}$. Taking the hermitian conjugate of the reality condition yields $(\mathcal{D}^\alpha W_\alpha)^\dagger = e^V(\mathcal{D}^\alpha W_\alpha)e^{-V}$, and substituting the expression for $\bar{W}_{\dot\alpha}$ shows that the reality condition is invariant under hermitian conjugation. To linear order in V the reality condition reduces to $D^\alpha\bar{D}^2D_\alpha V = \bar{D}^{\dot\alpha}D^2\bar{D}_{\dot\alpha}V$ which is true because $D^\alpha[\bar{D}^2, D_\alpha] = [\bar{D}^{\dot\alpha}, D^2]\bar{D}_{\dot\alpha}$ In the Wess-Zumino gauge the reality condition states that the auxiliary field $D^\alpha(\text{aux})$ is real. In general, D (aux) is given by $e^{V/2}(\mathcal{D}^\alpha W_\alpha)e^{-V/2}$ which is a real superfield.

7 The matter action in $N=1$ superspace

We consider chiral fields ϕ^i satisfying $\bar{D}_{\dot{\alpha}}\phi^i=0$, and antichiral fields $\bar{\phi}_i\equiv(\phi^i)^*$ satisfying $D_\alpha\bar{\phi}_i=0$. It is advantageous to introduce fields $\bar{\phi}_j(e^V)^j{}_i$ satisfying $\mathcal{D}_\alpha(\bar{\phi}e^V)=0$. Recall that $\mathcal{D}_\alpha=D_\alpha+\Gamma^a_\alpha T_a$ with $\Gamma_\alpha=e^{-V}(D_\alpha e^V)$. Thus

$$\begin{aligned}\mathcal{D}_\alpha(\bar{\phi}e^V)&=D_\alpha(\bar{\phi}e^V)-(\bar{\phi}e^V)\Gamma_\alpha\\&=\bar{\phi}(D_\alpha e^V)-(\bar{\phi}e^V)e^{-V}(D_\alpha e^V)=0\end{aligned}\tag{14.7.1}$$

Hence $D_\alpha\bar{\phi}=0$ is indeed equivalent to $\mathcal{D}_\alpha(\bar{\phi}e^V)=0$.

The x-space components are defined by using covariant derivatives in the chiral representation.

$$\begin{aligned}&\phi_|=\varphi,\mathcal{D}_\alpha\phi_|=\psi_\alpha,\mathcal{D}^\alpha\mathcal{D}_\alpha\phi_|=F\\&\bar{\phi}e^V{}_|=\varphi^*,\bar{D}_{\dot{\alpha}}(\bar{\phi}e^V)=\bar{\psi}_{\dot{\alpha}},\bar{D}^2(\bar{\phi}e^V)_|=F^*\end{aligned}\tag{14.7.2}$$

Recalling that $\bar{D}_{\dot{\alpha}}(\bar{\phi}e^V)$ stands for $[\bar{D}_{\dot{\alpha}},\bar{\phi}e^V]$ and $(D_\alpha)^\dagger=-\bar{D}_{\dot{\alpha}}$ according to (14.5.15), we see that $\bar{\psi}_{\dot{\alpha}}=(\psi_\alpha)^*$ so that ψ is a Majorana spinor, except that there is the extra factor e^V. Furthermore, although $\bar{\phi}_i=(\phi^i)^*$, we still identify $\phi^i{}_|=\varphi^i$ but $\bar{\phi}_ie^V{}_|=\varphi^*_i$. This requires an explanation.

We use at this point the fact that the action we shall shortly introduce is gauge invariant, and choose a gauge (the Wess-Zumino gauge) in which all $\theta,\bar{\theta}$-independent terms of V and the terms linear and quadratic in either θ or $\bar{\theta}$ vanish (but the terms with $\theta\sigma_\mu\bar{\theta}$ in V do not vanish and yield A_μ). Then $\bar{\phi}e^V=\bar{\phi}$ and $\bar{D}_{\dot{\alpha}}(\bar{\phi}e^V)_|=\bar{D}_{\dot{\alpha}}\bar{\phi}_|$ so that $\psi_{\dot{\alpha}}=(\psi_\alpha)^*$. Finally $\bar{D}^2\bar{\phi}e^V_|=\bar{D}^2\bar{\phi}_|$ justifying (14.7.2) for gauge invariant objects such as the action. One could also transform from the chiral representation back to the vector representation, but since we have not discussed the properties of the vector representation, we provide a third justification of (14.7.2) which does not require us to leave the chiral representation. Namely, all terms with e^V and derivatives of e^V which appear in (14.7.2) should cancel in the action. This is, of course, due to gauge invariance, and thus this justification is equivalent to the first explanation. We begin with $\bar{\phi}e^V{}_| = \varphi^* e^C$, $\bar{D}_{\dot{\alpha}}(\bar{\phi}e^V) = \bar{\psi}_{\dot{\alpha}} e^C + \varphi^*(\bar{D}_{\dot{\alpha}} e^V)_|$ and $\mathcal{D}_\alpha \phi_| = (D_\alpha e^V)\phi + \psi_\alpha$ etc., and should then find that all terms with $e^C, (\bar{D}_{\dot{\alpha}} e^V)_|$ etc. cancel.

The advantage of using covariant derivatives to define components is that all gauge covariantizations come out automatically; using ordinary D derivatives one would have to combine results into covariant expressions by hand.

The action is given by (we suppress the indices i and j and the trace symbol tr)

$$\begin{aligned}\mathcal{L} &= D^2 \bar{D}^2(\bar{\phi}e^V \phi) = \mathcal{D}^2[(\bar{D}^2(\bar{\phi}e^V))\phi] \\ &= [\mathcal{D}^2 \bar{D}^2(\bar{\phi}e^V)]\phi + 2[\mathcal{D}^\alpha \bar{D}^2(\bar{\phi}e^V)]\mathcal{D}_\alpha \phi + [\bar{D}^2(\bar{\phi}e^V)]\mathcal{D}^2\phi \end{aligned} \tag{14.7.3}$$

We stress again that the $\mathcal{D}^2$ and $\bar{D}^2$ derivatives are really defined in terms of (anti) commutators.

The last term in (14.7.3) yields the term $\mathbf{F}_i^* \mathbf{F}^i$ in (14.4.14). For the second term we use

$$\begin{aligned}&\mathcal{D}^\alpha \bar{D}^2(\bar{\phi}e^V) = [\mathcal{D}^\alpha, \bar{D}^2](\bar{\phi}e^V) = \{\mathcal{D}^\alpha, \bar{D}^{\dot{\beta}}\}\bar{D}_{\dot{\beta}}(\bar{\phi}e^V) - \bar{D}^{\dot{\beta}}\{\mathcal{D}^\alpha, \bar{D}_{\dot{\beta}}\}(\bar{\phi}e^V) \\ &= 2\{\mathcal{D}^\alpha, \bar{D}^{\dot{\beta}}\}\bar{D}_{\dot{\beta}}(\bar{\phi}e^V) - [\bar{D}^{\dot{\beta}}, \{\mathcal{D}^\alpha, \bar{D}_{\dot{\beta}}\}]\bar{\phi}e^V = \\ &= 4i(\partial^{\alpha\dot{\beta}} + A^{\alpha\dot{\beta}})\bar{\psi}_{\dot{\beta}} - \lambda^\alpha \varphi^* \end{aligned} \tag{14.7.4}$$

The first term in this result yields than the Dirac action and the second term yields according to (14.6.2) half the Yukawa terms, namely those of the form $\lambda^\alpha \varphi^* \psi_\alpha$.

Finally we consider the first term in (14.7.3). It yields

$$\begin{aligned}&\mathcal{D}^2 \bar{D}^2(\bar{\phi}e^V) = \mathcal{D}^\alpha[\mathcal{D}_\alpha, \bar{D}^2](\bar{\phi}e^V) = \\ &\mathcal{D}^\alpha\{\mathcal{D}_\alpha, \bar{D}^{\dot{\beta}}\}\bar{D}_{\dot{\beta}}(\bar{\phi}e^V) - \mathcal{D}^\alpha \bar{D}^{\dot{\beta}}\{\mathcal{D}_\alpha, \bar{D}_{\dot{\beta}}\}\bar{\phi}e^V \\ &= [\mathcal{D}^\alpha, \{\mathcal{D}_\alpha, \bar{D}^{\dot{\beta}}\}]\bar{D}_{\dot{\beta}}(\bar{\phi}e^V) - 2\{\mathcal{D}^\alpha, \bar{D}^{\dot{\beta}}\}\{\mathcal{D}_\alpha, \bar{D}_{\dot{\beta}}\}(\bar{\phi}e^V) \\ &+\bar{D}^{\dot{\beta}}[\mathcal{D}^\alpha, \{\mathcal{D}_\alpha, \bar{D}_{\dot{\beta}}\}]\bar{\phi}e^V \\ &= [\mathcal{D}^\alpha, \{\mathcal{D}_\alpha, \bar{D}^{\dot{\beta}}\}]D_{\dot{\beta}}(\bar{\phi}e^V) - 2\{\mathcal{D}^\alpha, \bar{D}^{\dot{\beta}}\}\{\mathcal{D}_\alpha, \bar{D}_{\dot{\beta}}\}(\bar{\phi}e^V)\end{aligned}$$

$$+\bar{D}^{\dot{\beta}}[\mathcal{D}^{\alpha},\{\mathcal{D}_{\alpha},\bar{D}_{\dot{\beta}}\}](\bar{\phi}e^{V})$$
$$=\{\bar{D}^{\dot{\beta}}[\mathcal{D}^{\alpha},\{\mathcal{D}_{\alpha},\bar{D}_{\dot{\beta}}\}]\}(\bar{\phi}e^{V})-2\{\mathcal{D}^{\alpha},\bar{D}^{\dot{\beta}}\}\{\mathcal{D}_{\alpha},\bar{D}_{\dot{\beta}}\}(\bar{\phi}e^{V}) \qquad (14.7.5)$$

The first term is proportional to $\varphi\bar{\lambda}_{\dot{\beta}}\bar{\psi}^{\dot{\beta}}$ and yields the other half of the Yukawa interaction. In the second term we recognize the Yang-Mills covariant Dalembertian acting on $\bar{\phi}e^{V}_{|}=\varphi^{*}$.

Hence we have also obtained the matter x-space action from the matter superspace action.

8 Field theories in x-space with rigid $N=2$ susy

The simplest $N=2$ model in $3+1$ dimensions is obtained by coupling an $N=1$ vector multiplet with field $(A^{a}_{\mu},\chi^{a},D^{a})$ to an $N=1$ scalar multiplet with fields $(M^{a},N^{a},\psi^{a},F^{a},G^{a})$. Since the fermions χ^{a} and ψ^{a} will appear symmetrically in this $N=2$ model, also the matter multiplet must be in the adjoint representation. The helicities of the physical fields are

$$(1,1/2)+(1/2,0,)\text{ plus for CPT }(-1,-1/2)+(-1/2,0) \qquad (14.8.1)$$

Taking the vector field massless, there is no central charge on-shell (there is no central charge which acts on the states), hence we expect a full rigid $U(2)$ symmetry, in addition to the local gauge group which is, as always, arbitrary in rigid susy.

Anticipating that χ^{a} and ψ^{a} will appear symmetrically, we introduce the spinor λ^{a}_{i}, with $\lambda^{a}_{1}=\chi^{a}$ and $\lambda_{2}=\psi^{a}$. Then the action reads after eliminating auxiliary fields

$$\mathcal{L}=tr\left[\frac{1}{2}F^{2}_{\mu\nu}+\bar{\lambda}_{i}\not{D}\lambda_{i}+(D_{\mu}M)^{2}+(D_{\mu}N)^{2}\right.$$
$$\left.-i\bar{\lambda}_{2}[\lambda_{1},M]-i\bar{\lambda}_{2}\gamma_{5}[\lambda_{1},N]+\frac{1}{2}g^{2}[M,N]^{2}\right] \qquad (14.8.2)$$

We can rewrite this in an $O(2)$ invariant way by rewriting $\bar{\lambda}_2^a \lambda_1^b f_{abc} = -\frac{1}{2}\epsilon_{ij}\bar{\lambda}_i^a \lambda_j^b f_{abc}$ and similarly for $\bar{\lambda}_2\gamma_5\lambda_1$. Then the action becomes

$$\mathcal{L} = tr\Big[\frac{1}{2}F_{\mu\nu}^2 + \bar{\lambda}_i \not{D}\lambda_i + (D_\mu M)^2 + (D_\mu N)^2 \\ -\epsilon_{ij}\bar{\lambda}_i[\lambda_j, M] - i\epsilon_{ij}\bar{\lambda}_i\gamma_5[\lambda_j, N] + \frac{1}{2}g^2[M,N]^2\Big] \tag{14.8.3}$$

where $trT_aT_b = -\frac{1}{2}\delta_{ab}$.

The transformation rules are

$$\delta A_\mu = \bar{\epsilon}_j\gamma_\mu\lambda_j; \delta M = \epsilon_{ij}\bar{\epsilon}_i\lambda_j; \delta N = i\epsilon_{ij}\bar{\epsilon}_i\gamma_5\lambda_j \\ \delta\lambda_i = -\frac{1}{2}\gamma^{\mu\nu}\epsilon_i F_{\mu\nu} - \epsilon_{ij}\gamma^\mu D_\mu(M + i\gamma_5 N)\epsilon_j - i\gamma_5\epsilon_i[M,N] \tag{14.8.4}$$

The susy algebra becomes

$$[\delta(\epsilon^{(1)}), \delta(\epsilon^{(2)})] = 2(\bar{\epsilon}^{(2)}\gamma^\mu\epsilon^{(1)})\partial_\mu + \delta_{\text{gauge}}(\lambda) \\ + \text{field equations for } \lambda_j \tag{14.8.5}$$

where the gauge parameters λ^a is given by $\bar{\epsilon}^{(2)}\gamma^\mu\epsilon^{(1)}A_\mu^a$.

Although the action has indeed two susy symmetries (ϵ_i with $i = 1, 2$), one only sees an explicit $O(2)$ symmetry rather than a $U(2)$ symmetry. The reason is that the 4-component spinor λ^i has two 2-component spinors $\lambda^{\alpha i}$ and $\bar{\lambda}_{\bar{\alpha}i} \equiv (\lambda^{\alpha i})^\dagger$, and the first transforms in the **2** of $U(2)$ but the second in the $\mathbf{2}^*$. We can raise the index i of $\lambda_{\dot{\alpha}i}$ with an ϵ^{ij} symbol, and obtain then a so-called modified Majorana spinor, also called $SU(2)$ Majorana spinor, or symplectic Majorana spinor; recall that $Usp(2) = SU(2)$.

$$\lambda^i = \begin{pmatrix} \lambda^{\alpha i} \\ c\epsilon^{ij}\bar{\lambda}_{\dot{\alpha}j} \end{pmatrix}, c \text{ a constant} \tag{14.8.6}$$

This spinor satisfies the following reality condition

$$\lambda^{i,T}C\epsilon_{ij} = (\lambda^\dagger)_j(i\gamma^0)\gamma_5 \tag{14.8.7}$$

To verify this relation we just write out both side of the equation

$$\begin{aligned}
\lambda^{i,T} C\gamma_5 \epsilon_{ij} &= (\lambda_{\beta j}, c\bar{\lambda}_j^{\dot{\beta}}) \\
\lambda^\dagger i\psi^0 &= (-\bar{\lambda}_j^{\dot{\alpha}}, c^* \epsilon_{4jk}\lambda_\alpha^k) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
&= (-c^* \lambda_{\alpha j}, -\bar{\lambda}_j^{\dot{\alpha}})
\end{aligned} \tag{14.8.8}$$

Clearly, for $c = -1$ we find agreement

$$\lambda^i = \begin{pmatrix} \lambda^{\alpha i} \\ -\epsilon^{ij}\bar{\lambda}_{\dot{\alpha} j} \end{pmatrix} \Leftrightarrow \lambda^{i,T} C\gamma_5 \epsilon_{ij} = \lambda_j^\dagger i\gamma^0 \tag{14.8.9}$$

[The matrix C satisfies $C\gamma^\mu C^{-1} = -\gamma^{\mu,T}$ but $C_+ \equiv C\gamma^5$ satisfies $C_+\gamma^\mu C^{-1} = +\gamma^{\mu,T}$. For that reason one sometimes denotes C by C_-. Then an ordinary Majorana spinor satisfies

$$\psi^T C_- = \psi^\dagger i\gamma^0 \tag{14.8.10}$$

while a symplectic Majorana spinor satisfies

$$\psi^{T^i} C_+ \epsilon_{ij} = \psi_j^\dagger i\gamma^0 \tag{14.8.11}$$

The extra γ_5 in C_+ compensates for the effects of the extra ϵ_{ij}. "Minus times minus is plus"].

The kinetic term can now be written as $\bar{\lambda}_i \not{D} \lambda^i$, and it is manifestly $U(2)$ invariant. Also the other terms in the action are $U(2)$ invariant.

9 The $N = 2$ hypermultiplet

The next model we discuss is the $N = 2$ hypermultiplet, the $N = 2$ generalization of the Wess Zumino model. It contains two complex scalars $\varphi_i (i = 1, 2)$ and their complex conjugates $\varphi^{*i} \equiv (\varphi_i)^*$, one complex Dirac spinor ψ, and two auxiliary fields F_i and $(F_i)^* \equiv F^{*i}$. The free field action with mass term reads

$$\begin{aligned}
\mathcal{L} = &-\partial_\mu \varphi^{*i} \partial^\mu \varphi_i - \bar{\psi}\not{\partial}\psi + F^{*i} F_i \\
&+ m(\varphi^{*i} F_i + \varphi_i F^{*i} - \bar{\psi}\psi)
\end{aligned} \tag{14.9.1}$$

The symbol $\bar{\psi}$ denotes of course $\psi^\dagger i\gamma^0$, there is no confusion possible. The susy rules read

$$\begin{aligned} \delta\varphi_i &= \bar{\epsilon}_i\psi, \delta F_i = \bar{\epsilon}_i\gamma^5\partial\!\!\!/\psi, \\ \delta\psi &= \partial\!\!\!/\varphi_i\epsilon^i + F_i(-\gamma^5)\epsilon^i; \delta\bar{\psi} = -\bar{\epsilon}_i\partial\!\!\!/\varphi^{*i} + \bar{\epsilon}_i\gamma^5 F^{*i} \end{aligned} \tag{14.9.2}$$

Note that ϵ^i is a symplectic Majorana spinor,

$$\bar{\epsilon}_i = \epsilon^{jT}C\gamma^5\epsilon_{ji} \tag{14.9.3}$$

To check these rules consider the terms in the variation of the massless action with φ^*, F^* and ψ

$$\delta\mathcal{L} = (\Box\varphi^{*i})\bar{\epsilon}_i\psi - (-\bar{\epsilon}_i\partial\!\!\!/\varphi^{*i} + \bar{\epsilon}_i F^{*i}\gamma^5)\partial\!\!\!/\psi + F^{*i}\bar{\epsilon}_i\gamma^5\partial\!\!\!/\psi \tag{14.9.4}$$

which indeed vanishes after partial integration. For the m-dependent terms we obtain

$$\frac{1}{m}\delta\mathcal{L}(m) = \varphi^{*i}\bar{\epsilon}_i\gamma^5\partial\!\!\!/\psi + \bar{\epsilon}_i\psi F^{*i} - (-\bar{\epsilon}_i\partial\!\!\!/\varphi^{*i} + \bar{\epsilon}_i F^{*i}\gamma^5)\psi = 0 \tag{14.9.5}$$

The susy commutator of φ_i yields

$$\begin{aligned} &[\delta(\epsilon_1), \delta(\epsilon_2)]\varphi_i = \bar{\epsilon}_i^{(2)}(\partial\!\!\!/\varphi_j\epsilon^{(1)j} + F_j\epsilon^{(1)j}) - 1 \leftrightarrow 2 \\ &= (\bar{\epsilon}_i^{(2)}\gamma^\mu\epsilon^{(1)j} - \bar{\epsilon}_i^{(1)}\gamma^\mu\epsilon^{(2)j})\partial_\mu\varphi_j + (\bar{\epsilon}_i^{(2)}\epsilon^{(1)j} - \bar{\epsilon}_i^{(1)}\epsilon^{(2)j})F_j \end{aligned} \tag{14.9.6}$$

For symplectic Majorana spinors $\epsilon^{(1)i}$ and $\epsilon^{(2)i}$ we have the identity

$$\begin{aligned} &\bar{\epsilon}_i^{(1)}\gamma^\mu\epsilon^{(2)j} = \epsilon^{(1)k,T}C\gamma^5\epsilon_{ki}\gamma^\mu\epsilon^{(2)j} \\ &= -\epsilon^{(2),jT}\gamma^{\mu,T}\epsilon_{ki}\gamma^{5,T}C^T\epsilon^{(1)k} \\ &= \epsilon_{ki}(\epsilon^{2,jT}\gamma^5 C\gamma^\mu\epsilon^{(1)k}) \text{ because } C^T = -C \\ &= \epsilon_{ki}\epsilon^{jj'}(\epsilon^{2,lT}\gamma^5 C\epsilon_{lj'}\epsilon^{(1)k}) \\ &= \epsilon_{ki}\epsilon^{jj'}(\bar{\epsilon}_j^2\gamma^\mu\epsilon^{(1)k}) \end{aligned} \tag{14.9.7}$$

If we then use

$$\epsilon_{ki}\epsilon^{jj'} = \delta_k^j\delta_i^{j'} - \delta_k^{j'}\delta_i^j \tag{14.9.8}$$

we see that only terms of the form $\bar{\epsilon}_j \ldots \epsilon^j$ remain. Hence (14.9.6) yields

$$[\delta(\epsilon_1), \delta(\epsilon_2)]\varphi_i = (\bar{\epsilon}_j^{(1)}\gamma^\mu \epsilon^{(1)j})\partial_\mu \varphi_i + (\bar{\epsilon}_j^{(2)}\epsilon^{(1)j})F_i \tag{14.9.9}$$

In the last term we used

$$\bar{\epsilon}_i^{(1)}\epsilon^{(2)j} = \epsilon_{ki}\epsilon^{jj'}\bar{\epsilon}_{j'}^{(2)}\epsilon^{(1)k} = \bar{\epsilon}_i^{(2)}\epsilon^{(1)j} - \bar{\epsilon}_{j'}^{2)}\epsilon^{(1)j'}\delta_i^j \tag{14.9.10}$$

The net result is the usual translation and a central charge transformations δ_z which is defined by

$$\delta_z \varphi_i = F_i, \delta_z \psi = \not{D}\psi, \delta_z F_i = \Box \varphi_i \tag{14.9.11}$$

Clearly $\delta_z^2 = \Box$, and $[\delta_z, \delta \text{ (susy)}] = 0$. For example, $[\delta_z, \delta \text{ (susy)}] \ \varphi_i = \bar{\epsilon}_i \not{\partial}\psi - \delta$ (susy) $F_i = 0$. On-shell (which is where the representation in terms of states lies) we find

$$\left.\begin{array}{ll} \delta_z \varphi_i & = -m\varphi_i \\ \delta_z \psi & = -m\psi \\ \delta_z F_i & = -mF_i \end{array}\right\} \delta_z \Phi_i = -m\Phi_i \tag{14.9.12}$$

The central charge satisfies the BPS bound, hence we have multiplet shortening.

There are no self interactions of the hypermultiplet! In $N = 1$ superspace we would need a prepotential cubic in Φ_i but since Φ_i forms an $SU(2)$ doublet we cannot construct an $SU(2)$ scalar cubic in ϕ_i. We can now write down a $S(2)$ invariant action

$$\begin{aligned} \mathcal{L} = tr \Big[& \frac{1}{2}F_{\mu\nu} + \bar{\lambda}_i \gamma^\mu D_\mu \lambda^i + (D_\mu M)^2 + (D_\mu N)^2 \\ & + \bar{\lambda}_i[\lambda^i, M] + \bar{\lambda}_i i\gamma_5[\lambda^i, N] + \frac{1}{2}[M, N]^2 \Big] \end{aligned} \tag{14.9.13}$$

The $U(1)$ transformations

$$A_\mu \to A_\mu, \lambda^i \to e^{i\varphi\gamma_5}\lambda^i, (M + iN) \to e^{-2i\varphi}(M + iN) \tag{14.9.14}$$

leave the action invariant and respect the Majorana condition

$$\begin{aligned}&\lambda^{Ti} C\gamma_5 \epsilon_{ij} \to (\lambda^{Ti} C\gamma_5 \epsilon_{ij}) e^{i\varphi\gamma_5}\\&\lambda_j^\dagger i\gamma^0 \to (\lambda_j^\dagger i\gamma^0) e^{i\varphi\gamma_5}\end{aligned} \tag{14.9.15}$$

The reason for the γ_5 in $e^{i\varphi\gamma_5}$ become clear if we go to 2-component spinors

$$\lambda^i = \begin{pmatrix} \lambda^{i\alpha} \\ \epsilon^{ij}\bar{\lambda}_{\dot{\alpha}j} \end{pmatrix} \begin{array}{l} \lambda^{i\alpha} \to e^{i\varphi}\lambda^{i\alpha} \\ \bar{\lambda}_{\dot{\alpha}j} \to e^{-i\varphi}\bar{\lambda}_{\dot{\alpha}j} \end{array} \tag{14.9.16}$$

The $SU(2)$ invariance of the action is manifest, fields transform as their indices indicate. As is well-known, $\lambda^{i\alpha}$ and $\epsilon^{ij}\bar{\lambda}_{\dot{\alpha}j}$ transform under $SU(2)$ in the same way - this was the reason for the definition in (14.9.3).

The susy transformation laws now become

$$\begin{aligned}&\delta A_\mu = \bar{\epsilon}_i \gamma_\mu \lambda^i = -\bar{\lambda}_i \gamma_\mu \epsilon^i = \text{ real}\\&\delta M = i\bar{\epsilon}_i \lambda^i = -i\bar{\lambda}_i \epsilon^i \text{ real}\\&\delta N = \bar{\epsilon}_i \gamma_5 \lambda^i = -\bar{\lambda}_i \gamma_5 \epsilon^i = \text{ real}\\&\delta\lambda^i = \gamma^{\mu\nu}\epsilon^i F_{\mu\nu} - \gamma^\mu D_\mu (M + iN\gamma^5)\epsilon^i - i\gamma^5\epsilon^i [M, N]\end{aligned} \tag{14.9.17}$$

Note that $i\gamma^5\epsilon^i$ is again a symplectic Majorana spinor, as is $i\gamma_\mu\epsilon^i$ and $\gamma_\mu\gamma_5\epsilon^i$.

An off-shell version exits; it contains a triplet of auxiliary fields $\vec{D}$ with $U(1)$ weight zero. One finds in $\delta\lambda^i$ an extra term $\vec{D}\cdot(\vec{\tau})^i{}_j\epsilon^j$. Further $\delta\vec{D}$ is proportional to the fermionic field equation, $\vec{\tau}_i{}^j\bar{\epsilon}_j\frac{\delta\mathcal{L}}{\delta\bar{\lambda}_i}$. In the action one finds a term $\frac{1}{2}(\vec{D})^2$, just as in the $N = 1$ case. Counting fields components (5 bosonic, 8 fermionic) suggests 3 bosonic auxiliary fields, and then the choice $\vec{D}$ is very natural.

As we already explained, one cannot construct an $SU(2)$ scalar which is cubic in Φ_i. We can, however, couple $N = 2$ vectors multiplets to the hypermultiplets. We can derive the action and transforamtion rules from our $N = 1$ results, or, more easily, by guessing and checking afterwards. The action is the sum of the action for an $N = 2$ vector multiplet and the following action

$$\begin{aligned}\mathcal{L}\ (\text{int}) &= -\frac{1}{2}D_\mu\varphi^{*i}D^\mu\varphi_i - \bar{\psi}\gamma^\mu D_\mu\psi + F^{*i}F_i \\ &+\varphi^{*i}\bar{\lambda}_i\psi - \bar{\psi}\lambda^i\varphi_i - \psi(M + i\gamma_5 N)\psi \\ &-\frac{1}{2}\varphi^{*i}(M^2+N^2)\varphi_i + \frac{1}{2}\varphi^{*i}\vec{D}\cdot(\vec{\tau})_i{}^j\varphi_j \end{aligned} \tag{14.9.18}$$

One can also write down a mass term

$$\mathcal{L}(m) = m(\varphi^{*i}F_i + F^{*i}\varphi_i - \bar{\psi}\psi + \varphi^{*i}M\varphi_i) \tag{14.9.19}$$

The transformation rules for A_μ, λ^i, M, N and $\vec{D}$ are unchanged, but those for the fields of the hypermultiplet receive corrections from the vector multiplet

$$\begin{aligned}\delta\varphi_i &= \bar{\epsilon}_i\psi \\ \delta\psi &= \not{D}\varphi_i\epsilon^i + \gamma^5 F_i\epsilon^i + (M + i\gamma_5 N)\epsilon^i\varphi_i \\ \delta F_i &= \bar{\epsilon}_i\gamma^5\not{D}\psi + \bar{\epsilon}_i(M - i\gamma_5 N)\gamma^5\psi + \bar{\epsilon}_i\gamma^5\lambda^j\varphi_j \end{aligned} \tag{14.9.20}$$

The $U(1)$ symmetry of the vector multiplet

$$M + iN \to e^{-2i\alpha}(M+iN);\ \lambda^i \to e^{i\alpha\gamma_5}\lambda^i;\ A_\mu \to A_\mu \tag{14.9.21}$$

is indeed lost when there is a mass, because the term $m\varphi^{*i}M\varphi_i$ breaks the $U(1)$ symmetry.

Note that the "gauged hypermultiplet" has only one coupling constant: the gauge coupling constant.

10 The $N = 4$ rigid susy model

To obtain a multiplet with $N = 4$ susy and spins $s \leq 1$, we need a massless multiplet or a multiplet with maximal central charge $z = m$. Since we are interested in a gauge theory for the vector fields, the multiplet is massless and hence there are no central charges, while $G = U(4)$. (As we shall see, G acts only as $SU(4)$ because the $U(1)$ is trivial in the $N = 4$ model). The particle content is one spin 1, 4 gauginos, 6 spin zero all in the adjoint representation. In terms of $N = 1$ multiplets we need one $N = 1$ vector multiplet and three $N = 1$ scalar multiplets.

$$(1, 1/2) + \left(\frac{1}{2}, 0, 0\right)^3 \tag{14.10.1}$$

The action for such an $N = 1$ model reads in general

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}F_{\mu\nu}^2 - \frac{1}{2}\bar{\lambda}\gamma^\mu D_\mu\lambda + \frac{1}{2}D^2 \\ &+ \sum_{i=1}^{3}\left\{-\frac{1}{2}(D_\mu A_i)^2 - \frac{1}{2}(D_\mu B_i)^2 - \frac{1}{2}\bar{\psi}_i\gamma^\mu D_\mu\psi_i + \frac{1}{2}F_i^2 + \frac{1}{2}G_i^2\right. \\ &\left.+\bar{\psi}_i[\lambda, (A_i + i\gamma_5 B_i)] + D[A_i, B_i]\right\} + V(\phi_1, \phi_2, \phi_3)\end{aligned} \tag{14.10.2}$$

Here $V(\phi_1, \phi_2, \phi_3)$ denotes all terms which follow from the superpotential.

A general $N = 2$ theory contains (i) an $N = 2$ multiplet (composed from an $N = 1$ vector multiplet and one scalar multiplet in the adjoint representation) (ii) hypermultiplets which can only couple to the $N = 2$ vector multiplet but not to themselves. Each hypermultiplet consists of a pair of $N = 1$ chiral multiplets ϕ^a and ϕ_a, with ϕ^a in some representation R of the gauge group and ϕ_a in R^*. (Note that cannot choose $(\phi^a)^*$ for ϕ_a because (ϕ^a) is not chiral). Thus one of the three chiral multipets in (14.10.2) must be in the adjoint representation, and the only gauge invariant superpotential is then

$$\begin{aligned}V(\phi_1^a, \phi_{2a}, {\phi_3}^a{}_b) &= b^a\phi_a + \phi_{a'}b^{a'} \\ &+{b_a}^b{\phi_b}^a - m\phi^a\phi_a + \lambda\phi_1^a(\phi_3)_a{}^b\phi_{2b}\end{aligned} \tag{14.10.3}$$

By requiring $N = 2$ susy one finds that only the mass term $m\phi^a\phi_a$ for the hypermultiplet and the trilinear coupling $\lambda\phi_1\phi_3\phi_2$ is allowed, and $\lambda = g$. Hence, $N = 2$ **susy allows only one coupling constant**.

The trilinear coupling $\phi_1^a(\phi_3)_a{}^b\phi^{2b}$ yields the minimal couplings of the $N = 1$ gaugino and auxiliary field D to the two $N = 1$ matter multiplets. In the $N = 4$

model all three $N=1$ chiral multiplets must occur symmetrically: they are all in the adjoint representation, hence the $N=1$ superpotential can be written as[15]

$$V = tr\phi_1[\phi_2,\phi_3], \phi_j \equiv \phi_j^a T_a \tag{14.10.4}$$

Hence the $N=4$ model (if it exists) should be given by an $N=1$ model with an $N=1$ vector multiplet coupled to three $N=1$ chiral multiplets ϕ_1,ϕ_2,ϕ_3, all the adjoint representations, and with a superpotential $\phi_1,[\phi_2,\phi_3]$. To demonstrate that this model has $N=4$ susy, we write the superpotential as $\epsilon^{ijk}tr\phi_i\phi_2\phi_3$ and find for the action

$$\begin{aligned}\mathcal{L} = tr\Bigg(&-\frac{1}{4}G_{\mu\nu}^2 - \frac{1}{2}\bar{\lambda}\gamma^\mu D_\mu\lambda + \frac{1}{2}D^2 \\ &+\sum_{i=1}^{3}\left[-\frac{1}{2}(D_\mu A_i)^2 - \frac{1}{2}(D_\mu B_i)^2 - \frac{1}{2}\bar{\psi}_i\gamma^\mu D_\mu\psi_i + \frac{1}{2}F_i^2 + \frac{1}{2}G_i^2\Bigg) \\ &+\bar{\psi}_i[\lambda,(A_i+i\gamma_5 B_i)] + D[A_i,B_i]\right] \\ &+\sum_{ijk}\epsilon_{ijk}\Big(\bar{\psi}_i[\psi_j,(A_k+i\gamma_5 B)] + F_i([A_j,A_k]+[B_j,B_k]) + 2G_i[A_j,B_k]\Big)\end{aligned} \tag{14.10.5}$$

To bring out the $O(4)$ symmetry of this model we define

$$\begin{aligned}&\lambda = \{\lambda_i = \psi_i \text{ for } i=1,2,3; \lambda_4 = \lambda\} \\ &A_{AB} = \{A_{ij} \equiv -\epsilon_{ijk}A_k, A_{4i} = A_i\} : \text{ selfdual} \\ &B_{AB} = \{B_{ij} = \epsilon_{ijk}B_k, B_{4i} = B_i : \text{ anti selfdual}\end{aligned} \tag{14.10.6}$$

Eliminating the auxiliary fields yields the following action.

$$\mathcal{L} = tr\Big(-\frac{1}{4}F_{\mu\nu}^2 - \frac{1}{2}\bar{\lambda}_A\gamma^\mu D_\mu\lambda_A - \frac{1}{8}(D_\mu A_{ij})(D^\mu A_{ij}) - \frac{1}{8}(D_\mu B_{ij})(D_\mu B_{ij})$$

[15] For $SU(N)$ gauge groups one may prefer to write $\phi_j^a T_a$ as $(\phi_j)^A{}_B T^B{}_A$ where $T^B{}_A$ denote the generators in any representation. For the fundamental representation $T^B{}_A$ has matrix elements

$$(T^B{}_A)^C{}_D = \delta_A{}^C\delta^B{}_D$$

Then $\phi_j^a(T_a)^C{}_D$ becomes $\phi_j{}^C{}_D$.

$$+\lambda_A[\lambda_B, A_{AB} + i\gamma_5 B_{AB}] + \frac{1}{32}[A_{ij}, B_{kl}]^2$$
$$+\frac{1}{64}[A_{ij}, A_{kl}]^2 + \frac{1}{64}[B_{ij}, B_{kl}]^2\Big) \qquad (14.10.7)$$

The $N = 4$ susy laws with parameter ϵ_A are now

$$\delta A_\mu = \bar{\epsilon}_A \gamma_\mu \lambda_A$$
$$\delta A_{AB} = \bar{\epsilon}_A \lambda_B - \bar{\epsilon}_B \lambda_A + \epsilon_{ABCD} \bar{\epsilon}_C \lambda_D$$
$$\delta B_{AB} = -i\bar{\epsilon}_A \gamma_5 \lambda_B + i\bar{\epsilon}_B \gamma_5 \lambda_A - i\epsilon_{ABCD} \bar{\epsilon}_C \gamma_5 \lambda_D$$
$$\delta \lambda_A = \gamma^{\mu\nu} \epsilon_A F_{\mu\nu} + \gamma^\mu D_\mu (A_{AB} + i\gamma_5 B_{AB}) \epsilon_B$$
$$+[A_{AB} - i\gamma_5 B_{AB}, A_{BC} + i\gamma_5 B_{BC}]\epsilon_C \qquad (14.10.8)$$

In two-component (chiral) notation the $SU(4)$ symmetry becomes clear. We define

$$\lambda_A^\alpha = \begin{pmatrix} \lambda_A^\alpha \\ \bar{\lambda}_{\dot{\alpha}}^A \equiv (\lambda_{\alpha A})^\dagger \end{pmatrix}; \phi_{AB} = A_{AB} + iB_{AB} \qquad (14.10.9)$$

Note that ϕ_{AB} satisfies a reality condition

$$\phi^{AB} \equiv (\phi_{AB})^\dagger = \frac{1}{2}\epsilon^{ABCD}\phi_{CD} \qquad (14.10.10)$$

The $N = 4$ action becomes then simply

$$\mathcal{L} = tr\Big(-\frac{1}{4}F_{\mu\nu}^2 - \bar{\lambda}_{\dot{\alpha}}^A (\sigma^\mu)^{\alpha\dot{\alpha}} \lambda_\alpha - \frac{1}{2}(D_\mu \phi_{AB})(D_\mu \phi^{AB})$$
$$+\lambda_A^\alpha [\lambda_{B,\alpha}, \phi^{AB}] + \bar{\lambda}^{A\dot{\alpha}} [\bar{\lambda}_{\dot{\alpha}}^B, \phi_{AB}] + \frac{1}{4}[\phi_{AB}, \phi_{CD}][\phi^{AB}, \phi^{CD}]\Big) \qquad (14.10.11)$$

The $SU(4)$ covariance is manifest. The transformation laws read

$$\delta A_\mu = \epsilon^A \sigma_\mu \bar{\lambda}_A - \bar{\epsilon}_A \sigma_\mu \lambda^A$$
$$\delta \phi_{AB} = \epsilon_A^\alpha \lambda_{B\alpha} - \epsilon_B^\alpha \lambda_{A\alpha} + \epsilon_{ABCD} \bar{\epsilon}^{C\dot{\alpha}} \bar{\lambda}_{\dot{\alpha}}^D$$
$$\delta \lambda_A^\alpha = \sigma^{\mu\nu} \epsilon_A^\alpha F_{\mu\nu} + \sigma^{\mu\alpha\dot{\beta}} (D_\mu \phi_{AB}) \bar{\epsilon}_{\dot{\beta}}^B$$
$$+[M_{AB}, M^{BC}]\epsilon_C^\alpha \qquad (14.10.12)$$

There is no $U(1)$ symmetry, hence the symmetry group of this action is $SU(4)$, not $U(4)$. The $U(1)$ symmetry of the vector multiplet

$$\begin{aligned}(M+iN) &\rightarrow e^{-2i\alpha}(M+iN)\\ \lambda &\rightarrow e^{i\alpha\gamma_5}\lambda\\ A_\mu &\rightarrow A_\mu\end{aligned}\tag{14.10.13}$$

cannot be extended to the $N=4$ action since the readily condition

$$\phi^{AB}=(\phi_{AB})=\frac{1}{2}\epsilon^{ABCD}\phi_{CD}\tag{14.10.14}$$

excludes a phase transformation. A phase transformation on $\lambda^{A\alpha}$ alone is also ruled out because it would not leave the Yukawa couplings invariant.

11 $N=2$ superspace

The coordinates of $N=2$ superspace are $x^{\alpha\dot{\alpha}}\equiv x^\mu\sigma_\mu^{\alpha\dot{\alpha}}$ and $\theta^{a\alpha},\bar{\theta}_a^{\dot{\alpha}}$, where the indices $a=1,2$ are indices[16] of the fundamental representation of the rigid symmetry group

[16] A few words about the indices a,α and $\dot{\alpha}$. Given a representation R of a Lie algebra, we can always construct two other representations, the complex conjugate representation R^* and the transposed representation $-R^T$. For $SU(2)$ all representations are pseudoreal or real ($R^*=SRS^{-1}$). It is customary to denote the fundamental representation of a compact group by v^a and the complex conjugate representation by v_a^* (where $v_a^*=(v^a)^*$).

$$\delta v_a=-v_b s^b{}_a,\ \delta(v^a)^*=(s^a{}_b)^*(v^b)^*$$

Clearly $(\theta^a)^*$ transforms like θ_a under $SU(2)$, and for that matter we write $(\theta^a)^*$ as $\bar{\theta}_a$. For the $U(1)$ part of $U(2),\theta_a=\theta^b\epsilon_{ba}$ and $\bar{\theta}_a=(\theta^a)^*$ transform oppositely, and by putting a bar on θ we indicate its $U(1)$ transformation properly.

For $Sl(2,C)$, the fundamental representation is $\delta v^\alpha=l^\alpha{}_\beta v^\beta$ with $l=i\vec{\omega}\cdot\vec{\sigma}+\vec{\Omega}\cdot\vec{\sigma}$ with 6 real parameters $\vec{\omega}$ and $\vec{\Omega}$. Now $-l^T$ and l^* are not equivalent, hence we distinguish between $\theta_\alpha=\theta^\beta\epsilon_{\beta\alpha}$ and $(\theta^\alpha)^*=\bar{\theta}^{\dot{\alpha}}$. The bar on θ is not necessary to distinguish between θ_α and $\theta^{\dot{\alpha}}$, but it is customary to write $\bar{\theta}^{\dot{\alpha}}$, and in the N=2 case it becomes even necessary in order to indicate the U(1) transformation. The representations θ^α and θ_α are again equivalent (because $\sigma_2\vec{\sigma}\sigma_2^{-1}=-\vec{\sigma}^T$), and $\bar{\theta}^{\dot{\alpha}}$ is equivalent to $\bar{\theta}_{\dot{\alpha}}$, but θ^α and $\bar{\theta}_{\dot{\alpha}}$ are inequivalent.

$U(2)$. The 2x2 matrices $\sigma^{\mu,\alpha\dot{\alpha}}$ are given by $\{I, \vec{\sigma}\}^{\alpha\dot{\alpha}}$ and the Majorana condition on the θ's reads[17]

$$(\theta^a_\alpha)^* = \bar{\theta}_{a\dot{\alpha}}; (\theta^{a\alpha})^* = -\bar{\theta}^{\dot{\alpha}}_a \tag{14.11.1}$$

As a consequence the matrix $\sigma_{\mu,\alpha\dot{\beta}}$ is then given by

$$\sigma_{\mu,\alpha\dot{\beta}} = \sigma^{\nu,\gamma\dot{\delta}}\eta_{\nu\mu}\epsilon_{\gamma\alpha}\epsilon_{\dot{\delta}\dot{\beta}} = -\sigma_2(-I, \vec{\sigma})\sigma_2 = (I, \vec{\sigma}^{,T})_{\alpha\dot{\beta}}$$

It is then clear that

$$\bar{\sigma}_{\mu,\dot{\alpha}\beta} \equiv (\sigma_{\mu,\alpha\dot{\beta}})^* = (\sigma^T_\mu)_{\dot{\alpha}\beta} = (I, \vec{\sigma})_{\dot{\alpha}\beta}$$

From $\{\partial/\partial\theta^{a\alpha}, \theta^{a\alpha}\} = 1$ (no sum) it follows that $(\partial/\partial\theta^{a\alpha})^\dagger = -\partial/\partial\bar{\theta}^{\dot{\alpha}}_a$, just as from $[\partial/\partial x^\mu, x^\mu] = 1$ (no sum) it follows that $(\partial/\partial x^\mu)^\dagger = -\partial/\partial x^\mu$. Hence $(x^{\alpha\dot{\beta}})^* = x^{\beta\dot{\alpha}}$.

The $N = 2$ rigid susy-covariant derivatives $D_{a\alpha}$ are defined by

$$\begin{aligned} D_{a\alpha} &= \partial/\partial\theta^{a\alpha} - i\sigma^\mu_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}_a\partial_\mu \\ \bar{D}^a_{\dot{\alpha}} &= \partial/\partial\bar{\theta}^{\dot{\alpha}}_a - i\sigma^\mu_{\beta\dot{\alpha}}\theta^{a\beta}\partial_\mu \end{aligned} \tag{14.11.2}$$

Clearly $\{D_{a\alpha}, D_{b\beta}\} = 0$ and $\{D_{a\alpha}, \bar{D}^b_{\dot{\beta}}\} = -2i\delta^a_b\partial_{\alpha\dot{\beta}}$. From $(\partial/\partial\theta^{a\alpha})^\dagger = -\partial/\partial\bar{\theta}^{\dot{\alpha}}_a$ and $(\partial_{\alpha\dot{\beta}})^\dagger = -\partial_{\beta\dot{\alpha}}$ we find the reality relations

$$(D_{a\alpha})^\dagger = -\bar{D}^a_{\dot{\alpha}},\ (D^\alpha_a)^\dagger = \bar{D}^{a\dot{\alpha}} \tag{14.11.3}$$

[17]Spinor indices are raised and lowered by $\epsilon^{\alpha\beta}$ and $\epsilon_{\dot{\alpha}\dot{\beta}}$ and numerically $\epsilon^{\alpha\beta} = \epsilon_{\alpha\beta} = -\epsilon^{\dot{\alpha}\dot{\beta}} = -\epsilon_{\dot{\alpha}\dot{\beta}}$. Thus we consider four spinors: $\theta^\alpha, \theta_\alpha = \theta^\beta\epsilon_{\beta\alpha}, \bar{\theta}^{\dot{\alpha}} = -(\theta^\alpha)^*$ and $\bar{\theta}_{\dot{\alpha}} = \bar{\theta}^{\dot{\beta}}\epsilon_{\dot{\beta}\dot{\alpha}} = (\theta_\alpha)^*$. Four-component complex (Dirac) spinors $\lambda^{\tilde{\alpha}}$ have given by $(\zeta^\alpha, \bar{\eta}_{\dot{\alpha}})$. With this definition, ζ and $\bar{\eta}$ transform the same way under the rotation subgroup $SU(2)$ of $Sl(2, C)$. The Dirac matrices are

$$\gamma^\mu = \begin{pmatrix} 0 & -i\sigma^\mu \\ i\bar{\sigma}^\mu & 0 \end{pmatrix};\ \begin{matrix} \sigma^\mu = (I, \vec{\sigma})^{\alpha\dot{\beta}} \\ \bar{\sigma}^\mu = (-I, \vec{\sigma})_{\dot{\alpha}\beta} \end{matrix}\ ; \gamma^5 \equiv \gamma^1\gamma^2\gamma^3\gamma^4 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The charge conjugation matrix C, satisfies $C\gamma^\mu C^{-1} = -\gamma^{\mu,T}$ and is proportional to $\gamma^4\gamma^2$ (because γ^1, γ^3 are antisymmetric, hence $C\gamma^1 = \gamma^1 C$ and $C\gamma^3 = \gamma^3 C$). We take it equal to $\gamma^4\gamma^2$. The Majorana condition $(\lambda)^\dagger i\gamma^0 = \lambda^T C$ with $C_{\tilde{\alpha}\tilde{\beta}} = \begin{pmatrix} \epsilon_{\alpha\beta} & 0 \\ 0 & \epsilon^{\dot{\alpha}\dot{\beta}} \end{pmatrix}$ leads then to (1).

In the $N = 2$ case we begin by introducing gauge-covariant susy-covariant derivatives $\nabla_{a\alpha}$, and postulate $SU(2) \otimes Sl(2,C)$ invariant constraints[18]

$$\{\nabla_{a\alpha}, \nabla_{b\beta}\} = i\epsilon_{\alpha\beta}\epsilon_{ab}\bar{W};\ \{\nabla_{a\alpha}, \bar{\nabla}^b_{\dot{\beta}}\} = -2i\delta^b_a \nabla_{\alpha\dot{\beta}} \tag{14.11.4}$$

We also require that $\nabla_{a\alpha}$ and $\nabla_{\alpha\dot{\alpha}}$ satisfy the same reality conditions as $D_{a\alpha}$ and $\partial_{\alpha\dot{\alpha}}$. (This means that we are working in the vector representation, as we shall later see.) The first constraint is necessary because we want to identify covariant derivations of $N = 1$ superspace with $\nabla_{1\alpha}$. So for $a = b = 1$ the first anti-commutator should vanish which requires the ϵ_{ab}. Then the right-hand side is antisymmetric in α, β hence we also need the $\epsilon_{\alpha\beta}$ in (4). Taking the hermitian conjugate of the first anticommutator shows that

$$\{\bar{\nabla}^a_{\dot{\alpha}}, \bar{\nabla}^b_{\dot{\beta}}\} = i\epsilon_{\dot{\alpha}\dot{\beta}}\epsilon^{ab}W,\ \boxed{(\bar{W})^\dagger = W} \tag{14.11.5}$$

where we used that numerically $\epsilon_{\dot{\alpha}\dot{\beta}} = -\epsilon_{\alpha\beta}$. We use antihermitian generators T_A for the gauge algebra, hence $\bar{W} = -(W^I)^\dagger T_I$.

In general torsions and curvatures are defined by

$$\{\nabla_{aA}, \nabla_{bB}\} = T^{cC}_{aA,bB}\nabla_{cC} + F^A_{aA,bB}T_A \tag{14.11.6}$$

where T_A are the generators of the group and A equals $\alpha, \dot{\alpha}$ or m. Hence the anti-commutators we imposed imply constraints on the curvatures.

From the Bianchi identity $[\{\nabla_{a\alpha}, \nabla_{b\beta}\}, \nabla_{c\gamma}]+$ cyclic $= 0$ we see that W is chiral (and $\bar{W}$ antichiral)

$$\boxed{\bar{\nabla}_{a\dot{\alpha}}W = 0,\ \nabla_{a\alpha}\bar{W} = 0} \tag{14.11.7}$$

Next consider the other Bianchi identity with 3 spinor derivatives

$$\begin{aligned}
[\{\nabla_{a\alpha}, \nabla_{b\beta}\}, \bar{\nabla}^c_{\dot{\gamma}}] &= [\nabla_{a\alpha}, \{\nabla_{b\beta}, \bar{\nabla}^c_{\dot{\gamma}}\}] + [\nabla_{b\beta}, \{\nabla_{a\alpha}, \bar{\nabla}^c_{\dot{\gamma}}\}] \\
-i\epsilon_{ab}\epsilon_{\alpha\beta}(\bar{\nabla}^c_{\dot{\gamma}}\bar{W}) &= [\nabla_{a\alpha}, -2i\delta^c_b\nabla_{\beta\dot{\gamma}}] + [\nabla_{b\beta}, -2i\delta^c_a\nabla_{\alpha\dot{\gamma}}]
\end{aligned} \tag{14.11.8}$$

[18] The first term in (4) is even invariant under $Sl(2,C) \otimes Sl(2,C)$ because ϵ_{ab} is an $Sl(2,C)$ invariant tensor, but the second term is only $U(2)$ invariant because δ^b_a is not an $Sl(2,C)$ invariant tensor.

Contraction with ϵ^{ab} and $\epsilon^{\alpha\beta}$ yields

$$\bar{\nabla}^c_{\dot{\gamma}}\bar{W} = [\nabla^{c\beta}, \nabla_{\beta\dot{\gamma}}]$$
$$\nabla_{c\gamma}W = [\bar{\nabla}^{\dot{\beta}}_c, \nabla_{\gamma\dot{\beta}}] \tag{14.11.9}$$

We substitute this result into the following Bianchi identity with one vector, one chiral and one antichiral derivative

$$\begin{aligned} 0 &= -2i\delta^\alpha_b[\nabla^{\alpha\dot{\alpha}}, \nabla_{\alpha\dot{\alpha}}] = [\{\nabla^\alpha_a, \bar{\nabla}^{b\dot{\alpha}}\}, \nabla_{\alpha\dot{\alpha}}] \\ &= \{\nabla^\alpha_a, [\bar{\nabla}^{b\dot{\alpha}}, \nabla_{\alpha\dot{\alpha}}]\} + \{\bar{\nabla}^{b\dot{\alpha}}, [\nabla^\alpha_a, \nabla_{\alpha\dot{\alpha}}]\} = \nabla^\alpha_a\nabla^b_\alpha W + \bar{\nabla}^{b\dot{\alpha}}\bar{\nabla}_{a\dot{\alpha}}\bar{W} \end{aligned} \tag{14.11.10}$$

Hence we find a second constraint on W

$$\boxed{\nabla^2{}_{ab}W = -\bar{\nabla}^2{}_{ba}\bar{W}} \tag{14.11.11}$$

where $\nabla^2{}_{ab} \equiv \nabla^\alpha_a\nabla_{b\alpha}$ and $\bar{\nabla}^2{}_{ba} \equiv \bar{\nabla}^{\dot{\alpha}}_b\bar{\nabla}_{a\dot{\alpha}}$.

Let us now find the $N=1$ content of the $N=2$ field strength W. We identify $D_{1\alpha}$ with the $N=1$ derivative D_α. Since $D_{1\alpha} = \partial/\partial\theta^{1\alpha} - i\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}_1\partial_\mu$, the natural $N=1$ coordinates are $\theta^{1\alpha}$ and $\bar{\theta}^{\dot{\alpha}}_1$. Hence, truncation from $N=2$ coordinates to $N=1$ coordinates means setting $\theta^{2\alpha}$ and $\bar{\theta}^{\dot{\alpha}}_2$ equal to zero. This we indicate by the symbol $|_{\theta^2}$, or just $|$.

We define the following $N=1$ superfields

$$W_{|\theta^2} \equiv \phi;\ \nabla_{2\alpha}W_{|\theta^2} \equiv W_\alpha \quad\Rightarrow\quad \bar{\nabla}^2_{\dot{\alpha}}\bar{W}_{|\theta^2} = \bar{W}_{\dot{\alpha}} = (W_\alpha)^\dagger \tag{14.11.12}$$

We shall presently show that there are no further $N=1$ superfields in W. The $N=1$ superfield ϕ is $N=1$ chiral, $\bar{\nabla}^1_{\dot{\alpha}}\phi = 0$, because W is $N=2$ chiral, $\bar{\nabla}_{a\dot{\alpha}}W = 0$, so $\bar{\nabla}^1_{\dot{\alpha}}W = 0$. Since $\bar{\nabla}^1_{\dot{\alpha}}$ does not depend on $\theta^{2\alpha}$ and $\bar{\theta}^{\dot{\alpha}}_2$, we have $(\bar{\nabla}^1_{\dot{\alpha}}W)_| = \bar{\nabla}_{\dot{\alpha}}(W_|) = 0$. The $N=1$ superfield W_α is also $N=1$ chiral, $\bar{\nabla}^1_{\dot{\alpha}}W_\alpha = 0$. This follows from $\{\nabla_{2\alpha}, \bar{\nabla}^1_{\dot{\alpha}}\} = 0$, which implies $\bar{\nabla}^1_{\dot{\alpha}}\nabla_{2\alpha}W = 0$, hence $0 = (\bar{\nabla}^1_{\dot{\alpha}}\nabla_{2\alpha}W)_| = \bar{\nabla}^1_{\dot{\alpha}}(\nabla_{2\alpha}W)_| = \bar{\nabla}_{\dot{\alpha}}W_\alpha$.

There are no further $N=1$ components in W. This follows from the constraint $\nabla^2_{ab}W = -\bar{\nabla}^2_{ba}\bar{W}$ as we now show. First we take $a=1, b=2$ in this relation

$$\begin{aligned} \nabla^2_{12}W = \nabla^{\alpha}_1(\nabla_{2\alpha}W) &= -\bar{\nabla}^{\dot{\alpha}}_2(\bar{\nabla}_{1\dot{\alpha}}\bar{W}) = \bar{\nabla}^{1\dot{\alpha}}(\bar{\nabla}^2_{\dot{\alpha}}\bar{W}) \\ \nabla^{\alpha}W_{\alpha} &= \bar{\nabla}^{\dot{\alpha}}\bar{W}_{\dot{\alpha}} \end{aligned} \quad (14.11.13)$$

This is the Bianchi identity of $N=1$ superspace. So we now know that W contains a vector and a chiral multiplet.

Next consider the case with $a=b=2$

$$\nabla^{\alpha}_2\nabla_{2\alpha}W = -\bar{\nabla}^{1\dot{\alpha}}\bar{\nabla}^1_{\dot{\alpha}}\bar{W} \quad \Rightarrow \quad \nabla^{\alpha}_2\nabla_{2\alpha}W_| = -\bar{\nabla}^{\dot{\alpha}}\bar{\nabla}_{\dot{\alpha}}\bar{\phi} \quad (14.11.14)$$

This shows that $\nabla_{2\alpha}\nabla_{2\beta}W = -\frac{1}{2}\epsilon_{\alpha\beta}(\nabla^{\gamma}_2\nabla_{2\gamma})W$ contains no new $N=1$ superfields.

The susy transformations in $N=2$ superspace are

$$\delta W = (\epsilon^{a\alpha}\nabla_{a\alpha} + \bar{\epsilon}^{\dot{\alpha}}_a\bar{\nabla}^a_{\dot{\alpha}})W \quad (14.11.15)$$

Hence the $N=1$ superfields ϕ and W_{α} transforms as usual under $N=1$ transformations[19]

$$\begin{aligned} \delta_1\phi &= (\epsilon^{1\alpha}\nabla_{1\alpha} + \bar{\epsilon}^{\dot{\alpha}}_1\bar{\nabla}^1_{\dot{\alpha}})W_| = \epsilon^{1\alpha}\nabla_{1\alpha}\phi \\ \delta_1 W_{\beta} &= (\epsilon^{1\alpha}\nabla_{1\alpha} + \bar{\epsilon}^{\dot{\alpha}}_1\bar{\nabla}^1{}_{\dot{\alpha}})W_{\beta|} = \epsilon^{1\alpha}\nabla_{\alpha}W_{\beta} \end{aligned} \quad (14.11.16)$$

The $N=2$ transformations (with $\epsilon^{2\alpha}$ and $\bar{\epsilon}^{\dot{\alpha}}_2$) act on the $N=1$ superfields as follows

$$\begin{aligned} &\delta_2\phi = [(\epsilon^{2\alpha}\nabla_{2\alpha} + \bar{\epsilon}^{\dot{\alpha}}_2\bar{\nabla}^2_{\dot{\alpha}})W]_| = \epsilon^{\alpha}_2 W_{\alpha} \\ &\delta_2 W_{\beta} = [(\epsilon^{2\alpha}\nabla_{2\alpha} + \bar{\epsilon}^{\dot{\alpha}}_2\bar{\nabla}^2_{\dot{\alpha}})\nabla_{2\beta}W]_| = [\epsilon^{2\alpha}(-\frac{1}{2})\epsilon_{\alpha\beta}(\nabla^2_{22}W) \\ &+\bar{\epsilon}^{\dot{\alpha}}_2(-2i\nabla_{\beta\dot{\alpha}})W_| = \frac{1}{2}\epsilon^2_{\beta}\bar{\nabla}^{1\dot{\alpha}}\bar{\nabla}^1_{\dot{\alpha}}\bar{\phi} + 2i\bar{\epsilon}_{2\dot{\alpha}}\nabla_{\beta}{}^{\dot{\alpha}}\phi \end{aligned} \quad (14.11.17)$$

[19]In 4-component spinor notation the contraction reads $\epsilon^{\tilde{\alpha}}Q_{\tilde{\alpha}}$ with $\epsilon^{\tilde{\alpha}} = (\epsilon^{\alpha}, \bar{\epsilon}_{\dot{\alpha}})$ and $Q_{\tilde{\beta}} = Q^{\tilde{\alpha}}C_{\tilde{\alpha}\tilde{\beta}}, = (Q^{\alpha}\epsilon_{\alpha\beta}, Q_{\dot{\alpha}}\epsilon^{\dot{\alpha}\dot{\beta}}) = (Q_{\beta}, -Q^{\dot{\beta}})$. In two-component spinor notation one obtains then $\epsilon^{\alpha}Q_{\alpha} - \epsilon_{\dot{\alpha}}Q^{\dot{\alpha}} = \epsilon^{\alpha}Q_{\alpha} + \epsilon^{\dot{\alpha}}Q_{\dot{\alpha}}$. Note that $\epsilon^{a\alpha}\nabla_{a\alpha} + \bar{\epsilon}^{\dot{\alpha}}_a\bar{\nabla}^a_{\dot{\alpha}}$ is antihermitian, so that also $\bar{\phi}$ and $\bar{W}_{\beta}$ transform as in (16) and (17).

where we used $\nabla^2_{ab} W = -\bar{\nabla}^2_{ba} \bar{W}$. Thus W and ϕ transform into each other. Note that we are still in the vector representation.

A natural candidate superfield for the $N = 2$ gauge action is the field strength W. Since it is chiral, we need a chiral measure $d^4x d^4\theta$ (where $d^4\theta = d^2\theta^1 d^2\theta^2$). The measure has dimensions $-4+2=-2$, and since W has dimensions $+1$ as follows from (4), we need two W's. This yields

$$S \text{ (gauge) } = \int d^4x d^4\theta \, Tr \frac{1}{4g^2} W^2 \tag{14.11.18}$$

(The factor $(4g^2)^{-1}$ is conventional). To rewrite this as the $N = 1$ action, use $d^2\theta^2 = D_2^\alpha D_{2\alpha}$ and then use gauge invariance of TrW^2 to replace this by $\nabla_2^\alpha \nabla_{2\alpha}$ to obtain

$$\begin{aligned}
S \text{ (gauge)} &= \int d^4x d^2\theta^1 \frac{2}{4g^2} Tr[(\nabla_2^\alpha W)(\nabla_{2\alpha} W) + W \nabla_{22}^2 W] = \\
&= \int d^4x d^2\theta^1 \frac{1}{2g^2} Tr[W^\alpha W_\alpha - \phi \bar{\nabla}^2 \bar{\phi}] \\
&= \int d^4x d^2\theta^1 \frac{1}{2g^2} Tr[W^\alpha W_\alpha - \bar{\nabla}^2 (\phi \bar{\phi})] \\
&= \int d^4x d^2\theta \frac{1}{2g^2} Tr W^\alpha W_\alpha - \frac{1}{2g^2} \int d^4x d^4\theta Tr \bar{\phi} \phi
\end{aligned} \tag{14.11.19}$$

These equations are valid in the vector representation where $\bar{\phi}^v = (\phi^v)^\dagger$. The chiral representation is obtained by a similarity transformation.[20] Since we shall often work in the chiral representation we introduce a new symbol

$$\tilde{\bar{\phi}} \equiv e^{-V} \bar{\phi}_{ch} e^V, \nabla_\alpha \tilde{\bar{\phi}} = 0, D_\alpha \bar{\phi}_{ch} = 0 \tag{14.11.20}$$

The $\bar{\phi}^{(v)} \phi^{(v)}$ in (19) can be rewritten as $\tilde{\bar{\phi}} \phi^{ch} = e^{-V} \bar{\phi}^{ch} e^V \phi^{ch}$ with $\bar{\phi}^{ch} = (\phi^{ch})^\dagger$. Thus we obtain the usual $N = 1$ action for a Yang-Mills and a chiral matter multiplet.

---

[20] Recall that in $N = 1$ superspace $e^V = e^\Omega (e^\Omega)^\dagger = e^\Omega e^{\bar{\Omega}}$ with $(e^\Omega)' = e^{\bar{\lambda}} e^\Omega e^{-K}$ and $(e^{-\bar{\Omega}})' = e^\lambda e^{-\bar{\Omega}} e^{-K}$. Further, $e^{\bar{\Omega}} = (e^\Omega)^\dagger$. In the vector representation

$$\begin{aligned}
&\phi^{(v)} = (e^{\bar{\Omega}} \phi^{ch}) = e^{\bar{\Omega}} \phi^{ch} e^{-\bar{\Omega}} \; ; \; \bar{\phi}^{(v)} = (\bar{\phi}^{ch} e^\Omega) = e^{-\Omega} \bar{\phi}^{ch} e^\Omega \\
&\bar{\nabla}^{(v)}_{\dot{\alpha}} = e^{\bar{\Omega}} D_{\dot{\alpha}} e^{-\bar{\Omega}} \; ; \; \nabla^{(v)}_\alpha = e^{-\Omega} D_\alpha e^\Omega = e^{\bar{\Omega}} \nabla^{ch}_\alpha e^{-\bar{\Omega}} = e^{\bar{\Omega}} (e^{-V} D_\alpha e^{+V}) e^{-\bar{\Omega}}
\end{aligned}$$

Clearly $\bar{\nabla}^{(v)}_{\dot{\alpha}} \phi^{(v)} = e^{\bar{\Omega}} D_{\dot{\alpha}} \phi^{ch} = 0$ and $\nabla^{(v)} \bar{\phi}^{(v)} = e^{-\Omega} D_\alpha \bar{\phi}^{ch} e^\Omega = 0$. Thus we find in the action $\bar{\phi}_v \phi_v = \bar{\phi}_{ch} e^\Omega e^{\bar{\Omega}} \phi_{ch} = \bar{\phi}_{ch} e^V \phi_{ch}$.

If there is a $U(1)$ factor in the gauge group one can also write down two Fayet-Iliopoulos terms. In $N = 1$ language they are[21]

$$\int d^2\theta\ \phi + h.c.,\ \int d^2\theta d^2\bar{\theta}\ V \tag{14.11.21}$$

To show that $\int d^2\theta\ \phi$ is $N = 2$ invariant we subtitute $\delta\phi = \epsilon^\alpha W_\alpha$. We obtain then $\delta S = \int d^4x d^2\theta \epsilon^\alpha \bar{D}^2 D_\alpha V = \int d^4x d^2\theta \bar{D}^2(\epsilon^\alpha D_\alpha V)$ which clearly vanishes.

To show that also the second term is $N = 2$ invariant for abelian V, we use $\int d^2\theta d^2\bar{\theta} V = D^\alpha \bar{D}^2 D_\alpha V = D^\alpha W_\alpha$ and then $\delta \int d^4x d^4\theta V$ becomes equal to $\int d^4x D^\alpha \delta W_\alpha$. Recalling that under $N = 2$ susy $\delta W_\alpha \sim \epsilon_\alpha \bar{\nabla}^{\dot{\alpha}} \bar{\nabla}_{\dot{\alpha}} \bar{\phi} + \epsilon_{\dot{\alpha}} \nabla_\alpha{}^{\dot{\alpha}} \phi$ we can commute the ∇^α to $\bar{\phi}$ to obtain zero. (For an abelian group $\nabla^{\alpha\dot{\alpha}}$ is equal to $\partial^{\alpha\dot{\alpha}}$).

We come now to the hypermultiplet. In $N = 2$ superspace it is difficult to describe, but in $N = 1$ superspace it consists of two chiral multiplets Q and $\widetilde{Q}$ in complex-conjugate respresentations of the gauge group. The action in $N = 1$ superspace reads in the chiral basis

$$\begin{aligned} S\ (\text{hyper})\ &= \int d^4x d^2\theta d^2\bar{\theta} [\bar{Q} e^V Q + \widetilde{Q} e^{-V} \overline{\widetilde{Q}}] \\ &+ \int d^4x d^2\theta (\widetilde{Q}\phi Q + h.c.) + \int d^4x d^2\theta (m\widetilde{Q} Q + c.c.) \end{aligned} \tag{14.11.22}$$

Gauge invariance follows from

$$Q' = e^\lambda Q, (e^V)' = e^{\bar{\lambda}} e^V e^{-\lambda}, \widetilde{Q}' = \widetilde{Q} e^{-\lambda}, \phi' = e^\lambda \phi e^{-\lambda} \tag{14.11.23}$$

($\widetilde{Q}$ is in the complex-conjugate representation, $\delta\widetilde{Q} = \lambda^a (T_a)^*$. So $\widetilde{Q}^T = -\lambda^a T_a$. So in $\widetilde{Q} e^{-V}$ the matrices $\exp -\lambda$ and $\exp \lambda$ cancel each other).

The $N = 1$ susy is manifest, but for the $N = 2$ susy we must determine δe^V. It is useful to introduce a superfield parameter satisfying $\partial_{\alpha\dot{\alpha}} \epsilon^\alpha = \bar{D}_{\dot{\alpha}} \epsilon = \bar{D}^2 \epsilon = 0$ and identify $\epsilon_\alpha = D_\alpha \epsilon$. (There will also be new symmetries with parameters $D^2\epsilon$ and ϵ which correspond to R symmetry and a central charge as we shall discuss).

[21] Recall that the action $\int d^4x d^4\theta\ V$ in $N = 1$ superspace is gauge invariant for abelian groups only, since $\delta V = \lambda - \bar{\lambda}$ with (anti) chiral $\bar{\lambda}(\lambda)$.

The transformation law of V under $N = 2$ susy.

We define in the chiral representation with $\phi = \phi_{ch}$ and $\tilde{\phi} = e^{-V}\bar{\phi}_{ch}e^V$, and dropping subscripts ch from now on,

$$e^{-V}\delta e^V = \frac{1}{2}(\bar{\epsilon}\phi + \epsilon e^{-V}\bar{\phi}e^V) \tag{14.11.24}$$

Thus $\bar{D}_{\dot{\alpha}}\phi = 0$ and $D_\alpha\bar{\phi} = 0$, where $\bar{\phi} = (\phi)^\dagger$. (The factors e^V and e^{-V} are needed for reality). From this result one can derive an expression for δV itself (the derivation starts from $0 = \delta[V, e^V]$) but we shall not need it.)

Let us first check that this proposal reproduces the result for δW_α, where $W_\alpha = \bar{D}^2 e^{-V}D_\alpha e^V$. First we construct the variation of $(e^{-V}D_\alpha e^V)$.

$$\begin{aligned}&\delta\nabla_\alpha = \delta(e^{-V}D_\alpha e^V) = e^{-V}D_\alpha e^V(e^{-V}\delta e^V) - e^{-V}\delta e^V e^{-V}D_\alpha e^V = \nabla_\alpha e^{-V}\delta e^V \\ &-e^{-V}\delta e^V\nabla_\alpha = [\nabla_\alpha, e^{-V}\delta e^V] = \nabla_\alpha\frac{1}{2}(\bar{\epsilon}\phi + \epsilon e^{-V}\bar{\phi}e^V) = \frac{1}{2}(\bar{\epsilon}\nabla_\alpha\phi + (\nabla_\alpha\epsilon)e^{-V}\bar{\phi}e^V)\end{aligned} \tag{14.11.25}$$

where we used that $\nabla_\alpha e^{-V}\bar{\phi}e^V = D_\alpha\bar{\phi} = 0$. Next we evaluate δW_α. We obtain

$$\begin{aligned}&\delta W_\alpha = \frac{1}{2}\bar{D}^2[\bar{\epsilon}\nabla_\alpha\phi + (\nabla_\alpha\epsilon)e^{-V}\bar{\phi}e^V] = \\ &\frac{1}{2}[(\bar{D}^2\bar{\epsilon})\nabla_\alpha\phi + 2(\bar{D}^{\dot{\alpha}}\bar{\epsilon})(\bar{D}_{\dot{\alpha}}\nabla_\alpha\phi) + \bar{\epsilon}\bar{D}^2\nabla_\alpha\phi \\ &+(\nabla_\alpha\epsilon)\bar{D}^2(e^{-V}\bar{\phi}e^V)]\end{aligned} \tag{14.11.26}$$

where we used that $\bar{D}^2 D_\alpha\epsilon = 0$. Using that $\bar{D}_{\dot{\alpha}}\phi = 0$, we can reduce this to

$$\begin{aligned}&\delta W_\alpha = \frac{1}{2}(\bar{D}^2\bar{\epsilon})\nabla_\alpha\phi - 2i\bar{\epsilon}^{\dot{\alpha}}\nabla_{\alpha\dot{\alpha}}\phi + \frac{1}{2}\bar{\epsilon}\bar{D}^2\nabla_\alpha\phi \\ &+\frac{1}{2}\epsilon_\alpha\bar{D}^2(e^{-V}\bar{\phi}e^V)\end{aligned} \tag{14.11.27}$$

To compare this result with (17), we must take into account that in (17) we used the vector representations while in (27) we used the chiral representation. The relation is

$$W_\alpha^{(v)} = e^{\bar{\Omega}}W_\alpha^{ch}e^{-\bar{\Omega}} \tag{14.11.28}$$

Hence we need $e^{-\bar{\Omega}}\delta e^{\bar{\Omega}}$. We claim that

$$e^{-\bar{\Omega}}\delta e^{\bar{\Omega}} = \frac{1}{2}\bar{\epsilon}\phi^{ch} \quad \Rightarrow \quad (\delta e^{\Omega})e^{-\Omega} = \frac{1}{2}\epsilon(\phi^{ch})^{\dagger} \tag{14.11.29}$$

Indeed, then in terms of the chiral representation

$$\begin{aligned} e^{-V}\delta e^{V} &= e^{-\bar{\Omega}}e^{-\Omega}\delta(e^{\Omega}e^{\bar{\Omega}}) = e^{-\bar{\Omega}}(e^{-\Omega}\delta e^{\Omega})e^{\bar{\Omega}} + e^{-\bar{\Omega}}\delta e^{\bar{\Omega}} \\ &= e^{-\bar{\Omega}}e^{-\Omega}(\delta e^{\Omega}e^{-\Omega})e^{\Omega}e^{\bar{\Omega}} + \frac{1}{2}\bar{\epsilon}\phi = \frac{1}{2}(\epsilon\tilde{\bar{\phi}} + \bar{\epsilon}\phi) \end{aligned} \tag{14.11.30}$$

So we can now evaluate $\delta W_{\alpha}^{(v)}$ and find

$$\begin{aligned} &\delta W_{\alpha}^{(v)} = e^{\bar{\Omega}}\delta W_{\alpha}^{ch}e^{-\bar{\Omega}} \to e^{\bar{\Omega}}\left[e^{-\bar{\Omega}}\delta e^{\bar{\Omega}}, W_{\alpha}^{ch}\right]e^{-\bar{\Omega}} \\ &= e^{\bar{\Omega}}\left[\frac{1}{2}\bar{D}^2\{\bar{\epsilon}\nabla_{\alpha}\phi + (\nabla_{\alpha}\epsilon)e^{-V}\bar{\phi}e^{-V}\}\right]e^{-\bar{\Omega}} \to e^{\bar{\Omega}}\left[\frac{1}{2}\epsilon\phi^{ch}, W_{\alpha}^{ch}\right]e^{-\bar{\Omega}} \end{aligned} \tag{14.11.31}$$

If the $\bar{D}^2$ hits $\nabla_{\alpha}\phi = D_{\alpha}\phi + [(e^{-V}D_{\alpha}e^{V}), \phi]$, the term $\bar{D}^2 D_{\alpha}\phi$ vanishes, and one is left with $[(\bar{D}^2(e^{-V}D_{\alpha}V)), \phi] = [W_{\alpha}^{ch}, \phi]$. Then this term cancels the last term in $\delta W_{\alpha}^{(v)}$. One is then left with

$$\delta W_{\alpha}^{(v)} = e^{\bar{\Omega}}\left[\frac{1}{2}(\bar{D}^2\bar{\epsilon})\nabla_{\alpha}\phi - 2i\bar{\epsilon}^{\dot{\alpha}}\nabla_{\alpha\dot{\alpha}}\phi + \frac{1}{2}\epsilon_{\alpha}\bar{D}^2(e^{-V}\bar{\phi}e^{V})\right]e^{-\bar{\Omega}} \tag{14.11.32}$$

If one now recalls that $e^{\bar{\Omega}}\nabla^{ch}e^{-\bar{\Omega}} = \nabla^{(v)}$ and idem for ϕ, one indeed recovers (17), except for the new term with $\bar{D}^2\bar{\epsilon}$.

Since we have new symmeties in δW_{α}, we must go back to the $N = 2$ gauge action and verify that it is also invariant under these new symmetries. First we write W_{α} in $W^{\alpha}W_{\alpha}$ as $\bar{D}^2 e^{-V}D_{\alpha}e^{V}$ and pull the $\bar{D}^2$ out to obtain a $d^2\theta d^2\bar{\theta}$ measure. With all fields in the chiral basis we obtain

$$\begin{aligned} &(2g^2)\delta_2 S(\text{gauge}) = Tr\int d^2\theta d^2\bar{\theta}[2W^{\alpha}(\delta\nabla_{\alpha}) - e^{-V}\bar{\phi}e^{V}e^{-V}\delta e^{V}\phi \\ &+(e^{-V}\delta e^{V})(e^{-V}\bar{\phi}e^{V}\phi) - e^{-V}\delta\bar{\phi}e^{V}\phi - e^{-V}\bar{\phi}e^{V}\delta\phi] \\ &= Tr\int d^2\theta d^2\bar{\theta}[2W^{\alpha}\delta\nabla_{\alpha} - [e^{-V}\bar{\phi}e^{V}, e^{-V}\delta e^{V}]\phi \\ &-e^{-V}((\bar{D}^{\dot{\alpha}}\bar{\epsilon})\bar{W}_{\dot{\alpha}})e^{V}\phi - e^{-V}\bar{\phi}e^{V}(D^{\alpha}\epsilon)W_{\alpha}] \end{aligned} \tag{14.11.33}$$

We used that $\delta\phi^{ch} = \delta\left(e^{-\bar{\Omega}}\phi^{(v)}e^{\bar{\Omega}}\right)$ is equal to

$$e^{-\bar{\Omega}}\delta\phi^{(v)}e^{\bar{\Omega}} + \left[e^{-\bar{\Omega}}\phi^{(v)}e^{\bar{\Omega}}, \left(e^{-\bar{\Omega}}\delta e^{\bar{\Omega}}\right)\right] \tag{14.11.34}$$

The first term gives $\delta\phi = \epsilon^\alpha W_\alpha$ with all fields in the chiral representation, while the second term is proportional to $\left[\phi^{ch}, \frac{1}{2}\bar{\epsilon}\phi^{ch}\right]$ and vanishes.

In the second term we substitute the result for $e^{-V}\delta e^V$ to obtain for this term

$$-\frac{1}{2}[e^{-V}\bar{\phi}e^V, \bar{\epsilon}\phi + \epsilon e^{-V}\bar{\phi}e^V]\phi \tag{14.11.35}$$

Using the Jacobi identities, the contribution from $\bar{\epsilon}\phi$ cancels because $[\phi,\phi]=0$, and the contribution from $\epsilon e^{-V}\bar{\phi}e^V$ cancels directly. Hence the second term in $\delta_2 S$ (gauge) vanishes. There remain three terms, each proportional to W_α or $\bar{W}_{\dot{\alpha}}$. Using the result derived previously for $\delta\nabla_\alpha = \delta(e^{-V}D_\alpha e^V)$ we obtain the following four terms

$$\begin{aligned}&W^\alpha[\bar{\epsilon}\nabla_\alpha\phi + (\nabla_\alpha\epsilon)e^{-V}\bar{\phi}e^V]\\ &-(\bar{D}^{\dot{\alpha}}\bar{\epsilon})e^{-V}\bar{W}_{\dot{\alpha}}e^V\phi - (D^\alpha\epsilon)e^{-V}\bar{\phi}e^V W_\alpha\end{aligned} \tag{14.11.36}$$

The terms with $D^\alpha\epsilon$ and $\nabla_\alpha\epsilon = D_\alpha\epsilon$ clearly cancel. So we are left with

$$\begin{aligned}&W^\alpha\bar{\epsilon}\nabla_\alpha\phi - e^{-V}(\bar{D}^{\dot{\alpha}}\bar{\epsilon})\bar{W}_{\dot{\alpha}}e^V\phi =\\ &= \bar{\epsilon}(\nabla_\alpha W^\alpha)\phi - (\bar{D}^{\dot{\alpha}}\bar{\epsilon})e^{-V}\bar{W}_{\dot{\alpha}}e^V\phi\\ &= \bar{\epsilon}(\bar{D}_{\dot{\alpha}}e^{-V}\bar{W}^{\dot{\alpha}}e^V)\phi - (\bar{D}^{\dot{\alpha}}\bar{\epsilon})e^{-V}\bar{W}_{\dot{\alpha}}e^V\phi = \text{ total derivative}\end{aligned} \tag{14.11.37}$$

where we used the $N=1$ Bianchi identity in (13).[22]

[22](The Bianchi identity in (13) is in the vector representation where $(W_\alpha)^\dagger = [D_\alpha, W]^\dagger = -[\bar{W}, \bar{D}_{\dot{\alpha}}] = (\bar{D}_{\dot{\alpha}}\bar{W}) = \bar{W}_{\dot{\alpha}}$. In the chiral representation $\nabla^\alpha W_\alpha = \bar{\nabla}^{\dot{\alpha}}\bar{W}_{\dot{\alpha}}$ becomes $\nabla^{ch,\alpha}W^{ch}_\alpha = \bar{\nabla}^{ch,\dot{\alpha}}\bar{W}^{ch}_{\dot{\alpha}}$ with $\nabla^{ch,\alpha} = e^{-V}D^\alpha e^V$ and $\bar{\nabla}^{ch,\dot{\alpha}} = \bar{D}^{\dot{\alpha}}$. Furthermore, $\bar{W}^{ch}_{\dot{\alpha}} = e^{-\bar{\Omega}}\bar{W}^{(v)}_{\dot{\alpha}}e^{\bar{\Omega}}$ and $\bar{W}^{(v)}_{\dot{\alpha}} = (W^v_\alpha)^\dagger = (e^{\bar{\Omega}}W^{ch}_\alpha e^{-\bar{\Omega}})^\dagger = e^{-\Omega}(W^{ch}_\alpha)^\dagger e^\Omega$. So one obtains

$$\nabla^{ch,\alpha}W^{ch}_\alpha = \bar{D}_{\dot{\alpha}}e^{-\bar{\Omega}}e^{-\Omega}W^{ch,\dagger}_\alpha e^\Omega e^{\bar{\Omega}} = \bar{D}_{\dot{\alpha}}e^{-V}(W^{ch}_\alpha)^\dagger e^V$$

This is the identity we used in (36).

Having checked that the gauge action is invariant under $N = 2$ susy, we must now show that also the action for the hypermultiplet is $N = 2$ invariant. We recall the action

$$\begin{aligned}
S\,(\text{hyper}) &= \int d^4x d^2\theta d^2\bar{\theta}(\bar{\phi}_+ e^V \phi_+ + \phi_- e^{-V}\bar{\phi}_-) \\
&+ \int d^4x d^2\theta[\phi_-\phi\phi_+ + m\phi_-\phi_+ + h.c.]
\end{aligned} \tag{14.11.38}$$

We study first the case with $m = 0$.

We claim that the $N = 2$ susy rules are

$$\begin{aligned}
\delta_2\phi_+ &= \frac{1}{2}\bar{D}^2(\bar{\epsilon}e^{-V}\bar{\phi}_-); \delta_2\bar{\phi}_+ = \frac{1}{2}D^2(\epsilon\phi_- e^{-V}) \\
\delta_2\phi_- &= -\frac{1}{2}\bar{D}^2(\bar{\epsilon}\bar{\phi}_+ e^V); \delta_2\bar{\phi}_- = -\frac{1}{2}D^2(\epsilon e^V \phi_+)
\end{aligned} \tag{14.11.39}$$

Note that these rules are gauge covariant. The variations of δe^V with $\bar{\epsilon}\phi$ cancel directly with the variations of ϕ_+ and ϕ_- in the potential

$$\begin{aligned}
&\frac{1}{2}\bar{\phi}_+ e^V(\bar{\epsilon}\phi)\phi_+ - \frac{1}{2}\phi_-(\bar{\epsilon}\phi e^{-V})\bar{\phi}_- \\
&+\frac{1}{2}\phi_-\phi\bar{\epsilon}e^{-V}\bar{\phi}_- - \frac{1}{2}\bar{\epsilon}\bar{\phi}_+ e^V\phi\phi_+ = 0
\end{aligned} \tag{14.11.40}$$

The remaining variation of e^V in $\bar{\phi}_+ e^V \phi_+$ cancels because it is antichiral

$$\bar{\phi}_+ e^V(\frac{1}{2}\epsilon e^{-V}\bar{\phi}e^V)\phi_+ = \frac{1}{2}\epsilon\bar{\phi}_+\bar{\phi}(e^V\phi_+) \tag{14.11.41}$$

Similarly the remaining variation of e^{-V} in $\phi_- e^{-V}\bar{\phi}_-$ cancels because the result is chiral. Now we must study the rest, which reads

$$\begin{aligned}
&\frac{1}{2}\bar{\phi}_+ e^V \bar{D}^2(\bar{\epsilon}e^{-V}\bar{\phi}_-) + \frac{1}{2}(D^2(\epsilon\phi_- e^{-V}))e^V\phi_+ \\
&-\frac{1}{2}(\bar{D}^2(\bar{\epsilon}\bar{\phi}_+ e^V))e^{-V}\bar{\phi}_- - \frac{1}{2}\phi_- e^{-V}D^2(\epsilon e^V\phi_+) \\
&+\bar{\phi}_+(-(\bar{D}^{\dot{\alpha}}\bar{\epsilon})e^V\bar{D}_{\dot{\alpha}}e^{-V})\bar{\phi}_- + \phi_-(D^\alpha\epsilon)(e^{-V}D_\alpha e^V)\phi_+
\end{aligned} \tag{14.11.42}$$

In the first term the $\bar{D}^2$ can hit $\bar{\epsilon}$, but then e^V and e^{-V} annihilate each other, and since the result $\bar{\phi}_+\bar{\phi}_-\bar{D}^2\bar{\epsilon}$ is antichiral, this term vanishes. If both $\bar{D}^2$ move past $\bar{\epsilon}$, this term cancels the term with $\bar{\epsilon}$ in the second line after partial integration. If one $\bar{D}^{\dot{\alpha}}$ hits $\bar{\epsilon}$ and the other hits $\bar{\phi}_-$ the e^V and e^{-V} again cancel, and since the result $\bar{\phi}_+(\bar{D}^{\dot{\alpha}}\bar{\epsilon})\bar{D}_{\dot{\alpha}}\bar{\phi}_-$ is linear (is annihilated by D^2), also this term vanishes. Finally, the contribution from the first term with $(\bar{D}^{\dot{\alpha}}\bar{\epsilon})$ and $(\bar{D}_{\dot{\alpha}}e^{-V})$ cancels the last term in $\delta_2 S$ (hyper) with $\bar{\epsilon}$. Hence the action for the hypermultiplet is indeed invariant.

The $N = 4$ model.

The $N = 4$ model can be very clearly understood by taking the superfields of the hypermultiplet in the adjoint representation:

$$\begin{aligned} \bar{\phi}_+ e^V \phi_+ &= Tr e^{-V}\bar{\phi}_+ e^V \phi_+ \\ \phi_- e^{-V}\bar{\phi}_- &= Tr e^{-V}\bar{\phi}_- e^V \phi_- \end{aligned} \tag{14.11.43}$$

From the gauge action we obtain

$$\bar{\phi} e^V \phi = Tr\ e^{-V}\bar{\phi} e^V \phi \tag{14.11.44}$$

Hence, we find a complete symmetry between the 3 superfields $\phi_i = \{\phi, \phi_+, \phi_-\}$. Also the potential can be written in a manifestly symmetric way

$$\phi_- \phi \phi_+ = Tr[\phi_-, \phi]\phi_+ = \epsilon^{ijk} Tr[\phi_i \phi_j \phi_k] \tag{14.11.45}$$

Thus there are now 3 susy, which together with the $N = 1$ susy give $N = 4$ susy. The $SU(3)$ symmetry is obvious. R symmetry completes this to $U(3)$. The $SU(4)$ symmetry only becomes obvious in terms of components (the spinors $\nabla_\alpha \phi_i$ and W_α form then a **4** of $SU(4)$).

The gauge multiplet contains all auxiliary fields needed for closure in $N = 2$ superspace (because we started from an $N = 2$ superfield W on which the susy algebra closed). However, the chiral multiplet does not have all auxiliary fields needed for $N = 2$ superspace (the transformation rules depend on the gauge multiplet). Thus,

$N=4$ Yang-Mills theory exists in $N=4$ superspace only on-shell. In fact, the $N=4$ field equations are the naive generalizations of the $N=2$ **off**-shell constraints.

The $N=2$ model with masses.

Having obtained the $N=1$ action for the $N=2$ vector multiplet and the hypermultiplet, we should now study the other symmetries hidden in the superfield ϵ and the $U(2)$ symmetry. We only studied the $N=2$ susy invariance for the massless case. We first extend the proof of $N=2$ invariance to the massive case. In the process we shall pick up extra terms in the transformation rules, and these we shall need to find the correct $U(2)$ transformation laws. We recall the actions and transformation laws

$$
\begin{aligned}
&S\ (\text{vector}) \ = Tr\int d^4x\left[-\int d^2\theta d^2\bar\theta\tilde{\bar\phi}\phi+\int d^2\theta W^\alpha W_\alpha\right]\,;\ \tilde{\bar\phi}=e^{-V}\bar\phi e^V\\
&\delta_2\phi=\epsilon_2^\alpha W_\alpha;e^{-V}\delta_2e^V=\frac{1}{2}(\bar\epsilon\phi+\epsilon\tilde{\bar\phi})\\
&\delta_2W_\alpha=-(\bar D^{\dot\alpha}\epsilon)\nabla_{\alpha\dot\alpha}\phi+\frac{1}{2}(D_\alpha\epsilon)\bar\nabla^2\tilde{\bar\phi}+\frac{1}{2}(\bar D^2\bar\epsilon)\nabla_\alpha\phi-\frac{1}{2}\bar\epsilon[\phi,W_\alpha]\\
&S\ (\text{hyper}) \ =\int d^4x\Big[\int d^2\theta d^2\bar\theta\{\bar\phi_+e^V\phi_++\phi_-e^{-V}\bar\phi_-\}\\
&+\int d^2\theta(\phi_-\phi\phi_++m\phi_-\phi_+)+\int d^2\bar\theta(\bar\phi_+\bar\phi\bar\phi_-+m\bar\phi_+\bar\phi_-)\Big]\\
&\delta_2\phi_+=\frac{1}{2}\bar D^2(\bar\epsilon e^{-V}\bar\phi_-);\delta_2\phi_-=-\frac{1}{2}\bar D^2(\bar\epsilon\bar\phi_+e^V)\\
&\delta^2\bar\phi_+=\frac{1}{2}\nabla^2(\epsilon\phi_-e^{-V});\delta_2\bar\phi_-=-\frac{1}{2}\nabla^2(\epsilon e^V\phi_+)
\end{aligned}
\tag{14.11.46}
$$

The mass terms vary into

$$
\begin{aligned}
\delta S\ (\text{hyper},m) \quad = \quad & -\frac{1}{2}m\bar\epsilon\bar\phi_+e^V\phi_++\frac{1}{2}m\phi_-\bar\epsilon e^{-V}\bar\phi_-\\
&+\frac{1}{2}m\epsilon\phi_-e^{-V}\bar\phi_--\frac{1}{2}m\bar\phi_+\epsilon e^V\phi_+
\end{aligned}
\tag{14.11.47}
$$

Clearly these four variations do not cancel. However, by replacing $\bar\epsilon$ in $\delta_2\phi_+$ and $\delta_2\bar\phi_-$ by $\bar\epsilon-\epsilon$, these variations cancel.

With these additions to $\delta_2\phi_+$ and $\delta_2\phi_-$ we must reanalyze the invariance of S

(hyper). The kinetic terms are still invariant

$$\delta_2(\text{new})\ S\ (\text{hyper, kinetic}) = \int d^2\theta d^2\bar{\theta}\left[-\frac{1}{2}D^2(\bar{\epsilon}\phi_- e^{-V})e^V\phi_+ + \frac{1}{2}\phi_- e^{-V}D^2(\bar{\epsilon}e^V\phi_+)\right] = 0 \tag{14.11.48}$$

because $D_\alpha\bar{\epsilon} = 0$. However, the ϕ^3 terms seem to offer a problem. They vary under the new susy laws as follows

$$\begin{aligned} &\delta_2\ (\text{new})\ \bar{\phi}_+)\bar{\phi}\bar{\phi}_- + \bar{\phi}_+\bar{\phi}\delta_2(\text{new})\bar{\phi}_- \\ &= -\frac{1}{2}(\bar{\epsilon}\phi_- e^{-V})\bar{\phi}\bar{\phi}_- + \frac{1}{2}\bar{\phi}_+\bar{\phi}(\bar{\epsilon}e^V\phi_+) \end{aligned} \tag{14.11.49}$$

These variations can be canceled by modifing the δ_2 rule for e^V

$$e^{-V}\delta_2(\text{new})e^V = -\frac{1}{2}\epsilon\phi - \frac{1}{2}\bar{\epsilon}\tilde{\phi}\ ;\ \tilde{\phi} = e^{-V}\bar{\phi}e^V \tag{14.11.50}$$

Indeed, when used in $\bar{\phi}_+ e^V\phi_+ + \phi_- e^{-V}\bar{\phi}_-$ these new variations are

$$\bar{\phi}_+ e^V\left(\frac{1}{2}\epsilon\phi - \frac{1}{2}\bar{\epsilon}\tilde{\phi}\right)\phi_+ - \phi_-\left(\frac{1}{2}\epsilon\phi - \frac{1}{2}\bar{\epsilon}\tilde{\phi}\right)e^{-V}\bar{\phi}_- \tag{14.11.51}$$

and the $\bar{\epsilon}$ terms clearly cancel. (We leave the ϵ terms as an exercise).

It might seem that with δ_2 (new) e^V also the invariance of the action $W^\alpha W_\alpha$ has to be reanalyzed, but this is fortunately not so. The transformation $e^{-V}\delta_2$ (new) $e^V = -\frac{1}{2}\epsilon\phi - \frac{1}{2}\bar{\epsilon}\tilde{\phi}$ is a sum of a chiral and an antichiral field. But this is also the form of a gauge transformation.

$$e^{-V}(\delta\ (\text{gauge})\ e^V) = e^{-V}(\bar{\Lambda}e^V - e^V\Lambda) = \tilde{\Lambda} - \Lambda \tag{14.11.52}$$

Hence δ_2 (new) e^V is a gauge transformation. Thus δ_2 (new) e^V leaves $W^\alpha W_\alpha$ invariant.

Finally, we come to $\tilde{\phi}\phi$. If we transform ϕ with the same gauge transformation as e^V, the extra term cancels

$$\delta_2\ (\text{new})\ \phi = \left[+\frac{1}{2}\epsilon\phi, \phi\right] = 0 \tag{14.11.53}$$

This proves the invariance of the $N = 2$ theory with masses for the hypermultiplet.

A few comments:

(i) For $U(1)$ factors in the gauge group we could have redefined $\phi + m \to \phi$ to absorb the mass terms into the Yukawa terms.

(ii) The deeper reason that the modifications δ_2 (new) for e^V and ϕ are gauge transformations is that the $\{e^V, \phi\}$ sector has all the auxiliary fields to make it fully $N = 2$ susy. Thus no modifications containing the "matter" variables $\phi_+, \phi_-, \bar{\phi}_+$ or $\bar{\phi}_-$ are allowed.

(iii) We could also have started from the observation that the combinations $\bar{\phi}_+ e^V$ etc. in $\delta_2 S$ (hyper, m) are (partial) field equations, and tried to obtain invariance by using the Noether method.

The $U(2)$ symmetries.

Let us now study the variations with the parameters $\bar{D}^2\bar{\epsilon} \equiv \bar{w}$. This is a complex parameter which we shall identify as the parameter for the T_+ part of the SU(2) group. First consider (27)

$$\delta W_\alpha = \frac{1}{2}(\bar{D}^2\bar{\epsilon})\nabla_\alpha \phi \quad \Rightarrow \quad \delta\lambda_\alpha = \frac{1}{2}\bar{w}\psi_\alpha \tag{14.11.54}$$

Since $\delta(\nabla_\beta W_\alpha) = \nabla_\beta \delta W_\alpha$ because $\delta\nabla_\beta = \delta(e^{-V} D_\beta e^V)$ contains no terms with $D^2\epsilon$ or $\bar{D}^2\bar{\epsilon}$, and also $\nabla_{(\beta}\delta W_{\alpha)}$ contains no $D^2\epsilon$ or $\bar{D}^2\bar{\epsilon}$ terms, $f_{\alpha\beta}$ is $w, \bar{w}$ inert, but

$$\delta(\nabla^\alpha W_\alpha) = \frac{1}{2}(D^2\epsilon)\bar{\nabla}^2\tilde{\bar{\phi}} + \frac{1}{2}(\bar{D}^2\epsilon)\nabla^2\phi \quad \Rightarrow \quad \delta D' = 2w\bar{F} + 2\bar{w}F \tag{14.11.55}$$

Furthermore $\delta\phi$ is inert under $w, \bar{w}$ transformations but

$$\begin{aligned} \delta(\nabla_\alpha\phi) &= -(D^2\epsilon)W_\alpha \quad \Rightarrow \quad \delta\psi_\alpha = -w\lambda_\alpha \\ \delta(\nabla^2\phi) &= (D^2\epsilon)\nabla^\alpha W_\alpha \quad \Rightarrow \quad \delta F = wD' \end{aligned} \tag{14.11.56}$$

Clearly, λ_α and ψ_α rotate into each other, and F and D' rotate into each other. In $N = 2$ notation these fields are

$$W, \nabla_{a\alpha}W, \nabla^2_{(ab)}W \tag{14.11.57}$$

The doublet of spinors $\nabla_{a\alpha}W \sim (\lambda_\alpha, \psi_\alpha)$ and the triplet of auxiliary fields $\nabla^2_{ab}W \sim (D', F, \bar{F})$ transform under SU(2) as the indices indicate. The T_3 piece of SU(2) gives λ_α and ψ_α opposite phases, but the U(1) piece gives them the same phase. Note that the U(1) group is chiral because $\nabla_{a\alpha}W$ and $\bar{\nabla}^a_{\dot{\alpha}}\bar{W} = (\nabla_{a\alpha}W)^\dagger$ transform oppositely. However, the SU(2) is vector like, as it should because the **2** of SU(2) is pseudo real.

The U(1) transformation properties of the fields in the $N=2$ gauge multiplet are very clear in the $N=2$ superspace formulation. The field W transforms with U(1) weight $\exp -2i\eta$, and the coordinates θ with U(1) weight $\exp -i\eta$ in order that $\int d^2\theta^1 d^2\theta^2 WW$ be invariant. Then

$$\nabla_{a\alpha} \to e^{i\eta}\nabla_{a\alpha}, W \to e^{-2i\eta}W, \bar{W} \to^{2i\eta} \bar{W} \tag{14.11.58}$$

and the scalars $A = W_|$ are SU(2) inert but $A \to (\exp -2i\eta)A$ under U(1). Further the two spinors transform as a SU(2) doublet with half the U(1) weight of A.

$$\nabla_{a\alpha}W \to e^{-i\eta}s_a{}^b\nabla_{b\alpha}W \tag{14.11.59}$$

Finally the auxiliary fields $\nabla^2_{ab}W$ are U(1) inert, in agreement with the reality condition, and transform as a triplet under $SU(2)$

$$\nabla^2_{ab}W \to s_a{}^c s_b{}^d \nabla^2_{cd}W \tag{14.11.60}$$

(Incidentally, $\nabla^2_{11}W \sim D^2\phi \sim F$ and $\nabla^2_{22}W \sim \bar{\nabla}^{11}\bar{\phi} \sim \bar{F}$ but $\nabla_{12}W \sim D' + i[A, \bar{A}]$. Namely, $(\nabla^\alpha_1\nabla_{2\alpha} + \nabla^\alpha_2\nabla_{1\alpha})W = 2\nabla^\alpha W_\alpha + \{\nabla^\alpha_2, \nabla_{1\alpha}\}W = 2D' + 2i[\bar{W}, W] \sim D' + i[\bar{A}, A]$). In the action one finds $(D' + i[A, \bar{A}])^2$, so that also in the $N=2$ formulation one recovers the $N=1$ result that $D' \sim [A, \bar{A}]$ on-shell).

Next we analyze the hypermultiplet. We begin again with the transformations with $D^2\epsilon$ and $\bar{D}^2\bar{\epsilon}$. First we see that the scalars of one chiral multiplet rotate into the scalars of the other chiral multiplet

$$\begin{aligned}\delta\phi_+ &= \frac{1}{2}(\bar{D}^2\bar{\epsilon})e^{-V}\bar{\phi}_- \equiv \frac{1}{2}\bar{w}\tilde{\bar{\phi}}_- \\ \delta\phi_- &= -\frac{1}{2}(\bar{D}^2\bar{\epsilon})\bar{\phi}_+e^V \equiv -\frac{1}{2}\bar{w}\tilde{\bar{\phi}}_+\end{aligned} \tag{14.11.61}$$

Note that these relations preserve chirality and gauge covariance. However, the spinors are inert(!)

$$\delta(\nabla_\alpha \phi_+) = \nabla_\alpha(\bar{D}^2\bar{\epsilon})(e^{-V}\bar{\phi}_-) = (\bar{D}^2\bar{\epsilon})\nabla_\alpha(e^{-V}\bar{\phi}_-) = 0 \tag{14.11.62}$$

because $e^{-V}\bar{\phi}_-$ is antichiral. The auxiliary fields F_+ and $\bar{F}_-$ rotate again into each other

$$\delta\nabla^2\phi_+ = -\frac{1}{2}(D^2\epsilon)\bar{\nabla}^2(e^{-V}\bar{\phi}_-) \quad\Rightarrow\quad \delta F_+ = -\frac{1}{2}w\bar{F}_- \tag{14.11.63}$$

Note that we need here the δ_2 (new) corrections to $\delta\phi_+$. Hence: $\delta\phi_+ = \frac{1}{2}\bar{w}\tilde{\bar{\phi}}_-$ but $\delta F_+ = -\frac{1}{2}w\bar{F}_-$: the scalars transform contragradiently to the auxiliary fields. This we can interpret also as follows: the θ's transform and the superfields have an overall transformation law, in such a way that the combined effect keeps the spinors invariant, but acts one way on the scalars and another way on the auxiliary fields. We now study the full $U(2)$.

Comment: In the absence of mass terms there is a second SU(2) under which the scalars transform, provided we are in a real representation.

The transformation laws of the hypermultiplet under U(2) are more complicated. Their $N=1$ superspace action

$$L\text{ (hyper) } = \int d^4\theta(\bar{\phi}_+e^V\phi_+ + \phi_-e^{-V}\bar{\phi}_-) + \int d^2\theta(\phi_-\phi\phi_+ + m\phi_-\phi_+) + h.c. \tag{14.11.64}$$

is for $m=0$ invariant under a full U(2). Namely, since $\int d^2\theta\phi$ is U(1) invariant by itself, ϕ_+ and ϕ_- must transform oppositely under $U(1)$, and then the kinetic terms are invariant, too. ($\int d^4\theta = \int d^2\theta d^2\bar{\theta}$ and $\exp V$ are separately invariant). The mass term clearly breaks this U(1) symmetry due to the measure $\int d^2\theta$. The action hs also a full $SU(2)$ symmetry. As we showed before, the two off-diagonal parts of the SU(2) symmetry are an invariance of the action with or without mass term, and the commutator $[T_+, T_-] \sim T_3$ leads to a third symmetry. This T_3 symmetry leaves ϕ and V invariant, but it transforms $\theta^{1\alpha}$ into $\theta^{2\alpha}$, and thus $d^2\theta$ is not T_3 invariant. The fields ϕ_+ and ϕ_- transform then the same way under T_3, contrary to what one might naively expect. The SU(2) doublet consists then of ϕ_+ and $\tilde{\phi}_-$

$$\begin{pmatrix} \phi_+ \\ \tilde{\phi}_- \end{pmatrix} \to s \begin{pmatrix} \phi_+ \\ \tilde{\phi}_- \end{pmatrix} \tag{14.11.65}$$

and under T_3 the fields ϕ_+ and $\tilde{\phi}_-$ acquire opposite phases. (Note that ϕ_+ and $\tilde{\phi}_-$ are in the same gauge representation, but have opposite chiralities. However, the spinors in ϕ_+ and $\tilde{\phi}_-$ are SU(2) inert, as we have seen, hence SU(2) transformations still commute with Lorentz transformations).

The U(2) symmetry appears in a more natural way in the x-component formulation of the $N = 2$ model. The components of the vector multiplet are $A, \lambda_{a\alpha}, \bar{\lambda}^{a\dot{\alpha}}, F_{ab}$ and $\bar{F}^{ab}$, and in the action the U(2) symmetry is manifest; for example

$$\bar{\lambda}^{a\dot{\alpha}} \nabla_{\alpha\dot{\alpha}} \lambda_a^\alpha ; \bar{\nabla}^{a\dot{\alpha}} \bar{A} \nabla_{a\dot{\alpha}} A ; \epsilon^{ab} \lambda_a^\alpha [\lambda_{b\alpha}, \bar{A}] \tag{14.11.66}$$

The U(1) symmetry requires in the last (Yukawa) term $\bar{A}$ to transform opposite to the two spinor λ_a^α. For the hypermultiplet one must combine ϕ_+ and $\bar{\phi}_-$ into a doublet Q_a, while ϕ_- and $\bar{\phi}_+$ form another doublet $\bar{Q}^a$ in the complex conjugate representation of the gauge group.

The two Fayet-Iliopoulos terms $c_1 \int d^4\theta V + (c_2 \int d^2\theta\phi + h.c.)$ break the U(2) symmetry down to only a U(1). To demonstrate this, we consider for abelian group factors U(1)

$$\begin{aligned} S(FI) &= c \int d^4x d^4\theta V + b \int d^4x d^2\theta\phi + \bar{b} \int d^4x d^2\bar{\theta}\bar{\phi} \\ \delta V &= i(\bar{\epsilon}\phi - \epsilon\bar{\phi}), \delta\phi = i(D^\alpha\epsilon)\bar{D}^2 D_\alpha V \end{aligned} \tag{14.11.67}$$

The cV term is U(2) invariant: $\delta \int cV = ic \int (\bar{\epsilon}\phi - \epsilon\bar{\phi})$. However, the terms with $b\phi$ and $\bar{b}\bar{\phi}$ vary as follows

$$\delta \int b d^4x d^2\theta\phi = \int d^4x d^4\theta bi(D\epsilon^\alpha)(D_\alpha V) = bi \int wV \tag{14.11.68}$$

Hence, only if $ib(D^2\epsilon)V + i\bar{b}(\bar{D}^2\bar{\epsilon})V$ vanishes there is an invariance. Thus only a U(1) (with parameters Rew or $Im\ w$) remains of the SU(2). The U(1) factor of the U(2) also remains.

Central charges.

It is a result of abstract susy Jacobi identities that the most general $N = 2$ susy algebra admits only one complex central charge Z,

$$\begin{aligned} \{Q_{a\alpha}, \bar{Q}^b_{\dot\beta}\} &= \delta^b_a P_{\alpha\dot\beta} \\ \{Q_{a\alpha}, Q_{b\beta}\} &= i\epsilon_{ab}\epsilon_{\alpha\beta} Z \end{aligned} \tag{14.11.69}$$

Furthermore, if $Z \neq 0$ then U(2) is broken down to $Usp(2) = SU(2)$ as is clear from (65). Since we already saw that the mass term in the action of the hypermultiplet breaks U(2) down to SU(2), one may suspect that in $N = 2$ models the central charge Z is proportional to m. This is indeed the case as we now show.

Recall the $N = 2$ susy transformations of the hypermultiplet, and consider θ-independent ϵ

$$\delta_2\phi_+ = \frac{1}{2}\bar{D}^2((\bar{\epsilon} - \epsilon)e^{-V}\bar{\phi}_-) = \frac{1}{2}(\epsilon - \bar{\epsilon})(\bar{D}^2 e^{-V}\bar{\phi}_-) \tag{14.11.70}$$

On shell, $\bar{D}^2 e^{-V}\bar{\phi}_-$ is proportional to $\phi\phi_+ + m\phi_+$. Since ϕ is chiral, the term $\delta\phi_+ \sim \phi\phi_+$ is a gauge transformation which does not concern us here, but $\delta_2\phi_+ = (\epsilon - \bar{\epsilon})m\phi_+$ is the action of the central charge. Hence, the constant parameter $(\epsilon - \bar{\epsilon})$ is the parameter for Z transformations.

To see Z realized in the $N = 2$ susy algebra, we evaluate the $[\delta_1(\eta), \delta_2(\epsilon)]$ commutator on ϕ_+

$$\begin{aligned} [\delta_1(\eta), \delta_2(\epsilon)]\phi_+ &= \bar{D}^2(\epsilon^\alpha\theta_\alpha - \bar{\epsilon}^{\dot\alpha}\bar{\theta}_{\dot\alpha})(\eta^\beta Q_\beta + \bar{\eta}^{\dot\beta}\bar{Q}_{\dot\beta})e^{-V}\bar{\phi}_- \\ &-(\eta^\beta\nabla_\beta + \bar{\eta}^{\dot\beta}\bar{D}_{\dot\beta})\bar{D}^2(\epsilon^\alpha\theta_\alpha - \bar{\epsilon}^{\dot\alpha}\bar{\theta}_{\dot\alpha})e^{-V}\bar{\phi}_- \end{aligned} \tag{14.11.71}$$

Pulling the Q's in the first term to the left and then replacing them by D's, there is only a contribution if the Q's hit the θ's in the $\epsilon\theta + \bar{\epsilon}\bar{\theta}$ term. The result is then proportional to

$$(\epsilon^{\alpha}\eta_{\alpha} - \bar{\epsilon}^{\dot{\alpha}}\eta_{\dot{\alpha}})\bar{D}^2 e^{-V}\bar{\phi}_- \qquad (14.11.72)$$

which is the central charge transformation after using the ϕ_+ field equation.

$N = 2$ representations and multiplet shortening.

The representations in terms of physical states of the $N = 2$ susy algebra consist of three classes: massless representations for which always $Z = 0$, massive representations for which $\mid Z \mid^2 + P^2 = 0$, and massive representations for which $\mid Z \mid^2 + P^2 \neq 0$ (including representations with $Z = 0$).

For massless representations, we can choose a Lorentz frame where $P^0 = P^3$. Then $P^{\alpha\dot{\beta}} = \sigma^{\mu,\alpha\dot{\beta}}P_{\mu} = P_0(\sigma^3 - \sigma^0)^{\alpha\dot{\beta}}$ has only $P^{-\dot{-}}$ nonvanishing, and hence we set $P_{+\dot{+}} = P$, while all other components of $P_{\alpha\dot{\beta}}$ vanish. In particular

$$\{Q_{a-}, \bar{Q}^a_{\dot{-}}\} = 0 \ \ \text{(no sum)} \qquad (14.11.73)$$

implies that the norm of $Q_{a-} \mid \psi >$ vanishes for any state $\mid \psi >$ in the (positive definite) Hilbert space of physical states. Hence we may set the operators Q_{a-} and $\bar{Q}^a_{\dot{-}}$ equal to zero. Then also Z vanishes. The Hilbert space is now obtained from a highest weight "vacuum" $\mid 0 >$ which satisfies $Q_{a+} \mid 0 >= 0$ by acting with $\bar{Q}_{a\dot{+}}$. This leads to 4-dimensional representations (combinations of two massless $N = 1$ representations), for example the vector multiplet with helicities $+1, (+\frac{1}{2})^2, 0$ and its CPT conjugate with $0, (-\frac{1}{2})^2, -1$.

For massive representations we can choose a Lorentz frame where $\vec{P} = 0$. Then $P^{\alpha\dot{\beta}} = \sigma^{0,\alpha\dot{\beta}}M$ and $P_{+\dot{+}} = P_{-\dot{-}} = M$ while $P_{+\dot{-}} = P_{-\dot{+}} = 0$. The susy algebra reduces to

$$\{Q_{1+}, Q_{2-}\} = -\{Q_{1-}, Q_{2+}\} = Z$$
$$\{Q_{1\pm}, \bar{Q}^1_{\dot{\pm}}\} = \{Q_{2\pm}, \bar{Q}^2_{\dot{\pm}}\} = M \qquad (14.11.74)$$

Now there are twice as many states. (We shall determine which Q's are now the creation and annihilation operators shortly).

However, when $M =\mid Z \mid$ we can remove the phase of Z by redefining $Q_{2\pm}$ and then we can achieve that $M = Z$. Then

$$0 = Z - M = \{Q_{1+}, Q_{2-}\} - \{Q_{1+}, \bar{Q}^1_{\dot{+}}\} = \{Q_{1+}, Q_{2-} - \bar{Q}^1_{\dot{+}}\} \tag{14.11.75}$$

Similarly

$$\{\bar{Q}^2_{\dot{-}}, Q_{2-} - \bar{Q}^1_{\dot{+}}\} = 0 \tag{14.11.76}$$

hence also

$$\begin{aligned}\{Q_{1+} - \bar{Q}^2_{\dot{-}}, Q_{2-} - \bar{Q}^1_{\dot{+}}\} = 0\\ \{Q_{1-} + \bar{Q}^2_{\dot{+}}, Q_{2+} + \bar{Q}^1_{\dot{-}}\} = 0\end{aligned} \tag{14.11.77}$$

In the massless case, we can set the operators $Q_{1+} - Q^2_{\dot{-}}$ and $Q_{1-} + Q^2_{\dot{+}}$ equal to zero (and also their hermitian conjugates $\bar{Q}^1_{\dot{+}} - Q_{2-}$, and $\bar{Q}^1_{\dot{-}} + Q_{2+}$, respectively). One obtains then a massive multiplet as short as a massless multiplet, and hence twice as short as the generic massive multiplet. This multiplet is generated by

$$Q_{1+} + Q^2_{\dot{-}}, \bar{Q}^1_{\dot{+}} + Q_{2-}, Q_{1-} - \bar{Q}^2_{\dot{+}}, Q_{2+} - Q^1_{\dot{-}} \tag{14.11.78}$$

(One can take $Q_{1+} + Q^2_{\dot{-}}$ and $Q_{1-} - \bar{Q}^2_{\dot{+}}$ as the annihilation operators since they anticommute. Since ψ^α and $\bar{\psi}_{\dot{\alpha}}$ transforms the same way under the rotation group, ψ_+ and $\bar{\psi}_{\dot{-}}$ transform the same way, hence $Q_{1+} + \bar{Q}^2_{\dot{-}}$ is a Lorentz covariant object). NB. $< Q >\neq 0$ gives mass to V and Z in $\delta W_\alpha = W_\alpha(\phi)\epsilon$. $< Q >\neq 0$ then equivalent to $V \sim< Q >$. Now $Z \neq< Q >$. Then always long susy multiplet).

Examples of these multiplets are obtained by considering spontaneous symmetry breaking in models with unbroken rigid $N = 2$ susy. The "Coulomb branch" was defined by those models in which all the "quarks" Q have vanishing expectation values $< Q >= 0$, by $< \phi> \neq 0$. In this case the vector multiplet "Higgses itself": the vector boson eats a scalar from its own multiplet, and as a consequence the number of states does not change, states only get shifted around. This is clearly a shortened multiplet.

The central change Z in this case is given by $\delta W_\alpha = (\bar{D}^2\nabla_\alpha)(\bar{\phi}\epsilon + \bar{\epsilon}\phi) = W_\alpha(<\bar{\phi}> \epsilon + \bar{\epsilon} <\phi>)$. (When $<\phi> \neq 0$, the vector gets a mass from $\bar{\phi}e^V\phi$ and this same mass appears in δW_α, as a central charge, thus allowing multiplet shortening.

On the other hand, in the "Higgs branch" one has $<Q> \neq 0$ and vector bosons eat scalars of the hypermultiplet. Now one gets a long massive multiplet (or massive shortened multiplets if $M = \mid Z \mid$).[23]

There is actually a nice covariant way of finding out which generators can be set to zero to find representations. Consider $P^{\alpha\dot{\beta}}\bar{Q}^a_{\dot{\beta}}$ and its hermitian conjugate $P^{\beta\dot{\alpha}}Q_{b\beta}$. The anticommutator is proportioned to $\delta^a_b P^{\alpha\dot{\beta}}P_{\beta\dot{\beta}}P^{\beta\dot{\alpha}} \sim \delta^a_b P^2 P^{\alpha\dot{\alpha}}$. Hence for $P^2 = 0$ the generators $P^{\alpha\dot{\beta}}\bar{Q}^a_{\dot{\beta}}$ are the ones one can set to zero. If $P^2 = \mid Z \mid^2$ one may instead consider

$$\bar{S}^{\dot{\beta}}_a = P^{\alpha\dot{\beta}}Q_{a\alpha} + \epsilon_{ab}\xi Z\bar{Q}^{b\dot{\beta}} \tag{14.11.79}$$

This generator anticommutes with its hermitian conjugate if $P^2 = \mid Z \mid^2$ and if one chooses the phase ξ such that ξZ is real

$$S^{a\beta} = P^{\beta\dot{\alpha}}\bar{Q}^a_{\dot{\alpha}} + \epsilon_{ab}\xi Z Q^b_\beta \tag{14.11.80}$$

Setting $S^{a\beta}$ and $\bar{S}^{\dot{\beta}}_a$ to zero is a consistent truncation, and for $P_{+\dot{+}} = P_{-\dot{-}} \neq 0$ one finds our earlier results back.

The vacuum structure of $N = 2$ theories.

To determine the supersymmetric vacua at the classical level, we must require that the F and D terms in the potential vanish. The effective potential reads schematically.

$$U = V_{,i}V^{,\bar{i}} + \frac{1}{2}(\bar{A}TA)^2 \tag{14.11.81}$$

where $\phi^i = \{\phi_{-|} \equiv \tilde{Q}, \phi_{+|} \equiv Q, \phi_| = A\}$ and $V_{,i} = \frac{\partial}{\partial\phi^i}V$. From the ϕ^3 terms in S (gauge) we find the conditions

$$\frac{\partial V}{\partial Q} = \tilde{Q}A + m\tilde{Q} = 0$$

[23] If $<Q> \neq 0$, the vector gets a mass from $\bar{Q}e^V Q$, but since the central charge still comes from $\delta W_\alpha = (\bar{D}^2\nabla_\alpha)(\bar{\epsilon}\phi + \bar{\phi}\epsilon)$ one always gets a long multiplet in the Higgs branch (true?).

$$\frac{\partial V}{\partial \tilde{Q}} = AQ + mQ = 0$$
$$\frac{\partial V}{\partial A} = \tilde{Q}TQ = 0$$
$$\bar{A}TA = \bar{Q}T_aQ - \tilde{Q}T_a\overline{\tilde{Q}} + [\bar{A}, A]_a = 0 \qquad (14.11.82)$$

In order to find a susy vacuum, all these equations should be satisfied. We now consider a few solutions.

The Coulomb phase. A solution in which (usually?) the group Q is broken down to an abelian subgroup is

$$< Q >=< \tilde{Q} >= 0, < A >\neq 0 \text{ but } [\bar{A}, A]_a = 0 \qquad (14.11.83)$$

(If also $< A >= 0$, then G remains unbroken, of course). For example, for SU(2) a nonvanishing $< A >$ can be mapped into the Cartan subalgebra, and then only an unbroken U(1) remains. For higher-dimensional groups, several U(1)'s may in general remain unbroken.

The Higgs phase: a solution in which (sometimes?) all of Q can be spontaneous broken is given by

$$< Q >\neq 0 \qquad (14.11.84)$$

If Q is in the fundamental representation, all of G is broken. We consider the mappless and massive cases separately.

$\boldsymbol{m \neq 0}$**.** Choose $< A >= 0$. Then we are left with

$$\tilde{Q}TQ = 0; \bar{Q}TQ = \tilde{Q}T\overline{\tilde{Q}} \qquad (14.11.85)$$

In SU(2) these equations can only be solved if there are more than one flavors; for example $\tilde{Q}$ (flavor 1) = 0 and Q (flavor 2) = 0 is then a solution.

$m \neq 0$. Choose $< A >$ diagonal with elements $+m, -m$ on the diagonal.

$$A = \begin{pmatrix} m & & & & & \\ & -m & & & & \\ & & m & & & \\ & & & -m & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix} \tag{14.11.86}$$

Clearly, in this case $[A, \bar{A}] = 0$, and some Q and $\tilde{Q}$ vanish (where A vanishes) while others must satisfy

$$\tilde{Q}TQ = 0, \bar{Q}TQ = \tilde{Q}T\overline{\tilde{Q}} \tag{14.11.87}$$

One needs in general at least $\frac{1}{2}\dim G$ flavors to satisfy these equations: 2 flavors for SU(2), 4 flavors for SU(3), etc.

Seiberg-Witten theory

Reviews: Alvarez-Gaumé and Hassan, hepth 9701 069

A. Bilal (pure YM only), hepth 9601 007

Ketov, Peskin (duality), Harvey (monopoles), Lerche hepth 9611190.

Articles: Seiberg and Witten: NPB **426** (1994) 19 (pure YM)

NPB **431** (1994) 484 (hypermultiplets).

Pure $N = 2$ super Yang-Mills theory.

The $N = 2$ action can be written as

$$S = Im \int d^4x d^2\theta_1 d^2\theta_2 \frac{1}{2}\tau_{cl} Tr W^2 \; ; \tau_{cl} = \frac{\theta}{2\pi} + i\frac{4\pi}{g^2} \tag{14.11.88}$$

The part with the θ- angle reads in terms of $N = 1$ superfields

$$\begin{aligned} S_\theta &= \frac{\theta}{4\pi} Im \int d^4x d^2\theta_1 Tr \left(2W^\alpha W_\alpha - 2\phi\bar{\nabla}^2\bar{\phi}\right) \\ &= \frac{\theta}{2\pi} Im \int d^4x d^2\theta_1 Tr W^\alpha W_\alpha - \frac{\theta}{2\pi} Im \int d^4x d^4\theta \phi\bar{\phi} \end{aligned} \tag{14.11.89}$$

Since $\phi\bar{\phi}$ is real, the second term vanishes, but the first term contains terms of the form $\nabla_{[\alpha} W_{\beta]} \nabla^{[\alpha} W^{\beta]} \sim (D')^2$ which is real, $W^\alpha \nabla^\beta \nabla_\beta W_\alpha \sim W^\alpha \nabla_\alpha \nabla^\beta W_\beta \sim$

$W^\alpha \nabla_\alpha \bar{\nabla}^{\dot\beta} W_{\dot\beta} \sim \lambda^\alpha \nabla_{\alpha\dot\beta} \bar\lambda^{\dot\beta}$ whose imaginary part is a total derivative $\nabla_{\alpha\dot\beta}(\lambda^\alpha \bar\lambda^{\dot\beta})$ which we drop as it presumably falls off fast enough, and finally

$$\nabla_{(\alpha} W_{\beta)} \nabla^{(\alpha} W^{\beta)} \sim f^{\alpha\beta} f_{\alpha\beta} \sim \left(\frac{F_{\mu\nu} - \frac{1}{2} i \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}}{2} \right)^2 \tag{14.11.90}$$

Hence

$$Im \int d^4x d^2\theta_1 \tau_{cl} Tr W^\alpha W_\alpha = \frac{\theta}{4\pi} \int d^4x Tr F_{\mu\nu} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} - \frac{4\pi}{g^2} \int Tr F^2 d^4x + \cdots \tag{14.11.91}$$

[In $2+1$ dimensions, the two total derivative terms give $N=1$ Chern-Simons theory, with the mass term $\bar\lambda\lambda$].

The $SU(2)$ symmetry of $N=2$ gauge theory is chiral (it acts differently on θ^α and $\bar\theta_{\dot\alpha}$), but since $SU(2)$ is pseudoreal, there are no anomalies for this $SU(2)$. Hence, the $SU(2)_R$ subgroup remains at the quantum level. The $U(1)$ R- symmetry is also axial, but it has anomalies

$$\theta^\alpha \to e^{iw} \theta^\alpha \; ; \; W \to e^{2iw} W \tag{14.11.92}$$

Our aim is to find "the low-energy effective action". There are (at least) 3 effective actions 1. The 1PI generating functional. It has infrared divergences if one goes on-shell.

2. The Wilsonian effective action: cut off all momenta at an IR cut-off.

3. Periwal's two-particle-irreducible generating functional.

"The IR divergences can spoil holomorphic properties and expansion into external momenta make only sense if there are no IR divergences".

Based on dimensional arguments, only assuming susy, the effective action in a low-energy approximation reads

$$\Gamma_{\text{eff}} = Im \int d^4x d^2\theta_1 d^2\theta_2 \mathcal{F}(W) \quad (N=2) \tag{14.11.93}$$

This contains 1-loop corrections plus all higher order leading nonperturbative corrections due to instantons. This result is similar to the $N=1$ nonrenormalization theorem

$$\Gamma_{\text{eff}} = \int d^4x d^2\theta \mathcal{V}(\phi) \quad (N=1) \tag{14.11.94}$$

but there only nonperturbative effects contribute to $\mathcal{V}(\phi)$.

It seems that the on-shell effective action is gauge-choice (α) independent - and this is claimed to be due to it being physical (?).

The tree graph action has a potential $V \sim (D')^2 + D'[\phi, \bar{\phi}] = [\phi, \bar{\phi}]^2$. (Recall that $\phi\bar{\phi} = \phi_{ch} e^{-V} \bar{\phi}_{ch} e^V$ in the chiral representation. This yields a term $Tr\phi_{ch}[\bar{\phi}_{cl}, D'] = Tr[\phi_{ch}, \bar{\phi}_{ch}]D'$). Hence the only minimum is at $[\phi, \bar{\phi}] = 0$, so ϕ and $\bar{\phi}$ lie in the Cartan subalgebra.[24]

The full form of the effective action will now be determined. In $N = 1$ language it reads

$$\begin{aligned} S = Im \int d^4x d^2\theta_1 d^2\theta_2 \mathcal{F}(W) = Im \int d^4x d^2\theta_1 \mathcal{F}_{AB}(\phi) W^{A\alpha} W^B_\alpha \\ + \int d^4x d^4\theta_1 \mathcal{F}_A(\phi) \bar{\phi}^A \end{aligned} \tag{14.11.95}$$

where A, B are group indices. The 1-loop (perturbative) contribution to S is given by

$$\mathcal{F}\ (\text{tree} + \text{one} - \text{loop}) \ = \frac{i}{2\pi} Tr W^2 \ln W^2/\Lambda^2 \quad (\text{for } U(1)) \tag{14.11.96}$$

where Λ^2 is the renormalization mass. Clearly, changes in Λ^2 are proportional to the classical action W^2, so the θ- term in the classical action can be transformed away by a suitable choice of Λ^2. To prove this 1-loop result one may

[24] On the usual complex basis with raising and lowering generators, the following result holds. To obtain a Cartan subalgebra, it must contain at least one "regular element" of the Lie algebra $\mathcal{L}$. A regular element has only r zero eigenvalues where r is the rank of $\mathcal{L}$. For example, in $SU(3)$, the hypercharge λ_8 is not regular since it has 4 zero eigenvalues (t_+, t_z, t_- and y), but t_z is regular (it commutes only with itself and λ_8). The subalgebra (t_+, λ_8) is **not** a Cartan algebra. This is clear from $[t_z, t_+] = t_+$. In fact, the only regular element of $SU(3)$ is t_z. Hence if one begins by choosing an element in $\mathcal{L}$ which does not commute with t_z (any element except t_z and λ_8), **one never gets a Cartan subalgebra**! However, we work in a real basis (with all T_A antihermitian) and then ϕ must lie in the Cartan subalgebra.

(i) check the $U(1)_R$ anomaly

(ii) compute the β function

(iii) do an explicit $(N=1)$ one-loop calculation

We briefly consider these checks.

(i) The $U(1)_R$ symmetry is $W \to e^{2iw}W, \theta \to e^{iw}\theta$. Then $\mathcal{F} \to e^{4iw}\mathcal{F} - \frac{2w}{\pi}W^2 e^{4iw}$. The function e^{4iw} cancels with $d^2\theta_1 d^2\theta_2$, so

$$\begin{aligned}\delta_R\Gamma(1\text{ loop}) &= \frac{-2w}{\pi}\int d^4x Im \int d^2\theta_1 d^2\theta_2 W^2 \\ &= -4w\,(\text{instanton} - \text{number}) \qquad (14.11.97)\end{aligned}$$

This is the correct anomaly. (The instanton number is given by $-\frac{1}{16\pi^2}\int d^4x Tr F_{\mu\nu}\tilde{F}^{\mu\nu}$ where $\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$ and $Tr T_a T_b = -\frac{1}{2}\delta_{ab}$, as usual).

(ii) The classical gauge action was $\tau_{cl} Tr W^\alpha W_\alpha$ and the 1-loop effective action in the gauge sector is $Tr\ \tau_{\text{eff}} W^\alpha W_\alpha$ where $\tau_{\text{eff}} = \mathcal{F}_{WW}$ for $U(1)$. Now $\mathcal{F}_W = \frac{i}{\pi}W \ln W^2/\Lambda^2 + \frac{i}{\pi}W$, hence $\mathcal{F}_{WW} = \frac{i}{\pi}(\ln W^2/\Lambda^2 + 3)$. The β function is then

$$\begin{aligned}\beta &= \frac{\partial}{\partial \ln \Lambda^2} g_{\text{eff}} = \frac{\partial}{\partial \ln \Lambda^2}\sqrt{\frac{4\pi}{Im\ \tau_{\text{eff}}}} = -\frac{1}{2}\frac{\sqrt{4\pi}}{(Im\ \tau_{\text{eff}})^{3/2}}\frac{\partial}{\partial \ln \Lambda^2} Im\ \tau_{\text{eff}} \\ &= \frac{-\frac{1}{2}\sqrt{\pi}}{(Im\ \tau_{\text{eff}})^{3/2}}\left(\frac{-1}{\pi}\right) = \frac{1}{2\sqrt{\pi}}\frac{1}{(\frac{4\pi}{g^2_{\text{eff}}})^{3/2}} = \frac{1}{16\pi^2}\ g^3_{\text{eff}} \qquad (14.11.98)\end{aligned}$$

This should be the beta function for $SU(2)$ gauge theory coupled to a complex scalar and a Weyl fermion in the adjoint representation. (Shifman and Vainstein have discussed that this definition of β is the same at one-loop as the usual one). In fact, one may note that

$$Im\frac{\partial}{\partial \ln \Lambda^2}\mathcal{F} = Im - \frac{1}{2\pi}TrW^2 = -\frac{1}{2\pi}TrW^2 \qquad (14.11.99)$$

Renormalizability requires that $\partial/\partial \ln \Lambda^2$ of the one-loop effective action be proportional to the classical action.

(iii) The 1-loop calculation of $\mathcal{F}_A(\phi)$. The $\mathcal{F}_A$ term is $\mathcal{F}_A(\phi)\bar{\phi} = \left(\frac{i}{\pi}W \ln W^2/\Lambda^2\right)\bar{W}$. So $S_{cl} + \mathcal{F}_A(\phi)\bar{\phi} = \int d^4\theta[\phi\bar{\phi}(1 + \ln \phi^2/\Lambda^2)]$. We shall now do the one-loop calculation which gives this result.

To compute $\mathcal{F}_A(\phi)\bar{\phi}$, one needs propagators for ϕ and W^α, and vertices with up to two quantum fields. So graphs of the form

As we shall derive, for the abelian $U(1)$ subgroup, the gauge choice (α dependence) drops out. In the susy Landau gauge there are no ϕ propagators. Then only gauge propagators and $\phi^2 VV$ vertices remain. Summing the loops with one, two, three etc. $\phi\bar{\phi}$ vertices gives then

$$\int d^4\theta \int \frac{d^4k}{k^2} \ln\left(1 + \frac{\phi\bar{\phi}}{k^2}\right) \tag{14.11.100}$$

This result looks like the Coleman-Weinberg formula, but it contains both the potential and the kinetic terms of the x- space approach, and further there is a prefactor $\frac{1}{k^2}$ which is needed to balance the $\int d^4\theta$.

The explicit 1-loop calculation.

We start from

$$\mathcal{L} = \frac{1}{4g^2}\int d^4x d^4\theta Tr\left[-\frac{1}{2}e^{-V}D^\alpha e^V \bar{D}^2 e^{-V} D_\alpha e^V + \bar{\phi}e^V\phi e^{-V}\right] \tag{14.11.101}$$

We want to obtain

$$\begin{aligned}\mathcal{L}\ (1-\text{loop})\ &= \frac{1}{16\pi}\int d^2\theta d^2\bar{\theta} Im\left(\frac{i}{\pi}W\ln\frac{W^2}{\Lambda^2} + \frac{i}{\pi}W\right)\bar{W}\\ &= \frac{1}{16\pi^2}\int d^2\theta d^2\bar{\theta} Re\left[\phi\bar{\phi}\ln\frac{\phi^2}{\Lambda^2} + \phi\bar{\phi}\right]\\ &= \frac{1}{16\pi^2}\int d^2\theta d^2\bar{\theta}\left(\phi\bar{\phi}\ln\frac{\phi\bar{\phi}}{\Lambda^2} + \phi\bar{\phi}\right) = \frac{1}{16\pi^2}\int d^2\theta d^2\bar{\theta}\phi\bar{\phi}\ln\frac{\phi\bar{\phi}}{\tilde{\Lambda}^2}\end{aligned} \tag{14.11.102}$$

where $\tilde{\Lambda}^2 = \Lambda^2/e$. We thus need $\bar{\phi}\phi V$ and $\bar{\phi}\phi VV$ vertices, and $\phi\bar{\phi}$ and VV propagators. The kinetic action is

$$\mathcal{L}\ (\text{kin})\ = -\frac{1}{2}D^\alpha V\bar{D}^2 D_\alpha V = \frac{1}{2}VD^\alpha\bar{D}^2 D_\alpha V \tag{14.11.103}$$

The gauge-fixing term is

$$\mathcal{L}\,(\text{fix}) = -\frac{1}{\alpha}(D^2V)(\bar{D}^2V) = -\frac{1}{2\alpha}V(D^2\bar{D}^2 + \bar{D}^2D^2)V \tag{14.11.104}$$

Then the sum yields

$$\mathcal{L}(V, \text{kinetic}) = -\frac{1}{2\alpha}V\left(\Box + (1-\alpha)D^\alpha\bar{D}^2D_\alpha\right)V \tag{14.11.105}$$

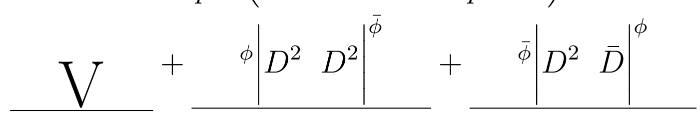
Since $\frac{D\bar{D}^2D}{-\Box}$ is a projection operator, the VV propagator is

$$\begin{aligned}\text{Prop}_{VV} &= \frac{1}{\Box}\left(\alpha - (1-\alpha)\frac{D^\alpha\bar{D}^2D_\alpha}{\Box}\right) \\ &= \frac{-i}{p^2}\left(\alpha + (1-\alpha)\frac{D^\alpha\bar{D}^2D_\alpha}{p^2}\right)\end{aligned} \tag{14.11.106}$$

The $\phi\bar{\phi}$ propagator is $\frac{1}{p^2}$. The vertices are $Tr\{\bar{\phi}[V,\phi] + \frac{1}{2}\bar{\phi}[V,[V,\phi]]\}$.

It is useful to combine the $V\phi\bar{\phi}V$ vertex with the tree graphs $V\phi\underbrace{\bar{\phi}\,\phi}\bar{\phi}V$ + $V\underbrace{\phi\bar{\phi}\phi\bar{\phi}}V$. One finds then, including the V propagator on the right-hand side,

$$\frac{1}{2}\left(\phi\bar{\phi} + \bar{\phi}\phi\right)\frac{-1}{p^2}\left(\alpha + (1-\alpha)\frac{D^\alpha\bar{D}^2D_\alpha}{p^2}\right)$$



$$\begin{aligned}&+\phi\bar{\phi}\frac{D^2\bar{D}^2}{p^2}\ \frac{-1}{p^2}\left\{\alpha + (1-\alpha)\frac{D^\alpha\bar{D}^2D_\alpha}{p^2}\right\} \\ &+\bar{\phi}\phi\frac{\bar{D}^2D^2}{p^2}\frac{-1}{p^2}\left\{\alpha + (1-\alpha)\frac{D^\alpha\bar{D}^2D_\alpha}{p^2}\right\}\end{aligned} \tag{14.11.107}$$

We shifted the $\bar{D}^2$ and D^2 from the left to the right, and then partially integrated them onto the V propagator. **We neglected terms with $\bar{D}^{\dot{\alpha}}\bar{\phi}$ and $D_\alpha\phi$ because we are not interested in them**. We can now drop the terms with three D's because $D^\alpha\bar{D}^2D_\alpha = \bar{D}^{\dot{\alpha}}D^2\bar{D}_{\dot{\alpha}}$, and three D's or $\bar{D}^2s$ vanish. **Using also that the external lines satisfy** $[\phi,\bar{\phi}] = 0$ we arrive at

$$\left(\frac{-\phi\bar{\phi}}{p^2}\right)\left[\alpha + (1-\alpha)\frac{D^\alpha\bar{D}^2D_\alpha}{p^2} + \frac{\alpha D^2\bar{D}^2}{p^2} + \frac{\alpha\bar{D}^2D^2}{p^2}\right] \tag{14.11.108}$$

All $\boldsymbol{\alpha}$ -dependence cancels! We are left with $\left(-\frac{\phi\bar{\phi}}{p^2}\right)\frac{D^\alpha\bar{D}^2D_\alpha}{p^2}$. The reason of the α- independence is that $\bar{D}_{\dot{\alpha}}\bar{\phi} = D_\alpha\phi = [\phi,\bar{\phi}] = 0$ imply the field equations $D^2\bar{D}^2(e^V\bar{\phi}e^{-V}) = 0$ and $\nabla^\alpha W_\alpha = [\phi,\bar{\phi}]$. Hence, as far as $\phi,\bar{\phi}$ lines are concerned, we are on-shell, and then gauge-invariance of the S matrix should do the trick (?).

The sum over "improved vertices" yields

$$\Gamma = \sum\frac{1}{n}\left[-\frac{1}{p^2}(\phi\bar{\phi})\right]^n\frac{D^\alpha\bar{D}^2D_\alpha}{p^2}\prod_j\delta^4(\theta_j-\theta_{j+1}) \tag{14.11.109}$$

where we used (again) that the one-but-last factor is a projection operator. Integration over θ's yields

$$\int\delta^4(\theta_1-\theta_2)D^\alpha\bar{D}^2D_\alpha\delta^4(\theta_1-\theta_2)d^4\theta_1d^4\theta_2 = \int\delta^4(\theta_1-\theta_2) \tag{14.11.110}$$

and one finds

$$\int d^4\theta\int\frac{d^4p}{(2\pi)^4p^2}\ln\left(1+\frac{\phi\bar{\phi}}{p^2}\right) \tag{14.11.111}$$

The p^{-2} in front comes from the factor $D^\alpha\bar{D}^2D_\alpha/p^2$. Doing the integral over angles first leads to

$$\begin{aligned}&= -\frac{1}{(4\pi)^2}\int d^4\theta\int_0^{\mu^2}dp^2\ln\left(1+\frac{\phi\bar{\phi}}{p^2}\right)\\ &= -\frac{1}{(4\pi)^2}\int d^4\theta\left[(p^2+\phi\bar{\phi})\{\ln(p^2+\phi\bar{\phi})-1\}-p^2\{\ln p^2-1\}\right]_0^{\mu^2}\end{aligned} \tag{14.11.112}$$

The $\int d^4\theta$ kills the $-p^2\ln(p^2-1)$, so we get

$$-\frac{1}{(4\pi)^2}\int d^4\theta(\mu^2+\phi\bar{\phi})\{\ln(\mu^2+\phi\bar{\phi})-1\}-\phi\bar{\phi}\ln\phi\bar{\phi}\} \tag{14.11.113}$$

Also $\mu^2\ln\mu^2$ is killed by $\int d^4\theta$, and we obtain

$$\begin{aligned}&-\frac{1}{(4\pi)^2}\int d^4\theta\mu^2\ln\left(1+\frac{\phi\bar{\phi}}{\mu^2}\right)+\phi\bar{\phi}\{\ln(\mu^2+\phi\bar{\phi})-\ln\phi\bar{\phi}\}\}\\ &= -\frac{1}{(4\pi)^2}\int d^4\theta\left[\phi\bar{\phi}\ln\mu^2-\phi\bar{\phi}\ln\phi\bar{\phi}\right]\\ &= \frac{1}{(4\pi)^2}\int d^4\theta\phi\bar{\phi}\left(\ln\frac{\phi\bar{\phi}}{\mu^2}-1\right)\end{aligned} \tag{14.11.114}$$

(Note that this result is gauge invariant, and obtained in the Landau gauge ($\alpha = 0$). This is perhaps Vilkovisky's gauge-invariant effective action).

The unbroken Z_8 subgroup of the $U(1)_R$ symmetry.

Recall the $U(1)_R$ symmetry at the classical level

$$W \to e^{2iw}W, d^4\theta \to e^{-4i\theta}d^4\theta \tag{14.11.115}$$

Since in the path integral we find

$$e^{i\int d^4xd^4\theta Im\mathcal{F}} \tag{14.11.116}$$

We find invariance if one obtains a factor $2\pi ni$ in the variation of the exponent. Now $\delta\int d^4\theta Im\ \mathcal{F} = -8w$ (instanton number), so $w = \pm\frac{\pi}{2}, \pi, 0$ are allowed. In fact, also $w = \frac{\pi}{4}$ (and multiples) are allowed, because then $W^2 \to -W^2$ and $d^4\theta \to -d^4\theta$ so that

$$\int d^4\theta ImW^2 \ln W^2/\Lambda^2 \to \int d^4\theta ImW^2 \ln W^2/\Lambda^2 - \frac{1}{2}Im\int d^4\theta W^2 \tag{14.11.117}$$

The last term is proportional to the instanton number and yields unity in the path integral. So, there is indeed a Z_8 invariance at the quantum level. In fact, the transformation $\theta \to e^{i\frac{\pi}{4}}\theta$ and $W \to {}^{i\frac{\pi}{4}\theta}$ yields the action back but with $\tau \to \tau - 1$. (Indeed $\int d^4\theta Im\ \tau W^2$ with $\tau = \frac{\theta}{2\pi} + i\frac{4\pi^2}{g^2}$, and adding $-\frac{1}{2}\int d^4\theta Im\ W^2$ is equivalent to $\tau \to \tau - 1$.) Recall now

$$\frac{\partial\mathcal{F}}{\partial W} = \frac{i}{\pi}\left(W\ln\frac{W^2}{\Lambda^2} + W\right)\ ;\frac{\partial^2\mathcal{F}}{\partial W^2} = \frac{1}{\pi}\left(\ln\frac{W^2}{\Lambda^2} + 3\right) \tag{14.11.118}$$

The simplest gauge invariant expression is TrW^2, and requiring that it be invariant under "the monodromy at infinity" means $W^2 \to e^{2\pi i}W^2$. This is the Z_4 group, so $W \to -W$. Then

$$\frac{\partial\mathcal{F}}{\partial W} \to -\frac{\partial\mathcal{F}}{\partial W} + 2W \quad (\text{due to } \ln e^{2\pi i} = 2\pi i) \tag{14.11.119}$$

We can write this as an $Sl(2, Z)$ transformation

$$\begin{pmatrix} \mathcal{F}_W \\ W \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \mathcal{F}_W \\ W \end{pmatrix} \text{ (“monodrony at infinity” due to large } t\text{)} \quad (14.11.120)$$

The transformation $\tau \to \frac{(-1)\tau+2}{-1} = \tau - 2$ is clearly a Mobius transformation.

There are no further loop corrections in the $N = 2$ theory, but there are instanton corrections. It is clear that one needs more than the 1 loop result: since $Im\ \tau_{\text{eff}} = \frac{4\pi}{g^2_{\text{eff}}} = \frac{1}{2\pi}(\ln \mid \frac{W^2}{\Lambda^2} \mid +6)$ we find that for W moving from $1 + \epsilon$ to $1 - \epsilon$ there is a zero in $g^{2\ -1}_{\text{eff}}$, and one enters a region with negative g^2_{eff}. (Phrased differently, for $\mid W^2 \mid < \Lambda^2 e^{-6}$ one finds that $4\pi/g^2_{\text{eff}}$ becomes negative).

Duality. To get a hold on the nonperturbative corrections to the effective action ($\mathcal{F}$), we use duality. This is only known for abelian groups, but since $SU(2)$ breaks down to $U(1)$ this is sufficient. We begin with QED, then $N = 1$ Yang-Mills theory, and finally end up with $N = 2$ Yang-Mills theory.

In QED we begin with $F_{\mu\nu}(A) = \partial_\mu A_\nu - \partial_\nu A_\mu$ and have the Bianchi identity $\partial^\mu * F_{\mu\nu}$ where $*F_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$. On-shell $\partial^\mu F_{\mu\nu} = 0$. Also $*(*F) = -F$.

The action reads

$$\begin{aligned} S &= \frac{1}{32\pi} Im \int d^4x\ \tau f^{\alpha\beta} f_{\alpha\beta} = \frac{1}{32\pi} Im\ \tau \int d^4x\ (F_{\mu\nu}(A) + i * F_{\mu\nu}(A))^2 \\ &= \frac{1}{4g^2} F^2_{\mu\nu}(A) + N, N = \frac{1}{32\pi^2} \int d^4x F_{\mu\nu}(A) * F^{\mu\nu}(A) \end{aligned} \quad (14.11.121)$$

Consider now the following parent action

$$S = \frac{1}{32\pi} Im \int d^4x\ \tau\ (F_{\mu\nu} + i * F_{\mu\nu})^2 + \frac{1}{4\pi} \int \tilde{A}_\mu \partial_\nu * F^{\mu\nu} \quad (14.11.122)$$

where $F_{\mu\nu}$ is an independent field. The last term can be rewritten as

$$\frac{1}{8\pi} F_{\mu\nu}(\tilde{A}) * F^{\mu\nu} = \frac{Im}{16\pi} \left(F_{\mu\nu}(\tilde{A}) + i * F_{\mu\nu}(\tilde{A}) \right) (F^{\mu\nu} + i * F^{\mu\nu}) \quad (14.11.123)$$

Completing squares one obtains

$$\begin{aligned} &\frac{1}{32\pi} Im\tau \left[(F + i * F) + \frac{1}{\tau} \left(F(\tilde{A}) + i * F(\tilde{A}) \right) \right]^2 \\ &+ \frac{1}{32\pi} Im \left(-\frac{1}{\tau} \right) \left(F(\tilde{A}) + i * F(\tilde{A}) \right)^2 \end{aligned} \quad (14.11.124)$$

Hence, the new action is related to the old action by $\tau \to -\frac{1}{\tau}$.

(More precisely: the $\tilde{A}$ field equation states that $F^{\mu\nu} = F^{\mu\nu}(A)$. Instead, the F field equation yields

$$\begin{aligned}&\frac{1}{2g^2}F + \frac{\theta}{16\pi^2} * F = -\frac{1}{8\pi} * F(\tilde{A})\\ &Im\ \tau(F + i * F) = - * F(\tilde{A})\end{aligned} \tag{14.11.125}$$

Solving for F and then substituting the result back into the action should yield the action for the dual field $\tilde{A}$ but with $-\frac{1}{\tau}$.)

One can understand the normalization of the Lagrange multiplier term as follows $\partial^\mu F_{\mu 0}(A) = q\delta^3(x)$ for an electric charge. The action is $\int[\frac{1}{4}F^2 + qA^0\delta^3(x)]$. For a magnetic charge $\tilde{q}$ we have $\partial^\mu * F_{\mu 0}(A) = \tilde{q}\delta^3(x)$. This charge should couple to the dual field as $\int \tilde{q}\tilde{A}^0\delta^3(x)$. Using $\partial^\mu * F_{\mu 0} = \tilde{q}$ we expect the coupling term $\int \partial^\mu * F_{\mu 0}\tilde{A}^0$. This explains the factor $\frac{1}{4}$.

For $N = 1$ gauge theory we start with $S = \frac{1}{8\pi}\int Im\ \ d^4xd^2\theta\tau W^\alpha W_\alpha$ where $W_\alpha = \bar{D}^2 D_\alpha(iV)$ (abelian case). The Bianchi identity reads $\nabla^\alpha W_\alpha = \bar{\nabla}^{\dot{\alpha}}\bar{W}_{\dot{\alpha}}$ and the field equation reads $\nabla^\alpha W_\alpha = 0$. (This is the field equation for the variation $(e^{-V}\delta e^V)$. It yields in the chiral representation $\int d^2\theta\bar{D}^2(\nabla_\alpha(e^{-V}\delta e^V))W^\alpha = \int d^4\theta\nabla_\alpha(e^{-V}\delta e^V)W^\alpha = \int d^4\theta(e^{-V}\delta e^V)\nabla^\alpha W_\alpha$. Setting $\nabla^\alpha W_{\alpha\,|\theta=0} = 0$ is the field equation for the auxiliary field. In the higher θ sectors we find the other field equations.) With the Bianchi identity we can also write the field equation as $\nabla^\alpha W_\alpha + \bar{\nabla}^\alpha\bar{W}_{\dot{\alpha}} = 0$. We now relax the Bianchi identity but keep W_α as a chiral field. (This is the simplest relaxation: letting W_α become nonchiral introduces many new fields, while dropping the Bianchi identity only means that we add the terms $\theta_\alpha iD''$ and $\theta^2\chi_\alpha$ to $W_\alpha = \lambda_\alpha + \theta^\beta f_{\beta\alpha} + \theta_\alpha D' + \theta^2\nabla_{\alpha\dot{\beta}}\bar{\lambda}^{\dot{\beta}}$).

The parent action now becomes

$$S = \frac{1}{8\pi}\int Im\ d^4xd^2\theta\ \tau W^\alpha W_\alpha + \frac{1}{4\pi}Im\int d^4xd^2\theta d^2\bar{\theta}i\tilde{V}D^\alpha W_\alpha \tag{14.11.126}$$

Variation of $\tilde{V}$ yields the reality condition on W_α back (due to Im we get two terms: $D^\alpha W_\alpha - (D^\alpha W_\alpha)^\dagger = 0$ hence $D^\alpha W_\alpha = \bar{D}^{\dot{\alpha}} \bar{W}_{\dot{\alpha}}$). Partially integrating the last term leads instead to

$$-\frac{1}{4\pi} Im \int d^4x d^2\theta W^\alpha(\tilde{V}) W_\alpha \tag{14.11.127}$$

Substituting the W_α field equation

$$\tau W_\alpha = W_\alpha(\tilde{V}) \tag{14.11.128}$$

back into the action leads to

$$S = \frac{1}{\theta\pi} \int Im\ d^4x d^2\theta \left(\frac{-1}{\tau}\right) W^\alpha(\tilde{V}) W_\alpha(\tilde{V}) \tag{14.11.129}$$

(Note that if $Im\ \tau > 0$ then also $Im - \frac{1}{\tau} > 0$).

Finally we come to the $N = 2$ case. The action reads

$$S = \frac{1}{16\pi} Im \int d^4\theta \mathcal{F}(W(V)); \bar{D}^a_{\dot{\alpha}} W = 0 \tag{14.11.130}$$

and $D^2_{ab} W = -\bar{D}^2_{ba} \bar{W}$ is the Bianchi identity. The field equation reads $D^2_{ab} W = 0$. (At $\theta = 0$ this is the field equation for the auxiliary field; the rest follows in the other θ sections). Thus the field equation can also be written as $D^2_{ab} W = \bar{D}^2_{ba} \bar{W}$. A parent action is

$$S = \frac{1}{16\pi} Im \int d^4\theta [\mathcal{F}(W) - \tilde{W} W] \tag{14.11.131}$$

where $\tilde{W} = \tilde{W}(\tilde{V})$. Then $\delta S / \delta \tilde{V} = 0$ leads to[25]

$$D^2_{ab} W = -\bar{D}^2_{ba} \bar{W} \tag{14.11.132}$$

and thus one finds then back $W = W(V)$. On the other hand, the W field equation yields

$$\frac{\delta S}{\delta W} = \mathcal{F}_{,W} - \tilde{W}(V) = 0 \tag{14.11.133}$$

[25]Use $\tilde{W}(\tilde{V}) = \bar{D}^4 D^2_{ab} \tilde{V}^{ab}$. The $\bar{D}^4$ leads to a $d^4\theta d^4\bar{\theta}$ integral. The $\tilde{V}$ equation leads to $Im\ D^2_{ab} \tilde{W} = 0$. This is equivalent to $D^2_{ab} \tilde{W} = -\bar{D}^2_{ab} \bar{\tilde{W}}$, see J. Gates and W. Siegel, *Nucl. Phys. B* **195** (1982) 39.

Defining $\tilde{W}(V) = \tilde{W}$, we **define duality by** $\tilde{F}(\tilde{W}) \equiv \mathcal{F}(W) - W\tilde{W}$ (a Legendre transformation). Hence, $\tilde{F}_W = 0$ and $\mathcal{F}_{\tilde{W}} = -W$. Further,

$$\tilde{F}_{\tilde{W}\tilde{W}} = \frac{\partial}{\partial \tilde{W}}(-W) = -\left(\frac{\partial \tilde{W}}{\partial W}\right)^{-1} = -\left(F_{WW}\right)^{-1}. \qquad (14.11.134)$$

Hence $\tilde{\tau} = -\frac{1}{\tau}$. Under this transformation

$$\begin{pmatrix} \tilde{F}_W \\ W \end{pmatrix} \to \begin{pmatrix} \tilde{F}_{\tilde{W}} \\ \tilde{W} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \mathcal{F}_W \\ W \end{pmatrix} \qquad (14.11.135)$$

Let the expectation value (minimum) of W be denoted by a $< W >= a$ (moduli space). In the action we find $\int Im\ \mathcal{F}(W) \sim (Im\mathcal{F}_{WW})\nabla\phi\nabla\bar{\phi} + \cdots$. Then we can introduce a metric on the moduli space

$$\begin{aligned} (dS)^2 &= Im\ \tau da d\bar{a} \\ &= Im\ \mathcal{F}_{aa} da d\bar{a} \\ &= Im\ (dF_a) d\bar{a} \text{ (because } F_W = \tilde{W}) \\ &+ Im\ d\tilde{a} d\bar{a} \end{aligned} \qquad (14.11.136)$$

References

[1] S. Coleman and J. Mandula, *Phys. Rev.* **159** (1967) 1251. The theorem in this paper is rigorously proved for local relativistic quantum field theory in four-dimensional Minkowski spacetime under the assumptions that there are no massless particles, and that for a given mass there are only a finite number of particles. We assume in the text that it also holds when there are massless particles.

[2] B. Zumino, *J. Math. Phys.* **3** (1962) 1055.

[3] H. Georgi, "Lie algebras in particle physics", Benjamin (1982), chapter XXI.

Chapter 15

Kinks, monopoles and other solitons

In this chapter we discuss solitons. We restrict our attention to relativistic field theories and define a soliton as a solution of the classical field equations in Minkowski spacetime which is static (time-independent) and has finite nonzero energy.[1] Having finite energy is equivalent to having finite mass because of the time independence of the fields, and these finite mass solutions can be interpreted as extended particles. We require that the energy be nonvanishing; solutions with vanishing energy we shall view as (nontrivial) vacua. In a moving Lorentz frame a static solution becomes time-dependent, but this is a trivial time-dependence. Nontrivially time-dependent soliton solutions also exist, the prime example being the "breather solutions" in the sine-Gordon model, but we shall not discuss them.

Quantum fluctuations around a soliton can be handled as in ordinary field theory, except that one then is in the situation of an x-dependent background (the soliton) instead of standard flat space. One should really treat both the soliton and the

[1] To avoid confusion, note that instantons are defined as solutions of the classical field equations in **Euclidean** space which have finite **action**. Then a soliton in $D+1$ dimensions can be viewed as an instanton in D space dimensions, since for static fields $H=-L$ and finite energy in $D+1$ dimension for static fields implies finite action in D space dimensions.

fluctuations as extended and point-like particles, but this is a very difficult problem which is perhaps better dealt with by string theory. In what follows below we view the soliton as a background, namely and x-dependent vacuum.

Most of the solitons have a topological origin: one can introduce the concept of a "winding number" for each configuration of the fields (by configuration we mean fields with a given space-time dependence which need not satisfy the field equations). Varying fields inside each configuration changes the energy of these configurations by a finite amount, but one cannot go from a sector with one winding number to a sector with another winding number by a finite change in the energy of the configurations in between: there is an infinite potential barrier. The solitons are then the lowest energy configurations inside a class with given winding number, and are topologically stable.

The topological nature of solitons leads to a concept of topological charge Z. The Hamiltonian H has the lower bound $|H| \geq Z$, and in the models we study the solitons saturate this bound, $H = |Z|$. This bound is called the BPS bound.

Having discussed classical aspects of solitons we move to their quantum properties. The quantization of solitons is a very interesting subject in quantum field theory. There are collective coordinates to be quantized, which leads to new order $\hbar^2$ terms in the action and Hamiltonian. An important question we shall discuss is whether the BPS bound remains saturated at the quantum level.

Supersymmetry gives a new perspective on soliton physics. We shall discuss the quantum BPS bound for susy solitons, and also discuss the multiplet structure of susy solitons.

We shall begin with the simplest soliton: the $1+1$ dimensional bosonic kink. Here we establish the main ideas. Then we move on to the susy kink, and discuss the relation of the susy algebra to the central charge and BPS bound.

Next we discuss solitons in some generality, using the simple but invaluable Derrick theorem. This leads us to monopoles and their susy extension. In many ways they resemble the kink and its susy extension, but we shall see that there are differences w.r.t. the BPS bound.

Finally we address a whole different set of topological field theories: Chern-Simons actions and Wess-Zumino-Witten models. They are intimately related to chiral anomalies which we discussed in chapter X. Here we focus on the topological nature of the effective actions for low-energy processes of mesons and baryons. We leave here the domain of renormalizable field theories, but they have their origin in renormalizable field theories such as QCD, as we shall explain.

1 The kink solution and the BPS bound

The simplest example of a soliton is "the kink", a solution of a relativistic field theory in $1+1$ dimensions with a real scalar field which exhibits spontaneous symmetry breaking. The Lagrangian density is given by

$$\mathcal{L} = \frac{1}{2}\dot{\varphi}^2 - \frac{1}{2}(\varphi')^2 - V; V = \frac{1}{2}U^2(\varphi) \tag{15.1.1}$$

where $\varphi' = \frac{\partial}{\partial x}\varphi, \dot{\varphi} = \frac{\partial}{\partial t}\varphi$ and $U(\varphi) = \sqrt{\frac{\lambda}{2}}(\varphi^2 - \frac{\mu^2}{\lambda})$. For later use we define $a^2 = \mu^2/\lambda$. The action has two Z_2 symmetries, $\varphi \to -\varphi$ and $\varphi(x) \to \varphi(-x)$. We call the latter reflection symmetry. There are two minima of the potential (thus solutions of the field equations) which break the symmetry $\varphi \to -\varphi$ spontaneously, one solution being $\varphi = \mu/\sqrt{\lambda}$ and the other being $\varphi = -\mu/\sqrt{\lambda}$. They preserve the reflection symmetry. The classical kink solution is a time-independent function $\phi_K(x)$ which interpolates between these solutions: it becomes $+\mu/\sqrt{\lambda}$ at $x = +\infty$, and $-\mu/\sqrt{\lambda}$ at $x = -\infty$. (The antikink solution has opposite boundary values and is given by minus the kink solution). The kink solition is odd under reflection symmetry, as we shall see. Rather than solve the field equation for φ with these boundary conditions directly, we shall first show that it also satisfies a first-order differential equation which follows from minimizing the Hamiltonian, and then solve this equation.

The Hamiltonian density can be written as a positive definite square plus a total derivative. This is a general feature of many models with solitons; for example, it also holds for monopoles. The total derivative term has a topological meaning and becomes in supersymmetric theories the central charge of the supersymmetry algebra. If the square vanishes the Hamiltonian becomes equal to the topological (central) charge, and in this case the so-called BPS bound is saturated.

To exhibit that the kink saturates the BPS bound we complete squares as follows for time-independent fields

$$\begin{aligned} \mathcal{H} &= \frac{1}{2}(\varphi')^2 + \frac{1}{2}U^2 = \frac{1}{2}(\varphi' \pm U)^2 \mp U\varphi' \\ H &= \int_{-\infty}^{\infty} \frac{1}{2}(\varphi' \pm U)^2 dx \mp W(\varphi) \mid_{x=-\infty}^{x=+\infty} \end{aligned} \tag{15.1.2}$$

where $\frac{\partial}{\partial x}W(\varphi) = U\varphi'$. More explicitly, the last term is given by

$$\begin{aligned} Z &\equiv \int U(\varphi)\partial_x\varphi dx = \int \partial_x \left(\int_0^{\varphi(x)} U(\varphi')d\varphi' \right) dx \\ &= \int \partial_x \left(\sqrt{\frac{\lambda}{2}} \left\{ \frac{1}{3}\varphi^3 - a^2\varphi \right\} \right) dx = \sqrt{\frac{\lambda}{2}} \left(\frac{1}{3}\varphi^3 - a^2\varphi \right) \mid_{x=-\infty}^{x=+\infty} \end{aligned} \tag{15.1.3}$$

This term only depends on the boundary values of the field $\varphi(x)$.

Finite energy requires that $\partial_x\varphi \pm U(\varphi)$ tend to zero for large $|x|$. Since the solution of $d\varphi \sim \mp\sqrt{\frac{\lambda}{2}}(\varphi^2 - a^2)dx$ is $\varphi(x) \sim \pm a\, tgh\left[\sqrt{\frac{\lambda}{2}}a(x - x_0)\right]$, we see that at the boundaries $\varphi = \pm a$, but $\varphi(+\infty)$ need not be the same as $\varphi(-\infty)$. In higher dimensions this will lead to the concept of a winding number, but in one space dimension one can have at most a flip between $+a$ and $-a$. Hence there is a topological[2] contribution to the energy H.

[2]By topological we mean that the values at $x = +\infty$ and $x = -\infty$ are different. The values themselves, $\pm a$, depend on the dynamics (on λ and μ^2). When we come to the monopole we shall see that the Higgs field behaves like the kink: it has topological winding but its value depends again on the dynamics.

Static minimum energy configurations are solutions of the Euler-Lagrange field equations because for them $H = -L$. We can thus obtain the kink solution by solving $\phi' + U = 0$ for all x, and not only for large x. The result is $\varphi_K(x) = a\ tgh\left(\sqrt{\frac{\lambda}{2}}ax\right)$ with $a = \mu/\sqrt{\lambda}$. We now discuss the relation between the various solutions in more detail.

The Hamiltonian is reflection symmetric and invariant under $\varphi \to -\varphi$, but the solutions break some of these symmetries. For fixed boundary conditions we get minimum energy H (and thus a solution) if

(i) φ is time-independent ($\dot{\varphi} = 0$), and
(ii) $\partial_x \varphi \pm U(\varphi) = 0$ everywhere. The solution for all x is either $\varphi(x) = \pm\ a$ or $\varphi(x) = \pm\ a\ tgh\left(\sqrt{\frac{\lambda}{2}}ax\right)$, with by definition a + sign for the kink solution and a − sign for the antikink solution.

So there are two kinds of vacua (= physical states with lowest energy for given boundary conditions)

(i) trivial vacua: $\varphi = \pm a$, for which $H = 0$ (symmetric under the reflection symmetry). These yield spontaneous symmetry breaking of the Z_2 symmetry in the sense that $\langle\varphi\rangle$ breaks the symmetry $\varphi \to -\varphi$ of the Hamiltonian.
(ii) (anti) kink vacua: $\varphi = \pm\ a\ tgh\sqrt{\frac{\lambda}{2}}ax$ with $H = \frac{4}{3}\sqrt{\frac{\lambda}{2}}a^3$, see (15.1.3) (antisymmetric under the reflection symmetry.) This solution does not describe ordinary spontaneous symmetry breaking because the background solution is here x-dependent. Rather, the solution interpolates between the two vacua in (i). The kink and antikink solution have the same energy $\frac{4}{3}a^3$; they are clearly related both by the Z_2 symmetry and by the reflection symmetry.

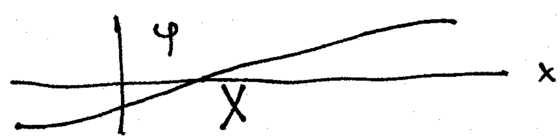
There is an obviously conserved current in this model, $j^\mu = \frac{1}{2}\varepsilon^{\mu\nu}\partial_\nu\varphi$. The corresponding conserved charge T is given by

$$T = \frac{1}{2}\int_{-\infty}^{\infty} \partial_x\varphi dx = \frac{1}{2}\varphi(\infty) - \frac{1}{2}\varphi(-\infty) = \begin{cases} a \text{ for kink} \\ -a \text{ for antikink.} \end{cases} \tag{15.1.4}$$

This current is conserved whether or not the field equation holds and does not seem to correspond to a (rigid) symmetry of the action. It clearly yields a topological charge. The topological charges Z in (15.1.3) and T in (15.1.4) are proportional to each other, and also Z in (15.1.3) can be associated with an identically conserved current, namely $j^\mu = \varepsilon^{\mu\nu}\partial_\nu(\int U d\varphi)$.

The solution for the kink has an arbitrary parameter X, namely $\varphi_K(x-X)$ is also a solution if $\varphi_K(x)$ is a solution. This is clearly due to the translational invariance of the action. We define the point X as the value of x where ϕ_K vanishes. As we shall discuss, X is a collective coordinate. The so-called "kink" and "anti-kink" solutions are given by

$$\phi_{K,\bar{K}} = \pm\frac{\mu}{\sqrt{\lambda}}\tanh\left(\mu(x-X)/\sqrt{2}\right)$$



(15.1.5)

Figure caption: The kink and antikink solutions.

The classical energy of the soliton at rest is given by

$$M_{cl} = \frac{1}{2}\int_{-\infty}^{+\infty}(\phi_{sol}{}')^2 dx + \int_{-\infty}^{\infty} V dx \tag{15.1.6}$$

From the field equation $\phi''_{sol} = \frac{\partial}{\partial\varphi}V(\phi_{sol})$ we find by multiplication by ϕ'_{sol} an equipartition theorem

$$\frac{1}{2}(\phi_{sol}{}')^2 = V(\phi_{sol}) + \text{ constant} \tag{15.1.7}$$

and the constant vanishes since for $x \to \pm\infty$ both ϕ'_{sol} and $V(\phi_{sol})$ tend to zero. Hence the classical mass of the kink at rest is given by

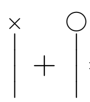
$$M_{cl} = \int_{-\infty}^{\infty}(\phi'_{sol})^2 dx = 2\sqrt{2}\mu^3/3\lambda \tag{15.1.8}$$

To obtain this result one may either use (15.1.3) or substitute $e^z = t$ to evaluate the integral.

In the corresponding quantum theory we have to relate bare and renormalized parameters through appropriate counter-terms. We shall now first discuss renormalization for the theory where the classical solution is $\varphi = \mu/\sqrt{\lambda}$, and later use the same counter terms for the theory with classical solution $\varphi_K(x)$. By power-counting one finds that only the tadpole graphs are divergent. We can make them finite by mass renormalization. This means that we write μ_0^2 instead of μ^2 in the action, and decompose the bare mass parameter μ_0^2 into a renormalized parameter μ^2 and a mass counter term $\delta\mu^2$, namely $\mu_0^2 = \mu^2 + \delta\mu^2$. We expand φ about one of the trivial vacua, $\varphi = \mu/\sqrt{\lambda} + \eta$. Then

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu \eta)^2 - \mu^2\eta^2 - \mu\sqrt{\lambda}\eta^3 - \frac{1}{4}\lambda\eta^4 + \frac{1}{2}\delta\mu^2\left(\eta^2 + 2\frac{\mu}{\sqrt{\lambda}}\eta\right) - (\delta\mu^2)^2/(4\lambda) \tag{15.1.9}$$

hence the renormalized mass parameter m^2 of the physical boson at tree-graph level is $m^2 = 2\mu^2$. We fix $\delta\mu^2$ by requiring that the one-loop tadpole vanishes exactly, which gives



$$= 0 \text{ hence } \delta m^2 = \frac{3\lambda\hbar}{2\pi}\int_0^\Lambda \frac{dk}{(k^2+m^2)^{1/2}} \tag{15.1.10}$$

where we have introduced an ultraviolet cutoff Λ for the logarithmic divergence.

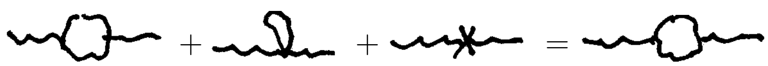
In addition to mass renormalization, we could also introduce coupling constant renormalization and wave function renormalization. From the background formalism for gauge theories we know that we must in general renormalize the background fields and the quantum fields separately. Background fields contain an infinite number of free constants (for example, the coefficients in the expansion into a complete set of functions), hence it is not surprising that one must renormalize the background fields separately. However, here we take the kink solution as background field which has no new arbitrary constants, hence we should not renormalize the background field separately from the quantum fluctuations. So there are only renormalizations possible of the mass μ, the coupling constant λ and the wave function $\eta(x,t)$. Since only the tadpole graphs are divergent, and their divergences can be removed by mass renormalization (as we showed for one-loop tadpoles above), we may choose a minimal renormalization scheme defined at all loops by

$$Z_\lambda = 1, \quad Z_\eta = 1, \quad Z_K = 1, \quad \mu_0^2 = \mu^2 + \delta\mu^2 \tag{15.1.11}$$

This renormalization scheme has the advantage of maximal simplicity, but one must not forget that there are still finite corrections if one is interested in physical definitions of the various parameters. Defining for instance the physical mass m_p of the meson through the pole of its propagator leads to an additional finite contribution at the one-loop level from the self-energy diagram

$$p^2 + m^2 + \Pi(p^2, m^2) = 0 \text{ at } p^2 + m_p^2 = 0 \tag{15.1.12}$$

When δm^2 is fixed by (20.0.10) the seagull graph (the one-loop graph with a four-point vertex) cancels the counter term. In diagrammatic notation we have



(15.1.13)

We can then iteratively solve for m_p^2, and find at the one-loop level

$$m_p^2 = m^2 + 9\lambda i\hbar \int \frac{d^2k}{(2\pi)^2} \frac{m^2}{(k^2+m^2)((k-p)^2+m^2)}_{|p^2 \to -m^2} = m^2 - \frac{\sqrt{3}}{2}\hbar\lambda\ . \tag{15.1.14}$$

However the **ratio** of the quantum mass of the kink and the physical mass of the meson is independent of any choice of $\delta\mu^2, Z_\lambda Z_\eta$, and Z_K.

Having renormalized the theory with a flat background, we turn to the sector with the kink as background. The normal modes of fluctuations $\eta(x,t) = \eta(x)\exp(-i\omega t)$ around φ_K are given by

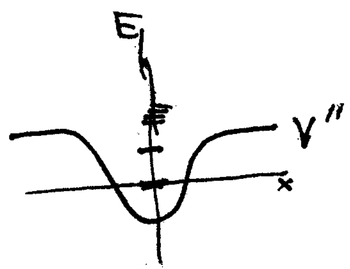
$$\left(-\frac{d^2}{dx^2} + V''(\varphi_K)\right)\eta_n(x) = \omega_n^2\ \eta_n(x) \tag{15.1.15}$$

and can be expressed in terms of elementary functions. There are two discrete eigenvalues, $\omega_{(0)} = 0$ corresponding to the translational zero mode, and $\omega_B = \sqrt{3}m/2$

which corresponds to a bound state (an excited state of the kink), followed by a continuum of eigenvalues $\omega = \sqrt{k^2+m^2}$ corresponding asymptotically to plane waves with a k-dependent phase shift,

$$\eta_k(x) \sim \exp\left(i[kx \pm \delta(k)/2]\right) \quad \text{for } x \to \pm\infty \quad \text{with } \delta(k) = -2\arctan\frac{3mk}{m^2-2k^2}\,. \tag{15.1.16}$$

Putting the system in a box of length L and imposing antiperiodic boundary conditions, the momenta k_n are solutions of $k_n L + \delta(k_n) = 2\pi n + \pi$.



$$\omega_B^2 = \frac{3}{4}m^2 = \frac{3}{4}(\sqrt{2}\mu)^2 = \frac{3}{2}\mu^2 \tag{15.1.17}$$

Figure caption: ω_B denotes a bound state: an excitation of the soliton. It has positive energy ($\omega_B > 0$). $\omega_{(0)}$ denotes the translational zero mode: the excitation which moves the soliton as a whole and thus does not change its energy ($\omega_{(0)} = 0$). ω_n denotes the continuum, mesons with energies $\omega_n = (k_n^2+m^2)^{1/2}$. Far away (large $\mid x \mid$) the effects of the soliton vanish, hence the mesons around the soliton have the same rest mass as the mesons without solitons ($m^2 = 2\mu^2$).

2 The supersymmetric kink

The supersymmetric (susy) extension of the kink is given by

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu\varphi)^2 - \frac{1}{2}U^2(\varphi) - \frac{1}{2}\bar{\psi}\gamma^\mu\partial_\mu\psi - \frac{1}{2}U'(\varphi)\bar{\psi}\psi \tag{15.2.1}$$

where $\bar{\psi} = \psi^\dagger\gamma^2 = \psi^T C$ since ψ is a Majorana spinor. The matrix C is the charge conjugation matrix in two dimensions which satisfies $C\gamma^\mu C^{-1} = -\gamma^{\mu,T}$. The fermionic terms in the action are clearly hermitian. We use a Majorana representation of the Dirac matrices with $\gamma^1 = \tau^3 = \tau_3$ and $\gamma^2 = \tau^2 = \tau_2$. Then $C = \tau^2$ and $\psi = \begin{pmatrix}\psi^+ \\ \psi^-\end{pmatrix}$ with real $\psi^+(x,t)$ and $\psi^-(x,t)$.[3] Furthermore, $\gamma^\mu\partial_\mu = \gamma^1\partial_x + \gamma^0\partial_t$ with $\gamma^2 = i\gamma^0$. Thus $(\gamma^1)^2 = (\gamma^2)^2 = +1$ but $(\gamma^0)^2 = -1$.

[3] The Lorentz algebra is normalized to $[L_{mn}, L_{rs}] = \eta_{nr}L_{ms} + 3$ terms. Using the vector representation of the Lorentz generators, $(L_{mn})^p{}_q = \delta^p_m\eta_{nq} - \eta_{mq}\delta_n{}^p$, we find that x^p transforms under $\lambda^{01}L_{01}$ as

The susy transformation rules which leave the action invariant are

$$\begin{aligned} \delta\varphi &= \bar{\varepsilon}\psi = -i(\varepsilon^+\psi^- - \varepsilon^-\psi^+) \\ \delta\psi &= \partial\!\!\!/\varphi\varepsilon - U(\varphi)\varepsilon \Rightarrow \begin{array}{l} \delta\psi^+ = -\dot{\varphi}\bar{\varepsilon} + (\partial_x\varphi - U)\varepsilon^+ \\ \delta\psi^- = \dot{\varphi}\varepsilon^+ - (\partial_x\varphi + U)\varepsilon^- \end{array} \end{aligned} \tag{15.2.2}$$

We shall discuss the invariance of the action in a moment. The terms without U and the U- dependent terms are then separately supersymmetric. Using $-\bar{\psi}\gamma^\mu\partial_\mu\psi = i\psi^+\partial_t\psi^+ + i\psi^-\partial_t\psi^- + i\psi^+\partial_x\psi^- + i\psi^-\partial_x\psi^+ = \frac{i}{2}(\psi^+ + \psi^-)(\partial_t + \partial_x)(\psi^+ + \psi^-) + \frac{i}{2}(\psi^+ - \psi^-)(\partial_t - \partial_x)(\psi^+ - \psi^-)$ and $\bar{\psi}\psi = -2i\psi^+\psi^-$, it is clear that $\psi^+ + \psi^-$ is left-moving and $\psi^+ - \psi^-$ is right-moving, and the mass term couples the left- and right-moving sector of the fermions. One could add an auxiliary field F to the action and transformation rules. Then $\delta\psi = \partial\!\!\!/\varphi\varepsilon + F\varepsilon, \delta F = \bar{\varepsilon}\partial\!\!\!/\psi$ and in the action $-\frac{1}{2}U^2$ is replaced by $\frac{1}{2}F^2 + FU$. With this auxiliary field the susy algebra closes (there are then no terms proportional to the ψ equation of motion on the right-hand side of the commutator of two susy transformations of ψ). Since we shall not need this auxiliary field, we do not introduce it.

The background solution $\varphi = \varphi_K(x), \psi = 0$ breaks half of supersymmetry. This is in general true for solitons, and is easy to prove in our case. We must show that $\delta\varphi_K = 0$ and $\delta\psi = 0$ if we substitute $\varphi = \varphi_K$ and $\psi = 0$ on the right-hand sides of $\delta\varphi$ and $\delta\psi$. The $\delta\varphi = 0$ is obvious but for $\delta\psi = 0$ we use that $\dot{\varphi}_K = 0$ and the BPS equation $\partial_x\varphi_K + U(\varphi_K) = 0$. Then it is clear that only the transformations with ε^-

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$\delta t = \lambda^{01}x$ and $\delta x = \lambda^{01}t$. More generally, $\delta v^p = \lambda^p{}_q v^q$. Then a covariant vector v_p (for example $\frac{\partial}{\partial x^p}x^2$) transforms as $\delta v_p = \lambda_p{}^q v_q$, so $\delta v_0 = -\lambda^{01}v_1$ and $\delta v_1 = -\lambda^{01}v_0$. If we denote $t+x$ by x^+ and $t-x$ by x^-, then $\delta x^+ = \lambda^{01}x^+$ and $\delta x^- = -\lambda^{01}x^-$. With $\eta^{+-} = \frac{\partial x^+}{\partial x^\alpha}\frac{\partial x^-}{\partial x^\beta}\eta^{\alpha\beta} = -2$ we find $\eta_{+-} = -\frac{1}{2}$ and $x_+ = -\frac{1}{2}x^- = \frac{1}{2}(t+x)$ and $x_- = \frac{1}{2}(t-x)$. Then $\delta x_+ = -\lambda^{01}x_+$ and $\delta x_- = \lambda^{01}x_-$.

The spin 1/2 Lorentz generators are given by $\frac{1}{2}\lambda^{mn}\gamma_m\gamma_n$ with $m < n$, since they, too, satisfy $[L_{mn}, L_{rs}] = \eta_{nr}L_{ms} + 3$ terms. A spinor transforms then under $\lambda^{01}L_{01}$ as $\delta\psi = \lambda^{01}\frac{1}{2}\gamma_0\gamma_1\psi$. Using $\gamma_0 = i\tau_2$ and $\gamma_1 = \tau_1$ this becomes $\delta\psi = \lambda^{01}\frac{1}{2}\tau_3\psi$. With $\psi = \begin{pmatrix} \psi^+ \\ \psi^- \end{pmatrix}$ this becomes $\delta\psi^+ = \frac{1}{2}\lambda^{01}\psi^+, \delta\psi^- = -\frac{1}{2}\lambda^{01}\psi^-$. This explains why the upper (lower) component of ψ are denoted by $\psi^+(\psi^-)$. Sometimes the notation x^{++} for $t+x$ and $x^=$ for $t-x$ is used, to indicate that x^{++} has twice the Lorentz weight of ψ^+. We shall not use this notation.

leave the background invariant, but not those with ε^+: the soliton breaks half of the susy.

We shall now construct the susy generators, and evaluate their anticommutators. The topological charge will be produced and susy will give additional information about it. To obtain the Noether current for susy, we vary the action, using (15.2.2) with local $\varepsilon(x,t)$

$$\begin{aligned}\delta S = \delta \int \mathcal{L} d^2x &= \int [\,(\Box\varphi)\bar{\varepsilon}\psi - \bar{\psi}\not{\partial}(\not{\partial}\varphi\varepsilon - U(\varphi)\varepsilon) \\ -UU'\bar{\varepsilon}\psi &- U'\bar{\psi}(\not{\partial}\varphi\varepsilon - U\varepsilon) - \frac{1}{2}U''\bar{\varepsilon}\psi\bar{\psi}\psi\,]\,d^2x \end{aligned} \tag{15.2.3}$$

The $\Box\varphi$ terms cancel since $\bar{\varepsilon}\psi = \bar{\psi}\varepsilon$, and the U' terms cancel, proving that we chose the sign of the term $U\varepsilon$ in $\delta\psi$ correctly. The UU' terms then fortunately cancel. Also the U'' term vanishes since $\psi_\alpha\psi_\beta\psi_\gamma = 0$ if α, β, γ can only be $+$ or $-$. There remain terms with $\partial_\mu\varepsilon$; these give the susy Noether current

$$\begin{aligned}\delta S &= \int \bar{j}^\mu \partial_\mu \varepsilon d^2x = \int -\bar{\psi}\gamma^\mu(\not{\partial}\varphi - U)\partial_\mu\varepsilon d^2x = \int \partial_\mu\bar{\varepsilon} j^\mu d^2x \\ j^\mu &= -(\not{\partial}\varphi + U)\gamma^\mu\psi \end{aligned} \tag{15.2.4}$$

This current is conserved when the equations of motion hold, $\Box\varphi - UU' - \frac{1}{2}U''\bar{\psi}\psi = 0$ and $\not{\partial}\psi + U'\psi = 0$.

The susy charges are the space integrals of the time component of the Noether current. Since the latter is a two-component spinor, we have two susy charges, Q^+ and Q^-. We should stress at this point that the $Q^\pm$ are composite operators, hence one should really apply point-splitting methods to evaluate the anticommutators. We shall begin by naively evaluating the anticommutators (but keeping track of the ordering in which the operators φ and ψ come), but later we shall come back to the issue of point splitting and other regularization schemes.

In one-component notation, the Dirac equation reads

$$(\partial_1 - \partial_0)\psi^- + U'\psi^+ = 0; (\partial_0 + \partial_1)\psi^+ + U'\psi^- = 0 \tag{15.2.5}$$

We also decompose j^0 into one-component parts[4] j^{0+} and j^{0-}. Using $\gamma^0 = -i\gamma^2$ and $\gamma^1\gamma^0 = \tau_3$ we get the following two conserved susy charges $Q = \int j^0 dx$ with $j^0 = (\partial_0\varphi - \tau_3\partial_1\varphi)\psi - U\gamma^0\psi$

$$\begin{array}{l} Q^+ = \int \left[\{(\partial_0 - \partial_1)\varphi\}\psi^+ + U\psi^-\right] dx \\ Q^- = \int \left[\{(\partial_0 + \partial_1)\varphi\}\psi^- - U\psi^+\right] dx \end{array}, \psi = \begin{pmatrix} \psi^+ \\ \psi^- \end{pmatrix}, Q = \begin{pmatrix} Q^+ \\ Q^- \end{pmatrix} \tag{15.2.6}$$

Both Q^+ and Q^- are hermitian. To find the anticommutators of Q^+ and Q^-, we use Dirac quantization for $\mathcal{L} = -\frac{1}{2}\bar{\psi}\partial\!\!\!/\psi = \frac{i}{2}(\psi^+\dot{\psi}^+ + \psi^-\dot{\psi}^- + \ldots)$.

The conjugate momenta of ψ^+ and ψ^- are given by $\pi_+ = \frac{\partial}{\partial\dot{\psi}^+}\int \mathcal{L} dx = -\frac{i}{2}\psi^+$ and $\pi_- = \frac{\partial}{\partial\dot{\psi}_-}\int \mathcal{L} dx = -\frac{i}{2}\psi_-$, respectively, and the Dirac brackets read

$$\begin{aligned} [p(x), \varphi(y)] &= -i\hbar\delta(x-y) \\ \left\{\psi^+(x), \psi^+(y)\right\} &= \hbar\delta(x-y); \text{idem for } \psi^- \end{aligned} \tag{15.2.7}$$

(The bracket $\{\pi_+, \psi_+\}$ gets an extra factor $\frac{1}{2}$ in the Dirac formalism because ψ^+ is proportional to π_+). We then find for $\{Q^+, Q^+\}$ the expected result

$$\begin{aligned} \left\{Q^+, Q^+\right\} &= \hbar \int \left[\{(\partial_0 - \partial_1)\varphi\}^2 - 2i\psi^+\partial_1\psi^+ + U^2 - 2iU'\psi^+\psi^-\right] dx \\ &= 2\hbar \int (T_{00} - T_{01})\, dx = 2\hbar P_- \end{aligned} \tag{15.2.8}$$

where $T_{\mu\nu}$ is the energy-momentum tensor[5], obtained by putting the action in curved space and varying w.r.t. the metric[6]

$$T_{\mu\nu} = -2\frac{\delta}{\delta g^{\mu\nu}} S_{|g_{\mu\nu}=\eta_{\mu\nu}} = \partial_\mu\varphi\partial_\nu\varphi + \frac{1}{2}\bar{\psi}\gamma_\mu\partial_\nu\psi + \eta_{\mu\nu}\mathcal{L}$$

[4]The current $j^{\mu\alpha}$ with vector index $\mu = 0, 1$ and spinor index $\alpha = +, -$ contains helicity $\pm 1/2$ parts and helicity $\pm 3/2$ parts. For example, $j^{+\alpha} \equiv j^{0\alpha} + j^{1\alpha}$ has a helicity $+3/2$ part $j^{++} \sim (\partial_0\varphi - \partial_1\varphi)\psi^+$ and a helicity $+1/2$ part $j^{+-} \sim U\psi^+$. In covariant notation the spin 1/2 part is given by $\gamma^\mu j_\mu = -2U\psi$ and the spin 3/2 part by $j^\mu - \frac{1}{2}\gamma^\mu\gamma \cdot \psi = -(\partial\!\!\!/\varphi)\gamma^\mu\psi$.

[5]The conservation of $T_{\mu\nu}$ on-shell follows easily and $T_{\mu\nu} = \frac{1}{2}\bar{\psi}\gamma_\mu\partial_\nu\psi$ is symmetric on-shell. (In the fermionic sector, $T_{01} - T_{10}$ is proportional to $\psi^+(\partial_0 + \partial_1)\psi^+ - \psi^-(\partial_0 - \partial_1)\psi^-$ which becomes on-shell equal to $-(\psi^+\psi^- + \psi^-\psi^+)U'$ which evidently vanishes. In covariant form, $\bar{\psi}\gamma_\mu\partial_\nu\psi - \bar{\psi}\gamma_\nu\partial_\mu\psi = \bar{\psi}\gamma_\mu\gamma_\nu\partial\!\!\!/\psi - \eta_{\mu\nu}\bar{\psi}\partial\!\!\!/\psi$ since $\gamma_{[\mu}\gamma_\nu\gamma_{\rho]} = 0$ in two dimensions).

[6]For fermions one must use vielbeins (zweibeins, "two-legs" in our case). They appear as $-\frac{e}{2}\bar{\psi}\gamma^m e_m{}^\mu\partial_\mu\psi$ where $e = \det e_\mu{}^m$, but no spin connection term $-\frac{e}{2}\bar{\psi}\gamma^m\frac{1}{4}\omega_\mu{}^{kl}\gamma_k\gamma_l\psi$ is needed since it vanishes in two dimensions for Majorana spinors. The stress tensor is then defined by $T_{\mu\nu} = -e_{m\nu}\frac{\delta}{\delta e_m{}^\mu}S$ and this yields (15.2.9).

$$= \quad \partial_\mu\varphi\partial_\nu\varphi - \frac{1}{2}\eta_{\mu\nu}\left\{(\partial\varphi)^2 + U^2\right\} + \frac{1}{2}\bar{\psi}\gamma_\mu\partial_\nu\psi \ \text{ on} - \text{shell} \qquad (15.2.9)$$

Then $T_{00} = \frac{1}{2}\dot{\varphi}^2 + \frac{1}{2}(\varphi')^2 + \frac{1}{2}U^2 + \frac{i}{2}\psi^T\dot{\psi}$ and $T_{01} = \dot{\varphi}\varphi' + \frac{i}{2}\psi^T\partial_x\psi$ so that

$$T_{00} - T_{01} = \frac{1}{2}(\dot{\varphi} - \varphi')^2 + \frac{1}{2}U^2 + \frac{i}{2}\psi^+(\partial_0 - \partial_1)\psi^+ + \frac{i}{2}\psi^-(\partial_0 - \partial_1)\psi^-. \qquad (15.2.10)$$

Using $\partial_0\psi^+ = -\partial_1\psi^+ - U'\psi^-$ and $(\partial_0 - \partial_1)\psi^- = U'\psi^+$ the fermionic terms in P_- become $-i\psi^+\partial_1\psi^+ - i\psi^+U'\psi^-$ which agrees with the result for $\{Q^+, Q^+\} = 2\hbar P_-$ in (15.2.8).

Similarly one may verify that $\{Q^-, Q^-\} = 2\hbar P_+$ with $P_+ = \int(T_{00} + T_{01})dx$ and the fermionic terms in P_+ become $i\psi^-\partial_1\psi^- - i\psi^+U'\psi^-$. Clearly, $P_1 = \frac{1}{2}(P_+ - P_-)$ contains as fermionic terms $\int\left(\frac{i}{2}\psi^+\partial_1\psi + \frac{i}{2}\psi^-\partial_1\psi^-\right)dx$ and generates translations. If we lower the spinorial indices of $Q^\pm$ by the ε symbol, $Q^+ = \varepsilon^{+-}Q_- = Q_-, Q^- = \varepsilon^{-+}Q_+ = -Q_+$, we finally obtain

$$\{Q_+, Q_+\} = 2\hbar P_+, \{Q_-, Q_-\} = 2\hbar P_- \qquad (15.2.11)$$

However, for the anticommutator $\{Q^+, Q^-\}$ we find a "central term". Using the following identity for two bosonic operators B and b, and two fermionic operators F and f

$$BFbf + bfBF = Bb\{F, f\} + [b, B]fF \qquad (15.2.12)$$

we find, using also $\psi^-(x)\psi^-(y)\delta(x - y) = 0$, by straightforward application of the canonical (anti) commutation relations without bothering about regularization of these composite operators

$$\left\{Q^+, Q^-\right\} = \hbar\int_{-\infty}^{\infty} 2U\partial_1\phi dx = 2\hbar\left[\int_0^{\phi(x)} U(\phi')d\phi'\right]_{-\infty}^{\infty} = \int_{\phi(-\infty)}^{\phi(+\infty)} U(\phi')d\phi' \equiv 2\hbar Z \qquad (15.2.13)$$

In the next section we shall use the full power of quantum field theory and regularize the composite operators $Q^\pm$; then extra terms will be found in Z which can be identified with anomalies, and which are necessary to maintain saturating the BPS bound at the quantum level since also H receives (equal) quantum corrections.

The bosonic kink solution is also the bosonic part of a solution of the susy field equations in which the fermion fields vanish. For the kink solution, $\partial_x\varphi + U = 0$, and $Z = -H = +\int U\partial_x\varphi dx$ because the fermionic terms in H are proportional to $\psi^T\dot{\psi}$ and vanish since the kink is a time-independent solution. For the antikink $\partial_x\varphi - U = 0$, and $Z = +H = \int U\partial_x\varphi dx$. Hence

$$\left\{Q^+, Q^-\right\} = 2\hbar Z = \mp 2\hbar M \tag{15.2.14}$$

where M is the value of the Hamiltonian for these solutions. It is clear that, for given $U(\varphi)$, i.e., for a given model, the central charge Z is a topological charge: it only depends on $\varphi(+\infty)$ and $\varphi(-\infty)$.

We shall now derive an expression for the total energy $\frac{1}{2}(P_+ + P_-) = \frac{1}{2}(P_0 + P_1) + \frac{1}{2}(P_0 - P_1) = P_0$ in terms of the susy generators.

$$\begin{aligned}\hbar(P_+ + P_-) &= Q_+Q_+ + Q_-Q_- \\ &= (Q_+ \mp Q_-)^2 \pm \{Q_+, Q_-\} = (Q_+ \mp Q_-)^2 \pm 2\hbar Z \end{aligned}\tag{15.2.15}$$

This looks very much like (15.1.3): the total energy $\frac{1}{2}(P_+ + P_-)$ is a sum of a boundary term Z and a positive definite term. Clearly, the minimum energy for fixed boundary values of $\varphi(x)$ is obtained when this positive definite term vanishes, $Q_+ \mp Q_- = 0$. Since

$$Q^+ + Q^- = \int \left\{\left[(\partial_0 - \partial_1)\varphi - U\right]\psi^+ + \left[(\partial_0 + \partial_1)\varphi + U\right]\psi^-\right\} dx \tag{15.2.16}$$

we see that *for static* φ the charge $Q^+ + Q^- = Q_- - Q_+$ vanishes for the kink solution. Hence, for the kink solution $P_+ + P_- = 2M$ agrees with $Z = -H$. For the antikink we find that $Q^+ - Q^-$ vanishes, and then again $P_+ + P_- = 2M$.

One can relax the property that the solutions are static, and derive the bound that the mass $M \geq |Z|$ in a manifestly Lorentz invariant way by rewriting $M^2 = \frac{1}{2}(P^+P^- + P^-P^+)$ in terms of susy charges by using the susy anticommutators (again omitting subtleties having to do with regularization)

$$\begin{aligned} 2\hbar^2 M^2 &= \left(Q^+Q^+Q^-Q^- + Q^-Q^-Q^+Q^+\right) \\ &= -Q^+Q^-Q^+Q^- - Q^-Q^+Q^-Q^+ + \left\{Q^+, Q^-\right\}^2 \\ &= -(Q^+Q^- - Q^-Q^+)^2 - Q^+Q^-Q^-Q^+ - Q^-Q^+Q^+Q^- + \left\{Q^+, Q^-\right\}^2 \end{aligned} \tag{15.2.17}$$

We used that $\{Q^+, Q^-\}$ commutes with Q^+ and Q^-. If one then rewrites one-half of the second term as $+\frac{1}{2}Q^-Q^+Q^-Q^+ - \frac{1}{2}\{Q^+, Q^-\}\,Q^-Q^+$ and similarly one-half of the third term as $\frac{1}{2}Q^+Q^-Q^+Q^- - \frac{1}{2}\{Q^-, Q^+\}\,Q^+Q^-$ one finds

$$2\hbar^2 M^2 = -(Q^+Q^- - Q^-Q^+)^2 + \frac{1}{2}\left\{Q^+, Q^-\right\}^2 = (\bar{Q}Q)^2 + 2\hbar^2 Z^2 \tag{15.2.18}$$

Since $\bar{Q}Q = -i(Q^+Q^- - Q^-Q^+)$ is hermitian and $(\bar{Q}Q)^2$ is positive, $M^2 \geq Z^2$, so the bound is saturated only for states $|\alpha>$ satisfying $\bar{Q}Q|\alpha>= 0$. In the rest frame where $P_+ = P_-$, we can rewrite $\bar{Q}Q = -i(Q^+Q^- - Q^-Q^+)$ as $\bar{Q}Q = i(Q^+ + Q^-)(Q^+ - Q^-) = -i(Q^+ - Q^-)(Q^+ + Q^-)$, and hence any state that is annihilated by $Q^+ + Q^-$ or $Q^+ - Q^-$ saturates the bound. The static kink and antikink solutions are just particular solutions, see the discussion below (15.2.16).

Although the minimum energy conditions $Q^+ \pm Q^- = 0$ have only led us back to the purely bosonic kink/antikink solutions, also ψ^+ and ψ^- are fixed because they should be *normalizable* solutions of the Dirac equation, for the same reasons that the zero modes of fermions in an instanton background should be normalizable. The ψ^+ and ψ^- must satisfy the field equations in order that Q^+ and Q^- be conserved (in operator language: all operators are Heisenberg operators and thus satisfy the field equations). The field equations for static ψ^+ and ψ^- can be combined into

$$(\partial_1 + U')(\psi^+ + \psi^-) = 0; (\partial_1 - U')(\psi^+ - \psi^-) = 0 \tag{15.2.19}$$

The solution of the first equation is either $\psi^+ + \psi^- = 0$ or

$$(\psi^+ + \psi^-)(x) = \exp - \int_0^x U'\left[\varphi(y)\right] dy \tag{15.2.20}$$

Similarly, the equation for $\psi^+ - \psi^-$ has as solution either $\psi^+ - \psi^- = 0$ or $\psi^+ - \psi^- = \exp \int_0^x U'[\varphi(y)]dy$. Since $U'(\varphi) = 2\lambda\varphi$ and $\varphi = \pm\, a\; tgh(\lambda a x)$ for the kink or antikink, we see that only one solution is normalizable. One finds then that for the kink $\psi^+ + \psi^- = \frac{1}{(\cosh \lambda a x)^2}$ and $\psi^+ - \psi^- = 0$ while for the antikink $\psi^+ - \psi^- = \frac{1}{(\cosh \lambda a x)^2}$ and $\psi^+ + \psi^- = 0$. Since ψ^+ is proportional to ψ^-, the fermionic term $-\frac{1}{2}U''\bar{\psi}\psi$ in the bosonic field equation vanishes by itself, and hence the kink/antikink solutions are the bosonic part of solutions of the susy system which also have nonvanishing fermionic parts.

Since for time-independent solutions of the Dirac equation the Hamiltonian is equal to minus the Lagrangian, the normalizable solution of the fermionic field equation in the background of the kink has zero energy. In second quantization, it corresponds to a creation and annihilation operator $b^\dagger$ and b. It is a zero mode because it drops out of the action. (Recall that the Dirac action is itself proportional to its field equations.) Considering a vacuum $\mid$ kink $>$ which corresponds to the kink as background, we may define $b \mid$ kink $>= 0$. Then we find an N=1 susy multiplet $\mid$ kink $>, b^\dagger \mid$ kink $>$ of two states with the same energy.[7] In general, $N = 1$ multiplets can be either massless, or massive without central charge Z, or massive with central charge Z. The massless multiplets consists of two states with helicity λ and $\lambda + 1/2$. In $3+1$ dimensions, the massive ones are twice as long, except when the mass M equals $\mid Z \mid$, in which case they are as short as the massless multiplets. In $1+1$ dimensions, all multiplets have the same length.[8] The zero modes of the kink

[7]There are also normalizable zero modes in the bosonic sector. They are due to the translational invariance of the kink solution and read $\tilde{\varphi}(x) = \frac{\partial}{\partial x}\varphi(x) = \lambda a^2 \cosh^{-2}(\lambda a x)$. Then all zero modes form a $N = 1$ multiplet.

[8]By helicity we mean in 2 dimensions the eigenvalue of the Lorentz generator L_{01}. "Multiplets" are irreducible representations of the susy algebra in terms of "states", i.e., vectors in a Hilbert space with a positive definite inner product. From (15.2.18) it is clear that massless representations cannot have a nonvanishing central charge Z. Consider all states of a given massless multiplet with two-momentum $q^\mu = (q, q)$. They are annihilated by $P^- = P^0 - P^1$, and hence in this space $\{Q^-, Q^-\} = 0$, so $Q^- = 0$. Further, $\{Q^+, Q^+\} = 2q$ and $\{Q^+, Q^-\} = 0$. The multiplets are then doublets which consist of a "vacuum" $\mid 0 >$ and the state $Q^+ \mid 0 >$. Since $[M_{12}, Q^\pm] = \pm\frac{1}{2}Q^\pm$, these states have helicities which differ

system form a massive multiplet with maximal central charge.

We conclude that the susy algebra in $1+1$ dimensions reads

$$\{Q_+, Q_+\} = 2\hbar P_+, \{Q_-, Q_-\} = 2\hbar P_-, \{Q_+, Q_-\} = 2\hbar Z \qquad (15.2.21)$$

The central charge Z is a topological charge and saturates the BPS bound. It is equal to minus the mass ($Z = -M$ for the kink and $Z = M$ for the antikink). The bosonic and fermionic zero mode form a massive $N = 1$ susy multiplet which is as short as a massless multiplet in $1+1$ dimensions. In $3+1$ dimensions massive multiplets are twice as long, and since one can prove using index theory (the Witten index for the sum of the number of bosonic and fermionic zero modes) that new zero modes cannot be created at the quantum level, the BPS bound $M = |Z|$ must hold in $3+1$ dimensions to all orders of perturbation theory. (This argument does not give information on whether the quantum correction are zero or nonzero, it only states that they are equal).

This concludes our discussion of the $1+1$ dimensional case at the classical level.

by $1/2$. An example of a massless multiplet is the action in (15.2.1) which indeed describes one bosonic and one fermionic state. (In $3+1$ dimensions, one must add to the $\lambda = (0, 1/2)$ multiplet its CTP conjugate multiplet $\lambda = (0, -1/2)$ but in $1+1$ dimensions states with a given helicity are eigenstates of CPT. The reason is that the charge conjugation matrix $C_{\alpha\beta}$ in $3+1$ dimensions ($\varepsilon_{\alpha\beta}$ on 2-component spinors) flips the helicity, but in $1+1$ dimensions $C_{\alpha\beta}$ maps a Majorana spinor into itself.) For massive multiplets in $1+1$ dimensions with $q^\mu = (M, 0)$ we obtain $\{Q^+, Q^+\} = \{Q^-, Q^-\} = M$ and $\{Q^+, Q^-\} = Z$. If $Z = M$ consider the operators $Q^+ + Q^-$ and $Q^+ - Q^-$. Clearly, $\{Q^+ - Q^-, Q^+ - Q^-\} = 0$, and $\{Q^+ + Q^-, Q^+ - Q^-\} = 0$ while $\{Q^+ + Q^-, Q^+ + Q^-\} = 4M$. Hence, $Q^+ - Q^- = 0$, and multiplets with $Z = M$ are as short as massless multiplets. (The same holds for $Z = -M$). An example is the kink multiplet. If $| Z | \neq M$, one still has $\{Q^+ + Q^-, Q^+ - Q^-\} = 0$ but now $\{Q^+ + Q^-, Q^+ + Q^-\} = 2(M + Z)$ while $\{Q^+ - Q^-, Q^+ - Q^-\} = 2(M - Z)$. This yields a two-dimensional Clifford algebra whose irreducible representations are again two-dimensional. An example with $Z = 0$ is obtained by adding a mass term to the action in (15.2.1).

3 Quantization of collective coordinates

Solitons (such as 2-dimensional kinks or 4-dimensional monopoles) and instantons are examples of classical solutions of the field equations which have the form

$$\phi(x, X_1, \ldots, X_N) \tag{15.3.1}$$

where x are the coordinates of spacetime, and $X_1, \ldots X_N$ are arbitrary constants on which the solution depends. If the energy of the classical solution of a soliton (the value of the classical Hamiltonian), or the value of the action of an instanton, does not depend on $X_1, \ldots, X_N$ these X_j are called collective coordinates or moduli. A change in one or more X_j is often due to the action of some continuous symmetry of the Hamiltonian. For example, the X_j could be the coordinates of the center of mass of the monopole; in this case the **continuous** symmetry which changes the collective coordinates is translational symmetry. Another example is the N-instanton solution for the gauge group $SU(2)$

$$\begin{aligned} A_\mu(x, X, \lambda) &= \frac{i}{2}[\gamma_\mu, \gamma^\nu]\partial_\nu \ln \phi \\ \phi &= 1 + \sum_{j=1}^{N} \lambda_j^2/(x^\mu - X_j^\mu)^2 \end{aligned} \tag{15.3.2}$$

in which case the collective coordinates are the centers of the instantons X_j^μ and their scale parameters λ_j. Now there is no symmetry of the action which changes only one X_j^μ or λ_j.

The problem we want to study is how to quantize field theories in the presence of a static (time independent) solution with collective coordinates. We consider from now on solitons. One might at first think that one could simply decompose the field $\varphi(x, t)$ into a background part $\phi_{sol}(x)$ and a quantum part $\eta(x, t)$

$$\varphi(x, t) = \phi_{sol}(x) + \eta(x, t) \tag{15.3.3}$$

and then proceed with the quantization of η as usual. This is not possible because, as we shall see, the presence of collective coordinates leads to “zero modes”. Zero modes are zero-energy solutions of the linearized field equations of the quantum field η in the background of the soliton, which are normalizable. If the Lagrangian has the generic form

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu \varphi)^2 - V(\varphi) \tag{15.3.4}$$

the field equations read $\Box\, \varphi - \frac{\partial}{\partial \varphi} V = 0$, and differentiating once w.r.t. the collective coordinate X_j we obtain

$$\left(\frac{\partial}{\partial x^2} - \frac{\partial^2}{\partial \varphi^2} V \right) \frac{\partial}{\partial X_j} \varphi_{sol} = 0 \tag{15.3.5}$$

As we now explain, this means that the $\frac{\partial}{\partial X_j}\varphi_{sol}(x, X_1 \ldots X_n)$ are zero modes.

One may put the system in a box in order that the spectrum of fluctuations is discrete. Setting $\eta(x,t) = \eta(x)e^{-i\omega t}$, these fluctuations are obtained by solving the equation

$$\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial \varphi^2} V \right) \eta = -\omega^2 \eta \tag{15.3.6}$$

and imposing suitable boundary conditions. We shall later discuss which boundary conditions one should impose, but for the time being we assume that a suitable set has been chosen. Comparing (15.3.5) with (15.3.6) shows that $\partial/\partial X_j \varphi_{sol}(x)$ are indeed zero modes. Expanding $\eta(x,t)$ into the complete orthonormal set of solutions $\eta_m(x)$ as follows

$$\eta(x,t) = \sum_m q^m(t) \eta_m(x) \tag{15.3.7}$$

the part of the action quadratic in η becomes a set of harmonic oscillators (use (15.3.9))

$$\begin{aligned} S &= \int \left[\frac{1}{2}\, \dot{\eta}^2 + \frac{1}{2}\, \eta \left\{ \frac{\partial}{\partial x^2} - \frac{\partial^2}{\partial \varphi^2} V \right\} \eta \right] dx dt \\ &= \int \sum_m \left(\frac{1}{2}\, \dot{q}_m^2 - \frac{1}{2}\, \omega_m^2 q_m^2 \right) dt \end{aligned} \tag{15.3.8}$$

Each harmonic oscillator has energy $\hbar\omega_m$, and the zero modes have by definition zero energy. The other fluctuations η_m (those with nonzero energy) are **all** orthogonal to

all zero modes and will be taken to be orthonormal to themselves

$$\int \eta_m(x, X_1, \dots X_n)\eta_n(x, X_1, \dots X_n)dx = \delta_{mn}$$
$$\int \eta_m(x, X_1, \dots X_n)\frac{\partial}{\partial X_j}\varphi_{sol}(x, X_1, \dots X_n) = 0. \tag{15.3.9}$$

That this is possible follows from the fact that eigenfunctions of the operator $\partial/\partial x^2 - \partial^2/\partial\varphi^2 V$ with different eigenvalues are orthogonal. We may view the continuous spectrum as distorted plane waves (distorted in x-space), and introduce creation and annihilation operators just as for the usual plane waves in field theory. For the bound states (if they exist) one can do the same thing because their energies $\hbar\omega_B$ are positive.

The Hamiltonian for the system $\varphi = \phi_{sol} + \eta$ would be

$$\begin{aligned} H &= M_{cl} + H^{(0)} + H_{int} \\ H^{(0)} &= \int_{-\infty}^{\infty} \left[\frac{1}{2}\,\dot{\eta}^2 + \frac{1}{2}(\partial_x\eta)^2 + \frac{1}{2}\left(\frac{\partial^2}{\partial\varphi^2}V\right)\eta^2\right] dx \end{aligned} \tag{15.3.10}$$

where M_{cl} is the energy of the classical soliton solution, and $H^{(0)}$ are the terms quadratic in η in the Hamiltonian.

It is now clear that zero modes lead to complications. The lowest-order quantum corrections $E^{(0)}$ to the classical energy M_{cl} of a soliton are still without problem. From ordinary quantum mechanics one obtains

$$H^{(0)} = \left(\sum_m \hbar\omega_m a_m^\dagger a_m\right) + \left(\sum_m \frac{1}{2}\hbar\omega_m\right) \tag{15.3.11}$$

Hence $E^{(0)} =< 0 \mid H^{(0)} \mid 0 >= \sum \frac{1}{2}\hbar\omega_m$ where $\mid 0 >$ denotes the vacuum in the sector with soliton (the state corresponding to the soliton solution). Similarly, $E^{(1)} =< 0 \mid H_{int} \mid 0 >$. However, the second-order corrections would be given by

$$E^{(2)} = \sum\nolimits_p' \frac{< 0 \mid H_{int} \mid p >< p \mid H_{int} \mid 0 >}{E_p - E_0} \tag{15.3.12}$$

where $\mid p >$ are all states in the Fock space obtained by acting with any number of $a_m^\dagger$ on $\mid 0 >$, and in the sum one should not include the vacuum state $\mid 0 >$. Since for one-particle states $E_p = E_0 + \hbar\omega_p$ with $E_0 = M_{cl} + \sum \frac{1}{2}\hbar\omega_m$, and for two-particle states $E_p = E_0 + \hbar(\omega_p + \omega_q)$ etc., the presence of a zero mode leads to a problem because $E_p - E_0$ vanishes if $\mid p >$ is the zero mode eigenvector. One could apply the formalism of perturbation theory for degenerate quantum mechanical systems, but there is a better approach. To explain this approach, consider the soliton as an extended particle with position X where X is the center of mass. Then one would expect that X becomes an operator $\hat{X}$ in quantum theory. Moreover, the final Hamiltonian should not depend on $\hat{X}$ since the system is translationally invariant. This suggests to replace (15.3.3) by

$$\varphi(x,t) = \phi_{sol}(x, X_1(t) \cdots X_N(t)) + {\sum}'_m q^m(t)\eta_m(x) \tag{15.3.13}$$

where the prime indicates that one should not include the zero modes in the sum. We shall explicity show that the Hamiltonian corresponding to (20.0.13) is indeed independent of $X_j(t)$. The alternative way of adding zero modes in the sum over m instead of incorporating them inside the soliton solution should give the same results in the end, and in instanton physics one sometimes uses the latter approach.

To obtain the Hamiltonian in the soliton sector, one must use the formalism for quantization of collective coordinates. Although the final formulas look somewhat complicated, the basic idea is very simple: one expands $\varphi(x,t)$ again into a sum of a background field (the soliton) and a complete set of small fluctuations about the background field, but instead of simply writing according to (15.3.3) $\varphi(x,t) = \phi_{sol}(x) + \sum q^m(t)\eta_m(x)$ where $\eta_m(x)$ stands for all modes (eigenfunctions of the linearized field equations), one deletes the zero mode for translations from the sum, and reintroduces it by replacing x by $x - X(t)$ on the right hand side of the expansion of φ. For small $X(t)$, the expansion of $\phi_{sol}(x - X(t))$ into a Taylor series gives $\phi_{sol} - X(t)\phi'_{sol}(x,t) + ...$, and since $\phi'_{sol}(x,t)$ is the translational zero mode (the normalizable solution of the linearized field equations with vanishing energy), one has not lost any degrees of freedom. Hence one substitutes

$$\varphi(x,t) = \phi_{sol}(x - X(t)) + \sum{}' q^m(t)\eta_m(x - X(t)) \tag{15.3.14}$$

into the action in (15.3.4), and using the chain rule, one finds an action of the form of a quantum mechanical nonlinear sigma model (but with infinitely many degrees of freedom)

$$L = \frac{1}{2}\dot{u}^I g_{IJ}(u)\dot{u}^J - \widetilde{V}(u); u^I = \{X(t), q^m(t)\} \tag{15.3.15}$$

where $\tilde{V}(u) = \int_{-\infty}^{\infty}\left[\frac{1}{2}(\varphi')^2 + \frac{1}{2}U^2(\varphi)\right]dx$. The metric g_{IJ} is given by

$$g_{IJ} = \int \frac{\partial\varphi(x,t)}{\partial u^I}\frac{\partial\varphi(x,t)}{\partial u^J}dx \tag{15.3.16}$$

and contains space integrals over expressions which depend on $q^m(t)$, with t fixed, $\eta_m(x)$ and $\phi_{sol}(x)$, but not on $X(t)$ due to the translational invariance of the integral over x. The Hamiltonian is then simply given by

$$H = \frac{1}{2}\pi_I g^{IJ}(u)\pi_J + \widetilde{V}(u); \pi_I = \{P(t), \pi_m(t)\} \tag{15.3.17}$$

where $g^{IJ}(u)$ is the matrix inverse of the metric $g_{IJ}(u)$ and $P(t)$ is the momentum conjugate to $X(t)$),while $\pi_m(t)$ are momenta canonically conjugate to $q^m(t)$.

Classically, this is the whole result. One may check that the equal-time Poisson brackets $\{Q,P\} = 1, \{q^m, \pi_n\} = \delta^m_n$ imply $\{\varphi(x,t), \Pi_0(y,t)\} = \delta(x-y)$ where $\Pi_0(x,t) = \dot{\varphi}(x,t)$, and vice-versa. Hence, the transition from $\varphi(x,t)$ and $\Pi_0(x,t)$ to $\{X(t), q^m(t)\}$ and $\{P(t), \pi_m(t)\}$ is a canonical transformation. In the next section we check this at the quantum level. It is useful to recast the many particle Hamiltonian in a form which resembles more the Hamiltonian of a $1+1$ dimensional field theory. To this purpose we introduce fields constructed from q^m and π_m as follows

$$\begin{aligned} \eta(x,t) &\equiv \sum{}' q^m(t)\eta_m(x - X(t)) \\ \pi(x,t) &\equiv \sum{}' \pi_m(t)\eta_m(x - X(t)) \end{aligned} \tag{15.3.18}$$

where the prime again indicates that no zero modes are to be included in the sum over m. By combining the π_m and q^m with the functions $\eta_m(x)$ which appear in $g^{IJ}(u)$, one can write the complete Hamiltonian only in terms of the momentum $P(t)$, the fields $\eta(x,t)$ and $\pi(x,t)$, and the background field $\phi_{\rm sol}(x)$. To simplify the notation, we introduce an inner product $(f,h) \equiv \int_{-\infty}^{+\infty} f^*(x)h(x)dx$. (The star is in the cases we consider not needed because all fields are real, but for example for the $N=(2,2)$ susy kink, one has complex scalar fields and Dirac fermions, and then one needs the star). Note that the functions η_m which parameterize the small fluctuations are orthogonal to the zero mode ${\phi_{sol}}'$ since they correspond to different eigenvalues of the linearized field equations

$$({\phi_{sol}}', \eta_m) = 0 \tag{15.3.19}$$

a result we shall use repeatedly.

Let us now work out the Hamiltonian. The matrix g_{IJ} is given by (subscripts o denote X)

$$\begin{aligned} g_{00} &= \int_{-\infty}^{\infty} (\phi'_{sol} + \sum_m{}' q^m \eta'_m)^2 dx = (\phi'_{sol} + \eta', \phi'_{sol} + \eta') \\ g_{0m} &= g_{m0} = -\int_{-\infty}^{\infty} (\phi'_{sol} + \sum_n{}' q^n \eta'_n)\eta_m dx = -(\eta', \eta_m) \\ g_{mn} &= \delta_{mn} \end{aligned} \tag{15.3.20}$$

Inverting this matrix leads to

$$g^{00} = \frac{1}{g}\ ;\ g^{0m} = g^{m0} = -g_{0m}/g; g^{mn} = \delta^{mn} + g_{0m}g_{0n}/g \tag{15.3.21}$$

where[9]

$$g = \det g_{IJ} = g_{00} - \sum_m g_{0m}g_{0m} \tag{15.3.22}$$

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[9]Recall that $\det \begin{pmatrix} AB \\ CD \end{pmatrix} = \det(A - BD^{-1}C)\det D = \det A \det(D\_CA^{-1}B)$ as follows from $\begin{pmatrix} AB \\ CD \end{pmatrix} = \begin{pmatrix} AO \\ CI \end{pmatrix}\begin{pmatrix} I & A^{-1}B \\ O & D\_CA^{-1}B \end{pmatrix} = \begin{pmatrix} IB \\ OD \end{pmatrix}\begin{pmatrix} A\_BO^{-1}C & O \\ D^{-1}C & I \end{pmatrix}$

The determinant $\det g_{IJ}$ has a simple form. Using that the η_m are orthonormal, and orthogonal to ϕ'_{sol}, we obtain

$$\det g_{IJ} = \int_{-\infty}^{\infty} (\phi'_{sol})^2 dx + 2\int_{-\infty}^{\infty} \phi'_{sol}\eta' dx + \int_{-\infty}^{\infty} (\eta')^2 dx \\ -\sum_n \left[\int_{-\infty}^{\infty} \eta'\eta_n dx\right]^2 \tag{15.3.23}$$

If one expands the function η' into the complete set $\{\eta_n(x), \eta_0(x)\}$ where the normalized zero mode is given by

$$\eta_0(x) = \phi'_{sol}(x)/\left(\int_{-\infty}^{\infty} \phi'^2_{sol} dx\right)^{\frac{1}{2}} = \phi'_{sol}(x)/(M_{cl})^{\frac{1}{2}} \tag{15.3.24}$$

then one obtains, using that ϕ'_{sol} is orthogonal to η_n,

$$g \equiv \det g_{IJ} = \int_{-\infty}^{\infty} (\phi'_{sol})^2 dx + 2\alpha\int_{-\infty}^{\infty} \phi'_{sol}\eta_0 dx + \alpha^2 \\ = M_{cl}\left[1 + \frac{(\eta', \phi'_{sol})}{M_{cl}}\right]^2 \tag{15.3.25}$$

where

$$\alpha = \int_{-\infty}^{\infty} \eta_0(x) q^m \eta'_m(x) dx = (\eta_0, \eta') \tag{15.3.26}$$

The classical Hamiltonian is according to (15.3.17) given by

$$H_{cl} = \frac{P^2}{2g} - \frac{P}{g}\sum_n g_{0n}\pi_n + \frac{1}{2}\sum_{m,n}\left(\delta_{mn} + \frac{g_{0m}g_{0n}}{g}\right)\pi_m\pi_n + \tilde{V}(q_n) \tag{15.3.27}$$

The term $\widetilde{V}(q_n)$ contains all contributions which did not involve time derivatives

$$\widetilde{V}(q_n) = \frac{1}{2}\int_{-\infty}^{\infty} (\phi'_{cl})^2 dx + \int_{-\infty}^{\infty} V(\phi_{cl}) dx + \frac{1}{2}\int_{-\infty}^{\infty} (\eta')^2 dx + \left\{\frac{1}{2!}\eta^2\frac{\partial^2}{\partial\varphi^2}V \cdots\right\} \tag{15.3.28}$$

Substituting the expressions for the metric and g, we arrive at the following expression for the classical Hamiltonian $H_{cl} = M_{cl} + H^{(0)}_{sol} + H^{cl}_{int,sol}$ in the sector with a soliton

$$H^{(0)}_{sol} = \int_{-\infty}^{+\infty}\left[\frac{1}{2}\pi(x,t)^2 + \frac{1}{2}\eta'(x,t)^2 + \frac{1}{2}\eta^2 V''(\phi_{sol})\right] dx \tag{15.3.29}$$

$$H^{cl}_{int,sol} = \frac{1}{2M_{cl}}\frac{[P + (\pi, \eta')]^2}{[1 + (\eta', \phi'_{sol})/M_{cl}]^2} + \int\left[\frac{1}{3!}\eta^3 V'''(\phi_{sol}) + \right. \\ \left. + \frac{1}{4!}\eta^4 V''''(\phi_{sol}) + \ldots\right] dx \tag{15.3.30}$$

All $X(t)$ dependence has disappeared from H_{cl} due to translational invariance of the integration over x. Note that there are no terms in H_{cl} which are linear in P, η or π.[10]

The canonical transformation to collective coordinates. We now prove that the transformation from $\varphi(x,t)$ and $\Pi_0(x,t) = \dot{\varphi}(x,t)$ to $u^I(t) = \{q^m(t), X(t)\}$ and $\pi_I(t) = \{\pi_m(t), P(t)\}$ is a canonical transformation. We shall do this at the quantum level where the canonical variables are operators. (For simplicity of notation we set $\hbar = 1$). This justifies the approach we have taken. If we had allowed the zero mode in the expansion of η, we would not have obtained a canonical transformation. We shall first show that if

$$[q^m(t), \pi_n(t)] = i\delta^m_n \quad \text{and} \quad [X(t), P(t)] = i \tag{15.3.31}$$

then

$$[\varphi(x,t), \Pi_0(y,t)] = i\delta(x-y). \tag{15.3.32}$$

The proof is straightforward, and at a given moment we shall use that the set of functions η_m and the zero mode $\eta_0 = \phi'_{sol}(M_{cl})^{-1/2}$ form a complete set

$$\sum_m \eta_m(x-X(t))\eta_m(y-X(t)) = \delta(x-y) - \phi'_{sol}(x-X(t))\phi'_{sol}(y-X(t))/M_{cl} \tag{15.3.33}$$

Afterwards we shall prove that $[\varphi(x,t), \varphi(y,t)] = 0$ and $[\Pi_0(x,t), \Pi_0(y,t)] = 0$.

We begin by writing $\varphi(x,t)$ and $\Pi_0(y,t)$ in terms of u^I and π_I

$$\begin{aligned}
\varphi(x,t) &= \phi_{sol}(x-X) + \eta(x-X) \\
\Pi_0(y,t) &= -(\phi'_{sol}(y-X) + \eta'(y-X))(g^{00}P + g^{0m}\pi_m) \\
&+ \sum \eta_m(y-X)(g^{m0}P + g^{mn}\pi_n)
\end{aligned} \tag{15.3.34}$$

[10]Of course, $\phi_{sol}(x-X(t))$ is **not** a solution of the field equations, but $\varphi(x,t) = \phi_{sol}(x)+$ terms at least linear in u^I, hence the action contains no terms linear in u^I. There are, however, terms proportional to $P^2\eta'$ in the Hamiltonian and terms proportional to $\dot{X}^2\eta'$ in the action. This leads to complications for the evaluation of the effective action of the collective coordinates if one first integrates over the oscillations η and π.

There are ordering ambiguities in $\Pi_0(y,t)$ which we will fix later; they do not matter for the commutator $[\varphi(x,t),\Pi_0(y,t)]$ because the ambiguities do not depend on π_I. We obtain then (with $\eta(x-X,t)$ defined in (15.3.18) and suppressing the t-dependence for notational simplicity)

$$\begin{aligned}&[\varphi(x,t),\Pi_0(y,t)] = (\phi'_{sol}(x-X)+\eta'(x-X))(\phi'_{sol}(y-X)+\eta'(y-X))ig^{00}\\&-\eta_m(x-X)(\phi'_{sol}(y-X)+\eta'(y-X))ig^{0m}\\&-(\phi'_{sol}(x-X)+\eta'(x-X))\eta_m(y-X)ig^{m0}\\&+\eta_n(x-X)\eta_m(y-X)ig^{mn}\end{aligned}\tag{15.3.35}$$

Substituting the expressions for the metric and using (15.3.33) we find

$$\begin{aligned}&(\phi'_{sol}(x-X)+\eta'(x-X))(\phi'_{sol}(y-X)+\eta'(y-X))\frac{i}{g}\\&-(\phi'_{sol}(y-X)+\eta'(y-X))\frac{i}{g}(\eta',\eta_m)\eta_m(x-X)\\&-(\phi'_{sol}(x-X)+\eta'(x-X))\frac{i}{g}(\eta',\eta_m)\eta_m(y-X)\\&+i\delta(x-y)-i\phi'_{sol}(x-X)\phi'_{sol}(y-X)/M_{cl}\\&+\frac{i}{g}(\eta',\eta_m)\eta_m(x-X)(\eta',\eta_n)\eta_n(y-X)\end{aligned}\tag{15.3.36}$$

Using once more the completeness relation, a great simplification occurs and we are left with

$$\begin{aligned}&i\delta(x-y)+2\frac{i}{g}\phi'_{sol}(x-X)\phi'_{sol}(y-X)(\eta',\phi'_{sol})/M_{cl}\\&+\frac{i}{g}(\eta',\phi'_{sol})(\eta',\phi'_{sol})\phi'_{sol}(x-X)\phi'_{sol}(y-X)/M_{cl}^2\\&+i\phi'_{sol}(x-X)\phi'_{sol}(y-X)\left(\frac{1}{g}-\frac{1}{M_{cl}}\right)\end{aligned}\tag{15.3.37}$$

Using (15.3.25) for g, we find for all terms except $i\delta(x-y)$ a result proportional to

$$2(\eta',\phi'_{sol})/M_{cl}+(\eta',\phi'_{sol})(\eta',\phi'_{sol})/M_{cl}^2+(1-g/M_{cl})=0\tag{15.3.38}$$

This completes the proof of the $[\varphi,\Pi_0]$ commutator.

Of course, $[\varphi(x,t),\varphi(y,t)]=0$ because $\varphi(x,t)$ does not contain any momenta P or π_m. However, at first sight it might seem that the proof that also $[\Pi_0(x,t),\Pi_0(y,t)]=0$ is more complicated. In fact, before evaluating this commutator, we should fix the ordering ambiguity of the operators in $\Pi_0(x,t)$. This we do by an argument which at the same time proves that $\Pi_0(x,t)$ commutes with $\Pi_0(y,t)$. Let $\Pi_0(x)$ be represented by $\frac{1}{i}\ \frac{\partial}{\partial\varphi(x)}$, and act on functions $F(\varphi(x))$ with inner product $(F,G)=\int F(\varphi(x))G(\varphi(x))\Pi_x d\varphi(x)$. Use then the chain rule to obtain

$$\frac{\partial}{\partial\varphi(x)}=\frac{\partial u^I}{\partial\varphi(x)}\,\frac{\partial}{\partial u^I}=\frac{\partial\varphi(y)}{\partial u^I}g^{IJ}\frac{\partial}{\partial u^J} \tag{15.3.39}$$

where we introduced the "vielbein" field ${E_x}^I=\frac{\partial u^I}{\partial\varphi(x)}$ and its inverse ${E_I}^x=\frac{\partial}{\partial u^I}\varphi(x)$. From (15.3.16) we find $g_{IJ}=\int {E_I}^x {E_J}^x dx$ and $g^{IJ}=\int {E_x}^I {E_x}^J dx$. The inner product for functions $\widetilde{F}(u)=F(\varphi(u))$ is then

$$(\widetilde{F},\widetilde{G})=\int \widetilde{F}(u)\widetilde{G}(u)\left(\det\frac{\partial\varphi}{\partial u}\right)\Pi_I du^I=\int \widetilde{F}(u)\widetilde{G}(u)\sqrt{g}\ \Pi_I du^I \tag{15.3.40}$$

With this inner product, the relation between $\frac{\partial}{\partial u^I}$ and π_I is not simply $\frac{1}{i}\ \frac{\partial}{\partial u^I}=\pi_I$ but rather, π_I is representated in terms of $\frac{\partial}{\partial u_I}$ by[11]

$$\pi_I=g^{-\frac{1}{4}}\ \frac{1}{i}\ \frac{\partial}{\partial u^I}\ g^{\frac{1}{4}} \tag{15.3.41}$$

[11] One proof is as follows. Consider in quantum mechanics $<x\mid\hat{p}\mid y>=\int <x\mid\hat{p}\mid p><p\mid y>dp$ where we used $\int\mid p><p\mid dp=I$. Since the inner product $<f\mid h>=\int f^*(x)h(x)\sqrt{g(x)}dx$ implies that $\mid x><x\mid\sqrt{g(x)}=I$ we find for $<x\mid p>$ the following result $<x\mid p>=\frac{1}{g^{1/4}(x)}\frac{e^{ipx}}{\sqrt{2\pi}}$. Then

$$\begin{aligned}<x\mid\hat{p}\mid y>&=\int\frac{1}{g^{1/4}(x)}p\frac{e^{ipx}}{\sqrt{2\pi}}\frac{e^{-ipy}}{\sqrt{2\pi}}\frac{1}{g^{1/4}(y)}dp\\&=\frac{1}{g^{1/4}(x)}\frac{1}{g^{1/4}(y)}\frac{1}{i}\frac{\partial}{\partial x}\int e^{ipx}e^{-ipy}\frac{dp}{2\pi}=\frac{1}{g^{1/4}(x)}\frac{1}{g^{1/4}(y)}\frac{1}{i}\frac{\partial}{\partial x}\delta(x-y)\end{aligned}$$

Moving $g^{1/4}(y)$ to the right (using that $\hat{p}_x$ and $g^{1/2}(y)$ commute) we obtain

$$\frac{1}{g^{1/4}(x)}\frac{\hbar}{i}\frac{\partial}{\partial x}g^{1/4}(y)\delta(x-y)=\frac{1}{g^{1/4}(x)}\frac{\hbar}{i}\frac{\partial}{\partial x}g^{1/4}(x)\delta(x-y).$$

We want to find the x-representation of $\hat{p}$, so $\hat{p}_x f(x)\sim\frac{\hbar}{i}\frac{\partial}{\partial x}f(x)$ where $f(x)=<x\mid f>$. Thus we want $<x\mid\hat{p}\mid y>=\hat{p}_x<x\mid y>=\hat{p}_x\frac{1}{g^{1/2}(y)}\delta(x-y)$. Clearly, the solution for $\hat{p}_x$ is (15.3.41).

Thus, the correct quantum operator $\Pi_0(x,t)$ is according to (15.3.39), (15.3.41) and 15.3.34)

$$\Pi_0(x,t) = -(\phi'_{cl}(x-X) + \eta'(y-X))g^{\frac{1}{4}}(g^{00}P + g^{0m}\pi_m)g^{-\frac{1}{4}}$$
$$+\eta_m(x-X)g^{\frac{1}{4}}(g^{m0}P + g^{mn}\pi_n)g^{-\frac{1}{4}} \tag{15.3.42}$$

Since the operators $\frac{\partial u^I}{\partial\varphi(x)}\frac{\partial}{\partial u^I}$ are equal to $\frac{\partial}{\partial\varphi(x)}$, they commute with each other, and hence also the operators $\frac{\partial u^I}{\partial\varphi(x)}g^{\frac{1}{4}}\pi_I g^{-\frac{1}{4}}$ commute with each other. These are the momenta $\Pi_0(x,t)$. Hence, the momenta indeed commute with each other. The reader who finds this proof too formal may check the relation $[\Pi_0(x,t), \Pi_0(y,t)]$ by direct calculation, using (15.3.42).

The quantum Hamiltonian. We must now discuss the subtle issue of operator ordering in H. We shall consider a soliton at rest, so we set $P=0$. Furthermore, due to $[q^m, \pi_n] = i\delta^m_n$ and $(\eta_m, \eta_m{}') = 0$ (if we work in a finite volume, and η_m and $\eta_m{}'$ have the same boundary conditions) one has the equality $(\pi, \eta') = (\eta', \pi)$, at least if one considers a finite number of modes in η and π. However, there are operator ordering ambiguities both in $(\pi,\eta')^2$ and also with respect to the term $(\eta', \phi_{sol}{}')/\sqrt{M_{cl}}$ in the denominator.

In general, one may require that the generators $H, P = \int T_{01}dx$ and $L = \int xT_{00}dx$ satisfy the Poincaré algebra. The expressions for these operators are quite complex, and in general it seems likely that the operator ordering which leads to closure of the Poincaré algebra is unique (in quantum gravity, such an ordering has never been found). There is, however, an ordering which guarantees closure, and this is the ordering we shall adopt. It is obtained by making the canonical transformation at the quantum level. One begins with the quantum Hamiltonian in "Cartesian coordinates" (i.e., in terms of the operators $\Pi_0(x)$ and $\varphi(x)$). In the Schrödinger representation the operator $\Pi_0(x)$ is represented by $\frac{1}{i}\,\partial/\partial\varphi(x)$, and making the change of coordinates from $\varphi(x)$ to X and q^m by applying the chain rule, one obtains the Laplacian in curved space[12]

$$\sum\left(\frac{\partial}{\partial\alpha^i}\right)^2=\frac{1}{\sqrt{g}}\frac{\partial}{\partial u^I}\sqrt{g(u)}g^{IJ}(u)\frac{\partial}{\partial u^J} \tag{15.3.43}$$

where α^i is the set $\varphi(x)$ and u^I the set X, q^m. Using again

$$\frac{\partial}{\partial u^J}=g^{1/4}(u)\pi_J g^{-1/4}(u) \tag{15.3.44}$$

the correct quantum Hamiltonian is given by

$$\hat{H}=\frac{1}{2}\frac{1}{g(u)^{1/4}}\pi_I\sqrt{g(u)}g^{IJ}(u)\pi_J\frac{1}{g(u)^{1/4}}+\widetilde{V}(u)+\Delta H \tag{15.3.45}$$

where ΔH are the counter terms needed for renormalization (see below). As a check note that $\hat{H}$ is hermitian.

It is useful to *rewrite* this Hamiltonian such that all expressions are Weyl ordered, because that yields the action to be used in the path integral. The result is

$$\hat{H}=\frac{1}{2}(\pi_I g^{IJ}\pi_J)_W+\widetilde{V}(u)+\Delta V+\Delta H \tag{15.3.46}$$

where ΔV is the difference between the first term in (15.3.45) and the first term in (15.3.46). The action to be used in the Hamiltonian path integral is then $S=\int(\dot{u}^I\pi_I-\frac{1}{2}\pi_I g^{IJ}\pi_J-\widetilde{V}(u)-\Delta V-\Delta H)dt$. After integration over π_I we find $S=\int(\mathcal{L}+\Delta\mathcal{L})dxdt-\int\Delta V dt$ where $\Delta\mathcal{L}=-\Delta\mathcal{H}$. This is not what one would naively have expected: there is an extra term ΔV of order $\hbar^2$. Furthermore there is a factor $(\det g_{IJ})^{1/2}$ in the measure[13] due to the Gaussian integration over π.

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[12]Although this is a result of elementary calculus, one can use methods of general relativity to prove this. The simplest proof is to use that the covariant derivative of a contravariant vector density is equal to the ordinary derivative. Since $\sqrt{g(u)}g^{IJ}(u)\frac{\partial}{\partial u^J}\varphi(u)$ for a scalar field φ is a contravariant vector density, (15.3.43) is generally covariant, and hence it should hold in any frame. This proves (15.3.43). More explicitly, one can also check (15.3.43) term by term. In that case we must show that $e_i{}^I\frac{\partial}{\partial u^I}e_i{}^J\frac{\partial}{\partial u^J}=(\det e_j{}^K)\frac{\partial}{\partial u^I}(\det e_L{}^k)e_i{}^I e_i{}^J\frac{\partial}{\partial u^J}$ where i is the flat index and I the curved index of vielbeins. The terms with two free derivatives match, while the terms with one free derivative lead to the condition $ee_i{}^I\left(\frac{\partial}{\partial u^I}e_i{}^J\right)=\frac{\partial}{\partial u^I}\left(ee_i{}^I e_i{}^J\right)$ with $e=\det e_I{}^i$. This relation holds provided $\frac{\partial}{\partial u^I}(ee_i{}^I)=0$. It can be rewritten as $ee_i{}^I e_j{}^J\left(\frac{\partial}{\partial u^I}e_J{}^j-\frac{\partial}{\partial u^J}e_I{}^j\right)=0$ which clearly holds since $e_I{}^j=\frac{\partial}{\partial u^I}\alpha^j$.

[13]The factor $(\det g_{IJ})^{1/2}$ in the measure can be exponentiated by using anticommuting ghosts b^I and

The operator $(1/2)\left(\pi_I g^{IJ}\pi_J\right)_W$ is given by[14] $(1/2)((1/4)\pi_I\pi_J g^{IJ}+(1/2)\pi_I g^{IJ}\pi_J+(1/4)g^{IJ}\pi_I\pi_J)$. One finds then after some tedious algebra[15]

$$\Delta V=\frac{\hbar^2}{8}\left[\partial_I\partial_J g^{IJ}(u)-4g^{-1/4}(u)\partial_I\{g^{1/2}(u)g^{IJ}(u)\partial_J g^{-1/4}(u)\}\right] \tag{15.3.47}$$

Substituting the expression for $g^{IJ}(u)=g^{IJ}(q^m)$ (recall that g^{IJ} does not depend on X) one finds

$$\begin{aligned}\partial_I\partial_J g^{IJ} &= \partial_{q^m}\partial_{q^n}\left\{\frac{(\eta_m,\eta')(\eta_n,\eta')}{(\eta_0,\phi_{sol}'+\eta')^2}\right\}\\ 4g^{-1/4}\partial_I\left\{g^{1/2}g^{IJ}\partial_J g^{-1/4}\right\} &= \frac{-1}{(\eta_0,\varphi')^{1/2}}\frac{\partial}{\partial q^m}\left[\frac{1}{(\eta_0,\varphi')^{3/2}}\frac{\partial}{\partial q^m}(\eta_0,\varphi')^2+\right.\\ &\left. +\frac{(\eta',\eta_m)(\eta',\eta_n)}{(\eta_0,\varphi')^{7/2}}\frac{\partial}{\partial q^n}(\eta_0,\varphi')^2\right]\end{aligned} \tag{15.3.48}$$

where $\varphi=\phi_{sol}+\eta$, and $\eta_0=\phi_{sol}'/\sqrt{M_{cl}}$ is the normalized zero mode. This leads to

$$\begin{aligned}\Delta V &= \frac{\hbar^2}{8}\left[-\frac{(\eta_0,\eta_m')(\eta_m',\eta_0)}{(\eta_0,\varphi')^2}\right.\\ &-2\frac{(\eta_0,\eta_m')(\eta_m,\eta_n')(\eta_n,\eta')+(\eta_0,\eta_m')(\eta_m,\eta')(\eta_n,\eta_n')}{(\eta_0,\varphi')^3}\\ &\left.+\frac{\{(\eta_0,\eta_m')(\eta_m,\eta')\}^2}{(\eta_0,\varphi')^4}+\frac{(\eta_m,\eta_n')(\eta_n,\eta_m')+(\eta_m,\eta_m')^2}{(\eta_0,\varphi')^2}\right]\end{aligned} \tag{15.3.49}$$

Further simplifications result by using the identities

$$\begin{aligned}&(\eta_0,\eta_0')=0,\ (\eta_0,\eta')=(\eta_0,\varphi'),\ (\eta_0,\eta_m')=-(\eta_0',\eta_m)\\ &(\eta_n,\eta')=(\eta_n,\varphi'),\ (\eta_m,\eta_m')=0,\ {\sum}'\eta_m(x)\eta_m(y)=\delta(x-y)-\eta_0(x)\eta_0(y)\end{aligned} \tag{15.3.50}$$

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c^I, and a commuting ghost a^I (see ref. 7)

$$(\det g_{IJ})^{1/2}=\int db^I dc^I da^I \exp\frac{i}{\hbar}\int_{-\infty}^{\infty}[b^I g_{IJ}c^J-a^I g_{IJ}a^J]dt$$

Using dimensional regularization, it is formally proportioned to $\delta(0)$, and is omitted.

[14]Weyl ordering of $q^m p^r$ means picking the term with $\alpha^m\beta^r$ from $(\alpha q+\beta p)^{m+r}$ and dividing by $\binom{m+r}{r}$. For $(qqp)_W$ one obtains $\frac{1}{3}(qqp+qpq+pqq)$ which can be rewritten as $\frac{1}{4}(qqp+2qpq+pqq)$. Similarly, for $(ppq^r)_W$ one finds after rewriting that it equals $\frac{1}{4}(ppq^r+2pq^rp+q^rpp)$.

[15]First commute $g^{-\frac{1}{4}}\pi_I$ in (15.3.45) to $\pi_I g^{-\frac{1}{4}}$, and then $\pi_J g^{-\frac{1}{4}}$ to $g^{\frac{1}{4}}\pi_J$. The order $\hbar$ terms cancel because $[g^{-\frac{1}{4}},\pi_I]+[\pi_I,g^{-\frac{1}{4}}]=0$.

The final answer for ΔV reads then

$$\begin{aligned}\Delta V &= \frac{\hbar^2}{8}\left[-\frac{(\eta_0{}', \eta_0{}')}{(\eta_0, \varphi')^2} + 2\left\{\frac{(\eta_0{}', \varphi'')}{(\eta_0, \varphi')^3} - \frac{(\eta_0{}', \eta_0{}')}{(\eta_0{}', \varphi')^2}\right\}\right.\\ &\left. + \frac{(\eta_0{}', \varphi')^2}{(\eta_0, \varphi')^4} - \sum_{m,n} \frac{|(\eta_m, \eta_n{}')|^2}{(\eta_0{}', \varphi')^2}\right] \end{aligned} \tag{15.3.51}$$

The total quantum Hamiltonian operator is then the sum of $M_{cl} + H^{(0)}_{\rm sol}$ in which no ordering problems are present, and $\left(H^{cl}_{\rm int,sol}\right)_W + \Delta V + \Delta H$ with $\left(H^{cl}_{int,sol}\right)_W$ given by (15.3.30) with the complicated momentum dependent term Weyl-ordered. This is the result in [2]. A drastic simplification is obtained by rewriting the latter term in a particular non-Weyl-ordered way in such a way that it absorbs all terms in ΔV except the first one. This leads to the final form of the interaction Hamiltonian [3]

$$\begin{aligned} H_{qu} &= M_{cl} + H^{(0)}_{sol} + H_{int,sol} + \Delta V + \Delta H \\ H_{int,sol} &= \frac{1}{2M_{cl}}\left\{(P + (\eta', \pi))\frac{1}{1 + (\eta', \varphi'_{sol})/M_{cl}}\right\}^2 \\ \Delta V &= -\frac{\hbar^2}{8M_{cl}^2}\int_{-\infty}^{+\infty} \frac{(\phi''_{sol})^2 dx}{[1 + (\eta', \varphi')/M_{cl}^{1/2}]^2} \\ V_{int} &= \int_{-\infty}^{+\infty}\left[\frac{1}{3!}\eta^3 V'''(\phi_{sol}) + \frac{1}{4!}\eta^4 V''''(\phi_{sol}) + \dots\right] dx \end{aligned} \tag{15.3.52}$$

Note that $H_{int,sol}$ is the square of a Weyl-ordered operator, but is not itself Weyl-ordered.

The complete quantum Hamiltonian operator which forms the basis for many quantum mechanical calculations of field theories with $\mathcal{L} = -\frac{1}{2}(\partial_\mu \varphi)^2 - V$ in the sector with solitons is thus

$$H = M_{cl} + H^{(0)}_{sol} + H_{int,sol} + V_{int} + \Delta V + \Delta H \tag{15.3.53}$$

where the soliton solution is denoted by ϕ_{sol}, M_{cl} is given by $\int_{-\infty}^{\infty}(\phi'_{sol})^2 dx$, the free part of H is given by $H^{(0)}_{sol}$ in (15.3.29), the interactions with the correct operator ordering are given in (15.3.52), and $\triangle H$ are the counter terms which are needed for renormalization. These ΔH are obtained from $\triangle\mathcal{L}(\varphi)$ by substituting $\varphi = \phi_{sol} + \eta$ and using $\triangle H = -\triangle\mathcal{L}$.

The final quantum Hamiltonian is again independent of the operator X. As one might expect, its conjugate momentum P is indeed the center of mass momentum. To prove this, note that $T_{\mu\nu} = \partial_\mu\varphi\partial_\nu\varphi - \eta_{\mu\nu}\mathcal{L}$, and $P = \int_{-\infty}^{\infty} T_{0x}dx$. Then

$$\begin{aligned} P &= \int_{-\infty}^{\infty} \frac{\partial\varphi}{\partial t}\frac{\partial\varphi}{\partial x}\,dx \\ &= \int_{-\infty}^{\infty} \left[(\phi'_{sol}(x-X)+\eta'(x-X))\dot{X} - \sum \dot{q}^m\eta_m\right](\phi'_{sol}(x-X)+\eta'(x-X))dx \\ &= g_{00}\dot{X} + g_{0m}\dot{q}^m = \frac{\partial L}{\partial\dot{X}} \end{aligned} \tag{15.3.54}$$

It is now clear why the approach we have followed with incorporating the collective coordinate $X(t)$ in ϕ_K as $\phi_K(x-X(t))$ is superior to the more naive approach of expanding $\varphi = \phi_{sol} + \eta$ and then including the zero mode into the expansion of η, and using degenerate perturbation theory. If we would have allowed an independent operator $q^0(t)$ in $\eta = q^0(t)\eta_0(x) + \sum{}' q^m\eta_m(x)$, and defined conjugate momenta as usual (by differentiating the action w.r.t. $\dot{q}$) this would still have corresponded to a canonical transformation. However, the final Hamiltonian would not have been independent of $X(t)$. By expanding $\varphi(x,t)$ as in (15.3.14) we have removed the zero mode from the quantum fluctuations η, but we put it back by making X a dynamical variable $\hat{X}(t)$. So we have not lost any degrees of freedom. Moreover, the final quantum Hamiltonian is translationally invariant (independent of X).

One can also study classical solutions which are time-dependent. A trivial example is obtained by Lorentz boosting a static solution, yielding for example a moving kink $\phi_K(\gamma x - \gamma vt)$ with $\gamma^{-1} = (1-v^2)^{1/2}$. One expands then as before, but with $x - X(t)$ replaced by $\gamma(x - X(t))$. However, less trivial examples also exist, for example the breather solutions in the sine-Gordon system, or nontopological solitons with a fixed nonvanishing charge Q or baryon number N.

4 Solitons in general

We have seen that the kink is a soliton in $1+1$ dimensions. We recall our definitions of a soliton: a time-independent nonsingular solution of the classical field equations in Minkowski spacetime **with finite nonzero energy**. Zero-energy solutions will occur in some examples below but we shall interpret them as vacua rather than particles. Hence we require that genuine solitons are extended lumps which are localized in a region where their energy is positive.

The question naturally arises what other solitons exists in $1+1$ and other dimensions. Derrick's theorem gives some information about this question. It is based on rigid scale transformations and goes as follows. Suppose there does exist a solitonic solution $\phi_0(x)$ in D space dimensions, where ϕ_0 denotes all fields, for example also vector fields. Then the energy functional $E(\phi)$ is stationary at ϕ_0 because for static solutions the equation $\frac{\partial}{\partial\phi_0(x)}E(\phi_0) = 0$ is equal to the Euler-Lagrange field equation ($L = p\dot{q} - H$ and for time-independent fields, $L = -H$). Consider next the field configuration $\phi_a(x) = a(\phi)\phi_0(ax)$, where $a(\phi)$ is a power of a which depends on ϕ. If the parameter a is near unity, $\phi_a(x) - \phi_0(x) \equiv \delta\phi_0(x)$ is a small variation of $\phi(x)$ about the solution $\phi_0(x)$, and since $\phi_0(x)$ is a solution of the field equations, the functional $E(\phi)$ should not change to first order in a. When it does change, the assumption that a solitonic solution $\phi_0(x)$ exists was false. We now apply this theorem to various models. First we consider models with only scalar fields, later we add vector fields. This will lead in a natural way to nonabelian monopoles.

Consider a set of scalar fields $\vec{\varphi}$ in $D+1$ spacetime dimensions with $\mathcal{L} = -\frac{1}{2}(\partial_\mu\vec{\varphi})^2 - V(\varphi)$. For static fields $E = \int\left[\frac{1}{2}(\partial_j\varphi)^2 + V(\varphi)\right]d^Dx$. Defining $\varphi_a(x) = \varphi_0(ax)$ (so we take $a(\varphi)$ equal to unity) we obtain

$$E(\varphi_a) = \int\left[\frac{1}{2}\left[\frac{\partial}{\partial x}\vec{\varphi}_0(ax)\right]^2 + V(\varphi_0(ax))\right]d^Dx$$

$$= \int \left\{ \frac{1}{2} a^2 \left[\frac{\partial}{\partial(ax)} \vec{\varphi}_0(ax) \right]^2 \frac{1}{a^D} d^D(ax) + V(\varphi_0(ax)) \frac{1}{a^D} d^D(ax) \right\}$$

$$= a^{2-D} \int \frac{1}{2} (\partial_y \varphi_0(y))^2 d^D y + a^{-D} \int V(y) d^D y \tag{15.4.1}$$

We are mostly interested in models where the potential of the scalars if of the form $\lambda \left(\varphi^2 - \frac{\mu^2}{\lambda} \right)^2$. Then $V(\varphi)$ is everywhere positive, and hence also $\int V(\varphi) d^D x$ is positive. We find

$$\frac{\partial}{\partial a} E(\varphi_a)_{|a=1} = (2-D) \int T(\varphi_0) - D \int V(\varphi_0) \tag{15.4.2}$$

where T is the kinetic term. Since both $\int T$ and $\int V$ are positive, there exists no solution φ_0 for $D > 2$. For $D = 1$ a solution is not ruled out, and indeed we found the kink. Note that we can relax the condition that $V(\varphi)$ is everywhere positive to the weaker condition that $\int V(\varphi_0) d^D x$ is positive. For $D = 2$ a solution could only exist if $\int V(\varphi_0) d^D x = 0$. The energy reads then $\int \frac{1}{2} (\partial_x \vec{\varphi})^2 d^D x$ and would seem to correspond to a free theory. However, one can introduce interactions by imposing the constraint $(\vec{\varphi})^2 = R^2$. The action reads

$$\mathcal{L} = -\frac{1}{2} (\partial_\mu \varphi)^2 + \lambda(\vec{x}, t)(\vec{\varphi} \cdot \vec{\varphi} - R^2), \varphi = (\varphi^1, \ldots, \varphi^N) \tag{15.4.3}$$

and has an $O(N)$ symmetry under rigid rotations.[16] The field equation for λ sets $\vec{\varphi}^2$ equal to R^2 and on-shell the Hamiltonian reads $H = \sum_{i,j=1}^{N-1} \frac{1}{2} g_{ij}(\varphi) \partial_x \varphi^i \partial_x \varphi^j$. We shall later show that this $O(N)$ model has nontrivial vacua with winding numbers in

[16]Solving the constraint $\phi^N = \pm[R^2 - (\phi^1)^2 \ldots - (\phi^{N-1})^2]^{1/2}$ leads to a nonlinear sigma model with only manifest $O(N-1)$ symmetry. The action reads $\mathcal{L}(\varphi) = -\frac{1}{2} \left(\delta_{ij} + \frac{\varphi^i \varphi^j}{R^2 - \varphi^2} \right) (\partial_\mu \varphi^i)(\partial^\mu \varphi^j)$ where now $\varphi^2 = (\varphi^1)^2 + \cdots + (\varphi^{N-1})^2$. The coset $O(N)/O(N-1)$ generates the remaining symmetries $\delta\varphi^j = \alpha\varphi^N, \delta\varphi^N = -\alpha\varphi^j$, which leave the original action and the constraint invariant. Substituting the expressions for φ^N, they become the spontaneously broken symmetries $\delta\varphi^j = \alpha[R^2 - (\varphi^1)^2 \ldots - (\varphi^{N-1})^2]^{1/2} = \alpha(R - \frac{1}{2}\varphi^2/R + \ldots)$. One can transform this action to $(\partial_\mu \chi)^2/(R^2 + \chi^2)$ by the field redefinition $\varphi^i = 2\chi^i(1+\chi^2)^{-1}$. (To obtain this start from $(\vec{y})^2 - y^+ y^- = 0$ with $y^\pm = y^0 \pm y^{N+1}$ and impose the constraint $y^0 = 1$. Set $(y^a, y^+, y^-) = e\left(x^a, 1, x^2\right)$. Then $(dy)^2 = e^2 (dw)^2$ with $w = (x^a, 1, x^2)$, because $w^2 = 0$ and $w dw = 0$ as well. Then $dy^2 = e^2 (dx)^2$. To find e we use $1 = y^0 = \frac{1}{2}(y^+ + y^-) = \frac{1}{2} e(1 + x^2)$ and find $e = 2/(1+x^2)$.)

$D \geq 3$ space dimensions (but no solitons). In 2 space dimensions it is known as the $O(3)$ model and has genuine solitons.

We considered models with $V(\varphi)$ positive, but one can also consider models with $V(\varphi)$ negative which lead to solitons. As an example (which has been used for constrained instantons) consider are real scalar field with action $\mathcal{L} = -\frac{1}{2}(\partial_\mu\varphi)^2 + \frac{1}{4}\lambda\varphi^4$ in $D+1$ Minkowski space. For time-independent solutions the energy is $E = \int \left(\frac{1}{2}(\partial_j\varphi)^2 - \frac{1}{4}\lambda\varphi^4\right) d^D x$, and solutions must satisfy $\partial_j^2\varphi + \lambda\varphi^3 = 0$. We try $\varphi(r) = \frac{c}{(r^2+\rho^2)^\alpha}$, c and α constants. One finds easily that this yields a solution for $\alpha = \frac{D}{2} - 1$ and $D = 4$ and $c^2 = 2\alpha D\rho^2/\lambda$. The energy of this solution is given by

$$\begin{aligned} E &= \int \frac{1}{4}\lambda\varphi^4 d^4x = \frac{\pi^2}{4}\lambda c^4 \int_0^\infty \frac{r^2 dr^2}{(r^2+\rho^2)^4} \\ &= \frac{\pi^2\lambda c^4}{24\rho^4} = \frac{8\pi^2}{3\lambda} \end{aligned} \tag{15.4.4}$$

This solution is a nontopological soliton.

Since scalars with a positive semidefinite potential $V(\varphi)$ cannot give solitons in $D > 2$, we next consider vector fields. Consider pure Yang-Mills theory. Derricks theorem for static solutions in D space dimensions with $A_0 = 0$ states that the energy functional $E = \int \frac{1}{4}F_{ij}^2 d^D x$ should be stationary for $A_\mu(x) = a^\lambda A_\mu(ax)$ at $a = 1$. If $\lambda \neq 1$ the terms in $\frac{1}{4}F_{ij}^2$ due to expanding a^λ and ax acquire different powers of al, and since they are not all positive definite, no information can be extracted from the case $\lambda \neq 1$. For $\lambda = 1, E(A,a) = E(A)a^{-D+4}$ hence in $D \neq 4$ space dimensions no solitons exist in pure Yang-Mills theory. The solitons of the $4+1$ dimensional pure YM theory are, of course, the instantons of 4 dimensional Euclidean space.

We give another proof which does not require $A_0 = 0$ that in pure Yang-Mills theory no solitons exist except in $4+1$ dimensions. Singular solutions exist, for example the Yang-Wu monopole. The proof proceeds as follows. The stress tensor is $T_{\mu\nu} = F_{\mu\alpha}F_{\nu\alpha} - \frac{1}{4}\eta_{\mu\nu}F^2$ hence

$$T_{\mu\nu}\eta^{\mu\nu} = \left(1 - \frac{1}{4}(D+1)\right) F_{\mu\nu}^2; T_{00} = \frac{1}{2}F_{0i}^2 + \frac{1}{4}F_{ij}^2 = \frac{1}{2}E^2 + \frac{1}{2}B^2 \tag{15.4.5}$$

For static fields

$$\frac{\partial}{\partial t}\int d^D x[x^i T_{oi}] = 0 \tag{15.4.6}$$

Conservation of translational symmetry ($\partial^\mu T_{\mu\nu} = 0$) for static fields implies

$$\int d^D x x^i \partial_j T_{ij} = 0 \tag{15.4.7}$$

Finite energy requires that **all** $F_{\mu\nu}$ tend to zero for large r faster than $r^{-D/2}$ because E is a sum of positive terms $(F_{\mu\nu})^2$. Then T_{ij} tends to zero faster than r^{-D}, hence

$$\int d^D x \partial_j (x^i T_{ij}) = \int (x^i T_{ij}) dS_j = 0 \tag{15.4.8}$$

It follows from (15.4.7) and (15.4.8) that

$$\int d^D x (\partial_j x^i) T_{ij} = \int d^D x T_{jj} = 0 \tag{15.4.9}$$

Substituting $T_{jj} = T_{\mu\nu}\eta^{\mu\nu} + T_{00}$ we arrive at

$$\int d^D x T_{jj} = \int \left[\left(\frac{5}{4} - \frac{D+1}{4}\right) F_{ij}^2 + \left(\frac{D+1}{2} - \frac{3}{2}\right) F_{0j}^2\right] d^D x = 0 \tag{15.4.10}$$

For $D = 3$ (our world) both F_{ij} and F_{0i} should vanish, hence then A_μ is pure gauge and the energy vanishes such field configurations. For field configurations which are pure gauge, $F_{\mu\nu}$ vanishes hence they are solutions of the field equations. Solutions with vanishing energy are vacua, hence if one could not gauge these solutions to the trivial solution $A_\mu = 0$, there would exist nontrivial vacua (vacua with "winding"). Actually, there do not exist field configurations which are everywhere pure gauge and which cannot be gauged to zero. To prove this we modify an argument of Coleman to the present case. Let $A_\mu = U^{-1}\partial_\mu U$ and write U as $U(r, \alpha_1, \alpha_2)$ where α_1, α_2 are angles. As r tends to zero, U tends to a particular group element $U(0, \alpha_1, \alpha_2)$ and this limit is smooth because we assumed that the solutions we consider are nonsingular. Hence $U(0, \alpha_1, \alpha_2)$ does not depend on α_1 and α_2 and is a constant group element. Constant group elements can be continuously deformed to the unity element. Hence $U(r, \alpha_1, \alpha_2)$ can be continuously deformed to the unit element. For $r = R$, all $U(r, \alpha_1, \alpha_2)$ can then be deformed to unity and hence there is no winding. Thus in pure YM theory in $D = 3$ there is only the trivial vacuum.

For $D \geq 5$ we can also rule out solitons if we assume that we can gauge A_0 away, or if we consider configurations with $A_0 = 0$. Then $F_{0i} = 0$, and (15.4.10) shows that $F_{ij} = 0$ everywhere. Hence again A_μ is pure gauge everywhere, and again only the trivial vacuum remains.

In $D = 4$, Derrick's theorem states that $\int \frac{1}{4} F_{ij}^2 - \frac{1}{2} \int F_{0i}^2$ should vanish for a solitonic solution, and the usual instanton solutions are examples of solitons in $D = 4$. (The $F_{\mu\nu}$ for instantons are proportional to $\sigma_\mu \bar{\sigma}_\nu - \sigma_\nu \bar{\sigma}_\mu$, and $\frac{1}{4}\sigma_{ij}^2 = -6 = \frac{1}{2}\sigma_i^2$).

Since no solitons exists in pure Yang-Mills theory with $A_0 = 0$, consider next Yang-Mills theory coupled to scalars, $\mathcal{L} = -\frac{1}{4}(G_{\mu\nu}^a)^2 - \frac{1}{2}(D_\mu \vec{\varphi})^2 - V(\vec{\varphi})$. The energy for static fields with $A_0 = 0$ reads

$$E = \int \left[\frac{1}{4} (G_{ij}^a)^2 + \frac{1}{2} (D_i \vec{\varphi})^2 + V(\varphi) \right] d^D x \qquad (15.4.11)$$

In order that E be finite, G_{ij}^a and $D_i \vec{\varphi}$ must tend to zero for large x. Then A_i^a becomes pure gauge and the components φ^I of $\vec{\varphi}$ must satisfy $\partial_i \varphi^I + A_i^a (T_a)^I{}_J \varphi^J = 0$. This suggests that for large radius r the gauge field A_i tends to zero as $\frac{1}{r}$, which fortunately yields a finite contribution $\int \frac{1}{4} G_{ij}^2 d^D x$ to the energy if $D \leq 3$. We consider from now on the case $D = 3$. Furthermore, $\varphi(r \to \infty)$ should tend to zero fast enough that $V(\varphi) = \frac{1}{4}\lambda \varphi^4 + \frac{1}{2}\mu^2 \varphi^2$ has finite energy. The pure gauges $A_i = U^{-1} \partial_i U$ at $r = \infty$ could then lead to a winding index which prevents the solution from unwinding, similar to the winding of instantons.

The problem with this scenario is that no solution with these properties exist. This is not immediately clear: Derrick's theorem with $a A_\mu(ax)$ and $a\vec{\varphi}(ax)$ does not rule out solitons. The first two terms in E scale with a^{4-D}, while also the $\lambda \varphi^4$ term scales with a^{4-D}, but $\frac{1}{2}\mu^2 \varphi^2$ scales with a^{2-D}. Hence, $\frac{\partial}{\partial a} E$ could vanish. However,

$\int \varphi^2 d^3x$ is finite only if φ tends to zero faster than $1/r^2$ at infinity, but then it would seem impossible to satisfy the φ field equation.

There is, however, a loop hole in this no-go theorem. Consider the following action

$$\mathcal{L} = -\frac{1}{4}G_{ij}^2 - \frac{1}{2}(D_i\vec{\varphi})^2 - \frac{1}{4}\lambda\left[(\vec{\varphi}^2) - \frac{1}{\lambda}\mu^2\right]^2, \mu^2 > 0 \tag{15.4.12}$$

Now $V(\varphi) = -\frac{1}{4}\lambda\left(\vec{\varphi}^2 - \frac{\mu^2}{\lambda}\right)^2$ does vanish for large r if we let $\vec{\varphi}^2$ tend to $F^2 \equiv \frac{\mu^2}{\lambda}$. We are in the familiar situation of spontaneous symmetry breaking.

Once we have deduced the correct action which could give a soliton, is not too difficult to prove that solitons exist. We must solve in 3-space dimensions

$$\left.\begin{array}{c} D_i\varphi^I = \partial_i\varphi^I + gA_i^a T_a(R)^I{}_J\varphi^J \to 0 \\ \varphi^I \to F^I \text{ with } (F^I)^2 = \mu^2/\lambda\ . \\ A_i^a \to U^{-1}(\theta,\varphi)\partial_i U/\theta,\varphi) \end{array}\right\} \text{ for } r \to \infty \tag{15.4.13}$$

Instead of explicitly solving these equations, we prove that a solution exists by using the concept of homotopy. The points in 3-dimensional space at $r \to \infty$ form a 2-sphere S_2 (space). On the other hand, the $\varphi^I = F^I$ at $r = \infty$ form vectors with fixed length which lie on S_{N-1} (internal) if $I = 1, \ldots N$. Thus the fields $\varphi^I(\vec{x})$ at large $|\ \vec{x}\ |$ yield a map from S_2 (space) $\to S_{N-1}$ (internal). These maps form equivalence classes (elements in one class can be continuously deformed into each other). It is a mathematical fact that the maps $\pi_n(S_m)$ of $S_n \to S_m$ are given by

$$\begin{aligned} &\pi_n(S_n) = Z; \pi_n(S_m) = 0 \text{ for } n < m \\ &\pi_n(S_1) = 0 \text{ for } n > 1; \pi_n(S_2) = \text{ for } n > 2 \end{aligned} \tag{15.4.14}$$

In our case we need $\pi_2(S_{N-1})$, and this is only nontrivial if $N - 1 = 2$. Thus we need precisely 3 scalars which should tend to a constant in each direction at spatial infinity.

There are various solutions for scalars in representations R of $SU(n)$ which lead to monopoles. The simplest is clearly the **3** of $SU(2)$. Then the various homotopy classes a labelled by the integers Z, i.e. there is a winding number $Q = 0, \pm 1, \pm 2$ etc. In this case we need $\pi_2(S_2) = Z$. In the Weinberg-Salam model, the scalars form a complex $SU(2)$ doublet $\varphi^\alpha = \begin{pmatrix} \varphi^+ \\ \varphi^0 \end{pmatrix}$ with potential $V = \frac{1}{4}\lambda \left[(\varphi^*_\alpha \varphi^\alpha)^2 - 2\frac{\mu^2}{\lambda} \right]^2$. In the usual case of spontaneous symmetry breaking one expands the scalars about the vacuum expectation value $\langle \varphi^\alpha \rangle = \begin{pmatrix} 0 \\ \varphi^0 \end{pmatrix}$ with real constant φ^0. However, to obtain a soliton (monopole) the $\langle \varphi^\alpha \rangle$ should become space-dependent, and then one needs $\pi_2(SU(2)/U(1))$.

Before going on with monopoles, we come back to the $O(N)$ models in D space dimensions, and also exhibit a winding number for them. We consider static fields. The constraint $\vec{\varphi} \cdot \vec{\varphi} = R^2$ is the λ field equation and holds at all points in the D space dimensions. The Hamiltonian for solutions is then $E = \int (\partial_j \varphi)^2 d^D x$ and is finite provided $|\partial_j \varphi|$ tends to zero for large radius r faster than $r^{-\frac{1}{2}D}$. This implies that $\vec{\varphi}(\vec{x})$ tends to a constant value φ_∞ as $r \to \infty$. We can then compactify the D dimensional plane into a sphere S_D (space). The fields $\varphi^I(x) I = 1, \dots, N$ subject to $\vec{\varphi} \cdot \vec{\varphi} = R^2$ form points on another sphere S_{N-1}, which we call again S_{N-1} (internal). The maps from S_D (space) into S_{N-1} (internal) form the homotopy classes $\pi_{N-1}(S_{N-1}) = Z$, hence there should be winding number $Q = 0, \pm 1, \pm 2 \dots$.

However, as in the case of pure YM theory, the field configurations with nontrivial winding cannot be solutions of the field equations if $D \neq 2$. Derrick's theorem states that $(2 - D)E(\vec{\varphi}_0) = 0$, hence for $D \neq 2$ the energy $E(\vec{\varphi}_0)$ would have to vanish. This would imply that $\partial_x \vec{\varphi} = 0$, and leave only the trivial solution $\vec{\varphi} =$ constant. The situation is similar to the constrained instantons: in both cases one has configurations with winding and one can make the energy (or action) in a given sector with winding smaller and smaller by choosing configurations which become more and more less regular, but there is no solution at the minimum.[17]

[17] For the constrained instantons the action for the scalars was proportional to ρ^2 while gauge action is $\delta\pi^2/g^2$, hence by taking ρ smaller the instanton gets more compressed and the energy gets lower. The expression for $F^2_{\mu\nu}$ is actually proportional to $\frac{\rho^4}{(x^2+\rho^2)^4}$ and as a function of x this is a regularized delta

We can exhibit this situation explicitly for the $O(2)$ model in one dimension where the 2 real scalar fields φ_1 and φ_2 satisfy $\varphi_1^2 + \varphi_2^2 = R^2$. Space has $D = 1$ and is thus the real line, and compactifying this real line to a circle S via stereographic projection, the following is an example of a field configuration with winding defined on S

$$\frac{\varphi_1 + i\varphi_2}{\sqrt{2}} = Re^{2in\,\mathrm{arctg\ ax}} \tag{15.4.15}$$

The energy is given by

$$\begin{aligned} \frac{1}{2}\int [(\partial_x\varphi_1)^2 + (\partial_x\varphi_2)^2]dx &= \int \partial_x\varphi^*\partial_x\varphi dx \\ = R^2 \int \left|\frac{2ina}{[1+(ax)^2]}\right|^2 dx &= 4R^2n^2a \int\limits_{-\infty}^{\infty} \frac{dy}{(1+y^2)^2}dy \end{aligned} \tag{15.4.16}$$

Clearly, for a tending to zero the energy decreases, but the point $a = 0$ is discontinuous because then $\varphi_1 + i\varphi_2$ becomes equal to R, and there is no longer winding.

5 The 't Hooft-Polyakov monopole

In $3 + 1$ dimensions, monopoles exist in the Georgi-Glashow model where the Higgs fields are in the adjoint representation of the group $SO(3)$ (the $SO(3)$ Higgs model). One can even construct "dyons", which are topological extended objects with both electric and magnetic charges, as shown by Julia and Zee. There are then electric and magnetic charges, defined by

$$\begin{aligned} 4\pi e &= \frac{1}{v}\int d^3x\partial_i \left(A^a G^a{}_{i0}\right) \ , < A^a >= \frac{x^a}{r}v \text{ at } r \to \infty. \\ 4\pi g &= \frac{1}{2v}\int d^3x\varepsilon_{ijk}\partial_i \left(A^a G^a{}_{jk}\right) \end{aligned} \tag{15.5.1}$$

where A^a is the Higgs fields. Let us first justify these definitions. In the gauge where $< A^a >= \delta^{a3}v$ everywhere, the electromagnetic field is identified as $F^3_{\mu\nu} =$

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function which for $\rho \to 0$ becomes the singular $\delta(x^2)$. However, at $\rho = 0$ the instanton disappears.

$\partial_\mu A_\nu{}^3 - \partial_\nu A_\mu{}^3$. (The mass term for the vector fields due to the Higgs effect is $\vec{A}_\mu \times \vec{A}$, hence only $A_\mu{}^3$ remains massless). Then the definitions in (15.5.1) are correct. A gauge-invariant definition is

$$F_{\mu\nu} = \frac{A^a}{|A|} G_{\mu\nu}{}^a - \frac{1}{e|A|^3} \varepsilon_{abc} A^a D_\mu A^b D_\nu A^c \tag{15.5.2}$$

because after a gauge transformation which puts $< A^a >$ along the 3-axis, $F_{\mu\nu}$ becomes $\partial_\mu A^3_\nu - \partial_\nu A_\mu{}^3$. Converseley, starting with $\partial_\mu A_\nu{}^3 - \partial_\nu A_\mu{}^3$ and $< A^a >= \delta^a{}_3 v$, a gauge transformation which maps $< A^a >$ along the radius leads to $F_{\mu\nu}$. We must now show why the second term in (15.5.2) has not been used in (15.5.1).

The asymptotic values of $A_\mu{}^a$ and A^a are

$$A^a_j \to -\varepsilon_{jab} \frac{x^b}{er^2} \ , \ A^a \to v \frac{x^a}{r} \tag{15.5.3}$$

Then

$$D_\mu A^a = \partial_\mu A^a + e\varepsilon^a{}_{bc} A_\mu{}^b A^c = \mathcal{O}\frac{1}{r^2} \text{ for } r \to \infty \tag{15.5.4}$$

and hence the definitions of e and g in (15.5.1) are also correct in a general gauge. Since

$$F_{ij} = -\varepsilon_{ija} \frac{x^a}{er^3} \tag{15.5.5}$$

we get a monopole charge

$$g = \frac{1}{e}. \tag{15.5.6}$$

In the limit that the coupling constant λ of the $\lambda\varphi^4$ terms vanishes, we can compute the mass of the monopole exactly. The Hamiltonian density can then be written for static fields as

$$\begin{aligned} \mathcal{H} &= \frac{1}{4}(G_{ij}{}^a)^2 + \frac{1}{2}(D_i A^a)^2 \\ &= \frac{1}{4}\left(G_{ij}{}^a - \varepsilon_{ijk} D_k A^a\right)^2 + \frac{1}{2}\varepsilon_{ijk} G_{ij}{}^a D_k A^a \\ &= \frac{1}{4}\left(G_{ij}{}^a - \varepsilon_{ijk} D_k A^a\right)^2 + \partial_k \left(\frac{1}{2}\varepsilon_{ijk} G_{ij}{}^a A^a\right) \end{aligned} \tag{15.5.7}$$

where we used the Bianchi identities. The mass M contains again a surface contribution $\int div\vec{B}d^3x = 4\pi gv$. So

$$M_{mon} = \pm 4\pi \frac{1}{e^2} M_W + \frac{1}{4}\left(G_{ij}{}^a \mp \varepsilon_{ijk} D_k A^a\right)^2, M_W \equiv ev. \tag{15.5.8}$$

A minimum mass is obtained for given monopole charge when

$$G_{ij}{}^a = \pm \varepsilon_{ijk} D_k A^a. \tag{15.5.9}$$

Identifying A^a as the time-component of $A_\mu{}^a$, we find *for static fields* a selfduality condition

$$G_{\mu\nu}{}^a = \pm \frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} G_{\rho\sigma}{}^a,\ \varepsilon_{ijk0} \equiv \varepsilon_{ijk} \tag{15.5.10}$$

because $G_{ko}{}^a = D_k A^a$. We can solve this equation by similar methods as used for instantons. One puts $A_\mu{}^a = \eta_{\mu ab} \partial_b \ln \rho$ where $\eta_{\mu ab}$ is the 't Hooft tensor. Time independence requires $\ln \rho = f(\vec{r}) + \alpha t$. The selfduality equation states $\Box \rho / \rho = 0$, hence $(\nabla^2 + \alpha^2) f(\vec{r}) = 0$. A solution which is regular at $\vec{r} = 0$ is then $f(\vec{r}) = (e^{\alpha r} - e^{-\alpha r})/r$.

3. Olive-Witten Monopoles in $N = 2$ susy YM theory with $G = SO(3)$.

We now extend the $SO(3)$ Georgi-Glashow model to a model with susy. We thus need a susy action with $SO(3)$ gauge fields $A_\mu{}^a$ and Higgs scalars A^a. This suggest to consider the coupling of the $N = 1$ vector (gauge) multiplet to the $N = 1$ Wess-Zumino (matter) multiplet, and to choose the matter fields in the adjoint representation. The resulting model turns out to possess a second susy. We therefore consider the $N = 2$ Yang-Mills model with gauge group $SO(3)$.

The most general susy algebra can only have central charges in $d = 3 + 1$ dimensions if $N \geq 2$. In the N=2 case, there are at most two central charges, and they appear as

$$\left\{ Q_i{}^\alpha, Q_j{}^\beta \right\} = \delta_{ij} \gamma^{\mu\alpha\beta} P_\mu + \varepsilon_{ij} (U C^{-1\alpha\beta} + V i \gamma_5{}^{\alpha\beta}) \tag{15.5.11}$$

where $(C^{-1})^{\alpha\beta}$ and $(\gamma_5)^{\alpha\beta} = (\gamma_5)^\alpha{}_\delta (C^{-1})^{\delta\beta}$ are antisymmetric and $(\gamma_\mu)^{\alpha\beta} = \gamma_\mu{}^\alpha{}_\delta (C^{-1})^{\delta\beta}$ is symmetric. (Since in $d = (1,1)$ the matrix $i\gamma_5{}^{\alpha\beta} = (\tau_1)^\alpha{}_\beta$ is symmetric, $N = 1$ theories can have one central charge V in $d = (1,1)$ but in $d = (3,1)$ there exists no symmetric Lorentz invariant tensor with two spinor indices). As we shall see, the monopole **solution**, even though it is constructed from the fields of the $N = 2$ vector multiplet, is itself part of another $N = 2$ multiplet: the hypermultiplet. We therefore begin by discussing the $N = 2$ vector- and hypermultiplets.

The $N = 2$ gauge action reads in superspace

$$S = \int d^4x d^2\theta_1 d^2\theta_2 F^a F^a \tag{15.5.12}$$

where F^a is $N = 2$ chiral, $\bar{\mathcal{D}}^{(1)}{}_{\dot\alpha} F = \bar{\mathcal{D}}^{(2)}{}_{\dot\alpha} F = 0$ and satisfies the reality constraint $D^2_{ij} F = \bar{D}^2_{ij} \bar{F}$ where $D^2_{ij} = D^{\dot\alpha}{}_i D_{\dot\alpha j}$ and $\bar{D}^2{}_{ij} = \bar{D}^{\dot\alpha}{}_i \bar{D}_{\dot\alpha j}$. (Since F^a carries a Yang-Mills index, the supersymmetric covariant derivatives contain also a Yang-Mills connection.) We expand F into $N = 1$ superfields as follows[18]

$$\begin{aligned} F^a &= \phi^a(\theta_1, \bar\theta_1, y) + \theta_2{}^\alpha W_\alpha{}^a(\theta_1, \bar\theta_1, y) \\ &+ (\theta_2{}^\alpha \theta_{2\alpha}) \bar{\mathcal{D}}^{\dot\alpha}_{(1)} \bar{\mathcal{D}}^{(1)}{}_{\dot\alpha} \left(\bar\phi^a(\theta_1, \bar\theta_1, y) e^V \right)^a \end{aligned} \tag{15.5.13}$$

where y is that combination of x and $\bar\theta_2 \gamma^\mu \gamma^5 \theta_2$ which is annihilated by $\bar{\mathcal{D}}^{(2)}{}_{\dot\alpha}$. The $N = 2$ chirality of F implies the $N = 1$ chirality of ϕ and W_α, namely $\bar{\mathcal{D}}^{(1)}{}_{\dot\alpha}\phi = \bar{\mathcal{D}}^{(1)}{}_{\dot\alpha} W_\alpha = 0$. Performing the θ_2 integration ($d^2\theta_2 = \mathcal{D}^\alpha_{(2)} \mathcal{D}_{(2)\alpha}$) yields

$$S = \int d^2\theta_1 W_\alpha W^\alpha + \int d^2\theta_1 d^2\bar\theta_1 \bar\phi e^V \phi \tag{15.5.14}$$

which is the action for $N = 1$ YM theory in superspace. (Use that $\phi \bar{\mathcal{D}}^2 \bar\phi = \bar{D}^2(\phi\bar\phi)$, and use that a covariantly antichiral scalar superfield is given by $e^V \bar\phi$ where $\bar\phi$ is

[18] The constraints of $N = 2$ superspace are $\left\{\mathcal{D}^i_\alpha, \bar{\mathcal{D}}_{\dot\alpha j}\right\} = i\delta^i_j \mathcal{D}_{\alpha\dot\alpha}$ and $\left\{\mathcal{D}^i_\alpha, \mathcal{D}^j_\beta\right\} = \varepsilon_{\alpha\beta}\varepsilon^{ij}\bar{F}^a T_a$ with $\mathcal{D}^i_\alpha = D^i_\alpha + i\Gamma^{ia}T_a$. They lead to $\left[\mathcal{D}^i_\alpha, \bar{F}\right] = 0$ and $\mathcal{D}^{\alpha i}\mathcal{D}_\alpha{}^j F = \varepsilon^{ik}\varepsilon^{jl}\bar{\mathcal{D}}^{\dot\alpha}{}_k \bar{\mathcal{D}}_{\dot\alpha l}\bar{F}$. Denoting $F^a{}_| = \phi^a, \mathcal{D}^\alpha_{(2)} F_| \equiv W^\alpha$ and $\mathcal{D}^\alpha_{(2)}\mathcal{D}_{(2)\alpha} F^a_| = \bar{\mathcal{D}}^{\dot\alpha}_{(1)} \bar{\mathcal{D}}_{(1)\dot\alpha} \bar{F}^a{}_| = \bar{\mathcal{D}}^2 \bar\phi^a$ we obtain (15.5.13). See Grimm, Sohnius and Wess, *Nucl. Phys. B* **133**, 275 (1978).

an ordinary antichiral field, whereas a covariantly chiral ϕ is also ordinary chiral (according to the chiral representation of covariant derivatives).

The $N = 2$ Yang-Mills multiplet contains one vector $A_\mu{}^a$, two spinors, a scalar A^a and a pseudoscalar B^a. It is obtained by coupling the N=1 vector multiplet $(A_\mu{}^a, \lambda^a)$ to the N=1 Wess-Zumino multiplet (ψ^i, A^i, B^i). In order to have a second susy, the matter particles ψ^i, A^i, B^i must also be in the adjoint representation of the gauge group. The action is then the sum of the gauge-covariantized kinetic terms, a coupling of the gaugino to its matter current, and (after elimination of the auxiliary fields) a four-scalar coupling. The two susys are related to each other by interchanging the gaugino and the Higgsino. In a manifestly $O(2)$ symmetric notation one introduces the spinor $\lambda_j{}^a$ with $\lambda_1{}^a = \lambda^a$ and $\lambda_2{}^a = \psi^a$. (We suppressed the spinor indices). Then under the first susy $\delta A_\mu{}^a = \bar{\varepsilon}_1 \gamma_\mu \lambda_1{}^a$ but under the second susy $\delta A_\mu{}^a = \bar{\varepsilon}_2 \gamma_\mu \lambda_2{}^a$. The complete action and transformation rules read

$$\begin{aligned}
\mathcal{L} = & - \frac{1}{4}(F_{\mu\nu}{}^a)^2 - \frac{1}{2}\bar{\lambda}_j{}^a \not{D} \lambda_j{}^a - \frac{1}{2}(D_\mu A^a)^2 - \frac{1}{2}(D_\mu B^a)^2 \\
& - \frac{1}{2}\varepsilon^{ij} g \bar{\lambda}_i{}^a (A^b + i\gamma_5 B^b)\lambda_j{}^c f_{abc} + \frac{1}{2} g^2 (f^a{}_{bc} A^b B^c)^2 \\
\delta A_\mu{}^a = & \ \bar{\varepsilon}_i \gamma_\mu \lambda_i{}^a, \delta A^a = \varepsilon_{ij} \bar{\varepsilon}_i \lambda_j{}^a, \delta B^a = \varepsilon_{ij} \bar{\varepsilon}_i i\gamma_5 \lambda_j{}^a \\
\delta \lambda_i{}^a = & -\frac{1}{2}\gamma^{\mu\nu} F_{\mu\nu}{}^a \varepsilon_i - \varepsilon_{ij} \not{D}(A^a + i\gamma_5 B^a)\varepsilon_j - g f^a{}_{bc} A^b B^c (i\gamma_5 \varepsilon_i)
\end{aligned} \tag{15.5.15}$$

The spinors are ordinary Majorana spinors, so $\bar{\lambda}_j = \lambda_j{}^T C$.

The action has a manifest $O(2)$ symmetry. In fact, it has even a $U(2)$ symmetry of which the $SO(2)$ is part, but the $SU(2)$ becomes only manifest after one rewrites this model in terms of "symplectic Majorana spinors". Instead of the usual Majorana condition $\bar{\lambda} = \lambda^T C = \lambda^\dagger i\gamma^0$, for **two** Majorana spinors χ_j one can also define a symplectic Majorana conditon

$$\bar{\chi}^j = (\chi_j)^\dagger i\gamma^0 = \chi_k{}^T C i\gamma_5 \varepsilon^{kj} \tag{15.5.16}$$

Then the consistency condition that $(\bar{\chi}^j)^\dagger = i\gamma^0\chi^j$ be equal to $({\chi_k}^T Ci\gamma_5\varepsilon^{kj})^\dagger$ after replacing in the latter expression ${\chi_k}^\dagger$ by $({\chi_\ell}^T Ci\gamma_5\varepsilon^{\ell k})i\gamma^0$, is satisfied, because the square of ε^{ij} (a factor -1) is compensated by the square of $i\gamma_5$ (another factor -1). (In any representation where γ^k and $i\gamma^0$ are hermitian, $C^\dagger C = \alpha I$ with $\alpha > 0$, and $C^T = -C$. This follows from taking the hermitian conjugate or the transposition of $C\gamma^\mu C^{-1} = -\gamma^{\mu,T}$. From Schur's lemma it follows that $C^T = \pm C$, and if $C^T = +C$ then $\tilde{C} = C\gamma_5$ satisfies $\tilde{C}^T = -\tilde{C}$. By rescaling C we can then achieve that $C^\dagger C = I$). Using symplectic Majorana spinors, all i, j indices are contracted like $\bar{\lambda}^j\lambda_j$, and the $SU(2)$ is now manifest. The original $SO(2)$ is now the $SU(2)$ symmetry with generator τ_3, while the $U(1)$ in $U(2)$ acts as follows

$$\delta A_\mu = 0, \delta\lambda_j = e^{i\alpha\gamma_5}\lambda_j, \delta(A + iB) = e^{2i\alpha}(A + iB) \tag{15.5.17}$$

This is also a symmetry of the action in (15.5.15). The auxiliary fields F and G of the Wess-Zumino model and D of the vector multiplet fuse into a real $SU(2)$ isovector D^a. For example, $\delta\lambda_j = \ldots + D_a{(\tau^a)_j}^k\varepsilon_k$, where τ^a are the Pauli matrices, and in the action one then finds the expected term $+\frac{1}{2}(D^a)^2$. We shall not need this $U(2)$ symmetry, and thus revert to the formulation in terms of ordinary Majorana spinors.

The susy algebra without auxiliary fields reads

$$\begin{aligned}&[\delta(\varepsilon_1), \delta(\varepsilon_2)]\phi^a = 2{\bar{\varepsilon}_2}^i\gamma^\mu{\varepsilon_1}^i\partial_\mu\phi^a + \delta_{YM}(\Lambda^a)\phi^a \\ &(\text{plus } \lambda - \text{ equations of motion if } \phi^a \text{ equals } {\lambda^a}_j)\end{aligned} \tag{15.5.18}$$

where the gauge parameter Λ^a is field dependent

$$\Lambda^a = 2{\bar{\varepsilon}_2}^i\gamma^\mu{\varepsilon_1}^i{A_\mu}^a + 2\varepsilon_{ij}{\bar{\varepsilon}_2}^i(A^a + i\gamma_5 B^a){\varepsilon_1}^j \tag{15.5.19}$$

There is no central charge term on the right-hand side. However, as we shall see, there are static solutions of the field equations with spontaneously broken gauge symmetry for which A becomes a constant $< A >= v$ at spatial infinity, and then the gauge transformation with parameter $2v^a\varepsilon_{ij}{\bar{\varepsilon}_2}^i{\varepsilon_1}^j$ becomes a central charge Z where Z acts on the group indices of the fields ϕ^a as $Z\phi^a = (2v^b \varepsilon_{ij} \bar{\varepsilon}_2{}^i \varepsilon_1{}^j) f^a{}_{bc} \phi^c$. (Hence there is **one** central charge of the form $v \cdot T$ where T is proportional to $f^a{}_{bc}$). Central charges commute with the generators of the susy algebra, but need not commute with the gauge generators.

Let us now demonstrate the presence of a central term in the hypermultiplet. The action of two free Wess-Zumino multiplets reads

$$\mathcal{L} = -\partial_\mu \phi^*{}_i \partial^\mu \phi^i - \bar{\psi} \partial\!\!\!/ \psi + F^*{}_i F^i \tag{15.5.20}$$

where ϕ^i and F^i are complex scalars and ψ a complex (Dirac) spinor. The susy rules with symplectic Majorana spinors ε_j but an ordinary Dirac spinor ψ, read

$$\begin{aligned} \delta\phi^i &= \bar{\varepsilon}^i \psi, \delta F^i = \bar{\varepsilon}^i \partial\!\!\!/ \psi, \delta\psi = \partial\!\!\!/ \phi^i \varepsilon_i + F^i \varepsilon_i \\ \bar{\varepsilon}^i &\equiv \varepsilon_j{}^T C i \gamma_5 \varepsilon^{ji} = \varepsilon^\dagger i \gamma^0 \end{aligned} \tag{15.5.21}$$

The susy algebra on ϕ^i becomes

$$\begin{aligned} [\delta(\varepsilon), \delta(\eta)]\phi^i &= \bar{\eta}^i (\partial\!\!\!/ \phi^j \varepsilon_j + F^j \varepsilon_j) - \varepsilon \leftrightarrow \eta \\ &= 2(\bar{\eta}^k \gamma^\mu \varepsilon_k) \partial_k \phi^i + (\bar{\eta}^k \varepsilon_k) F^i \end{aligned} \tag{15.5.22}$$

The last term corresponds to the central charge V in the $N = 2$ susy algebra of (15.5.11). This central charge acts like

$$Z\phi^i = F^i, Z\psi = \partial\!\!\!/ \psi, ZF^i = \Box \phi^i \tag{15.5.23}$$

and the parameter $\bar{\eta}^k \varepsilon_k$ in (15.5.22) is purely imaginary. The general $N = 2$ susy algebra (15.5.11) contains two central charges, but in 2-component notation it reads

$$\{Q^{Ai}, Q^{Bj}\} = \varepsilon^{AB} \varepsilon^{ij} z \tag{15.5.24}$$

with z a complex number, and one can remove the phase $e^{i\delta}$ of z by redefining Q^{Ai} and Q^{Bj} by phase factors $e^{\frac{1}{2}i\delta}$ (a unitary transformation). Thus (15.5.22) represents actually the general case. The central charge Z clearly vanishes on-shell, and further it satisfies

$$[Z, Q^{\alpha}{}_{i}] = 0, Z^2 = \Box \tag{15.5.25}$$

There is no general superfield formalism for hypermultiplets. If one treats the central charge as an extra coordinate ζ in a coset approach, expansion into a power series in this extra coordinate leads to an infinity of fields. If one sets the central charge equal to zero, the fields satisfy $\Box\varphi = \partial\!\!\!/\psi = F = 0$, hence they are on-shell.[19]

If one adds mass terms $\mathcal{L}_m = m(\phi_i^* F^i + F_i^* \phi^i - \bar{\psi}\psi)$ to the hypermultiplet, one finds after using the field equations that on-shell Z is proportional to the mass

$$Z\phi^i = -m\phi^i, Z\psi^i = -m\psi^i, ZF^i = -mF^i \tag{15.5.28}$$

Hence, on states the central charge is equal to the mass. This is an example of "multiplet shortening". If one studies the representations of the N-extended susy algebras in 3 + 1 dimensions, one finds that always $\mid Z \mid \leq M$ in order that the representation be unitary; but if $\mid Z \mid < M$ multiplets are twice as long as massless multiplets, while if $\mid Z \mid = M$ the multiplets are as short as massless multiplets.

[19]It is an interesting and open question whether there exist covariant constraints involving the extra bosonic coordinate ζ which truncate the infinite series to a finite one without bringing one on-shell. The following $N = 2$ action suggests this approach: $S = \int d^4x d^2\theta_{(1)} d^2\bar{\theta}_{(1)} [\phi\bar{\phi} - \Sigma\bar{\Sigma} + \frac{1}{2}X^2]$ where ϕ is $N = 1$ chiral $(\bar{D}^{\dot{\alpha}}_{(1)}\phi = 0), \Sigma$ is $N = 1$ linear $(\bar{D}^2_{(1)}\Sigma = 0)$ and X is arbitrary. The $N = 1$ susy is obvious. The action of the $N = 2$ susy generators (Q) is expressed in terms of $N = 1$ covariant derivatives (D) as follows

$$Q\bar{\phi} = -D\bar{\Sigma}, Q\bar{\Sigma} = -DX, QX = D\Sigma, Q\Sigma = D\phi, Q\phi = 0 \tag{15.5.26}$$

These relations can be written as a constraint on a ζ dependent superfield

$$\begin{aligned} \nabla_\zeta \eta &= 0, \bar{\nabla}_\zeta \eta = 0, \eta = \bar{\phi} + \zeta\bar{\Sigma} + \zeta^2 X - \zeta^3\Sigma + \zeta^4\phi \\ \nabla_\zeta &\equiv D + \zeta Q; \bar{\nabla}_\zeta \equiv \bar{Q} - \zeta\bar{D}; \{\nabla_\zeta, \nabla_{\bar{\zeta}}\} = 0 \end{aligned} \tag{15.5.27}$$

The action can be written as $\int d^4\theta_{(1)} \oint \frac{d\zeta}{2\pi i\zeta} \left(\frac{\eta}{\zeta^2}\right)^2$. If one were to begin with a superfield η which is an arbitrary series in ζ, the condition $\zeta^4\eta^\star(-\frac{1}{\zeta}) = \eta(\zeta)$ would reproduce the field η given above. Similarly ∇_ζ and $\bar{\nabla}_\zeta$ are related by $\bar{\nabla}_\zeta = \zeta\nabla^*_{-\frac{1}{\zeta}}$.

We shall now again evaluate the susy algebra, and study the relation between the central charge and monopoles. The susy current follows again from the susy transformation laws by letting ε^j become local, and collecting terms with $\partial_\mu \varepsilon^j$ from the Dirac action. Since

$$\delta\lambda_i = -\frac{1}{2} F_{\mu\nu}\gamma^{\mu\nu}\varepsilon_i - \varepsilon_{ij} \not{D}(A + i\gamma_5 B)\varepsilon_j - ig_5[A,B]\varepsilon_i \tag{15.5.29}$$

we find

$$j^{\mu,i} = tr(\gamma^{\rho\sigma} F_{\rho\sigma}\gamma^\mu\lambda^i + \varepsilon^{ij}\not{D}(A + i\gamma_5 B)\gamma^\mu\lambda^j + ig\gamma_5[A,B]\gamma^\mu\lambda^i) \tag{15.5.30}$$

Using the representation $\gamma^k = \begin{pmatrix} 0 & -i\sigma^k \\ i\sigma_k & 0 \end{pmatrix}$ and $\gamma^0 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$, the $N = 2$ susy charges are of the form

$$\begin{aligned}
\begin{pmatrix} Q^{i+} \\ Q^{i-} \end{pmatrix} &= \begin{pmatrix} 0 & \sigma^k(E_k - iB_k) + ig[A,B] \\ -\sigma^k(E_k + iB_k) - ig[A,B] & 0 \end{pmatrix} \begin{pmatrix} \lambda^{\alpha i} \\ \lambda^{\dot\alpha i} \end{pmatrix} \\
&+ \varepsilon^{ij} \begin{pmatrix} \sigma^m D_m(A - iB) + D_0(A - iB) & 0 \\ 0 & -\sigma^m D_m(A + iB) + D_0(A + iB) \end{pmatrix} \begin{pmatrix} \lambda^{\alpha j} \\ \lambda^{\dot\alpha j} \end{pmatrix} \\
&\equiv M\psi^i + \varepsilon^{ij} N\psi^j
\end{aligned} \tag{15.5.31}$$

Likewise we define the row vector

$$(\bar{Q}^{i+}, \bar{Q}^{i-}) = \lambda^{i,T} M^T + \varepsilon_{ij}\lambda^{j,T} N^T \tag{15.5.32}$$

We now evaluate the anticommutator $\{Q^{i\alpha}, Q^{j\beta}\}$. We shall explicitly compute the terms without fermion fields. These are obtained by taking first the product

$$(M\lambda^i + \varepsilon^{ik} N\lambda^k)(\lambda^{j,T} M^T + \varepsilon^{j\ell}\lambda^{\ell,T} N^T) \tag{15.5.33}$$

and then replacing $\lambda^j\lambda^{k,T}$ by the anticommutator $\{\lambda^j, \lambda^{k,T}\} = i\gamma^0 C^{-1}\delta^{jk}$. (The Dirac bracket is $\{\psi_i(x), {\psi_j}^\dagger(y)\} = \delta_{ij}\delta(\vec{x} - \vec{y})$ and ${\psi_j}^\dagger = {\psi_j}^T C i\gamma^0$). Moving the matrix $\gamma^0 C^{-1}$ through M^T and N^T, the transpositions disappear. The terms with δ^{ij} should be proportional to $\gamma^\mu P_\mu$. One finds indeed this structure

$$\begin{pmatrix} 0 & \begin{array}{l} {E_i}^2 + {B_i}^2 + (D_0 A)^2 + (D_0 B)^2 + (D_m A)^2 + (D_m B)^2 + g^2[A,B]^2 \\ +\sigma_m\,(\varepsilon^{mij} E_i B_j + D_0 A D_m A + D_0 B D_m B) \end{array} \\ \text{same with } -\sigma_m & 0 \end{pmatrix} \tag{15.5.34}$$

The terms with ε_{ij} are of diagonal form

$$\begin{pmatrix} \left\{\sigma_k(E^k - iB^k) + ig[A,B]\right\}\{\sigma^m D_m(A-iB) + D_0(A-iB\} & 0 \\ 0 & \text{same with } E^k+iB^k \text{ and } A+iB \end{pmatrix} \tag{15.5.35}$$

From (15.5.11) we expect that the ε^{ij} terms are a topological charge. The terms in the left-upper corner of the 2×2 matric turn out to be

$$\begin{aligned} &E^k D_k A - B^k D_k B - i(E^k D_k B - B^k D_k A) \\ &+ig[A,B]D_0 A + g[A,B]D_0 B \end{aligned} \tag{15.5.36}$$

Partially integrating D_k in $B^k D_k B$ we get simply $\partial_k(trB^kB)$ due to the Bianchi identity $D_k B^k = 0$. However, in the term $E^k D_k A$ we get $\partial_k tr(E^k A) - (D_k E^k)A$ and the last term in (15.5.36) is just the source term which cancels $(D_k E^k)A$. (This is the Gausz constraint which we may use in Hamiltonian formalism to simplify the generators because it is a first class constraint.) Similarly for $-iE^k D_k B$ and the one but last term in (15.5.36). Hence we find the expected structure

$$\left\{Q^i_\alpha, Q^j{}_\beta\right\} = i\delta^{ij}\gamma^\mu_{\alpha\beta}P_\mu + \varepsilon^{ij}\left(C_{\alpha\beta}U + i(\gamma_5)_{\alpha\beta}V\right) \tag{15.5.37}$$

where U and V are real constants

$$\begin{aligned} U &= \int d^3x \partial_k \left(E_k{}^a A^a - B_k{}^a B^a\right) \\ V &= \int d^3x \partial_k \left(E_k{}^a B^a - B_k{}^a A^a\right) \end{aligned} \tag{15.5.38}$$

The generators P_μ generate actually covariant translations; for example they transform A_μ into $\xi^\nu F_{\nu\mu} = \xi^\nu \partial_\nu A_\mu + \partial_\mu(\xi^\nu A_\nu)$ where the last term is a gauge transformation. (Since Q is gauge-invariant, one must get a gauge-invariant result in the commutator, and the combination of translations and gauge transformations in (15.5.34) and (15.5.35) is manifestly gauge invariant. Also (15.5.38) is gauge invariant.)

A nonvanishing expectation value for A and/or B in the vector multiplet implies spontaneous symmetry breaking, but for fields which vanish at infinity, there is no central charge generated: the central charge we have found is of topological origin. In the spontaneously broken theory, the VEV for A^a leads to an ordinary nontopological central charge in the susy anticommutator, which is not the central charge due to total-derivative effects at infinity, but rather like the central charge proportional to a mass (which can also be generated by reduction from 5 to 4 dimensions).

To derive again a bound on the mass M of the monopole, we go to the rest frame where $P_i = 0$, and $P_0 = M$, and use the chiral invariance of the theory to rotate V(or U) away. (The $N = 2$ action for the vector multiplet in (15.5.15) is invariant under (15.5.17). The matrices $\exp i\alpha\gamma_5$ transform $CU + i\gamma_5 V$ into $(\cos 2\alpha + i\gamma_5 \sin 2\alpha)(U + i\gamma_5 V)$, hence yield an $O(2)$ rotation on the vector (U, V)). Since the fermions are vector-like coupled to the gauge fields, there are no anomalies in this chiral symmetry). Then we choose a Majorana representation for the Dirac matrices in which $i\gamma^0 = C$ and find

$$\left\{Q_\alpha{}^i, Q_\beta{}^j\right\} = \delta^{ij}\delta_{\alpha\beta}M + \varepsilon^{ij}C_{\alpha\beta}U \tag{15.5.39}$$

Since $\{Q_\alpha{}^i, Q_\beta{}^j\}$ viewed as a 8×8 matrix is positive definite and $\varepsilon^{ij}C^{-1}{}_{\alpha\beta}$ has eigenvalues ± 1, we find $M \geq |U|$. Rotating back to $V \neq 0$, we then get

$$M^2 \geq (U^2 + V^2) \tag{15.5.40}$$

Since we can always rotate B away by a chiral $U(1)$ rotation,

$$M^2 \geq < A >^2 (e^2 + g^2) \tag{15.5.41}$$

For the kink and the bosonic monopole, the bound on M is saturated by fields which satisfy the field equation. In the susy case, this is also true but the value of $< A >$ is undetermined. This is not due to a Prasad Sommerfeld limit where both λ and m^2 tend to zero, but rather $V \sim g^2 tr[A, B]^2 \geq 0$ and the absolute minimum at

$V = 0$ is given by $< B^a >$ proportional to $< A^a >$. One can rotate $< B^a >$ to zero, but since $\lambda \sim g^2$ is nonzero, the potential itself does not vanish (only its value at the minimum). Solutions to field equation are usually associated with particles. Hence, there are particles, namely monopoles, for which the mass is of topological origin and given by $M = v\sqrt{e^2 + g^2}$. The same mass formula holds for the ordinary particles in susy Higgs models. For example, vector bosons and their associated gauginos acquire a mass due to spontaneous symmetry breaking which is given by $M = ev$ and photons satisfy $M = 0$. In the scalar sector, the radial components $x^a \cdot A^a/r$ are the neutral Higgs bosons which are indeed massless in this model, while the two remaining transversal components of A^a are the would-be Goldstone bosons. In the sector with B^a fields, the neutral radial B^a field is massless, while the two transversal components of B^a describe a charged particle whose mass is indeed equal to gv.

When the bound on the mass is satisfied, the left-hand side of (15.5.39) must vanish. This means that some of the $Q^{\alpha i}$ must vanish in this representation. This phenomenon is well-known to occur in the theory of unitary irreducible representations of susy algebras. If $M = Z$ (where $Z = \sqrt{U^2 + V^2}$), "multiplet shortening" occurs: multiplets are as short as massless ones whereas if $M > Z$, multiplets are longer (have higher spin) than the massless multiplets. The usual Higgs effect gives masses to some particles: for example, vector bosons eat a would-be Goldstone boson and become massive, and fermions acquire a mass proportional to the vacuum expectation value of the Higgs boson. However, the **number** of states after spontaneous symmetry breaking is the same as before. One expects this to remain true at the quantum level. (This is really an assumption, namely that nonperturbative effects will not change the number of states). If $N = 2$ susy is unbroken at the quantum level, multiplets must remain "short multiplets", and this implies that the bound on the mass must remain satisfied. Hence, in $N = 2$ susy theories, the masses of monopoles and dyons remain given by $M =< A > \sqrt{e^2 + g^2}$ at the quantum level (but the mass of vector bosons may get radiative corrections to $e < A >$ where in both cases $< A >$ itself may receive radiative corrections).

The equality of nontopological and topological masses has led to some interesting conjectures by Montonen and Olive. They observed that in the BPS limit of the 't Hooft-Polyakov monopole the massless Higgs field satisfies the same $1/r^2$ law as electromagnetism, so that a W^+ and W^- particle attract each other with twice the EM strength whereas a W^+ and W^+ do not exert a $1/r^2$ force on each other at all. This is the same result as derived by Manton (*Nucl. Phys. B* **126** (1977)) for monopole-antimonopole or monopole-monopole interactions. So they conjecture that W^+ and W^- particles can be described in a dual theory by monopoles and antimonopoles. A problem is that since $W^\pm$ has a magnetic moment, the monopole should have an electric dipole moment. Now, the classical 't Hooft-Polyakov monopole has no electric dipole moment (it is symmetric) so it should be produced by quantum effects. This seems far-fetched, but a classical dual picture is actually possible. There is actually a whole $N = 2$ monopole multiplet, just like the usual $N = 2$ electric multiplet. The monopole with vanishing electric dipolemoment would then corespond to a scalar particle, not to W^+ and W^-.

The triangle inequality applied to masses of multi-monopole-dyon configurations which satisfy the bound yields

$$M(q + q_1, e_1 + e_2) \leq M(q_1, e_1) + M(q_2, e_2) \tag{15.5.42}$$

This shows that a bound state of a monopole and a dyon (if it exists) is stable. One can plot the q_i and e_i on a lattice, and then the triangle inequality shows that a dyon cannot decay into other dyons (there is no energy released, hence there is no phase space for this decay). The same holds for monopoles, and explains to some extent Manton's result.

N.B. The $N = 2$ model is exactly solvable in superspace (with Lax pairs), but these Lax pairs seem rather formal and of little use for its quantization and the determination of its exact spectrum.

6 Chern-Simons terms and WZW effective actions

Chern-Simons (CS) actions can be introduced by considering in $2n$ dimensions the $2n$-form trF^n and observing that it is closed

$$d\,trF^n = trdF^n = trDF^n = 0 \tag{15.6.1}$$

We used that $tr\,dF^n = trDF^n$ because $tr[A, F^n] = 0$. The reason for $DF^n = 0$ is that D is a derivative: $DF^n = (DF)F^n + F(DF)F^{n-2} + \cdots + F^{n-1}DF$ and $DF = dF + [A, F] = 0$. (With $F = dA + AA$ one obtains $DF = d(dA + AA) + [A, dA + AA]$ and this vanishes since $d(AA) + AdA - dAA = 0$ and $[A, AA] = 0$). The statement that DF vanishes is called the Bianchi identity.

For polynomial forms it is true that closure implies exactness: when $d\Omega = 0$ then $\Omega = d\Omega'$. Thus $trF^n = d\omega_{2n-1}$ and the $2n-1$ form ω_{2n-1} is the Chern-Simons form. To give an example, consider trF^2, the Pontrjagin invariant $trF_{\mu\nu}{}^*F_{\mu\nu}$. As one knows from instanton physics, it is a total derivative. We can easily prove this using forms

$$trF^2 = d\omega_3, \omega_3 = tr\left(FA - \frac{1}{3}A^3\right) = tr\left(dAA + \frac{2}{3}A^3\right) \tag{15.6.2}$$

Indeed, $d\omega_3$ equals trF^2 (use $trA^4 = 0$).

One application of Chern-Simons terms is that they can be used to give Yang-Mills fields in 3 dimensions a topological mass. The action is then

$$\mathcal{L} = tr\left(-\frac{1}{4}F^2_{\mu\nu} + \text{ CS term }\right) \tag{15.6.3}$$

The CS term is gauge-invariant under small gauge transformations but under large gauge transformations it transforms into a total derivative. If one quantizes the coefficient in front of the CS term the action is also invariant under large gauge transformations. Then one finds mass quantization due to topological considerations.

A very interesting relation exists between CS actions and low-energy effective actions for pseudoscalar mesons. CS actions produce Wess-Zumino-Witten (WZW) models in one dimension lower as we now discuss. These WZW models are nonrenormalizable but local field theories with pseudoscalar fields, whose action varies under a local gauge variation into the consistent chiral anomaly. They are used in low-energy phenomenology where one writes the effective action Γ as a sum of an anomaly-free but nonlocal functional Γ_{reg} and a local functional Γ_{WZW} which carries the anomaly

$$\Gamma = \Gamma_{\text{reg}} + \Gamma_{\text{WZW}}, \delta_{\text{gauge}}(\Lambda)\Gamma_{\text{WZW}} = \int \Lambda^a G^a_{\text{cons}} d^4x \tag{15.6.4}$$

For some processes, Γ_{reg} does not contribute and then one can only describe these processes by Γ_{WZW}. The functional Γ_{reg} is hard to determine, but one can construct Γ_{WZW} rather easily in terms of pseudoscalar fields. It can be shown that there does not exist a local functional Γ_{WZW} in terms of only gauge fields satisfying (15.6.4).

The fundamental relation which incorporates all other relations is

$$\begin{aligned} \omega^0_{2n-1}(A_g, F_g) &= \omega^0_{2n-1}(A, F) + d\alpha_{2n-2}(A, F, g) + \omega^{(0)}_{2n-1}(g^{-1}dg, 0) \\ &= \omega^0_{2n-1}(A, F) + d(\omega^1_{2n-2}(A, F, g)) \end{aligned} \tag{15.6.5}$$

where $A^g = g^{-1}(d + A)g$ and $F^g = g^{-1}Fg$ are the finite gauge transformations of A and F, respectively. The various terms have the following meaning:

$$\begin{aligned} &\omega^0_{2n-1}(A, F) = \text{ CS Lagrangian} \\ &\omega^0_{2n-1}(g^{-1}dg, 0) = \text{ WZW term in odd dimensions} \\ &\alpha_{2n-2}(A, F, g) \text{ contains the consistent anomaly in } 2n-2 \text{ dimensions} \\ &\omega_{2n-2}(A, F, g) = \text{ gauged WZ action in even dimensions} \end{aligned} \tag{15.6.6}$$

As an example we work out the case $n = 2$. We find the following sequence of results (omitting the trace symbol tr)

$$\begin{aligned} &F^2 = d\left(FA - \frac{1}{3}A^3\right) \\ &F^g A^g - \frac{1}{3}(A^g)^3 = (g^{-1}Fg)(g^{-1}dg + g^{-1}Ag) - \frac{1}{3}(g^{-1}dg + g^{-1}Ag)^3 \end{aligned}$$

$$= \left(FA - \frac{1}{3}A^3\right) + (Fdgg^{-1} - dgg^{-1}A^2 - dgg^{-1}dgg^{-1}A) - \frac{1}{3}(g^{-1}dg)^3$$
$$= \left(FA - \frac{1}{3}A^3\right) - d[(dgg^{-1})(A)] - \frac{1}{3}(g^{-1}dg)^3 \tag{15.6.7}$$

Clearly $\alpha_2 = -(dgg^{-1})A$. The consistent anomaly[20] G in $d = 2$ is proportional to dA. Expanding $dgg^{-1} = d\lambda + \cdots$, we indeed find $\alpha_2 = \lambda dA + \cdots$. The last term $-\frac{1}{3}(g^{-1}dg)^3$ is closed because $d\left[-\frac{1}{3}(g^{-1}dg)^3\right] = (g^{-1}dg)^4$ which vanishes inside the trace ($g^{-1}dg$ anticommutes with $(g^{-1}dg)^3$). We shall show that it is exact, $-\frac{1}{3}(g^{-1}dg)^3 = d\omega^1_{2n-2}(A_2 = F = 0, g)$ and we shall construct ω^1_{2n-2}.

For $n = 3$ one finds $F^3 = d\omega^0_5(A, F)$ and $\omega^{(0)}_5(A_g, F_g) = \omega^{(0)}_5(A, F) + d\alpha_4(A, F, g) + \omega^{(0)}_5(g^{-1}dg, 0)$. The usual (by which we mean ungauged) WZW model in four dimensions is given by $\omega^1_4(A = F = 0, g)$ where $\omega^0_5(g^{-1}dg, 0) = \frac{1}{10}(g^{-1}dg)^5$ is equal to $d\omega^1_4(A = F = 0, g)$. As we show below

$$\begin{aligned} \alpha_4 &= Tr - \frac{1}{2}dgg^{-1}[AdA + dAA + A^3] \\ &+ Tr\left[\frac{1}{4}(dgg^{-1})A(dgg^{-1})A + \frac{1}{2}(dgg^{-1})^3A\right] \end{aligned} \tag{15.6.8}$$

and the first term yields the consistent anomaly $G = \frac{1}{2}d(AdA + dAA + A^3)$ in four dimensions if one sets $g^{-1}dg = d\Lambda$. The remaining terms in α_4 are needed for gauging the WZW model as we now discuss. The ungauged WZW model describes low-energy processes between pseudoscalar mesons, but the gauged WZW model can also describe processes between vector fields and pseudoscalar mesons.

The WZW term in $2n - 2$ dimensions is $\omega^{(1)}_{2n-2}(A = F = 0, g)$. It is gauged by α_{2n-2}, but it is not gauge invariant; rather, its gauge variation is the consistent chiral anomaly. To see this we return to $\omega_{2n}(F) = d\omega_{2n-1}(A, F)$ and note

[20] Consistency of G requires that the BRST variation of $tr\int cdG = tr\int cA$ vanishes. Since $\delta c = c^2\Lambda, \delta A = d(c\Lambda) + [A, c\Lambda]$ we find after partial integration and using the cyclicity of the trace

$$\begin{aligned} \delta(cdA) &= c^2\Lambda dA + cd(dc\Lambda) - dcAc\Lambda + (dc)c\Lambda A \\ &= (c^2dA - (dc)Ac + (dc)cA)\Lambda = [c^2dA + (dc^2)A]\Lambda = d(c^2A)\Lambda \end{aligned}$$

Hence $\int cdA$ is indeed BRST invariant.

that $\omega_{2n}(F) = \omega_{2n}(F^g)$. We conclude that $d(\omega_{2n-1}(A,F) - \omega_{2n-1}(A^g,F^g)) = 0$. We simplify the notations and view only A and g as independent variables. Then $\omega_{2n-1}(A) = \omega_{2n-1}(A^g) + d\omega_{2n-2}(A,g)$. We write this result three times

$$\begin{aligned} -\omega_{2n-1}(A) &= -\omega_{2n-1}(A^{hg}) - d\omega_{2n-2}(A,hg) \\ \omega_{2n-1}(A) &= \omega_{2n-1}(A^h) + d\omega_{2n-2}(A,h) \\ \omega_{2n-1}(A^h) &= \omega_{2n-1}(A^{hg}) + d\omega_{2n-2}(A^h,g) \end{aligned} \tag{15.6.9}$$

In the sum all terms without an overall exterior derivative cancel, and the rest yields

$$\omega_{2n-2}(A,hg) = \omega_{2n-2}(A,h) + \omega_{2n-2}(A^h,g) + d\omega_{2n-3} \tag{15.6.10}$$

Replacing g by $h^{-1}g$ yields

$$\omega_{2n-2}(A,g) = \omega_{2n-2}(A,h) + \omega_{2n-2}(A^h,h^{-1}g) + d\omega_{2n-3} \tag{15.6.11}$$

We recall that $\omega_{2n-2}(A,g)$ is the gauged WZW model, and under gauge transformations $g \to h^{-1}g$ while $A \to A^h = h^{-1}(d+A)h$. Note that $\omega_{2n-2}(A,h)$ vanishes when h is unity (one may check this with (15.6.8).

If we then take h near unity, we can interpret this equation as follows:
Theorem: the gauge variation of the gauged WZW model, $\int \omega_{2n-2}(A^h,h^{-1}g)$, is equal to the gauged WZW model itself, $\int \omega_{2n-2}(A,g)$, minus the consistent anomaly, $\int -\omega_{2n-2}(A,h) = -\int (h^{-1}dh)G$. Note that the gauged WZW model $\omega_{2n-2}(A,g)$ transforms into $\omega_{2n-2}(A,h)$. So the anomaly of the action is proportional to the action to first order in scalars if one replaces the remaining scalars by the gauge parameters. All scalars have disappeared after the gauge transformation, but they have done their job: they have produced the consistent chiral anomaly.

The effective action of Wess and Zumino is usually written as

$$W[A,\xi] = \int d^2x \int_0^1 dt\, tr(\xi G_t) \tag{15.6.12}$$

where $\xi G_t = \xi dA_t$ and $A_t = g_t^{-1}dg_t + g_t^{-1}Ag_t, g_t = e^{t\xi}$. We now show that it is the same as obtained from higher dimensions.

$$WA, g = \int d^2x\alpha_2(dgg^{-1}, A) - \frac{1}{3}\int\limits_{B_3}^{g(x)} [g^{-1}(x,t)(d+\delta)g(x,t)]^3 \tag{15.6.13}$$

where $d = dx^\mu \frac{\partial}{\partial x^\mu}$ and $\delta = dt\frac{\partial}{\partial t}$. Substituting $\alpha_2(dgg^{-1}, A) = -dgg^{-1}A$ we obtain

$$W[A,g] = \int d^2x(-dgg^{-1})A - \int\limits_0^1 dt \int d^2x\xi(g_t^{-1}dg_t)(g_t^{-1}dg_t) \tag{15.6.14}$$

where we choose the parametization $g(x,t) = g_t(x) = e^{t\xi}$.

$$\begin{aligned} -dgg^{-1}A &= -\int\limits_0^1 dt\frac{\partial}{\partial t}[(dg_t)g_t^{-1}A] \\ &= -\int\limits_0^1 dt[d(g_t\xi)g_t^{-1}A - dg_t(-\xi)g_t^{-1}A] \\ &= -\int\limits_0^1 dt[d\xi(g_t^{-1}Ag_t)] = -\int\limits_0^1 dtd\xi(A_{g_t} - g_t^{-1}dg_t) \end{aligned} \tag{15.6.15}$$

we arrive at

$$\begin{aligned} W[A,g] &= -\int\limits_0^1 dtd\xi(A_{g_t} - g_t^{-1}dg_t) - \int\limits_0^1 \xi(g_t^{-1}dg_t)(g_t^{-1}dg_t) \\ &= -\int dtd\xi A_{g_t} \end{aligned} \tag{15.6.16}$$

we find agreement.

The physically more interesting case is, of course $d = 4$, corresponding to $2n = 6$. We start from the form $\omega_6 = trF^3$ and find the Chern-Simons 5-form from $\omega_6 = d\omega_5$. It reads

$$\begin{aligned} \omega_5 &= tr\left(dAdAA + \frac{3}{2}dAA^3 + \frac{3}{5}A^5\right) \\ &= tr\left(F^2A - \frac{1}{2}FA^3 + \frac{1}{10}A^5\right) \end{aligned} \tag{15.6.17}$$

A finite gauge transformation ($F^g = g^{-1}F_g$ and $A^g = g^{-1}dg + g^{-1}Ag$) yields then the functional α_4 according to

$$\omega_5(F^g, A^g) = \omega_5(F,A) + d\alpha_4(F, A, dgg^{-1}) + \omega_5(0, g^{-1}dg) \tag{15.6.18}$$

It is easiest to evaluate the left-hand side if it is written as $\omega_5(F, A + dgg^{-1})$. Introducing the notation

$$V = dgg^{-1} \tag{15.6.19}$$

we find $d\alpha_4$ as the terms in $\omega_5(F, A+V)$ which are both V and A (or F) dependent

$$\begin{aligned} d\alpha_4 = tr\Big[& F^2V - \frac{1}{2}(FA^2 + AFA + A^2F)V \\ & -\frac{1}{2}(FA + AF)V^2 - \frac{1}{2}FV^3 - \frac{1}{2}FVAV \\ & +\frac{1}{2}VA^4 + \frac{1}{2}V^2A^3 + \frac{1}{2}VAVA^2 + \frac{1}{2}V^3A^2 \\ & +\frac{1}{2}V^2AVA + \frac{1}{2}V^4A\Big] \end{aligned} \tag{15.6.20}$$

It is not obvious that this expression is indeed a total derivative. To prove this, and to obtain α_4 in explicit form, we replace F by $dA + A^2$ and find

$$\begin{aligned} d\alpha_4 = tr\Big[& (dA + A^2)(dA + A^2) - \frac{1}{2}V(dAA^2 + AdAA + A^2dA + 3A^4) \\ & -\frac{1}{2}V^2(dAA + AdA + 2A^3) - \frac{1}{2}(dA + A^2)V^3 - \frac{1}{2}V(dA + A^2)VA \\ & +\frac{1}{2}VA^4 + \frac{1}{2}V^2A^3 + \frac{1}{2}VAVA^2 + \frac{1}{2}V^3A^2 + \frac{1}{2}V^2AVA + \frac{1}{2}VA^4 \end{aligned} \tag{15.6.21}$$

All terms proportional to VA^4 cancel, and the rest can be written as

$$\begin{aligned} d\alpha_4 = tr\Big[& V\left(dAdA + \frac{1}{2}dAA^2 - \frac{1}{2}AdAA + \frac{1}{2}A^2dA\right) \\ & -\frac{1}{2}V^2(dAA + AdA + A^3) - \frac{1}{2}VdAVA - \frac{1}{2}V^3dA + \frac{1}{2}V^2AVA + \frac{1}{2}VA^4\Big] \end{aligned} \tag{15.6.22}$$

This is indeed a total derivative

$$d\alpha_4 = d\,tr\left[-\frac{1}{2}V(AdA + dAA + A^3) + \frac{1}{4}VAVA + \frac{1}{4}VAVA + \frac{1}{2}V^3A\right] \tag{15.6.23}$$

To prove that (15.6.22) is equal to (15.6.23) use $dV = V^2$, and note that $dV^3 = V^4$.

The terms in α_4 which are linear in V yield the consistent anomaly G. For small gauge transformations, $V \simeq d\lambda$ and $\alpha_4 = \lambda G$. With $\alpha_4 = -\frac{1}{2}V(AdA + dAA + A^3) + \mathcal{O}(V^2)$ we find for the consistent anomaly

$$G = \frac{1}{2}d(AdA + dAA + A^3) \qquad (15.6.24)$$

The consistent anomaly should satisfy the consistency condition that the BRST variation of $\int trcGd^4x$ vanishes. Let us check this. Omitting the symbols $\int$ and tr for the time being we obtain

$$\begin{aligned}
&\delta_{BRST}cG = c^2d(AdA + dAA + A^3) \\
&+cd(\{dc + [A,c]\}dA + Ad[A,c]) \\
&+cd(\{d[A,c]\}A + dA\{dc + [A,c]\} \\
&+cd[\{dc + [A,c]\}A^2 + A\{dc + [A,c]\}A + A^2\{dc + [A,c]\}]
\end{aligned} \qquad (15.6.25)$$

We first analyze the terms with two A fields, and then the terms with three A fields

The term with two A's read

$$\begin{aligned}
&2c^2dAdA - 2dc[A,c]dA + 2cdAd[A,c] \\
&-dcdcA^2 - dcAdcA - dcA^2dc \\
&= 2c^2dAdA + 2dccAdA - 2dcAcdA \\
&+2cdcdAA + 2dcdAcA - dcdcA^2 \\
&-dcAdcA - dcdcA^2
\end{aligned} \qquad (15.6.26)$$

a) the terms with two c's next to each other are

$$= (+2dccAdA + 2cdcdAA - 2dcdcA^2) \qquad (15.6.27)$$

Writing the first term as $-2dc^2AdA$ all terms cancel

$$\begin{aligned}
&= \quad -2cdcAdA + 2cdcdAA - 2dcdcA^2 \\
&= \quad = -d(2cdcA^2)
\end{aligned} \qquad (15.6.28)$$

b) The terms with two c's not next to each other read

$$\begin{aligned} &-2dcAcdA + 2dcdAcA \\ &-dcAdcA \end{aligned} \tag{15.6.29}$$

The terms $dcAdcA$ equals minus itself and this vanishes. The first two terms also cancel after partial integration ($-2dcAcdA = -2dcdAcA$ because $dcAdcA$ vanishes again by itself).

The terms with three A's read

$$\begin{aligned} &(-dc^2)A^3 - dc(Ac - cA)A^2 \\ &-dcA(Ac - cA)A - dcA^2(Ac - cA) \end{aligned} \tag{15.6.30}$$

All terms cancel by themselves. Hence, the consistent anomaly contracted with the ghost field is indeed BRST invariant.

We now construct the Wess-Zumino term W. It is given by

$$d\alpha_4 + \omega_5(0, g^{-1}dg) = dW \tag{15.6.31}$$

By integrating over a five-ball B_5 which contains the origin in group space ($g = 1$), and is parametrized by four coordinates x^μ and a radius t which runs from 0 to 1, we find

$$W = \alpha_4(A, dgg^{-1}) + \frac{1}{10} \int_{B_5}^{g(x)} tr(g^{-1}dg)^5 \tag{15.6.32}$$

The notation $\int_{B_5}^{g(x)}$ denotes a 5-dimensional integral $\int_0^1 dt \int d^4x$ where $g(x,t) = 1$ at $t = 0$ and $g(x,t) = g(x)$ at $t = 1$. Since the integrand is invariant under general coordinate transformations, we may parametrize $g(x,t)$ anyway we like, and a particularly useful parametrization is

$$g(x,t) = e^{t\xi}; \xi = \xi^a(x)T_a \tag{15.6.33}$$

We take $A = A(x)$ independent of t; since all terms in α_4 contain a factor dgg^{-1} we can then write

$$\alpha_4 = \int_0^1 dt \frac{\partial}{\partial t} \alpha_4(A(x), dg(x,t)g^{-1}(x,t)) \tag{15.6.34}$$

This leads to

$$\begin{aligned} W_4 &= \alpha_4 + \frac{1}{10} \int_{B_5}^{g(x)} tr(g^{-1}dg)^5 \\ &= \int_0^1 dt \frac{\partial}{\partial t} \Big[\left(-\frac{1}{2} dg_t g_t^{-1} \right) (AdA + dAA + A^3) \\ &+ \frac{1}{4}(dg_t g_t^{-1}) A (dg_t g_t^{-1}) A + \frac{1}{2} (dg_t g_t^{-1})^3 A \Big] \\ &+ \frac{1}{2} \int_0^1 dt \xi (g_t^{-1} dg_t)^4 \end{aligned} \tag{15.6.35}$$

Using

$$\frac{\partial}{\partial t} V_t = \frac{\partial}{\partial t}(dg_t g_t^{-1}) = d(g_t \xi) g_t^{-1} - dg_t \xi g_t = g_t d\xi g_t^{-1} \tag{15.6.36}$$

we claim that W_4 becomes equal to

$$W_4 = \int_0^1 dt \left(-\frac{1}{2} d\xi \right) (A_t dA_t + dA_t A_t + A_t^3) \tag{15.6.37}$$

It is clear that (15.6.35) can be written as

$$\begin{aligned} W_4 = \int_0^1 dt \left(-\frac{1}{2} d\xi \right) \Big[& \tilde{A} \tilde{dA} + \tilde{dA} \tilde{A} + \tilde{A}^3 \\ & - \tilde{A} g_t^{-1} dg_t \tilde{A} - g_t^{-1} dg_t g_t^{-1} dg_t \tilde{A} + g_t^{-1} dg_t \tilde{A} g_t^{-1} dg_t \\ & - \tilde{A} g_t^{-1} dg_t g_t^{-1} dg_t \Big] + \frac{1}{2} \int_0^1 dt \xi (g_t^{-1} dg_t)^4 \end{aligned} \tag{15.6.38}$$

where $\tilde{A} \equiv g_t^{-1} A g_t$ and $\tilde{dA} \equiv g_t^{-1} dA g_t$. If we then write

$$\begin{aligned} \tilde{A} &= A_t - g_t^{-1} dg_t \\ \tilde{dA} &= dA_t + g_t^{-1} dg_t A_t + A_t g^{-1} dg_t - g^{-1} dg_t g^{-1} dg_t \end{aligned} \tag{15.6.39}$$

it is clear that all terms in (15.6.38) with only A_t and dA_t but no $g_t^{-1}dg_t$ match. The remaining terms all depend on $g_t^{-1}dg_t$ and should cancel. To simply the notation, we denote $g_t^{-1}dg_t$ by U

$$U = g_t^{-1}dg_t \tag{15.6.40}$$

and find then for the U dependent terms in the integrand

$$\begin{aligned}
&\left(-\frac{1}{2}d\xi\right)[A_t(A_t + A_tU - U^2)\\
&-U(dA_t + UA_t + A_tU - U^2)\\
&+(UA_t + A_tU - U^2)(A_t - U) + dA_t(-U)\\
&-A_tA_tU - A_tUA_t - UA_tA_t\\
&+A_tUU + UA_tU + UUA_t - U^3\\
&-(A_t - U)U(A_t - U) - U^2(A_t - U) + U(A_t - U)U\\
&+(A_t - U)U^2\Big] + \frac{1}{2}\xi U^4
\end{aligned} \tag{15.6.41}$$

The U^3 terms inside the square brackets sum up to $+\frac{1}{2}d\xi U^3$, and combine with the last term $\frac{1}{2}\xi U^4$ into a total derivatives (recall that $dU^3 = U^4$). The terms with one A field are proportional to $(-U^2A_t + A_tU^2) - (UdA_t + dA_tU)$ and do not contribute either. Finally the terms with two A fields cancel straightforwardly.

Hence we have seen that a finite gauge variation of the Chern-Simons action is equal to the Chern Simons term itself, plus the exterior derivative of the Wess-Zumino term. The Wess-Zumino term is the integrated consistent chiral anomaly.

7 The winding of the Wess-Zumino term

The functional

$$W = c\int_{B_5}^{g(x)} tr(g^{-1}dg)^5, c \text{ a constant to be fixed} \tag{15.7.1}$$

The symbol $\int_{B_5}$ denotes an integral over the 5-ball B_5. It is invariant under $e^{i\lambda_R \cdot T} g e^{-i\lambda_L \cdot T}$ where T_a are the generators. It is also independent of the choice of five-dimensional coordinates y^i because it is a scalar density in general relativity. We claim that it only depends on the values of $g(x)$ on the surface of the ball (spacetime). To show this we make a small variation of g somewhere in the interior of B_5. Then

$$\begin{aligned}
&\delta tr(g^{-1}dg)^5 = 5\varepsilon^{ijhlm} tr(g^{-1}\partial_i g)(g^{-1}\partial_j g)(g^{-1}\partial_k g)(g^{-1}\partial_l g)\delta(g^{-1}\partial_m g) d^5x \\
&= 5\varepsilon^{ijhlm} tr(g^{-1}\partial_i g)\cdots(g^{-1}\partial_l g)[-g^{-1}\delta g g^{-1}\partial_m g + g^{-1}\partial_m \delta_g] d^5x \\
&= 5\varepsilon^{ijhlm} tr(g^{-1}\partial_i g)\cdots(g^{-1}\partial_l g)[g^{-1}\partial_m(\delta g g^{-1})g] \\
&= 5\varepsilon^{ijklm}\partial_m tr(g^{-1}\partial_i g)\cdot\cdot(g^{-1}\partial_l g)(g^{-1}\delta g)
\end{aligned} \tag{15.7.2}$$

We partially integrated ∂_m and used that terms such as $\partial_m\partial_i g$ cancel due to the ε symbol, while terms due to ∂_m hitting the four factors g^{-1} in $(g^{-1}\partial g)$ cancel because there are two terms with a plus sign and two terms with a minus sign. Since the surface of B_5 is closed (namely S_4), the total derivative vanishes. Thus $W(g(x))$ is an a group-invariant term which one can add to the action.

The coefficient appearing in W is quantized! To understand the reason for his surprising and interesting fact, consider the map of the compactified spacetime S_4 into the group manifold G.

(15.7.3)

The map of S_4 (space) into the group manifold is S_4 (internal) and forms the boundary of a fiveball B_5. However, one can also consider another five-ball $B_5{}'$ of which S_4 (space) is the boundary. The difference of the integrals over B_5 minus $B_5{}'$ is the integral over S_5. In order that $\exp iS$ (WZ) be independent of the choice of B_5, the change in S should be $2\pi m$. This leads to the quantization condition on c.

$$c\varepsilon^{ijhlm}\int_{S_5} tr(g^{-1}\partial_i g g^{-1}\partial_j g \ldots g^{-1}\partial_m g) = 2\pi n \tag{15.7.4}$$

The integral is proportional to the winding number of the map of S_5 (space) into the group manifold G,[21] and since

$$\pi_5(G) = Z \tag{15.7.5}$$

for $U(N)$, there is indeed a quantization condition on c.

8 $SU(3) \times SU(3)$ symmetry in QCD and the WZW term

The rigid $U(3)_{\text{left}} \times U(3)_{\text{right}}$ symmetry between the 3 highest quark **flavours** (up, down, strange) in the quark-gluon interactions

$$\begin{pmatrix} u \\ d \\ s \end{pmatrix}' = e^{i(\theta_V^a \lambda^a + \theta_A^a \lambda_a \gamma_5)} \begin{pmatrix} u \\ d \\ s \end{pmatrix} \tag{15.8.1}$$

is **presumably** spontaneously (dynamically due to the QCD interactions with the colour group $SU(3)$) broken down to $SU(3)_{\text{vector}} \times U(1)_{\text{vector}}$, where $SU(3)_{\text{vector}}$ is the approximate $SU(3)$ of Gell-Mann and Neeman, and $U(1)_{\text{vector}}$ is the baryon symmetry (leading to conservation of baryon number).

Because the axial $SU(3)$ generators $\lambda_a \gamma_5$ are spontaneously broken, there should appear Goldstone bosons with the same quantum numbers as these generators, namely pseudoscalars in octets of $SU(3)$ vector. The $U(1)_{\text{axial}}$ is also broken, but instead of leading to a Goldstone boson, one gets instantons:

$$\partial^\mu j_\mu \text{ (singlet)} \sim \int F_{\mu\nu} \tilde{F}^{\mu\nu} \tag{15.8.2}$$

Coupling to **fictitious** vector fields A_μ, V_μ which gauge the rigid $SU(3) \times SU(3)$ and fictitious fermions to cancel anomaly, one finds at low energy

[21] If one fills up the interior of S_4 one obtains a 4-ball B_4 which is mapped to the half-sphere S_5 whose boundary is S_4 (internal).

high-energy: elementary particles	low-energy: bound states
quarks	Goldstone bosons ξ^a of $SU(3)_{\text{axial}}$
qluons (which play here no role)	in $SU(3)_{\text{vector}}$ multiplets.
fictitious vectors V_μ^a	fictitious vectors V_μ^a
fictitious axial vectors A_μ^a	fictitious axial vectors A_μ^a
fictitious spinors (which cancel anomalies)	fictitious spinors

The anomaly in the effective action $\Gamma(\xi^a, V, A)$ should be same as in the fundamental theory.

$$\delta \text{ (axial) } \Gamma(\xi^a, V, A) = \int \xi^a G^a(V, A) d^4x \tag{15.8.3}$$

The Wess Zumino term is proportional to the integrated consistent chiral anomaly. For theories with vector fields V and axial vector fields A the chiral anomaly is the Bardeen anomaly

$$\begin{aligned}
&G^a = \frac{1}{16\pi^2}\varepsilon^{\mu\nu\rho\sigma} tr\lambda^a \left[V_{\mu\nu}V_{\rho\sigma} + \frac{1}{3}A_{\mu\nu}A_{\rho\sigma} + \frac{32}{3}A_\mu A_\nu A_\rho A_\sigma\right. \\
&-\frac{8}{3}(V_{\mu\nu}A_\rho A_\sigma + A_\mu V_{\nu\rho}A_\sigma + A_\mu A_\nu V_{\rho\sigma}) \\
&V_{\mu\nu} = \partial_\mu V_v - \partial_\nu V_\mu + [V_\mu, V_\nu] + [A_\mu, A_\nu] \\
&A_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [V_\mu, A_\nu] + [A_\mu, V_\nu]
\end{aligned} \tag{15.8.4}$$

The infinitesimal gauge transformation are

$$\begin{aligned}
\delta V_\mu &= \partial_\mu \lambda_\nu + [V_\mu, \lambda_\nu] + [A_\mu, \lambda_A] \\
\delta A_\mu &= \partial_\mu \lambda_A + [V_\mu, \lambda_A] + [A_\mu, \lambda_n u]
\end{aligned} \tag{15.8.5}$$

and one may check that $V_{\mu\nu}$ and $A_{\mu\nu}$ transform each homogeneously both under λ_V and λ_A transformations.

$$\begin{aligned}
\delta V_{\mu\nu} &= [V_{\mu\nu}, \lambda_V] + [A_{\mu\nu}, \lambda_A] \\
\delta A_{\mu\nu} &= [A_{\mu\nu}, \lambda_V] + [V_{\mu\nu}, \lambda_A]
\end{aligned} \tag{15.8.6}$$

In fact, if one sets

$$\begin{aligned}
\lambda_V &= \frac{1}{2}(\lambda_L + \lambda_R) & \lambda_A &= \frac{1}{2}(\lambda_L - \lambda_R) \\
V_\mu &= \frac{1}{2}(W_\mu^L + W_\mu^R) & A_\mu &= \frac{1}{2}(W_\mu^L - W_\mu^R)
\end{aligned} \tag{15.8.7}$$

then the transformation rules decouple

$$\begin{aligned} \delta W_\mu^L &= \partial_\mu \lambda_L + [W_\mu^L, \lambda_L] \\ \delta W_\mu^R &= \partial_\mu \lambda_R + [W_\mu^R, \lambda_R] \end{aligned} \tag{15.8.8}$$

Finite gauge transformations are given by

$$\begin{aligned} &(W_\mu^L)' = e^{-\lambda_L}(\partial_\mu + W_\mu^L)e^{\lambda_L} \\ &(W_\mu^R)' = e^{-\lambda_R}(\partial_\mu + W_\mu^R)e^{\lambda_R} \\ &\begin{pmatrix} V_\mu{}' \\ A_{\mu'} \end{pmatrix} = \frac{1}{2}(W_\mu^{L'} \pm W_\mu^{R'}) = \\ &\frac{1}{2}e^{-(\lambda_V+\lambda_A)}(\partial_\mu + V_\mu + A_\mu)e^{\lambda_V+\lambda_A} \\ &\pm\frac{1}{2}e^{-\lambda_V+\lambda_A}(\partial_\mu + V_\mu - A_\mu)e^{\lambda_V-\lambda_A} \end{aligned} \tag{15.8.9}$$

For pure gauge fields which are due to an axial gauge transformation we get

$$\begin{aligned} V_\mu{}' &= \frac{1}{2}(e^{-\lambda_A}\partial_\mu e^{\lambda_A} + e^{\lambda_A}\partial_\mu e^{-\lambda_A}) \\ A_\mu{}' &= \frac{1}{2}(e^{-\lambda_A}\partial_\mu e^{\lambda_A} - e^{\lambda_A}\partial_\mu e^{-\lambda_A}) \end{aligned} \tag{15.8.10}$$

If we write this as $V' = \frac{1}{2}(g^{-1}dg + gdg^{-1})$ and $A' = \frac{1}{2}(g^{-1}dg - gdg^{-1})$ we can easily check that the curvatures

$$dV + V^2 + A^2 \text{ and } dA + VA + AV \tag{15.8.11}$$

vanish for pure gauge fields.

The Wess-Zumino term becomes then for pure gauge fields

$$\begin{aligned} W &= \frac{c}{16\pi^2}\int_0^1 dt\, tr\left(\frac{1}{2}\xi^a\right) G^a(V_t, A_t) \\ &= \frac{c}{16\pi^2}\int_0^1 dt\, tr\left(\frac{1}{2}\xi\right)\left[\frac{32}{3}\varepsilon^{\mu\nu\rho\sigma} A_\mu{}' A_\nu{}' A_\rho{}' A_\sigma{}'\right] \end{aligned} \tag{15.8.12}$$

where $A' = \frac{1}{2}g^{-1}dg - gdg^{-1} = \frac{1}{2}g^{-1}(dg^2)g^{-1}$. The constant c is known from the chiral anomaly. Denoting g^2 by U we have

$$W = \frac{c}{16\pi^2}\int_0^1 dt\, tr\left(\frac{1}{2}\xi\right)\frac{32}{3}\varepsilon^{\mu\nu\rho\sigma}\frac{1}{16}(U^{-1}\partial_\mu U U^{-1}\partial_\nu U U^{-1}\partial_\rho U U^{-1}\partial_\sigma U) \tag{15.8.13}$$

Hence the part of the effective action for the interactions between Goldstone bosons which takes into account the anomalies in the $SU(3)\times SU(3)$ flavour group, is given by

$$\begin{aligned}\Gamma(\xi^a) &= \frac{1}{16\pi^2}\cdot\left(\frac{32}{3}\right)n\int_0^1 dt\varepsilon^{\mu\nu\rho\sigma}(n = \text{number of colours})\\ &\frac{1}{16}\int tr\left(\frac{1}{2}\xi\right)(U^{-1}\partial_\mu U)\ldots U^{-1}\partial_\sigma U)\\ &= \frac{1}{24\pi^2}ne^{\mu\nu\rho\sigma}\int_0^1 dt\, tr\frac{1}{2}\xi[U^{-1}\partial U]\left(\text{ with } \frac{1}{2}\xi = U^{-1}\partial_t U\right)\end{aligned} \tag{15.8.14}$$

This functional W gives a good description at low-energy of the interactions between the $SU(3)$ Goldstone fields (pions, bosons). If one expands $U = e^{2i\xi^a\pi_a}$, one finds to lowest order

$$\begin{aligned}\mathcal{L} &= c\varepsilon^{\mu\nu\rho\sigma}(TrT_aT_bT_cT_dT_e)\\ &\int d^4x\xi^a(x)\partial_\mu\xi^b(x)\partial_\nu\xi^c(x)\partial_\rho\xi^d(x)\partial_\sigma\xi^e(x)\end{aligned} \tag{15.8.15}$$

This term describes $K^+K^- \to \pi^+\pi^-\pi^0$. The nontopological low-energy action $\mathcal{L} = tr(U^{-1}\partial_\mu U U^{-1}\partial^\mu U)$ cannot produce terms which are odd in the number of Goldstone fields, but processes such as $KK \to \pi\pi\pi$ are observed in nature, so one needs the WZW term.

One can get information on the low-energy spectrum of bound states, and in particular understand why the axial $SU(3)$ symmetry is broken (leading to Goldstone bosons) by using an argument due to 't Hooft on "anomaly matching". The idea is to promote the rigid $SU(3)_{\text{vector}}\times SU(3)_{\text{axial}}$ symmetry to a local symmetry, with new fictitious gauge fields. (Actually, for the left-handed $SU(2)$ subgroup, these gauge fields are not at all fictitious; they correspond to the W and Z bosons). These gauge fields are supposed to couple with very small coupling constants thus guaranteeing that perturbation theory is reliable. The idea is to first concentrate on the high-energy part of the theory, with quarks and gluons and asymptotic freedoms and to introduce fictitious fermions which only couple to the fictitious gauge fields such that all chiral anomalies (of quarks and fictitious fermions) cancel.

Next we look at the low-energy part of the theory.[22] Here we do not find quarks (because we assume confinement, for example by the $SU(3)$ colour group) but we still find the fictitious fermions. In addition there could be massless fermionic bound states which could give anomalies in triangle graphs. There are now only two possibilities
(i) the gauge group $SU(3)_{\text{vector}} \otimes SU(3)_{\text{axial}}$ is broken. Since $SU(3)$ is not broken perturbatively (in fact, not even nonperturbatively, see C. Vafa and E. Witten, *Nucl. Phys.* 198?) one need only consider the breaking of $SU(3)_{\text{axial}}$. If one of the generators of $SU(3)_{\text{axial}}$ is broken, the presence of the full unbroken $SU(3)_{\text{vector}}$ implies that the whole $SU(3)_{\text{axial}}$ is broken. This leads to an octet of Goldstone bosons.
(ii) the gauge group $SU(3)_{\text{vector}} \times SU(3)_{\text{axial}}$ is not broken. In this case the anomalies due to the bound states (due to quark confinement) and fictitious fermions must cancel. The fictitious fermions were, however, introduced precisely to cancel the anomalies of the quarks. It follows that the total anomaly from the quarks must be equal to the total anomaly due to all massless fermionic bosonic states.

Summarizing so far: if G is broken one gets Goldstone bosons. If G is unbroken one needs (in general) massless fermions at low energy to cancel the chiral anomaly.

Let us now apply this anomaly matching approach to the real world. First assume there are only two flavours. Later we discuss the case with three flavours, which leads to opposite conclusions. As symmetry group G we have $SU(2)_L \otimes SU(2)_R \otimes$

[22]See M. Peskin in "Les Houches, session 39, 1982".

$U(1)_{\text{vector}}$ because the $U(1)_{\text{axial}}$ is certainly broken by instantons. (Note that these symmetries are rigid flavour symmetries of the strong interactions). The quarks must be massless because otherwise the rigid axial symmetries would be broken. We shall show that in this case $G = SU(3)_{\text{vector}} \times SU(3)_{\text{axial}}$ can remain unbroken at low energies provided the proton and neutron are massless. In the high energy sector there are by assumption only two quarks, the u and d quark. (There are, of course, also leptons but they are present both at high and low energy; they could be part of the fictitious fermions and do not play a role in the matching of anomalies between quarks and massless colour-singlet bound-state fermions). The chiral anomalies are due to one-loop triangle graphs with external $SU(2)_L$ or $SU(2)_R$ or $U(1)_{\text{vector}}$ gauge fields. Since $SU(2)$ is pseudoreal, we only need consider diagrams with at least one $U(1)$ field. This leaves only three graphs

(15.8.16)

The first graph has no anomaly because the $U(1)$ group we consider is vectorial. The anomaly of the second graph is proportional to $3e_{U(1)} tr\, \tau_a \tau_b$ where $e_{U(1)}$ is the coupling constant of the fermions to the $U(1)_{\text{vector}}$ gauge field and the factor 3 is due to the three colors of quarks. The proton and neutron are color-singlets but they contribute the same anomaly because their $U(1)$ coupling $e_{U(1)}$ is 3 times bigger (since they contain 3 quarks. Explicitly the proton has the same quantum numbers as the combination $\varepsilon^{abc}(\psi_a^{i,\alpha}\psi_b^{j,\beta}\psi_c^{k,\gamma})_{\beta\gamma}\varepsilon_{jk}$. In fact, the antisymmetry of the color factor ε_{abc} and the charge conjugation matrix $\varepsilon_{\beta\gamma}$ implies that j and k couple only to a $SU(2)$ singlet, as indicated by ε_{jk}).

Hence the anomalies of the massless u and d quark have been matched with those of the proton and neutron. The conclusion is that in two-flavour QCD one need not get Goldstone bosons.

The situation is drastically different for three-flavour QCD because although

$SU(2)_L$ and $SU(2)_R$ are pseudoreal, $SU(3)_L$ and $SU(3)_R$ are complex. We need now consider two additional triangles, with three $SU(3)_L$ or three $SU(3)_R$ vertices

(15.8.17)

It is sufficient to focus on the graphs with L vertices, the analysis for the graphs with R vertices being identical. For the diagram with one $U(1)$ vertex and two $SU(3)_L$ vertices one finds the following quark anomaly

$$3e_{U(1)} tr\ t_a^F t_b^F \tag{15.8.18}$$

where t_a^F are Gell-Mann flavour matrices. The composite massless color-singlet fermions are now of the form $\varepsilon^{abc}(\psi_a^{i\alpha}\psi_b^{j\beta}\psi_c^{\beta\gamma})$. Restricting our attention to spin 1/2 fermions, we contract with $\varepsilon_{\beta\gamma}$, but then the antisymmetry in jk leads to a $\bar{3}$ combination under $SU(3)_L$. These composite fermions still have $U(1)$ charge 3, but they are now in a $3 \otimes \bar{3}$ of $SU(3)_L$. The anomaly is thus

$$3xtr\ t_a^{adj} t_b^{adj} = 3(6) tr\ t_a^F t^F \tag{15.8.19}$$

where the factor 3 refers to the $U(1)$ charge, and the factor 6 relates the Dynkin label $t(R)$ of the adjoint representation and the fundamental representations. One can also build other composite fermions out of three quarks, for example using one left-handed and two right-handed quarks. In all cases the anomalies are too large for the composite fermions. The conclusion is thus that for QCD with three quarks, the axial $SU(3)$ symmetry must be broken and this explains why Goldstone bosons appear in nature.

9 Skyrmions

Skyrmions are solitons in the effective action for Goldstone bosons that represent baryons. The action is given by

$$\mathcal{L} = \frac{1}{16\pi^2 F_\pi^2} Tr(\partial_\mu U^{-1} \partial_\mu U) + \mathcal{L}\ (4 - \text{derivatives}) + \mathcal{L}_{\text{WZ}} \tag{15.9.1}$$

where $\mathcal{L}_{\text{WZ}}$ is the WZ action written in 5 dimensions, while U is given by

$$\begin{aligned} U &= e^{2i\xi^a(x)\lambda_a} \\ \lambda_a \xi^a(x) &= \frac{\sqrt{2}}{F_\pi} \begin{pmatrix} \frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta^0 & \pi^+ & K^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta^0 & K^0 \\ K^- & \bar{K}^0 & -\sqrt{\frac{2}{3}}\eta^0 \end{pmatrix} \end{aligned} \tag{15.9.2}$$

The matrix U contains the fields $\xi^a(x)$ of the 0^- octet and expanding U as $U = 1 + \frac{2i}{F_\pi}\sum_{a=1}^{8} \lambda_a \pi^a (tr\lambda^a\lambda^b = 2\delta^{ab})$ we find the kinetic terms.

$$\begin{aligned} \mathcal{L}\ (\text{kin}) &= -\frac{1}{16} F_\pi^2 Tr \partial_\mu U \partial_\mu U^+ \\ &= -\frac{1}{2}(\partial_\mu \pi^0)^2 - \partial_\mu \pi^+ \partial_\mu \pi^- - \partial_\mu K^+ \partial_\mu K^- \\ &\quad -\partial_\mu \overline{K^0} \partial_\mu K^0 - \frac{1}{2} \partial_\mu \eta^0 \partial_\mu \eta^0 \end{aligned} \tag{15.9.3}$$

The term $\mathcal{L}$ (4-derivatives) is an $SU(3) \times SU(3)$ invariant action with 4 derivatives, needed to obtain solutions which avoid Derrick's no-go theorem.

In order that $\int \partial_\mu U^{-1} \partial_\mu U d^3x$ yields a finite energy solution (the soliton), the matrix U should tend to a fixed element $U(\infty)$ in all 3 directions. Hence R^3 compactifies to S_3, and there exist configurations with winding since

$$\pi_3(SU(N)) = Z \tag{15.9.4}$$

The solutions in a given sector with winding are the lowest-energy configurations in that sector, but the energy does not satisfy the BPS bound, just as in the case of monopoles with a $\lambda\varphi^4$ coupling when $\lambda \neq 0$. These solitons by themselves are, of course, a nonperturbative solution, and a quantum theory with such solitons is nonrenormalizable. Despite this draw-back, they are being used in conjunction with monopoles, to study low-energy quantum corrections.

Under global charge rotations U transforms as follows

$$U \to U + i\varepsilon[Q, U]; Q = \begin{pmatrix} 2/3 & & \\ & -1/3 & \\ & & -1/3 \end{pmatrix} \tag{15.9.5}$$

because $\xi' = e^{-i\varepsilon Q}\xi e^{i\varepsilon Q}$, so $(\exp \xi)' = U' = e^{-i\varepsilon Q} e^{\xi} e^{i\varepsilon Q} = U + i\varepsilon[Q, U]$. Gauging this one finds

$$\begin{aligned} \Gamma(U \text{ and } A_\mu) = \int d'' x A_\mu J^\mu_{\text{Noether}} \\ + \int d^4x (\partial_\mu A_\nu) A_\rho [\ldots]_\sigma e^{\mu\nu\rho\sigma} \end{aligned} \tag{15.9.6}$$

The last term describes $\pi^0 \to 2\gamma$, and agress with QCD if n (winding) $= n$ (colours) $= 3$. Note that so far we only discussed flavor $SU(3) \times SU(3)$, so color $SU(3)$ is a result!

The baryon current is the piece with the $U(1)_V$ current. If quarks have baryon number $\frac{1}{n\ (\text{color})}$, then a Skyrme soliton has baryon number one. To prove that these solitons with baryon number 1 are fermions, one may consider a soliton at rest with time dependence e^{-iMt} and show one that is adiabatically rotated through 2π gives $e^{-iMt} e^{-in\ (\text{colour})\ \pi}$.

10 The normalization of the WZW terms

The normalization of the WZW term is needed if one wants to determine the value of the quantized coupling constant. The fact that the coupling constant is quantized can be understood from the fact that there is winding. We must evaluate the following integral

$$\int_{S_n} Tr(g^{-1}dg)^n =$$
$$= (TrT_{a_1}\dots T_{a_n})\int\left(g^{-1}\frac{\partial}{\partial\varphi^{a_1}}g\right)^{a_1}\dots\left(g^{-1}\frac{\partial}{\partial\varphi^{a_n}}g\right)^{a_n}\partial_{\mu_1}\varphi^{a_1}\dots\partial_{\mu_n}\varphi^{a_n}\varepsilon^{\mu_1\dots\mu_n}d^nx \tag{15.10.1}$$

We first consider $SU(2)$ and S_3. This is the case which corresponds to the Wess-Zumino-Witten terms in 2 dimensions. The homotopy group is $\pi_3(SU(2)) = Z$.

(15.10.2)

The φ^i are coordinates on the group manifold, $g = \exp(\varphi^i\delta_i^a T_a)$, and $\varphi^i(\sigma,\tau)$ is map from the compactified worldsheet S_2 into the group. The WZW term can be written as an integral over a 3-ball B_3, obtained by filling in the interior of the S_2.

$$\Big\}B_3 \text{ with coordinates } \sigma,\tau,t \text{ radius } t \text{ with } 0\le t\le 1. \tag{15.10.3}$$

One can always fill in the interior of S_2 because $\pi_2\ (SU(n)) = 0$ for all n. The group coordinates on B_3 are $\varphi^i(\sigma,\tau,t)$. The WZW term becomes then proportional to

$$tr(T(R)_aT(R)_bT(R)_c)\int_0^1 dt\int d\sigma\, d\tau\left(g^{-1}\frac{\partial}{\partial\varphi^i}g\right)^a\left(g^{-1}\frac{\partial}{\partial\varphi^j}g\right)^b\left(g^{-1}\frac{\partial}{\partial\varphi^k}g\right)^c$$
$$\partial_\mu\varphi^i\partial_\nu\varphi^j\partial_\rho\varphi^k\varepsilon^{\mu\nu\rho}\equiv\int^{B_3}(g^{-1}dg)^3 \tag{15.10.4}$$

One can fill in S_2 in different ways

$$\Big\}\text{ difference is an } S_3 \tag{15.10.5}$$

and the difference of both integrals is an integral over S_3

$$\int_{B_3}(g^{-1}dg)^3-\int_{B_3{}'}(g^{-1}dg)^3=\int_{S_3}(g^{-1}dg)^3 \tag{15.10.6}$$

We begin with $G = SU(2)$ and $T(R)_a$ the fundamental 2×2 representation. For other representations one finds then an integer multiplet of this result. For other groups we shall use a theorem by Bott which states that one can always deform a S_3 in G such that in the end all points of the deformed S_3 lie in an $SU(2)$ subgroup of G. This will give then the normalization of the WZW term in 2 dimensions for other groups as well.

$$\begin{aligned} T_a &= \frac{-i\sigma_a}{2}; [T_a, T_b) = f_{ab}{}^c T_c; f_{ab}{}^c = \varepsilon_{abc} T(R) = \frac{1}{2} \\ \gamma_{ab} &= -f_{ab}{}^q f_{bq}{}^p = 2\delta_{ab}; TrT_aT_b = -\gamma_{ab}T(R) \end{aligned} \tag{15.10.7}$$

Using Euler angles $\varphi^i = (\varphi, \theta, \psi)$

$$\begin{array}{c} 0 \leq \varphi \leq 2\pi \\ 0 \leq \theta \leq \pi \\ 0 \leq \psi \leq 4\pi \end{array} \tag{15.10.8}$$

to parametrize group elements as $g = e^{T_3\psi} e^{T_1\theta} e^{T_3\varphi}$ we find

$$\begin{aligned} & g^{-1}\frac{\partial}{\partial\varphi}g = T_3 \Rightarrow \left(g^{-1}\frac{\partial}{\partial\varphi}g\right)^3 = 1 \\ & g^{-1}\frac{\partial}{\partial\theta}g = T_1\cos\varphi + T_2 \text{ since } \Rightarrow \left(g^{-1}\frac{\partial}{\partial\theta}g\right)^1 = \cos\varphi; \left(g^{-1}\frac{\partial}{\partial\theta}g\right)^2 = \sin\varphi \\ & g^{-1}\frac{\partial}{\partial\psi}g = T_3\cos\theta - \sin\theta(T_2\cos\varphi - T_1\sin\varphi) \\ & \Rightarrow \left(g^{-1}\frac{\partial}{\partial\varphi}g\right)^1 = \sin\varphi; \left(g^{-1}\frac{\partial}{\partial\varphi}g\right)^2 = -\sin\theta\cos\varphi; \left(g^{-1}\frac{\partial}{\partial\varphi}g\right)^3 = \cos\theta \end{aligned} \tag{15.10.9}$$

Hence

$$e_i{}^a(\varphi) = \left(g^{-1}\frac{\partial}{\partial\varphi^i}g\right)^a = \begin{pmatrix} 0 & 0 & 1 \\ \cos\varphi & \sin\varphi & 0 \\ \sin\theta\sin\varphi & -\sin\theta\cos\varphi & \cos\theta \end{pmatrix} \tag{15.10.10}$$

Then

$$\begin{aligned} & (TrT_aT_bT_c)\int\left(g^{-1}\frac{\partial}{\partial\varphi^i}g\right)^a \cdots \left(g^{-1}\frac{\partial}{\partial\varphi^k}g\right)^c \partial_\mu\varphi^i \cdots \partial_\rho\varphi^k \varepsilon^{\mu\nu\rho} \\ & = \left(-\frac{1}{4}\varepsilon_{abc}\right)\int\left(g^{-1}\frac{\partial}{\partial\varphi^i}g\right)^a \cdots \left(g^{-1}\frac{\partial}{\partial\varphi^h}g\right)^c \varepsilon^{ijk} d^3\varphi \end{aligned}$$

$$
\begin{aligned}
&= \left(-\frac{1}{4}\varepsilon_{abc}\right)\int \varepsilon^{abc}(\det e_i{}^a)d^3\varphi \\
&= \left(-\frac{3}{2}\right)\int(-\sin\theta)d^3\varphi = \int_0^{2\pi} d\varphi \int_0^{\pi} dt \int_0^{4\pi} d\varphi \left(\frac{3}{2}\sin\theta\right) = 24\pi^2
\end{aligned}
\tag{15.10.11}
$$

Hence

$$
S = \frac{1}{12\pi}\int_{B_3}^{g(x)} (g^{-1}dg)^3 \tag{15.10.12}
$$

is unamibiguous in the path integral. This action can be written as an action in 2 dimensions by doing the t-integral and choosing $g(\sigma,\tau,t) = \exp t\varphi(\sigma,\tau) \equiv g_t$

$$
\begin{aligned}
S(\lambda) &= \frac{1}{4\pi}\int_0^1 dt \int d\sigma\, d\tau (g_t^{-1}\partial_\sigma g_t)(g_t^{-1}\partial_\tau g_t)(g_t^{-1}\partial_t g_t) \\
&= \frac{1}{4\pi}\int_0^1 dt \int d\sigma\, d\tau tr \lambda(e^{-t\lambda}\partial_\sigma e^{t\lambda})(e^{-t\lambda}\partial_t e^{t\lambda}) \\
&= \frac{1}{4\pi}\int_0^1 dt \int d\sigma\, d\tau tr \left(\sum \frac{1}{n!}t^n[[\partial_\sigma\lambda,\lambda]\ldots\lambda]\right)\left(\sum \frac{1}{m!}t^m[\cdots[\partial_t\lambda,\lambda]\cdots\lambda]\right)
\end{aligned}
\tag{15.10.13}
$$

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Chapter 16

Renormalization of composite operators

Composite operators contain products of field operators at the same point in spacetime. In a path integral approach, the operator aspect plays a minor role, but we shall retain the term composite operators for products of fields and derivatives of fields at the same point in spacetime. A composite operator at a point x can be thought of as a Green's function for a set of fields at spacetime points $x^1, \ldots, x^n$ in which these spacetime points come together at a point x; this leads in general to extra divergences. Conversely, one method of regularization of composite operators consists of pulling the constituent fields apart ("point splitting"), the distance between points serving as the inverse of the regulator mass.

Composite operators occur in many places in particle physics: as stress tensors which couple a renormalizable quantum field theory to external gravitation, in the operator product expansion of two (or more) QED or QCD currents (the Wilson expansion), as fermion condensates $< \bar{\psi}\psi >$ in attempts to explain chiral symmetry breaking, etc. The renormalization of such composite operators is the subject of this chapter. (For the renormalization of gauge theories we also introduced the composite operators $D_\mu c$ and *ctimesc*, but by coupling them to external currents K^μ and L, we made them part of the quantum action, and proved that the theory with these extra composite operators is renormalizable). We shall only consider Green functions with one composite operator insertion but not Green's functions with two or more composite operators inserted. Green's functions with two insertions of composite operators may be reduced to the case with only one insertion when the points of the insertions become close by the method of operator product expansions.

A new aspect in the renormalization of composite operators (new with respect to the renormalization of ordinary quantum field theories without composite operators such as Yang-Mills theory) is "operator mixing": the divergences in the proper graphs with one insertion of a composite operator $O_1(x)$ at a point x may contain in general local divergences which are proportional to another composite operator $O_2(x)$. The complete set of composite operators must be renormalized simultaneously (loop-by-loop), and multiplicative renormalization of the fields and coupling constant in the underlying gauge field theory together with operator mixing leads to the following relation between unrenormalized and renormalized composite operators

$$O_j[A_\mu{}^a, \dots, g] = \sum_k Z_j{}^k O^{ren}{}_k \left[Z_3{}^{1/2} A_\mu{}^{a,ren}, \dots, \frac{Z_1}{Z_3{}^{3/2}} u \right] \tag{16.0.1}$$

For the definition of Z_1 and Z_3 we refer back to chapter III. The matrix $Z_j{}^k$ is determined order by order in the number of loops by the divergences in these proper graphs with one insertion of a composite operator. Hence there are counter terms proportional to $Z_j{}^k - 1$, but these counter terms are not produced by rescaling of the fields and parameters. Rather, they are only fixed by requiring that they remove the divergences in these proper graphs. For this reason the renormalization of composite operators is sometimes called additive renormalization. Clearly additive renormalization is a more general concept than multiplicative renormalization.

Certain composite operators such as conserved currents "do not renormalize": for them the matrix $Z_j{}^k$ is unity (but the elementary fields A_μ, b, c and matter fields must still be renormalized as indicated in (16.0.1).) In general, however, the matrix $Z_j{}^k$ is not equal to unity. The operators O_j which mix with each other have all the same dimension, Lorentz and group index structure, ghost number (and other quantum numbers if they are conserved), and this implies that there are in general only a finite number of composite operators which mix with each other. Since in Yang-Mills theory different fields have different quantum numbers, no mixing occurred there.

We shall mainly discuss the operator mixing in nonabelian gauge theories with gauge fixing term $-\frac{1}{2\xi}(\partial \cdot A^a)^2$, and by first analyzing the divergences which are produced by gauge invariant composite sources, we shall identify an interesting class of composite operators which has the following structure

$$\begin{aligned} O[A,b,c] &= G(A,c)(x) + \delta_B F(x) \\ &- \left(\frac{\partial}{\partial A_\mu{}^a(x)} S\right) \frac{\partial}{\partial \partial^\mu b_a(x)} F(x) + \left(\frac{\partial}{\partial c^a(x)} S\right) R^a(x) \end{aligned} \tag{16.0.2}$$

From the derivation it will be clear that these operators are independent of the BRST sources $K_a{}^\mu$ and L_a. The $G(A,c)(x)$ denote the set of BRST invariant operators which only depend on $A_\mu{}^a$ and c^a (if O has ghost number zero, $G(A)$ is gauge invariant). Further, the last two terms contain the complete $A_\mu{}^a$ and c^a field equations (including contributions from the gauge fixing term but without BRST sources); δ_B is defined in (16.1.16) and generates the BRST variations of $A_\mu{}^a, c^a$ and b_a, while $F(x)$ and $R^a(x)$ depend on $b_a(x)$ only through $\partial^\mu b_a(x)$. We shall later rewrite (16.0.2) by replacing S by $\hat{S}$ and δ_B by G_0 where G_0 yields the BRST variations of only $A_\mu{}^a$ and c^a. Since $\delta_B \partial_\mu b_a = -\frac{1}{\xi}\partial_\mu \partial \cdot A_a = -\frac{\partial}{\partial A_\mu{}^a} S$ (fix) the terms in the difference cancel. The operators with F and R^a are sometimes called alien operators, and the operators with F are called class I operators, while those with R are called class II operators.

The main result of section 1 is the proof that the set of operators in (16.0.2) is closed under renormalization, and that the matrix $Z_j{}^k$ in (16.0.1) has a triangular form. The closure is not obvious because the last two terms in (16.0.2) are not δ_B invariant in general. The sum of the last three terms is also not Q-exact[1] There are,

[1]If F would be equal to $-L_a R^a$, the sum of these three terms would be Q-exact, but F (and R^a) are

of course, other composite operators not of this form. For example, some authors include all operators which are proportional to the complete equations of motion of A_μ, b and c, and not only the last two terms in (16.0.2). The energy momentum tensor of quantum Yang-Mills theory belongs to this extended class of composite operators, since it contains a term proportional to the antighost field equation, as we show in section 1. We shall restrict our discussion to the set in (16.0.2) since this is the set which is generated by gauge invariant operators if one uses a gauge-invariant operator as an insertion in a one-loop graph. We shall in particular analyze the structure of the Z matrix in (16.0.1) for composite operators of this form in section 2, and derive theorems concerning physical matrix elements of gauge invariant operators and of these BRST exact or equation-of-motion operators. In section 3 a crucial theorem is proven, giving the general solution of the equation $Q\Gamma_{N(x)}(div) = 0$ for the divergences in proper graphs with one insertion of an arbitrary composite operator coupled to an external source $N(x)$ at the point x. In the older literature a rather complicated (and, as some people claim, incomplete) proof of this theorem was given, but we follow the more modern approach of cohomology, which leads to a simpler and complete proof. Finally in section 4 we consider conserved currents, and derive nonrenormalization and finiteness theorems.

1 Examples of composite operators

Before plunging into the complications of gauge theories, it may be useful to first consider a simpler example without the complications of gauge fixing terms and ghosts. Consider Yukawa theory with massless spinors in $3+1$ dimensions

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu \varphi)^2 - \frac{1}{2}\mu^2\varphi^2 - \lambda\varphi^4 - \bar{\psi}\gamma^\mu\partial_\mu\psi + g\varphi\bar{\psi}\psi \qquad (16.1.1)$$

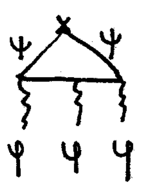
K- and L-independent. Some authors consider for this reason as a mathematical problem only Q-exact operators, but as we shall see, starting from gauge-invariant composite operators, one obtains the set (16.0.2).

The operator $O_1 = \varphi^3$ can be viewed as a composite operator with dimension 3, which mixes with the other composite operator of dimension 3, namely $O_2 = \bar{\psi}\psi$. The operator $O_3 = \partial^2\varphi$ has also dimension 3; it cannot be used as an insertion in proper graphs because it is linear in fields so it does not produce proper graphs. Moreover, it cannot be produced as a divergence when the fermions are massless. However, if one adds a mass term $-m\bar{\psi}\psi$ to the action, O_2 starts mixing with O_3, and also with $O_4 = m^2\varphi$.

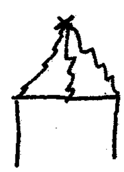
The degree of divergence of a proper graph with one insertion of O_1 or O_2 is

$$D = 4 - E_\varphi - \frac{3}{2}(E_\psi + E_{\bar{\psi}}) - 1 \tag{16.1.2}$$

Simple one- and two-loop graphs show already that operator mixing indeed occurs



The cross denotes the insertion of the composite operator O_j.



After a Fourier transform, momentum p^μ flows in at the vertex of the composite operator, and simple power counting shows that divergences induced by O_i may be proportional O_j for $j \neq i$. The reader may construct the Z matrix for this example through one-and two-loop order using dimensional regularization.

One may introduce the composite operator simply as a new vertex in the theory by adding the following term to the action

$$S(N) = \int N^j(x) O_j(x) d^4x \tag{16.1.3}$$

For gauge theories, $N^j(x)$ is treated on the same footing as the external BRST sources ${K_a}^\mu(x)$ and $L_a(x)$, hence $N^j(x)$ does not play a role in the Legendre transformation, and the effective action depends on the following fields

$$\Gamma = \Gamma[A, b, c, K, L, N] \tag{16.1.4}$$

If matter is present, one has further dependence on the matter fields and their external sources for BRST transformations. The proper graphs with precisely one insertion of the composite operator O_j are given by

$$\Gamma_{N(x)} \equiv \frac{\partial}{\partial N^j(x)} \Gamma_{|N=0} \tag{16.1.5}$$

The divergences in Γ_N will then be *local* polynomials in the fields (and derivatives thereof), not integrals over spacetime as in the case of quantum field theories without composite operators. (Sometimes one may contract Γ_N with, for example, a lepton current and then integrate over spacetime; this is, of course, equivalent to standard perturbation theory to first order in the perturbation). We shall only discuss the theory of insertions of such local operators. The theory of insertions of integrated composite operators is vastly more difficult.

To identify a class of interesting composite operators we begin by constructing the stress tensor for quantum Yang-Mills theory. We shall use the "gravitational stress tensor" although the canonical stress tensor would give the same answer in this case. (In general, one needs to add extra terms in the canonical approach to make the stress tensor symmetric on-shell while the gravitational approach always yields a symmetric stress tensor). The action in curved spacetime reads

$$\begin{aligned} S[g_{\mu\nu}] &= \int \Big[-\frac{1}{4}\sqrt{-g} g^{\mu\rho} g^{\nu\sigma} F_{\mu\nu}{}^a F_{\rho\sigma}{}^a \\ &- \frac{1}{2\xi}\frac{1}{\sqrt{-g}} (\partial_\mu \sqrt{-g} g^{\mu\nu} A_\nu{}^a)^2 - (\partial_\mu b_a)\sqrt{-g} g^{\mu\nu} (D_\nu c)^a \Big] d^4x \end{aligned} \tag{16.1.6}$$

It is invariant under general coordinate transformations. (Recall that the covariant divergence of a contravariant vector density in general relativity (such as $\sqrt{-g}g^{\mu\nu}A_\nu$ or $\sqrt{-g}g^{\mu\nu}(D_\nu c)^a$
where $D_\nu c^a = \partial_\nu c^a + g f^a{}_{bc} A^b{}_\nu c^c$) coincides with the ordinary derivative.) The stress tensor is then given by differentiation of the action w.r.t. the external metric, and then returning to flat space

$$T_{\mu\nu} \equiv -\frac{2^\delta}{\delta g^{\mu\nu}} S[g_{\mu\nu}]_{|g_{\mu\nu}=\eta_{\mu\nu}} = T_{\mu\nu}{}^{GI} + T_{\mu\nu}{}^{GV} \tag{16.1.7}$$

Here ${T_{\mu\nu}}^{GI}$ is the gauge invariant piece, ${F_{\mu\rho}}^a {F_\nu}^{a\rho} - \frac{1}{4} g_{\mu\nu} F^2$, while ${T_{\mu\nu}}^{GV}$ is the gauge variant piece due to $\mathcal{L}(\text{fix}) + \mathcal{L}(\text{ghost})$

$$\begin{aligned} \frac{1}{2}{T_{\mu\nu}}^{GV} &= \frac{1}{\xi}\left(-{A_{(\mu}}^a \partial_{\nu)} \partial \cdot A^a + \frac{1}{2}\eta_{\mu\nu} A^a \cdot \partial\partial \cdot A^a + \frac{1}{4}\eta_{\mu\nu}(\partial \cdot A^a)^2\right) \\ &+ \partial_{(\mu} b^a D_{\nu)} c^a - \frac{1}{2}\eta_{\mu\nu} \partial b^a \cdot D c^a \end{aligned} \tag{16.1.8}$$

The notation $A_{(\mu}\partial_{\nu)}$ means that one should take the part symmetric in μ and ν, so $A_{(\mu}\partial_{\nu)} = \frac{1}{2}(A_\mu \partial_\nu + A_\nu \partial_\mu)$. We claim that the gauge variant piece has a particular structure which is generic for the composite operators we want to study. Namely, it is a sum of a BRST exact term and an equation of motion term

$$\begin{aligned} \frac{1}{2}{T_{\mu\nu}}^{GV} &= \delta_B\left[{A_{(\mu}}^a \partial_{\nu)} b^a - \frac{1}{2}\eta_{\mu\nu} A^a \cdot \partial b^a - \frac{1}{4}\eta_{\mu\nu} b^a \partial \cdot A^a\right] \\ &+ \frac{1}{4}\eta_{\mu\nu} b_a \partial \cdot D c^a \end{aligned} \tag{16.1.9}$$

Note that with auxiliary field d^a, the complete gauge variant piece of the stress tensor is BRST exact. Since the BRST rules with auxiliary field in curved space are still metric independent, δ_B and $\delta/\delta g^{\mu\nu}$ commute. Use then

$$\begin{aligned} &S(fix) + S(ghost) = \delta_B \int b_a(\partial_\mu \sqrt{-g} g^{\mu\nu} {A_\nu}^a + \frac{1}{2}\xi\sqrt{-g} d^a) d^4x \\ &\frac{1}{2}{T_{\mu\nu}}^{GV}(\text{with } d^a) = \delta_B[{A^a}_{(\mu}\partial_{\nu)} b^a - \frac{1}{2}\eta_{\mu\nu} A^a \cdot \partial b^a + \frac{1}{4}\eta_{\mu\nu}\xi b^a \cdot d^a] \\ &= {A_{(\mu}}^a \partial_{\nu)} d^a - \frac{1}{2}\eta_{\mu\nu} A^a \cdot \partial d^a + \frac{1}{4}\eta_{\mu\nu}\xi d_a d^a + \partial_{(\mu} b^a D_{\nu)} c^a - \frac{1}{2}\eta_{\mu\nu}\partial b^a \cdot D c^a \end{aligned} \tag{16.1.10}$$

Note that the gravitational field $g^{\mu\nu}$ is treated here as an external field that does not transform under BRST variations. Substitution of the d field equation $d = -\frac{1}{\xi}\partial.\ A$ reproduces (16.1.8).

This example has produced a composite operator which is a sum of a gauge invariant operator, a δ_B exact operator and a term proportional to the antighost field equation. This example falls outside the class of operators in (16.0.2), since in (16.0.2) only the gauge field and ghost field equations are allowed. Some authors omit the last term in (16.1.9) because it can be written as $b(x)\frac{\partial}{\partial b(x)}S$; inside a path integral one can partially integrate $b(x)\frac{\partial}{\partial b(x)}e^{\frac{i}{\hbar}S}$ and in dimensional regularization $\delta^4(0)$ vanishes. However, in correlation functions $\frac{\partial}{\partial b(x)}$ can act on other antighost fields at other points, and if one considers external fields with given momentum instead of different x-space values, one finds correction terms. Other authors have considered the contracted operator $\Delta^\mu\Delta^\nu T_{\mu\nu}$ with $\Delta^2 = 0$; then the last term in (16.1.9) vanishes and this operator falls inside the set of (16.0.2). Others have considered a more general set of operators than (16.0.2), which also contains the antighost field equation. We shall restrict our attention to the class in (16.0.2), because this is the set of operators which mixes with gauge invariant operators as we now discuss.

Consider the divergences which are produced by a gauge invariant composite operator $G(x)$ which depends only on the classical fields. In this case the action with $S + \int N(x)G(x)d^4x$ is still BRST invariant, and we can as usual derive a Ward identity by making a change of the integration variables A_μ, b, c which amounts to an infinitesimal BRST transformation. This Ward identity is in form equal to the Ward identity found before in ordinary quantum gauge field theories. Taking the logarithm of Z, and then performing a Legendre transform, we find for theories with linear gauge fixing terms and without auxiliary fields, the usual pair of Ward identities

$$\int \left[\partial\hat{\Gamma}[N]/\partial K_a{}^\mu(x)\ \frac{\partial}{\partial A_\mu{}^a(x)}\hat{\Gamma}[N] + \partial\hat{\Gamma}[N]/\partial L_a(x)\frac{\partial}{\partial c^a(x)}\hat{\Gamma}[N]\right] d^4x = 0$$

$$\left(\frac{\partial}{\partial b_a(x)} - \partial^\mu\frac{\partial}{\partial K_a{}^\mu(x)}\right)\hat{\Gamma}[N] = 0 \tag{16.1.11}$$

Assume now that the theory without composite sources (pure quantum Yang-Mills theory) has been renormalized to all loop order, so that $\hat{\Gamma}^{ren}[N=0]$ is finite to all order in $\hbar$, and consider the one-loop divergences in $\hat{\Gamma}_N^{ren}$. Differentiating the Ward identities w.r.t. $N(y)$, and then setting $N(y)$ to zero, and considering all terms of order $\hbar$, the only terms which can possibly be divergent are the one-loop terms in $\hat{\Gamma}_{N(y)}{}^{\text{ren}}$. These divergences satisfy then the equations

$$Q\hat{\Gamma}_{N(y)}{}^{ren}(div, \hbar) = 0$$

$$\left(\frac{\partial}{\partial b_a(x)} - \partial^\mu \frac{\partial}{\partial K_a{}^\mu(x)}\right) \hat{\Gamma}_{N(y)}{}^{ren}(div, \hbar) = 0 \tag{16.1.12}$$

where $\hat{\Gamma}^{\text{ren}}{}_{N(y)}(\text{div}, \hbar)$ are *local* polynomials in the renormalized fields and external sources (and derivatives thereof). Also the two operators Q and $\partial/\partial b - \partial K$ are written in terms of renormalized objects. The second relation is the usual antighost field equation, and it is unchanged since the gauge invariant composite operator $G(x)$ does not depend on the antighost. This relation states that $\hat{\Gamma}_N{}^{\text{ren}}(\text{div})$ only depends on the combination

$$k_a{}^\mu \equiv K_a{}^\mu(x) - \partial^\mu b_a(x) \tag{16.1.13}$$

(Even though (16.1.12) depends both on x and on y, it still only depends on the difference $K_a{}^\mu - \partial^\mu b_a$ as can be seen by writing each $K^\mu{}_a(z)$ in $\hat{\Gamma}^{\text{ren}}{}_N$ as $(K^\mu{}_a(z) - \partial^\mu b(z)) + \partial^\mu b(z)$.) The solution to the equation $Q\hat{\Gamma}_{N(y)}{}^{\text{ren}}(\text{div}, \hbar) = 0$ has been studied in [1,2], and, as we show in section 3, it is given by

$$\hat{\Gamma}_{N(y)}{}^{\text{ren}}(\text{div}, \hbar) = \alpha^k G_k(y) + Q(\beta^j F_j(y)) \tag{16.1.14}$$

where $G_k(x)$ are all BRST invariant composite operators which depend only on $A_\mu{}^a$ and c^a and have the same quantum numbers as $G(x)$ (one of the $G_k(x)$ may be $G(x)$ itself), while $F_j(y)$ is a local polynomial which only depends on $K_a{}^\mu - \partial^\mu b_a$ (and $A_\mu{}^a, c^a$ and L_a, of course), and α^k and β^j are coefficients proportional to $\hbar(n-4)^{-1}$. Since we inserted a gauge invariant operator, (16.1.14) has ghost number zero and hence $G_k(y)$ are gauge-invariant in this case. Recalling the definition of Q

$$\begin{aligned} Q &= G_o + \int \left[\left(\frac{\partial}{\partial A_\mu{}^a}\hat{S}\right)\frac{\partial}{\partial K_a{}^\mu} - \left(\frac{\partial}{\partial c^a}\hat{S}\right)\frac{\partial}{\partial L_a}\right] d^4x \\ G_o &= \int \left[(D_\mu c)^a \frac{\partial}{\partial A_\mu{}^a} - (\frac{1}{2} g c \times c) \cdot \frac{\partial}{\partial c}\right] d^4x \end{aligned} \tag{16.1.15}$$

with $\hat{S} = S(\text{class}) + S(\text{ghost}) + S(\text{extra})$, we may add $S(\text{fix})$ to $\hat{S}$, and find then

$$\begin{aligned} Q &= \delta_B + \int \left[\left(\frac{\partial}{\partial A_\mu{}^a}S\right)\frac{\partial}{\partial K_a{}^\mu} - \left(\frac{\partial}{\partial c^a}S\right)\frac{\partial}{\partial L_a}\right] d^4x \\ \delta_B &= \int \left[(D_\mu c^a) \cdot \frac{\partial}{\partial A_\mu{}^a} - \left(\frac{1}{2} g c \times c^a\right) \cdot \frac{\partial}{\partial c^a} + \frac{1}{\xi}(\partial^\nu A_\nu{}^a)\partial^\mu \frac{\partial}{\partial K_a{}^\mu}\right] d^4x \end{aligned} \tag{16.1.16}$$

Since the divergences depend only on $K_a{}^\mu - \partial^\mu b_a$, we may replace the last term by $\frac{1}{\xi}\partial\cdot A^a \frac{\partial}{\partial b^a}$ when Q acts on these polynomials. Inserting this result for Q into (16.1.14), we find that the one-loop divergences due to a gauge invariant composite operator are a sum of gauge-invariant operators, δ_B- exact operators, and terms proportional to the complete gauge and ghost quantum field equations. This is the same structure as we encountered in the set of operators given in (16.0.2) if we use that F depends only on $K^\mu - \partial^\mu b$ and replace $\frac{\partial}{\partial K^\mu}F$ by $\frac{-\partial}{\partial(\partial^\mu b)}F$, put $K = L = 0$ and identify $\frac{\partial}{\partial L}F$ with $-R$. It is clear that this function R depends only on $\partial^\mu b$ but not on b. At this point it is not yet necessary to go to the case $K = L = 0$, but in the next section we consider further composite operators where one must consider the case $K = L = 0$. This is the set of composite operators we shall now study in more detail. We prefer to work with the operator Q in the form (16.1.15) and not in the form (16.1.16) because the latter would be modified in the case of spontaneous symmetry breaking with an off-diagonal gauge fixing term and divide the Q exact operators into class I operators and class II operators $O_j = O_j[A_\mu, \partial^\mu b, c]$ where

$$
\begin{aligned}
\text{class I}: O_j &= -\left(\frac{\partial}{\partial A_\mu{}^a}\hat{S}\right)\frac{\partial}{\partial\partial^\mu b_a}F_j + G_o F_j \text{ for } F_j = F_j(A_\mu, \partial^\mu b, c) \\
\text{class II}: O_j &= \left(\frac{\partial}{\partial c^a}\hat{S}\right) R_j{}^a \text{ for } R_j{}^a = R_j{}^a(A_\mu, \partial^\mu b, c)
\end{aligned} \tag{16.1.17}
$$

The operator G_0 consists of the first two terms of δ_B in (16.1.16). In the next section we show that the same set of composite operators is produced by renormalization at higher loop levels. This is not obvious because these composite operators are no longer invariant under a BRST change of integration variables (which is generated by the operator δ_B, not Q). So we shall prove that if the underlying Yang-Mills theory has been renormalized to all loop order, and the theory with composite sources has been made finite through $(n-1)$ loop order (by subtracting divergences and putting the counter terms into the matrix form with $Z_j{}^k$), then the n-loop divergences are again of the form (16.1.14). Actually, it is not necessary to have renormalized the Yang-Mills theory to all order in $\hbar$; it is sufficient to have renormalized to n-loop order and in practice this is what one uses. We only assumed in our discussion that it was renormalized to all loop order to simplify the discussion.

One comment: if we would have started with the insertion of a composite δ_B-invariant operator such as $\delta_B B(x)$ where $B(x)$ does not depend on b, K and L, instead of the insertion of a gauge invariant composite operator $G(x)$, nothing would have changed since $S(N)$ in (16.1.3) would still be BRST invariant, and the same structure of one-loop divergences would have been found. Note, however, that in general $\delta_B B(x)$ is not the same as $QB(x)$, and it is the latter type of operators which appear in (16.1.14) and which give rise to the operators in (16.1.16).

2 Closure under renormalization and structure of the Z matrix

We saw that the insertion of a gauge invariant operator produced one-loop divergences which are either gauge invariant or BRST exact in the sense of QF_j. Let us now study which n-loop divergences are produced by the insertion of such a BRST exact operator. Since

$$O_j = QF_j = G_0 F_j + \left(\frac{\partial}{\partial A_\mu{}^a}\hat{S}\right)\frac{\partial}{\partial K_a{}^\mu}F_j - \left(\frac{\partial}{\partial c^a}\hat{S}\right)\frac{\partial}{\partial L_a}F_j \tag{16.2.1}$$

where $O_j = O_j[A_\mu{}^a, K_a{}^\mu - \partial^\mu b_a, c^a, L_a]$, the action $\int N^j(x)O_j(x)d^4x$ is no longer invariant under a BRST change of the integration variables $A_\mu{}^a, b_a$ and c^a of the path integral. (For this to be true, O_j should satisfy $\delta_B O_j = G_0 O_j + \frac{1}{\xi}\partial\cdot A^a\frac{\partial}{\partial b^a}O_j = 0$ and not $QO_j = 0$.) Hence, there are changes w.r.t. the analysis performed for the gauge invariant operators. In fact, the result will be that we find a somewhat wider class of composite operators in the divergences which are K- and L- independent. (The BRST exact operators at $K = L = 0$ are a subset). To find this wider class, we simply go through all steps once again.

Adding a term $S(N) = \int N^j O_j d^4x$ to the action $\hat{S} + S(fix)$, we can make the sum δ_B invariant to first order in N^j by modifying the BRST rules with terms linear in N^j

$$\begin{aligned}
\delta'_B A_\mu{}^a &= \left(D_\mu c^a + N^j \frac{\partial}{\partial K_a^\mu} O_j \right) \Lambda \\
\delta'_B c^a &= \left(\frac{1}{2} g c \times c + N^j \frac{\partial}{\partial L_a} O_j \right) \Lambda \\
\delta'_B b_a &= \delta_B b_a = -\frac{1}{\xi} \partial \cdot A_a \Lambda
\end{aligned} \tag{16.2.2}$$

Indeed, $\hat{S} + S(fix)$ is BRST invariant for $N^j = 0$, and the variations to order N^j of $\hat{S} + S(fix) + S(N)$ read

$$\begin{aligned}
&\int [\, \partial \hat{S}/\partial A_\mu{}^a \; N^j \frac{\partial}{\partial K_a{}^\mu} O_j + \partial \hat{S}/\partial c^a \; N^j \frac{\partial}{\partial L_a} O_j - \frac{1}{\xi} \partial \cdot A \partial^\mu \left[N^j \frac{\partial}{\partial K_a{}^\mu} O_j \right] \\
&+ N^j \left(\frac{1}{\xi} \left(\partial^\mu \partial \cdot A \right) \frac{\partial}{\partial (\partial^\mu b_a)} O_j + G_0 O_j \right)]\, d^4x
\end{aligned} \tag{16.2.3}$$

where we assumed that O_j is commuting[2] and used that O_j only depends on b_a through $\partial^\mu b_a$. Since $Q O_j = 0$ due to $Q^2 = 0$, we can eliminate $G_0 O_j$ from

$$G_0 O_j(y) + \int \left[\left(\frac{\partial}{\partial A_\mu{}^a} \hat{S} \right) \frac{\partial}{\partial K_a{}^\mu} O_j(y) - \left(\frac{\partial}{\partial c^a} \hat{S} \right) \frac{\partial}{\partial L_a} O_j(y) \right] d^4x = 0 \tag{16.2.4}$$

This already cancels all terms in the variation of the action except the two terms with $\partial \cdot A$. (Recall that the left-derivative $\frac{\partial}{\partial c^a} \hat{S}$ equals minus the right-derivative $\partial \hat{S}/\partial c^a$ since c^a is anticommuting.) However, recalling that O_j depends only on $K_a{}^\mu - \partial^\mu b_a$, these terms also cancel. Thus $\delta'_B(\hat{S} + S(\text{fix}) + S(N)) = \mathcal{O}(N^2)$.

We define the path integral as usual, but with the modified transformation laws

$$\begin{aligned}
&Z\left[J, \beta, \gamma, K, L, N\right] = \int dA db dc \exp \frac{i}{\hbar} \int [\, \mathcal{L}(\text{class}) + \mathcal{L}(\text{ghost}) \\
&+ \mathcal{L}(fix) + K_a{}^\mu \left(D_\mu c^a + N^j \frac{\partial}{\partial K_a{}^\mu} O_j \right) + L_a \left(\frac{1}{2} g c \times c + N^j \frac{\partial}{\partial L_a} O_j \right) \\
&+ N^j O_j + J_a{}^\mu A_\mu{}^a + \beta_a c^a + b_a \gamma^a \,]\, d^4x
\end{aligned} \tag{16.2.5}$$

[2] If O_j is anticommuting, the second line in (16.2.3) acquires an overall minus sign, and then the terms in the BRST variations which are linear in N should also get an extra minus sign.

We recall that the operator O_j is defined by $O_j = QF_j$ and may depend on $K - \partial b$ and L. Further, $\mathcal{L}$(ghost) is the usual ghost action and does not contain any N-dependent terms. The new terms of the form $KN\frac{\partial}{\partial K}O_j$ and $LN\frac{\partial}{\partial L}O_j$ will lead to variations of order KN and LN. Hence the action in the path integral, except for the terms with Schwinger sources, is invariant up to terms of order N^2, KN, LN

$$\begin{aligned} &\delta'_B S' = \mathcal{O}(N^2, KN, LN) \\ &S' = S(\text{class}) + S(\text{fix}) + S(\text{ghost}) + S'(\text{extra}) + S(N) \end{aligned} \tag{16.2.6}$$

where S'(extra) contains the terms linear in KN and LN. Making a change of variables which amounts to an infinitesimal *modified* BRST transformation, and assuming that the Jacobian still equals unity[3], we find the usual Ward identity

$$\int \left(J_a{}^\mu \frac{\partial}{\partial K_a{}^\mu} + \beta_a \frac{\partial}{\partial L_a} + \frac{1}{\xi}\partial^\mu \cdot \frac{\partial}{\partial J_a{}^\mu}\gamma^a \right) d^4x Z = \mathcal{O}(N^2, KN, LN) \tag{16.2.7}$$

The antighost field equation yields

$$< \partial^\mu D_\mu c(x) + \gamma^a(x) + \frac{\partial}{\partial b_a(x)} \int N^j(y) O_j(y) d^4y >= \mathcal{O}(KN, LN) \tag{16.2.8}$$

---

[3]If this Jacobian is not unity there are anomalies. One could begin with the $\Gamma\Gamma$ equation and study its anomalies in the presence of an external source $N(x)$, and then transform back to W or Z. The $\Gamma\Gamma$ equation with a possible anomaly reads

$$\Gamma(N) \times \Gamma(N) = A(N)0\Gamma$$

where $A(N)$ is the anomaly, which is a local polynomial at least linear in N (assuming the original theory without N has no anomaly). Acting with $\Gamma(N)$, differentiating w.r.t. $(N(x)$, and then setting all N to zero yields the consistency condition

$$QA(x) = 0$$

The general solution of $A(x)$ is, as we show in section 3, either Q-exact or a sum of products of invariants in gauge fields and invariants in ghosts. The latter are of the form Trc^n and correspond $1-1$ to the Casimir operators. Hence for each trc^3 the index is odd (or zero). For simple groups, there are no anomalies generated by Q-exact composite operators because these anomalies should be c-independent or odd in c, but the anomaly would then be linear in c or even in c. The latter are Q-exact, whereas for a simple group $trc = 0$. For semisimple groups, for example $SU(5) \otimes SU(5)'$, candidate anomalies can be constructed. For example, the composite operator Trc^5 leads to a candidate anomaly $(Trc^3)Tr(c')^3$. In order to show that in these cases there still are no anomalies, one would have to prove that they are not produced by loop graphs.

where the terms of order KN and LN are due to a possible $\partial_\mu b$ dependence of the terms $KN\frac{\partial}{\partial K}O$ and $LN\frac{\partial}{\partial L}O$ in the action. Recalling that O^j depends only on ${K_a}^\mu - \partial^\mu b_a$, we can write the third term in (16.2.8) as $\partial^\mu \frac{\partial}{\partial {K_a}^\mu(x)}(\int N^j O_j)$, and then, as one may check, we find the same antighost field equation as before

$$\left(\partial^\mu \frac{\partial}{\partial {K_a}^\mu} + \gamma^a\right) Z = \mathcal{O}(KN, LN) \tag{16.2.9}$$

In the usual way we now go from Z to W, from W to Γ, and from Γ to $\hat{\Gamma} = \Gamma - S(fix)$. This yields

$$\begin{aligned}
&\int \left[\left(\frac{\partial}{\partial {A_\mu}^a}\hat{\Gamma}\right)\frac{\partial}{\partial {K_a}^\mu}\hat{\Gamma} - \left(\frac{\partial}{\partial c^a}\hat{\Gamma}\right)\frac{\partial}{\partial L_a}\hat{\Gamma}\right] d^4x = \mathcal{O}(N^2, KN, LN) \\
&\left(\partial^\mu \frac{\partial}{\partial {K_a}^\mu} - \frac{\partial}{\partial b_a}\right)\hat{\Gamma} = \mathcal{O}(KN, LN)
\end{aligned} \tag{16.2.10}$$

The last Ward identity implies the following parametrization of the effective action

$$\begin{aligned}
&\hat{\Gamma}\,(A, b, c, K, L, N) = \hat{\Gamma}_{N=0}(A, \partial b - K, c, L) \\
&+ \int \Big[N^j Q_j\,(A, \partial b - K, c) + N^j L_a {X_j}^a(A, b, c) \\
&+ \mathcal{O}(K^2 N, L^2 N)\,\Big]\, d^4x
\end{aligned} \tag{16.2.11}$$

To obtain this result, first replace every ∂b by $\partial b - K$ in all terms which are linear in N^j and L_a-independent and add terms with NK, NK^2, NK^3 etc. to correct for this replacement. Then (16.2.10) states that no correction terms proportional to NK are needed. For later use we explicitly wrote down the terms which are linear in N^j and L_a.

Thus the terms linear in N in $\hat{\Gamma}$ still only depend on $\partial^\mu b - K$, but the terms of order K^2N and LN are no longer restricted. We now assume that the theory without composite operators has been completely renormalized, while the theory with composite operators has been renormalized at $(n-1)$ loop level. Differentiating (16.2.10) once w.r.t. $N^j(y)$, and then setting $N^j(y) = 0$, using that $\hat{\Gamma}_{N=0}$ is finite while by induction $\frac{\partial}{\partial N(y)}\hat{\Gamma}_{|N=0}$ can only contain divergences of order $\hbar^n$, leads to

$$G_0\left(\frac{\partial}{\partial N^j(y)}\hat{\Gamma}(div)\right) = \int\left[\left(-\frac{\partial}{\partial A_\mu{}^a}\hat{S}\right)\frac{\partial}{\partial K_a{}^\mu}\frac{\partial}{\partial N^j(y)}\hat{\Gamma}(\mathrm{div})\right] + \left[\left(\frac{\partial}{\partial c^a}\hat{S}\right)\frac{\partial}{\partial L_a}\frac{\partial}{\partial N^j(y)}\hat{\Gamma}(div)\right]d^4x + \mathcal{O}(K,L) \tag{16.2.12}$$

Due to the extra terms denoted by $\mathcal{O}(K,L)$ we must now set $K = L = 0$. This will lead to composite operators which are K and L independent, and of the form (16.0.2).

Using the parametrization of $\hat{\Gamma}$ in (16.2.11) and putting $N = K = L = 0$, we find the following equation for $\mathcal{O}_j(A,\partial b,c,y) \equiv \hat{\Gamma}^{div}_{N_j(y)}(A,\partial b,c,K=L=0)$

$$\begin{aligned} G_0\mathcal{O}_j(A,\partial b,c,y) &= \int\left[-(D_\nu G^{\nu\mu} + \partial^\mu b\times c)(\frac{\partial}{\partial(\partial^\mu b)}\mathcal{O}_j(A,\partial b,c,y)\right]d^4x \\ &- (D_\mu(A)\partial^\mu b_a)X^a{}_j(A,b,c,y) \end{aligned} \tag{16.2.13}$$

Hence, the composite operators $\mathcal{O}_j(A,\partial b,c,y)$ which are generated by the insertion of the Q-exact operator in (16.2.1) at the point y satisfy (16.2.13). Since (16.2.13) is an equation for the local polynomials $\mathcal{O}_j(A,\partial b,c,y)$ and $X^a{}_j(A,b,c,y)$, it is clear that $X^a{}_j(A,b,c,y)$ can only depend on ∂b

$$X^a{}_j = X^a{}_j(A,\partial b,c,y) \tag{16.2.14}$$

Hence, although the second Ward identity in (16.2.10) by itself does not imply that X only depends on $\partial_\mu b$, the first Ward identity in (16.2.10) supplies the extra information that X only depends on $\partial_\mu b$. From now on we shall stop writing explicitly the dependence on y except when confusion might arise.

We shall now first show that an insertion of any local operator $O_j(A,\partial b,c)$ for which a local $X^a{}_j(A,b,c)$ (and hence $X^a{}_j(A,\partial b,c)$ as we have argued) can be found such that (16.2.13) holds, will produce composite operators of the same kind. This shows that the set of $O_j(A,\partial b,c)$ satisfying (16.2.13) closes under renormalization. Then we shall solve (16.2.13) and find the general form of these composite operators. They are, as expected, of the form given in (16.0.2). Finally we shall analyze the Z

matrix and show that it is triangular. Since we no longer view these operators O_j as the limit $K = L = 0$ of K, L dependent operators, this set of composite operators is more general than the sets considered before.

Consider a local composite operator $O_j(A, \partial b, c)$ satisfying (16.2.13), and consider the action

$$\begin{aligned} S = S(\text{class}) + S(\text{fix}) + S(\text{ghost}) + \int [\, K_a{}^\mu \left(D_\mu c^a - N^j \frac{\partial}{\partial(\partial^\mu b)} O^j \right) \\ + L_a \left(\frac{1}{2} g c \times c + N^j X^a{}_j(A, \partial b, c) \right)] \end{aligned} \tag{16.2.15}$$

The only difference with the analysis for BRST exact operators in (16.2.5) is that we replaced $\frac{\partial}{\partial K^\mu} O^j$ by $-\frac{\partial}{\partial \partial^\mu b} O^j$, and $\frac{\partial}{\partial L} O_j$ by $X^a{}_j$ and assume that O_j is independent of K and L. It follows that the action S is still invariant under the modified BRST transformations up to terms of order N^2, KN, LN. Hence the same Ward identities as before are found, and this at once proves the closure under renormalization.

We shall now show that the general solution of the Ward identity (16.2.13) for O_j is given by

$$\begin{aligned} O_j(A, \partial b, c) &= O_j^I + O_j{}^{II} \\ O_j{}^I &= -\left(\frac{\partial}{\partial A_\mu{}^a} \hat{S} \right) \frac{\partial}{\partial(\partial^\mu b_a)} F_j + G_0 F_j \ ; \ F_j = F_j(A, \partial b, c) \\ O_j{}^{II} &= \left(\frac{\partial}{\partial c^a} \hat{S} \right) R_j{}^a \ ; \ R_j{}^a = R_j{}^a(A, \partial b, c) \end{aligned} \tag{16.2.16}$$

The operators $O_j{}^I$ are called "class I operators", while those of the form $O_j{}^{II}$ are called "class II operators". Comparison with (16.0.2) or (16.1.17) shows that there is an ambiguity in this definition which must be fixed. Operators of the form $D_\nu G^{\nu\mu}(A) P_\mu(A, c)$ where $P_\mu(A, c)$ transforms as a vector under G_0 (for example, $P_\mu(A) = D^\rho G_{\rho\mu}$) are both G_0 invariant and part of class I operators. We shall always consider such operators as part of class I operators. Thus, the G_0 invariant operators we consider do not vanish when the classical gauge field equations are satisfied.

Let us first check that the class I and class II operators are indeed solutions of the Ward identity. Consider first a class II operator $\hat{O}_j = \left(\frac{\partial}{\partial c^a}\hat{S}\right) R_j{}^a(A, \partial b, c)$ at a point y. We must show that there exists a function $X_j{}^a(A, \partial b, c)$ at the point y such that

$$G_0\hat{O}_j(y) + \int \left(\frac{\partial}{\partial A_\mu{}^a(x)}\hat{S}\right)\left(\frac{\partial}{\partial\partial^\mu b_a(x)}\hat{O}_j(y)\right) d^4x = \left(\frac{\partial}{\partial c^a(y)}\hat{S}\right) X_j{}^a(y) \quad (16.2.17)$$

where $\hat{S} = S(\text{class}) + S(\text{ghost})$ since we have set $K = L = 0$. Substituting the explicit expressions for S (class) and S (ghost), this relation indeed holds. (Use $D_\mu(D_\nu G^{\nu\mu}) = 0$. The terms proportional to $D_\mu c \times \partial^\mu b \cdot R_j$ cancel in a nontrivial way while a term $-(D_\mu\partial^\mu b) \times c$. R is proportional to $\frac{\partial}{\partial c^a}\hat{S}$ and contributes to $X_j{}^a$).

Next we consider a class I operator. In (16.2.17), terms proportional to $G_0\frac{\partial}{\partial\partial^\mu b}F_j$ or $G_0{}^2$ cancel, and double derivatives of F_j w.r.t. $\partial^\mu b$ cancel for reasons of symmetry. The remainder cancels if one uses that $G_0 D_\nu G^{\nu\mu} = D_\nu G^{\nu\mu} \times c$ and $G_0\partial^\mu b \times c = \partial^\mu b \times (\frac{1}{2}gc \times c) = -cx(\partial^\mu bxgc)$. Hence, also the first class operators are solutions of the equation (16.2.17).

To prove that the most general solution of (16.2.13) is indeed the set of class I and class II operators, we note that insertion of a composite operator $Q^j(A, \partial b - K, c)$ satisfying (16.2.13) into a path integral with action (16.2.15) will lead to an effective action satisfying (see (16.2.11))

$$Q\hat{\Gamma}(N, \text{div}) = \mathcal{O}(N^2, NK, NL) \quad (16.2.18)$$

Note that we replaced $O^j(A, \partial b, c)$ by $O^j(A, \partial b - K, c)$; this is of course always possible, but it is crucial for what follows. Hence, after differentiation w.r.t. $N(y)$ and then setting $N(y)$ to zero, we obtain

$$Q\hat{\Gamma}_N(\text{div}) = \alpha K + \beta L \quad (16.2.19)$$

where α and β are local polynomials in all fields and sources and derivatives thereof. Since $Q^2 = 0$, we find

$$Q(\alpha K + \beta L) = 0 \quad (16.2.20)$$

This is a problem in local cohomology. In the next section we show that "there is no nontrivial cohomology in the sector with K and L", which means that the general solution of this equation is

$$\alpha K + \beta L = QX, X = \alpha' K + \beta' L \tag{16.2.21}$$

It follows that

$$Q(\hat{\Gamma}_N(\text{div}) - X) = 0 \tag{16.2.22}$$

The solution of this latter equation is

$$\hat{\Gamma}_N(\text{div}) - X = G + QF \tag{16.2.23}$$

Taking $K = L = 0$ in (16.2.23) leads then to

$$\hat{\Gamma}_N(\text{div})\,|_{K=L=0} = G + (QF)_{K=L=0} \tag{16.2.24}$$

This is indeed the set of operators in (16.2.17). Namely the terms linear in $L, O = L_a R$, produced the class II operators, while the L-independent terms in F produce the class I operators because F depended only on $k^\mu = K^\mu - \partial^\mu b$.

We shall now show that the Z-matrix for the renormalization of the complete set of G_0 invariant and alien operators is triangular. Namely, the insertion of a gauge invariant operator can produce all three kinds of divergences, but alien operators go only into alien operators, and class II operators mix only with themselves. (We recall that alien operators are class I and class II operators, while G_0 invariant operators with ghost number zero are gauge invariant).

Of course, this result is basis dependent, and the basis on which it holds is given in (16.2.16). It is, in general, not possible to modify the G_0 invariant operators G_j by adding suitable linear combinations of G_0 noninvariant operators, such that these redefined operators G'_j (which then are no longer G_0 invariant) only mix with themselves *in every n-point function.* We begin with the class II operators, as this is the simplest case, and afterwards analyze insertions of a class I operator.

We must show that class II operators produce only class II divergences. We consider therefore a general class II operator $O_j^{II} = R_j{}^a \left(\frac{\partial}{\partial c^a}\hat{S}\right)$ and insert it into the path integral

$$Z = \int dAdbdce^{i\int[\hat{\mathcal{L}}+\mathcal{L}(fix)+JA+\beta c+b\gamma+N^j O_j{}^{II}]d^4x} \tag{16.2.25}$$

We change the integration variable c^a such that the equation of motion term $N^j O_j{}^{II}$ disappears

$$c^a = (c^a)' - N^j R_j{}^a \tag{16.2.26}$$

Provided the Jacobian is unity, we then find the following equivalent expression for the path integral (dropping primes)

$$Z = \int dAdbdce^{i\int[\hat{\mathcal{L}}+\mathcal{L}(fix)+JA+\beta c+b\gamma-\beta_a N^j R_j{}^a]d^4x} \tag{16.2.27}$$

Thus the generating functional for connected graphs with precisely one insertion of the composite operator is given by (assuming that N^j is commuting)

$$< O_j^{II} >_{\text{conn.}} = -i\left(\frac{\partial}{\partial N^j}\ln Z\right)_{N=0} = -\beta_a < R_j{}^a >_{\text{conn.}} \tag{16.2.28}$$

Next we go over to proper graphs by using that $-i\frac{\partial}{\partial N^j}\ln Z$ equals $\frac{\partial}{\partial N^j}\Gamma$ and replacing β by $-\frac{\partial}{\partial c}\Gamma$. Then the symbol $< R_j{}^a >$ which is a functional of J, is replaced by $R^a{}_{j,eff}$ which is a functional of the fields, and BRST-sources, and which is one-particle irreducible [4]

$$\frac{\partial}{\partial N^j}\Gamma = \left(\frac{\partial}{\partial c^a}\Gamma\right) R^a{}_{j,eff} = \frac{1}{\sqrt{Z_{gh}}}\left(\frac{\partial}{\partial c^a{}_{ren}}\Gamma^{\text{ren}}\right) R^a{}_{j,eff} \tag{16.2.29}$$

Since the theory without composite source has been renormalized, Γ^{ren} is finite, and by induction the n-loop divergences can only come from the terms of order $\hbar^n$ in $< Z_{gh}^{-1/2} R^a{}_{j,eff} >$. Hence, if we consider the order $\hbar^n$ divergences we can replace

[4]The equality of W_N and Γ_N does not mean, of course, that connected graphs with one insertion are equal to proper graphs with one insertion. Rather, in the Green's functions there are extra factors with $\partial\phi/\partial J = \partial^2 W/\partial J\partial J$ etc. These factors turn connected graphs into proper graphs.

$\frac{\partial}{\partial c^a{}_{\rm ren}}\Gamma^{\rm ren}$ by $\frac{\partial}{\partial c^a{}_{\rm ren}}\hat{S}$. The divergences in $< Z_g^{-1/2} R^a_{j,eff} >$ are the products of local polynomials $\tilde{R}^a_j$, a factor $\hbar^n$ and powers of $(n-4)^{-1}$.

$$\hat{\Gamma}_N(\text{div}) = \left(\frac{\partial}{\partial c^a}\hat{S}\right)\tilde{R}^a_j \tag{16.2.30}$$

This shows that the divergences are all proportional to class II operators, which we set out to prove.

Next we consider the divergences which are generated by the insertion of a class I composite operator. We shall only insert class I operators with vanishing ghost number; the theory for class I operators with nonvanishing ghost number seems unknown. We intend to show that no divergences proportional to gauge invariant operators are produced, and therefore we shall consider only the set of proper graphs with external $A_\mu{}^a$ lines, but no b_a or c^a lines. For this purpose we only need to perform the Legendre transformation from $J_a{}^\mu$ to $A_\mu{}^a$, but we do not introduce Schwinger sources for the ghosts and antighosts. We shall show that the divergences with only external $A_\mu{}^a$ lines are of the form $(D_\mu G^{\mu\nu})_a$ times some polynomial $P_\nu{}^a(A)$ which contains coefficients with divergences (powers of $(n-4)^{-1}$). This is the b, c independent part of the following class I operator

$$O_j{}^I = -\left(\frac{\partial}{\partial A_\mu{}^a}\hat{S}\right)\ \frac{\partial}{\partial(\partial^\mu b_a)}\ [\partial^\nu b_c P_\nu{}^c] + G_0\,[\partial^\nu b_c P^c{}_\nu] \tag{16.2.31}$$

To prove this, we follow a technique which is frequently used terms proportional to field equations in an action or transformation rules can often be removed by field redefinitions or by redefinitions of the transformation rules. Although one can in principle perform these redefinitions order by order in some coupling constant or field, we shall only go to first order in the source $N(x)$ for the composite operator.

Consider the action with a first class composite operator $O_j{}^I$ and with only Schwinger sources for $A_\mu{}^a$

$$\mathcal{L} \;=\; \hat{\mathcal{L}} + \mathcal{L}(\text{fix}) + J_a{}^\mu A_\mu{}^a + N^j O^I_j$$

$$\begin{aligned}
\hat{S} &= S(\text{class}) + S(\text{ghost}) + S(\text{extra}) \\
O^I_j(x) &= -\int \left(\frac{\partial}{\partial A_\mu(y)} \hat{S} \right) \left(\frac{\partial}{\partial \partial^\mu b(y)} F_j(x) \right) d^4y + G_o F_j(x) \\
F_j(x) &= F_j(A, \partial b, c, x)
\end{aligned} \tag{16.2.32}$$

We can remove the terms in O_j proportional to the field equations by redefining ${A_\mu}^a$

$$\begin{aligned}
{A_\mu}^a(x) &= ({A_\mu}^a)'(x) + \int N^j(y) \, \frac{\partial}{\partial \partial^\mu b_a(x)} \, F_j(y) d^4y \\
c^a &= c^{a\prime}, \; b^a = b^{a\prime}
\end{aligned} \tag{16.2.33}$$

Then, to first order in $N, \hat{S} + \int N^j O^I_j = \hat{S}' + \int N^j G_0 F_j$.

The Jacobian for this change of integration variables is

$$J = 1 - \int Tr N^j(z) \left(\frac{\partial}{\partial A^b_\nu(y)} \, \frac{\partial}{\partial(\partial^\mu b_a(x))} \right) \, F_j(z) d^4z \tag{16.2.34}$$

One should exponentiate this Jacobian with ghosts, just as the Faddeev-Popov determinant, but we shall assume that we may put $J = 1$.

Performing the same change of variables on the remaining terms in S, we find

$$\begin{aligned}
&\mathcal{L} \to \hat{\mathcal{L}}' - \frac{1}{2\xi} \left[\partial^\mu \left({A_\mu}'^{\,a} - N^j \frac{\partial}{\partial(\partial^\mu b_a)} F_j \right) \right]^2 + {J_a}^\mu ({A_\mu}'^{\,a} + N^j \frac{\partial}{\partial(\partial^\mu b_a)} F_j) \\
&+ N^j G_0 F_j
\end{aligned} \tag{16.2.35}$$

Dropping primes, we find an expression for the path integral Z without equation of motion terms. Since we are interested in the divergences due to an insertion of O^I_j, we evaluate $< O^I_j >= -i \frac{\partial}{\partial N^j} \ln Z \mid_{N=0}$, and find that $< O^I_j >= -i \left(\frac{\partial}{\partial N^j} \ln Z \right)$ at $N = 0$ is given by

$$\begin{aligned}
< O^I_j(x) > \;&=\; \int \left[\frac{1}{\xi} < \partial^\mu \partial \cdot A^a(y) \frac{\partial}{\partial(\partial^\mu b_a)(y)} F_j(x) > + {J_a}^\mu(y) < \frac{\partial}{\partial(\partial^\mu b_a)(y)} F_j(x) > \right] d^4y \\
&+ < G_0 F_j(x) >
\end{aligned} \tag{16.2.36}$$

We can obtain an expression for $< G_0 F_j >$ by making the usual BRST change of integration variables on $< F_j >$, the path integral average of F_j, at $N^j = 0$

$$< G_0 F_j + \frac{1}{\xi} \partial_\mu \partial \cdot A^a \left(\frac{\partial}{\partial(\partial^\mu b_a)} F_j \right) + \left[\int i {J_a}^\mu (D_\mu c)^a \right] F_j >= 0 \tag{16.2.37}$$

Then $< O_j^I >$ simplifies to

$$< O_j^I(x) > = < \left[\int J_a{}^\mu(y) \frac{\partial}{\partial(\partial^\mu b_a)(y)} d^4y \right] F_j(x) > -i < \left[\int J_a{}^\mu(y) D_\mu c^a(y) d^4y \right] F_j(x) > \tag{16.2.38}$$

As always, we can use the Legendre transform to express $J_a{}^\mu$ in terms of derivatives of Γ, but in this case where we only consider external $A_\mu{}^a, \Gamma$ is only a function of $A_\mu{}^a$. Moreover, according to the Legendre transformation

$$J_a{}^\mu = \partial\Gamma/\partial A_\mu{}^a = \partial\hat{\Gamma}[A]/\partial A_\mu{}^a + \frac{1}{\xi}\partial^\mu(\partial \cdot A^a) \tag{16.2.39}$$

and the last term can be dropped when inserted into (16.2.38) due to the following identity (obtained after partial integration)

$$\begin{aligned} &\frac{\partial}{\partial y_\mu} < -\frac{\partial}{\partial(\partial^\mu b_a(y))} F_j(x) + i\,(D_\mu c)^a\,(y) F_j(x) > = \\ &\int dA e^{i[S(\text{class})+S(fix)+J\cdot A} \int dbdc \frac{\partial}{\partial b_a(y)} \left[F_j(x) e^{iS(\text{ghost})} \right] = 0 \end{aligned} \tag{16.2.40}$$

Hence we obtain

$$\begin{aligned} < O_j^I(x) > &= \int \left(\frac{\partial}{\partial A_\mu{}^a(y)} \hat{\Gamma}[A] \right) < \left\{ \frac{\partial}{\partial(\partial^\mu b_a)(y)} F_j(x) - i D_\mu c^a(y) F_j(x) \right\} > d^4y \\ &= \int \left(\frac{\partial}{\partial A_\mu{}^{a,\text{ren}}} \hat{\Gamma}_{\text{ren}}[A^{\text{ren}}] \right) \mathcal{F}_j d^4y; \mathcal{F}_j = < \frac{\partial}{\partial\partial^\mu b_a(x)} F_j - i D_\mu c^a F_j(x) > Z_3^{-1/2} \end{aligned} \tag{16.2.41}$$

If the divergences at $(n-1)$ loops have been renormalized away by a suitable Z matrix, one finds at the n loop level (all terms proportional to $\hbar^n$) only a divergence in the $\hbar^n$ term of $\mathcal{F}_j$. So $\mathcal{F}_j{}^{(n)}(\text{div}) = \sum_k a_k{}^{(n)} F_j{}^k$, where $F_j{}^k$ is a complete set of operators (gauge invariant as well as gauge noninvariant operator). Hence

$$\Gamma_N^{\text{div}} = < O_j >^{\text{div}} = Z_3^{-1/2}(D_\rho G^{\rho\mu}) a_k^{(n)} F_j^k \tag{16.2.42}$$

This set of divergences is, however, of the form

$$\left(\frac{\partial}{\partial A_\mu{}^a} \hat{S} \right) \frac{\partial}{\partial(\partial^\mu b)} \left[\sum_k a_k{}^{(n)} F_j{}^k \partial_\mu b \right] \text{ at } b = c = 0 \tag{16.2.43}$$

This expression is indeed proportional to type I operators. There can also be type II operators since they vanish at $b = c = 0$. Hence, type I operators mix only with type I and type II operators.

3 The general solution of $QX = 0$ from cohomology [2]

We saw in the analysis of divergences in proper graphs of renormalizable gauge theories that a crucial role is played by the equation $QX = 0$ where X is a spacetime integral of a polynomial in the fields and sources $({A_\mu}^a, b_a, c^a, {K_a}^\mu, L_a)$ and a finite number of derivatives of these. Further, Q is the BRST operator

$$\begin{aligned} Q &= G_0 + \left(\frac{\partial}{\partial {A_\mu}^a}\hat{S}\right)\frac{\partial}{\partial {K_a}^\mu} - \left(\frac{\partial}{\partial c^a}\hat{S}\right)\frac{\partial}{\partial L_a} \\ G_0 &= (D_\mu c)^a \frac{\partial}{\partial {A_\mu}^a} - \frac{1}{2} g {f^a}_{bc} c^b c^c \frac{\partial}{\partial c^a} \end{aligned} \tag{16.3.1}$$

where we recall that $\hat{S}$ is the quantum action with BRST sources but without gauge fixing term. The polynomial X depends on ${K_a}^\mu$ and b_a only through ${K_a}^\mu - \partial^\mu b_a$, and Q is nilpotent, $Q^2 = 0$.

In some cases (for example when one uses a nonlinear gauge fixing term) it is useful to keep the auxiliary field d_a in the theory and one encounters then a slightly different problem, namely $Q(d)Y = 0$ where Y depends also on an auxiliary field d_a, and

$$Q(d) = Q - d_a \frac{\partial}{\partial b_a} \tag{16.3.2}$$

[2] In mathematics one considers functions of the Lie algebra like $f = \lambda_a c^a$ such that $f(T_a) = \lambda_a$. Both λ_a and c_a are constant. Thus the ghosts are viewed as duals to the generators of the Lie algebra: $(c^a, T_b) = \delta^a_b$, or $c^a = (T_a)^*$. In this way ghosts entered mathematics already in the 1930's. Nowadays we are dealing with x-dependent ghost fields, but as we shall show, the derivatives of ghost fields are BRST exact.

The polynomial Y may now depend both on $K_a{}^\mu$ and b_a, and not only through $K_a{}^\mu - \partial^\mu b_a$. The operator $G_0 - d_a \frac{\partial}{\partial b_a}$ generates BRST transformations of the fields $A_\mu{}^a, c^a, b_a$ and d_a and is nilpotent. Also both $Q(d)$ and Q are nilpotent.

In the theory of renormalization of composite operators, one encounters a third problem, but now X and Y are local polynomials, not spacetime integral of such polynomials. Let us denote by $H^*(Q)$ the set of solutions of the equation $QX = 0$ (or $Q(d)Y = 0$) which are not themselves of the form QX'(or $Q(d)Y'$). The sets $H^*(Q)$ are actually groups, the group action being addition, and in mathematics they are called cohomology groups. The star indicates that one allows any ghost number; one can also consider $H^p(Q)$ which is the cohomology group of the "differential Q" with ghost number p. The operators Q and $Q(d)$ act on products of polynomials as derivations, i.e., they satisfy the (graded) Leibniz rule. We distinguish between cohomology in the linear vector space of polynomials integrated over spacetime and cohomology in the linear vector space of local polynomials. The former case is called relative cohomology (relative to d) and denoted by $H^*(Q \mid d)$ where d indicates that one studies polynomials modulo total derivatives (This d has nothing to do with the auxiliary field d in $Q(d)$; since we shall no longer discuss $H^*(Q \mid d)$ no confusion is possible.) We shall only study local polynomials in this section. The restriction to local polynomials is not a serious drawback because we solved the cohomology of integrated polynomials of dimension 3 with ghost number -1 when we studied divergences in the effective action by brute force: first using power counting to narrow down the set of possible counter terms, and then requiring that this set be annihilated by Q. Also BRST anomalies in the Ward identities can be found by this brute force method, but now one needs integrated polynomials of ghost number $+1$ and dimension 5, and this is in practice a lot more tedious. Alternative methods based on relative cohomology are preferable [28] (but not unavoidable). (local expressions modulo total derivatives) In principle we could execute a similar problem for the local polynomials, but since we are interested in composite operators of any dimension and any ghost number, we would have to do the analysis for all these cases separately, and an approach which obtains all these results in one stroke is clearly preferable. Such an approach is cohomology.

To solve the equations $Q(d)Y = 0$ and $QX = 0$ we begin by reducing the former problem to the latter. Next we show that Q can be decomposed into $\beta+\gamma$ where β and γ are each nilpotent, and anticommute with each other. Then we derive two lemmas: (1) all terms which are annihilated by γ (γ closed) and depend on K_a^μ and/or L_a are γ-exact ("there is no nontrivial γ cohomology with nonvanishing antighost number"), and (ii) all terms which are β closed are either β exact or sums of group invariants built from ${K_a}^\mu, L_a$ and ${G_{\mu\nu}}^a$ and group invariants built from c^a. The second lemma is a celebrated theorem Lie algebra cohomology. Finally we combine both lemmas to solve the equation $QX = 0$. The general solution is $X = G + QZ$ where G is a G_0 invariant polynomial. If X has ghost number zero, G is gauge invariant, and then this result (but not its proof) is well-known. This solution was conjectured long ago by Zuber and Kluberg-Stern, and a proof was proposed by Joglekar and Lee. We construct a proof by combining and simplifying refs (3, 11).

a. The cohomology of $Q(d)$ is isomorphic to Q. We shall show that for every element of $H^*(Q(d))$ there is a unique element of $H^*(Q)$, where the operator $Q(d)$ acts on polynomials in the fields and sources $({A_\mu}^a, b_a, c^a, d^a, {K_a}^\mu, L_a)$ and the operator Q acts on polynomial in same set of fields and sources but with the field d^a removed and the source K and the field b appearing only in the combination $K^\mu - \partial^\mu b$. In other words we shall show that for every nontrivial solution Y of the equation $Q(d)Y = 0$ there is a nontrivial solution Y_0 of the equation $QY_0 = 0$ and vice versa.

Consider the equation $Q(d)Y = 0$. Define an operator ρ ("the contracting homotopy operator")

$$\rho d_a = -b_a, \rho b_a = 0, \rho {A_\mu}^a = \rho c^a = \rho {K_a}^\mu = \rho L_a = 0 \tag{16.3.3}$$

It is easy to check that the operator $N_{b,d} = \{\rho, Q(d)\}$ counts the sum of the number of b and d fields in a given monomial. For example

$$\begin{aligned}\{\rho, Q(d)\}d_a &= Q(d)\rho d_a = -Q(d)b_a = d_a \\ \{\rho, Q(d)\}b_a &= \rho Q(d)b_a = -\rho d_a = b_a \\ \{\rho, Q(d)\}A_\mu{}^a &= \rho Q(d)A_\mu{}^a = \rho D_\mu c^a = 0, \text{etc.}\end{aligned} \tag{16.3.4}$$

It is also clear that $N_{b,d}$ commutes with $Q(d)$ since $Q(d)$ is nilpotent. Decompose now Y into terms with definite eigenvalues of $N_{b,d}$

$$Y = Y_0 + Y_1 + Y_2 + \dots \tag{16.3.5}$$

So, Y_0 has no b nor d fields, Y_1 has either one b or one d field etc. Then $Q(d)Y_n$ is also an eigenvector of $N_{b,d}$ with the same eigenvalue as Y_n (because $Q(d)$ and $N_{b,d}$ commute). Hence, if $Q(d)Y = 0$, then $Q(d)Y_n = 0$ for each $n = 0, 1, 2 \dots$. Consider now a Y_n with $n \neq 0$. Then we have the following series of identities

$$\begin{aligned}N_{b,d}Y_n &= nY_n = \{\rho, Q(d)\}Y_n = Q(d)(\rho Y_n) \\ Y_n &= Q(d)(\frac{1}{n}\rho Y_n) = Q(d)Z_n \text{ for } n > 0.\end{aligned} \tag{16.3.6}$$

So, each term in Y with at least one b or d field is $Q(d)$ exact. Then the general solution of $Q(d)Y = 0$ is given by the general solution of

$$Q(d)Y_0(A_\mu, c, K^\mu, L) = 0 \tag{16.3.7}$$

to which one should add any terms of the form $Q(d)Z$ for any Z. On Y_0, however, the term $d\frac{\partial}{\partial b}$ vanishes, as Y_0 does not depend on b (and d). Moreover, Q only depends on $k^\mu = K^\mu - \partial^\mu b$ as we shall see. Hence we find

$$QY_0(A_\mu, c, k^\mu, L) = 0 \tag{16.3.8}$$

where Q is given in (16.3.1).

This proves that $H^*(Q(d)) \subset H^*(Q)$: given an element Y of $H^*(Q(d))$, the representative without b and d field and with the source K replaced by k is an element Y_0 of the cohomology group $H^*(Q)$. It is very easy to see that the converse is also true, namely that $H^*(Q) \subset H^*(Q(d))$. Indeed, given a solution $Y_0(A, c, k, L)$ of the equation $QY_0 = 0$ one obtains a solution Y of the equation $Q(d)Y = 0$ by simply replacing k by K in Y_0, $Y(A, c, K, L) = Y_0(A, c, k = K, L)$, as formula (16.3.7) shows.

The construction of "contractible pairs" as in (16.3.3) will be repeatedly used. It allows one to remove all fields which are not essential for the cohomology.

b. Q as a double complex. The operator Q can be decomposed as $Q = \beta + \gamma$ where β and γ are nilpotent and anticommute with each other

$$Q = \beta + \gamma, \beta^2 = \gamma^2 = \beta\gamma + \gamma\beta = 0 \tag{16.3.9}$$

One such decomposition, which is particularly useful, is obtained by introducing an antighost number n_a as follows

$$n_a(b_b) = n_a({K_b}^\mu) = 1, n_a(L_b) = 2, n_a({A_\mu}^b) = n_a(c^b) = 0 \tag{16.3.10}$$

Note that this antighost number is not minus the ghost number. If one takes for β the terms in Q with antighost number zero, and γ denotes the terms with antighost number -1, it is clear that $\beta^2 = \beta\gamma + \gamma\beta = \gamma^2 = 0$. An explicit expression for β and γ is obtained by working out the terms in Q in (16.3.1) with field equations

$$\begin{aligned}
\beta &= (D_\mu c)^a \frac{\partial}{\partial {A_\mu}^a} - \frac{1}{2} g (c \times c)^a \frac{\partial}{\partial c^a} \\
&+ \left(\frac{\partial}{\partial {A_\mu}^a} S(\text{ghost}) \right) \frac{\partial}{\partial {K_a}^\mu} + \left(\frac{\partial}{\partial {A_\mu}^a} \int K^\nu D_\nu c \right) \frac{\partial}{\partial {K_a}^\mu} - \left(\frac{\partial}{\partial c^a} \int L \cdot \frac{1}{2} g c \times c \right) \frac{\partial}{\partial L_a} \\
\gamma &= \left(\frac{\partial}{\partial {A_\mu}^a} S(\text{class}) \right) \frac{\partial}{\partial {K_a}^\mu} - \left(\frac{\partial}{\partial c^a} S(\text{ghost}) + \frac{\partial}{\partial c^a} \int K^\nu D_\nu c \right) \frac{\partial}{\partial L_a}
\end{aligned} \tag{16.3.11}$$

From these expressions one sees what the introduction of the antighost number has done: it splits kinematics from dynamics. It allows to separate the BRST transformations of the minimal fields A_μ and c from the field equations for A_μ and c. More explicitly, in terms of $k^\mu \equiv K^\mu - \partial^\mu b$, replacing $\frac{\partial}{\partial K^\mu}$ by $\frac{\partial}{\partial k^\mu}$ (which is allowed in the space of polynomials $X(A, c, K^\mu - \partial^\mu b, L)$, we find

$$\begin{aligned}\beta &= (D_\mu c)\frac{\partial}{\partial A_\mu} - \frac{1}{2}g(c\times c)\frac{\partial}{\partial c} - g(c\times k^\mu)\frac{\partial}{\partial k^\mu} - g(c\times L)\frac{\partial}{\partial L} \\ &= G_0 - g(c\times k^\mu)\frac{\partial}{\partial k^\mu} - g(c\times L)\frac{\partial}{\partial L} \end{aligned} \tag{16.3.12}$$

while the "Koszul-Tate differential" γ is given by

$$\gamma = (D_\nu G^{\nu\mu})_a \frac{\partial}{\partial {k_a}^\mu} - (D_\mu {k_a}^\mu)\frac{\partial}{\partial L_a} \tag{16.3.13}$$

The nilpotency of γ follows directly from $D_\mu(D_\nu G^{\nu\mu}) = 0$, while the nilpotency of β follows from the nilpotency of G_0 and the Jacobi identities ($\frac{1}{2}(c\times c)\times L - (c\times L)\times c = 0$, and idem for K). The anticommutativity of β and γ follows from the observation that $D_\nu G^{\nu\mu}$ and $D_\mu k^\mu$ transform as isovectors under β transformations, as do $\frac{\partial}{\partial k^\mu}$ and $\frac{\partial}{\partial L}$.

In mathematics, the precise definition of a double complex is more involved, but instead of pursuing these general aspects further, we shall work with our particular Q which has many special properties.

c. Removing $\partial_{(\mu}\partial_\nu \ldots A_{\rho)}$ and $\partial_{(\mu}\partial_\nu \ldots D_{\rho)}c$. We can in a unique way write any polynomial $X(A, c, k, L)$ as a polynomial in the following building blocks

(i) symmetric derivatives of ${A_\mu}^a({A_\mu}^a, {\partial_{(\mu}A_{\nu)}}^a, {\partial_{(\mu}\partial_\nu A_{\rho)}}^a, \ldots)$

(ii) symmetric derivatives of $D_\mu c^a(D_\mu c^a, \partial_{(\mu}D_{\nu)}c^a, \partial_{(\mu}\partial_\nu D_{\rho)}c, \ldots)$

(iii) covariant derivatives of $G_{\mu\nu}, k^\mu$ and L
$(G_{\mu\nu}, D_\mu G_{\nu\rho}, D_\mu D_\nu G_{\rho\sigma}, \ldots, k^\mu, D_\nu k^\mu, D_\nu D_\rho k^\mu, \ldots, L, D_\mu L, D_\mu D_\nu L, \ldots)$

(iv) undifferentiated ghosts fields (c^a)

Clearly Q maps (i) into (ii), and (ii) into zero, while each element of (iii) rotates as a vector into itself times c, and finally the ghost fields in (iv) transform into $\frac{1}{2}gc\times c$. It is clear how to achieve this parametrization. For example a term $\partial_\mu A_\nu$ can be written as $(\partial_{(\mu} A_{\nu)} - \frac{1}{2}[A_\mu, A_\nu]) + \frac{1}{2}G_{\mu\nu}$, while $\partial_\mu c$ is written as $-[A_\mu, c] + D_\mu c$.

To solve the equation $QX = 0$, we first remove all terms in X which depend on symmetric derivatives of A_μ or $D_\mu c^a$ (the terms in (i) and (ii)) by a similar trick as we used to get rid of b_a and d_a. Namely, we observe that the pairs $x = \partial_{(\mu}\partial_\nu \ldots A_{\rho)}$ and $y = \partial_{(\mu}\partial_\nu \ldots D_{\rho)}c$ form "contractible pairs"

$$Qx = y, Qy = 0 \tag{16.3.14}$$

We therefore define an operator ρ which acts on these building blocks in the reverse way

$$\rho y = x, \rho x = 0, \rho(\text{rest}) = 0 \tag{16.3.15}$$

Note that ρ cannot be promoted to an operator which commutes with ordinary derivatives and satisfies the Leibniz rule. For example, $\rho(A_\mu) = \rho(c) = 0$, but $\rho(D_\mu c) = A_\mu$.

We can then repeat the arguments: $N_s = \{Q, \rho\}$ counts the number of all factors $\partial_{(\mu}\partial_\nu \ldots A_{\rho)}$ and $\partial_{(\mu}\partial_\nu \ldots D_{\rho)}c$ in a given monomial. For example

$$\begin{aligned}
\{Q,\rho\}G_{\mu\nu} &= \rho Q G_{\mu\nu} = \rho(G_{\mu\nu} \times c) = 0 \\
\{Q,\rho\}A_\mu &= \rho Q A_\mu = \rho D_\mu c = A_\mu \\
\{Q,\rho\}D_\mu c &= Q A_\mu = D_\mu c
\end{aligned} \tag{16.3.16}$$

Decomposing X into terms which are eigenvectors of N_s with eigenvalues $n = 0, 1, 2, \ldots$, we find again that $X_n = QZ_n$ for $n \neq 0$. Hence, the general solution of $QX = 0$ is equal to the general solution of $QX(D^p G_{\mu\nu}, D^q k^\mu, D^r L, c^a) = 0$ where X may contain *covariant* derivatives of $G_{\mu\nu}$,k and L but not of c, plus a term QZ with arbitrary Z. In other words, only the terms in (iii) and (iv) are left.

d. Removing k and L (γ cohomology). Next we consider the k and L dependence of X. The source k introduces classical field equations into the cohomology of γ via

$\gamma k^\nu = D_\mu G^{\mu\nu}$, and if there were no source L one would find k-dependent nontrivial γ cohomology, for example $a = D_\mu k^\mu$ (which indeed satisfies $\gamma a = 0$). The sources L remove this nontrivial γ cohomology: $\gamma L = -D_\mu k^\mu$, so $a = D_\mu k^\mu$ is, in fact, γ exact, namely $a = -\gamma L$. Summarizing

$$\gamma k^\mu = D_\nu G^{\nu\mu}, \gamma L = -D_\mu k^\mu \tag{16.3.17}$$

We claim that in general all terms which depend on k and L and which are annihilated by γ are γ exact: **there is no nontrivial γ cohomology at non-zero antighost number**. Of course, we are interested in the cohomology of Q and not that of γ. However, at the end of this section we shall use the results of the analysis of the γ cohomology to solve the Q cohomology.

For the proof, we first refine the basis for the polynomials as follows. Consider the linear vector space spanned by $G_{\mu\nu}$ and all its the covariant derivatives. This space is called a "jet space" in the mathematical literature. We want to change basis in this space such that the field equations become new coordinates. To this end we replace $D_\nu G^{\nu\mu}$ (no sum) by the field equation $\mathcal{L}^\mu \equiv D_\nu G^{\nu\mu}$ (with a sum over ν) while for the rest we choose a basis as follows: A complete basis in the space of $D_\mu G_{\nu\rho}$ is the set $\mathcal{L}^\mu$ and $D_{((\mu} G_{\nu))\rho}$ where $((\quad))$ means symmetrization and tracelessness. (This care of the Bianchi identities $D_{[\mu} G_{\nu\rho]} = 0$). In the space of more covariant derivatives of curvatures are generalizes this basis choice as follows: $D_{((\mu_1} D_{\mu_2} \dots D_{\mu_k} G_{\mu_{k+1}))^\lambda}$ and $(D_{\alpha_1} \dots D_{\alpha_k} \mathcal{L}^\beta)$ traceless.

Because of the Bianchi identity $D_\nu \mathcal{L}^\nu = 0$ we want to remove certain terms which are linear combinations of other terms, in order to be left with a set of linearly independent basis vectors. To this purpose, we remove $D_0 \mathcal{L}^0$. Next we choose the following basis in the space spanned by $D_\mu D_\nu G_{\rho\sigma}$

$$D_i \mathcal{L}_j, D_0 \mathcal{L}_j, D_j \mathcal{L}_0, \text{ the parts of } D_\mu D_\nu G_{\rho\sigma} \text{ traceless in } \mu\nu\rho\sigma \tag{16.3.18}$$

More generally, we remove all traces from $D_{\mu_1} \dots D_{\mu_k} G_{\rho\sigma}$, and the traces $D_\nu G^{\nu\rho}$ are replaced by the symbol $\mathcal{L}^\rho$, while any terms with $D_0\mathcal{L}^0$ are omitted. The remaining terms are then covariant derivatives acting on the set in (16.3.18).

In the space of $k^\mu, D_\nu k^\mu, \ldots$, we single out the trace $D_\mu k^\mu$ and call it k. Then $D_\mu k_\nu$ is decomposed into a traceless part and its trace k. The rest of the terms is obtained by acting with covariant derivatives on this basis set. Again we make terms like $D_\mu D_\nu k^\rho$ symmetric in $\mu\nu$ and traceless both in (ν, ρ) and in (μ, ρ) by using

$$D_\mu D_\nu k_\rho = D_\nu D_\mu k_\rho + [G_{\mu\nu}, k_\rho]$$

and $[G_{\mu\nu}, k_\rho]$ lies again in the space of polynomials we consider. (The tensors $D_\mu D_\nu k_\rho$ etc. are representations of the Lorentz group and the irreducible representations are traceless). For L no special basis need be constructed.

We now define an operator ρ which is in some sense an inverse of γ. Namely, corresponding to

$$\gamma k^\mu = \mathcal{L}^\mu, \gamma\mathcal{L}^\mu = 0; \gamma L = -k, \gamma k = 0 \tag{16.3.19}$$

(where we recall the definitions $\mathcal{L}^\mu = D_\nu G^{\nu\mu}$ and $k = D_\mu k^\mu$), we define

$$\rho\mathcal{L}^\mu = k^\mu, \rho k^\mu = 0; \rho k = -L, \rho L = 0 \tag{16.3.20}$$

On all other elements, the action of ρ is obtained by requiring that ρ commute with the covariant derivative.

$$\begin{aligned} [\rho, D_\mu] &= 0 \Rightarrow \rho(D_0\mathcal{L}_i) = D_0 k_i, \rho(D_i\mathcal{L}_0) = D_i k_0, \\ \rho(D_i\mathcal{L}_j) &= D_i k_j, \rho(D_0 k_i) = 0, \rho(D_i k_0) = 0, \rho(D_i k_j) = 0 \\ \rho(D_\mu L) &= 0, \rho(G_{\mu\nu}) = 0 \end{aligned} \tag{16.3.21}$$

So, in particular, $\rho(D_\mu G_{\nu\rho}) = 0$ if $D_\mu G_{\nu\rho}$ has no trace part, but $\rho(D_\mu G^{\mu\rho}) = k^\rho$.

The operator $\{\gamma, \rho\}$ counts the total number of L, k^μ and $\mathcal{L}^\mu$ which appear in a given monomial. For example

$$\{\rho, \gamma\}k^\mu = k^\mu, \{\rho, \gamma\}D_i k_j = \rho(D_i\mathcal{L}_j) = D_i k_j$$

$$\begin{aligned} \{\rho,\gamma\}D_\mu L &= -\rho(D_\mu k) = D_\mu L, \{\rho,\gamma\}k = k \\ \{\rho,\gamma\}G_{\mu\nu} &= 0, \{\rho,\gamma\}\mathcal{L}^\mu = \gamma(\rho\mathcal{L}^\mu) = \gamma k^\mu = \mathcal{L}^\mu. \end{aligned} \tag{16.3.22}$$

Since γ commutes with $\{\gamma,\rho\}$ (due to $\gamma^2=0$), we can use the same arguments as for the X and Y sector, and conclude: any polynomial with positive antifield number is γ exact. Moreover, if $X=\gamma Y$ and X has zero or nonzero antighost number, then Y has always nonzero antighost number because γ has negative antighost number. Hence Y is proportional to k and/or L.

e. Lie algebra cohomology (β cohomology). We set the results for the γ cohomology derived in the last subsection aside for a moment, and first derive some properties of the β cohomology. We consider now local polynomials p in undifferentiated ghost fields c and the tensors $G_{\mu\nu}{}^a, k_a^\mu$ and L_a with any number of covariant derivatives (including none) which are annihilated by the BRST charge β. We need a special property of such polynomials which can be proven by using what is called Lie algebra cohomology. [24] The polynomials may have any ghost number. We do not restrict ourselves to forms only; in fact, both the Lorentz indices and the group indices need not be contracted and may have any symmetry (including none). We shall prove that the cohomology of β is factorized: it is a sum of products of invariants constructed from $G_{\mu\nu}, kL$ and their covariant derivatives, and invariants constructed from the ghost fields c^a. What follows is a rather technical discussion, but one may proceed to (16.3.37) if one believes the answer.

In the linear vector space of polynomials, the BRST charge β can be written as

$$\beta = c^a\delta_a - \frac{1}{2}c^a\delta_{gh,a} \tag{16.3.23}$$

where

$$\delta_a = \left\{\frac{\partial}{\partial c^a},\beta\right\} \tag{16.3.24}$$

while $\delta_{gh,a}$ is that part of δ_a which only acts on ghosts. The δ_a act on the indices a of vectors in the adjoint representation and span the Lie algebra, $[\delta_a,\delta_b]=f_{ab}{}^c d_c$.

So $c^a\delta_a c = -gc \times c$ and $c^a\delta_a k^\mu = -gc \times k^\mu$. Further $-\frac{1}{2}c^a\delta_{gh,a}c^b = \frac{1}{2}g(c \times c)^b$ but $-\frac{1}{2}c^a\delta_{gh,a}k^\mu = 0$. To prove (16.3.23) one may verify that it holds for each of the fields and sources separately. Or one may note that, except on c^b, β can be written as $c^a\delta_a = c^a\frac{\partial}{\partial c^a}\beta$, since β introduces one ghost field in the transformation laws while $c^a\frac{\partial}{\partial c^a}$ is the ghost counting operator. The second term in (16.3.23) is needed to obtain the factor 1/2 in the BRST transformation law of the ghosts.

It is clear that δ_a and β commute

$$[\beta, \delta_a] = 0 \tag{16.3.25}$$

because $\beta^2 = 0$. Furthermore, δ_a acts both on tensors and on ghosts by the adjoint action. Hence, as far as δ_a is concerned, we need not distinguish between ghosts and other fields.

Consider the space Z of all polynomials p satisfying $\beta p = 0$.

$$Z = \{p \mid \beta p = 0\} \tag{16.3.26}$$

Clearly, δ_a maps Z into itself since $\beta\delta_a p = \delta_a\beta p = 0$. Hence

$$\delta_a Z \subset Z \tag{16.3.27}$$

The space Z can be decomposed into a δ-invariant subspace, a δ-exact subspace whose elements are of the form $\delta_a z$ with $z\epsilon Z$, and a remainder

$$Z = Z_{inv} \oplus Z_\delta \oplus Z_{rest} \tag{16.3.28}$$

Clearly, Z_{inv} and Z_δ are mapped into themselves by δ. Furthermore, since the representations of semisimple groups are completely reducible, δ maps also Z_{rest} into itself. This space Z_{rest} is empty, because any element in $z\epsilon Z_{rest}$ is transformed under δ into δz, which is nonvanishing (since Z_{rest} is not part of Z_{inv}), but lies in Z_δ. Hence

$$Z = Z_{inv} \oplus Z_\delta \tag{16.3.29}$$

We conclude that each p with $\beta p = 0$ can be decomposed into

$$p = p_{inv} + \delta_a \tilde{p},\ \beta\tilde{p} = 0, \delta_a p_{inv} = 0 \tag{16.3.30}$$

Then we also have

$$p = p_{inv} + \beta\eta, \eta = \frac{\partial}{\partial c^a}\tilde{p} \tag{16.3.31}$$

because $\delta_a = \{\frac{\partial}{\partial c^a}, \beta\}$ and $\beta\tilde{p} = 0$. So only the polynomials which are scalars under the adjoint action of the group, can have nontrivial β cohomology. Polynomials with indices can always be written as the adjoint rotation of another polynomial.

We now study the δ invariant sector Z_{inv} further. The δ invariant polynomials p_{inv} satisfy $\delta_a p_{inv} = 0$ but also $\beta p_{inv} = 0$ since p_{inv} lies in Z. Since $\delta_a Z_{inv} = 0$, we have

$$\beta Z_{inv} = \beta_{gh} Z_{inv} = -\frac{1}{2}c^a \delta_{gh,a} Z_{inv} = 0 \tag{16.3.32}$$

where β_{gh} is the second term in β in (16.3.23). We can now repeat the same steps as before, the only difference being that β now acts only on the ghost fields but not on the tensors. In other words, we can forget about the tensors, and concentrate on the ghosts.

The δ-invariant subspace Z_{inv} is mapped by $\delta_{gh,a}$ into itself because $\beta_{gh} \equiv -\frac{1}{2}c^a\delta_{gh,a}$ commutes with $\delta_{gh,a} = \{\frac{\partial}{\partial c^a}, \beta_{gh}\}$ due to ${\beta_{gh}}^2 = 0$. (Recall that $\beta_{gh} c^b = \frac{1}{2}g(c\times c)^b$ and $c^a\delta_a c^b = -g(c\times c)^b$). It follows that the space $\delta_{gh,a}Z_{inv}$ is annihilated by β_{gh}. To show that $\delta_{gh,a}Z_{inv}$ lies in Z_{inv}, we must show that δ_b annihilates $\delta_{gh,a}Z_{inv}$. Now $\delta_{gh,a}Z_{inv} = \{\frac{\partial}{\partial c^a}, \beta_{gh}\}Z_{inv} = \beta_{gh}\frac{\partial}{\partial c^a}Z_{inv} = \beta\frac{\partial}{\partial c^a}Z_{inv}$. (We used in the last step that $(\beta - \beta_{gh})\frac{\partial}{\partial c^a}Z_{inv} = 0$. This follows from $c^b\delta_b\frac{\partial}{\partial c^a}Z_{inv} = \{c^b\delta_b, \frac{\partial}{\partial c^a}\}Z_{inv} = \delta_a Z_{inv} = 0$). Since δ_b commutes both with β (because $\delta_b = \{\frac{\partial}{\partial c^b}, \beta\}$ and $\beta^2 = 0$) and with $\frac{\partial}{\partial c^a}$ (because δ_b does not depend on c), and $\delta_b Z_{inv} = 0, \delta_b$ indeed annihilates $\delta_{gh,a}Z_{inv}$.

Hence, Z_{inv} decomposes into a δ_{gh} invariant subspace $Z_{inv,inv}$, and a δ_{gh} exact subspace $Z_{inv,\delta_{gh}}$ with elements of the form $\delta_{gh}\tilde{p}$ with $\tilde{p}$ in Z_{inv}. (We use again that the representations of $\delta_{gh,a}$ are completely reducible because they are representations of a semisimple algebra.)

$$\begin{aligned} Z_{inv} &= Z_{inv,inv} \oplus Z_{inv,\delta_{gh}} \\ \delta Z_{inv,inv} &= \delta_{gh} Z_{inv,inv} = 0 \\ Z_{inv,\delta_{gh}} &= \{z | z = \delta_{gh}\tilde{p}, \tilde{p}\epsilon Z_{inv}\} \end{aligned} \tag{16.3.33}$$

Next we show that $Z_{inv,\delta_{gh}}$ is β exact, and thus it will play no further role. To prove this, we first demonstrate that $Z_{inv,\delta_{gh}}$ is β_{gh} exact. This follows from

$$\delta_{gh,a}\tilde{p} = \left\{\frac{\partial}{\partial c^a}, \beta_{gh}\right\}\tilde{p} = \beta_{gh}\left(\frac{\partial}{\partial c^a}\tilde{p}\right) \tag{16.3.34}$$

because $\beta_{gh}\tilde{p} = (\beta - c^a\delta_a)\tilde{p} = 0$. (Recall that all polynomials lie in Z, so $\beta\tilde{p} = 0$, whereas all polynomials in Z_{inv} satisfy $\delta_a Z_{inv} = 0$). To prove that $Z_{inv,\delta_{gh}}$ is also β exact, we note that $\beta_{gh} = \beta - c^b\delta_b$, and

$$c^b\delta_b\frac{\partial}{\partial c^a}\tilde{p} = c^b\frac{\partial}{\partial c^a}\delta_b\tilde{p} = 0 \tag{16.3.35}$$

because $\left[\delta_b, \frac{\partial}{\partial c^a}\right] = 0$ due to $f_{ab}{}^b = 0$.

We are left with the sector $Z_{inv,inv}$ which are scalars in the ghost sector, and hence also scalars in the tensor sector. We shall first show that none of the elements of $Z_{inv,inv}$ is β exact, hence $Z_{inv,inv}$ contains all the nontrivial cohomology of β. Next we shall show that the scalars built from ghost fields can be written as "primitive forms" trc^n (where $c = c^aT_a$) which are closely related to Casimir invariants. To prove that no element z in $Z_{inv,inv}$ is β exact, we first show that no element in $Z_{inv,inv}$ is β_{gh} exact. To prove this assume the contrary

$$z = \beta_{gh}\zeta,\ z\epsilon Z_{inv,inv} \tag{16.3.36}$$

Using $\beta_{gh} = \frac{1}{2}c^a\delta_{gh,a} = \frac{1}{2}\delta_{gh,a}c^a$ because $f_{ab}{}^b = 0$, we see that $\beta_{gh}\zeta = \delta_{gh,a}(\frac{1}{2}c^a\zeta)$ which is a contradiction. Next we show that the $z\epsilon Z_{inv,inv}$ are not β exact either. Using that also $c^a\delta_a = \delta_a c^a$, we see that if by assumnption $z = \beta\eta$, then $z = \delta z' + \delta_{gh} z''$. Since $\delta z'$ does not lie in Z_{inv}, and $\delta_{gh} z''$ does not lie in $Z_{inv,inv}$ also this assumption leads to a contradiction. Hence we have shown that no element z in $Z_{inv,inv}$ is β exact.

We have thus shown the following theorem: if $\beta p = 0$ then $p = \sum c^{\alpha\beta} I_\alpha(T) I_\beta(c) + \beta p'$ where $c^{\alpha\beta}$ are constants and $I_\alpha(T)$ are invariant polynomials in $G_{\mu\nu}, k, L$ and their covariant derivatives, labeled by α, while $I_\beta(c)$ are invariant polynomials in the ghost fields, labeled by β.

We now discuss the invariants in ghost fields further. An example of an invariant $I_\beta(c)$ is $f_{abc}c^a c^b c^c$. This can be written as $\theta_1(c) = trc^3 = \frac{1}{2} f_{ab}{}^c tr(T_c T_d) c^a c^b c^d = \frac{1}{2} f_{ab}{}^c \gamma_{cd} c^a c^b c^d$. It is invariant because $\beta c = c^2$ and $trc^4 = 0$. For $SU(3)$ another example is

$$\theta_2(c) = trc^5 = f_{ab}{}^c f_{pq}{}^r (trT_c T_r T_s)(c^a c^b c^p c^q c^s) \tag{16.3.37}$$

where $trT_c T_r T_s$ in this expression is symmetric in c and r, and cyclic, hence the totally symmetric symbol d_{crs}. Again its invariance is obvious: $\delta\theta_2(c) \sim trc^6 = 0$. Thus in both examples the invariant is proportional to a Casimir invariant tensor (γ_{ab} and d_{crs}.) This is a general result. Any invariant polynomial is a sum of products of primitive polynomials $\theta_\alpha(c)$ where $\alpha = 1, \ldots, r$ with r the rank of the simple Lie algebra. These $\theta_\alpha(c)$ are independent, meaning that one cannot express one of them as a sum of products of others. Of course $trc^{2n} = 0$, but can also determine the power of the remaining invariants

$$\begin{aligned} \theta_\alpha(c) &= trc^{2m(\alpha)-1} = f_{a_1 b_1}{}^{c_1} \ldots f_{a_{m(\alpha)-1} b_{m(\alpha)-1}}{}^{c_{m(\alpha)-1}} \\ &\times tr(T_{c_1} \ldots T_{c_{m(\alpha)-1}} T_d) c^{a_1} c^{b_1} \ldots c^{a_{m(\alpha)-1}} c^{b_{m(\alpha)-1}} c^d \end{aligned} \tag{16.3.38}$$

This trace is proportional to the α-th totally symmetric (Casimir) invariant tensor $d_{c_1 \ldots c_{m(\alpha)-1} d}$. So, $\theta_\alpha(c)$ has $2m(\alpha) - 1$ ghost fields where $m(\alpha)$ is dimension of the corresponding Casimir operator. (A simple Lie algebra has r Casimir invariant. The lowest one is always γ_{ab}. For SU(3) the other one is d_{abc}). To prove that the $\theta_\alpha(c)$ are independent building blocks for invariants, one may show that the product $\prod_{\alpha=1}^{r} \theta_\alpha(c)$ contains precisely as many ghost fields as the dimension of the group, and is non-vanishing. Since the square of any $\theta_\alpha(c)$ vanishes and two $\theta_\alpha(c)$ anticommute, $\theta_\alpha(c)$ satisfy a Grassmann algebra (one could view them as the fermionic coordinates of some suitable superspace).

f. The general solution of $QR(A, c, k, L) = 0$. In this subsection we combine the results on the γ cohomology and the Lie algebra cohomology (β cohomology). Consider a polynomial R which is annihilated by Q, and decompose it w.r.t. antighost number

$$R = R_0 + R_1 + \ldots + R_N \tag{16.3.39}$$

Since $Q = \beta + \gamma$, we get a hierarchy of relations

$$\beta R_0 + \gamma R_1 = 0, \beta R_1 + \gamma R_2 = 0, \ldots, \beta R_{L-1} + \gamma R_L = 0, \beta R_N = 0 \tag{16.3.40}$$

If $N = 0, R_L = R_N$ does not depend on k, L, and $\beta R = 0$ means that R is G_0 invariant, see (16.3.12). (In particular, at ghost number zero, R is then gauge invariant).

If $N > 0$, the relation $\beta R_N = 0$ implies according to Lie algebra cohomology

$$R_N = \sum S^J(t) E^J(c) + \beta S_N \tag{16.3.41}$$

where $S^J(t)$ are group-invariant polynomials depending on G, k, L and their covariant derivatives. We can then remove S_N by subtracting the BRST exact terms QS_N from R

$$\begin{aligned} R - QS_N \equiv R' &= R_0 + R_1 + \ldots + (R_{N-1} - \gamma S_N) + (R_N - \beta S_N) \\ &= R_0 + R_1 + \quad + R_{N-1}' + R_N' \\ R_N' &= \sum S^J(t) E^J(c) \end{aligned} \tag{16.3.42}$$

It follows that $\gamma R_N' = \sum(\gamma S^J(t)) E^J$, and this is again a group-invariant polynomial because in $S^J(t)$ all indices were contracted ($S^J(t)$ was a group invariant), while γ

only replaces some tensors by other tensors but does not undo the contraction of the indices.

The next equation in (16.3.40)

$$\beta {R_{N-1}}' + \gamma {R_N}' = 0 \tag{16.3.43}$$

states that $\gamma {R_N}'$ must be β exact. This is impossible: as we have shown, a group-invariant can never be written as the adjoint transformation of some other polynomial. Hence $\gamma {R_N}' = 0$. Since the γ cohomology at nonvanishing antighost number is trivial,

$${R_N}' = \gamma T_{N+1} \tag{16.3.44}$$

Since ${R_N}'$ is a group invariant according to (16.3.42), and γ does not change the representations of tensors, T_{N+1} is again a group-invariant of the form (16.3.42). Hence, $\beta T_{N+1} = 0$, and combining this with (16.3.44), we finally reach our goal

$${R_N}' = QT_{N+1} \tag{16.3.45}$$

This shows that we can omit ${R_N}'$ from R.

We can now repeat the analysis for R_{N-1}, and so on, until we reach R_0. Hence

$$R = G + QX. \tag{16.3.46}$$

where G is a G_0 invariant polynomial.

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Chapter 17

The effective potential at the one-loop level

The effective action can be expanded into terms without derivatives, terms with one derivative, terms with two derivatives, etc. The terms without derivatives are called the effective potential. These are polynomials in fields, logarithms of fields, logarithms of logarithms of fields, etc. If we take all fields as constants, the minimum of the effective potential determines whether spontaneous symmetry breaking occurs at the quantum level. The calculation of the sum of all one-particle irreducible one-loop graphs with constant external scalar fields is not too difficult, and we shall discuss three applications: the Coleman-Weinberg mechanism for quantum induced spontaneous symmetry breaking, the effective potential for supersymmetric theories, and a lower bound on the mass of the Higgs boson. However, the effective action, and in particular the effective potential, are Green's functions, and as such they are in general gauge-dependent. We shall therefore have to study whether the issue of susy breaking, or the value of the vacuum energy are gauge-dependent. This we shall achieve by using Ward identities for the effective potential which give information about the dependence on the gauge parameters.

1 The Coleman-Weinberg mechanism

To break gauge theories spontaneously, one has to choose potentials and mass terms for the Higgs scalars. In order that the original gauge symmetry group breaks down to a particular subgroup H, one must carefully choose the potentials and mass terms. It would be nicer if the theory determined itself these potentials and mass terms. One step in this direction is a mechanism considered by Coleman and Weinberg, who started from a classical massless theory, but found that at the one-loop level the quantum corrections add an order $\hbar$ term to the potential such that the minimum of the effective potential to order $\hbar$ occurs away from the origin. If there is only one coupling constant in the theory, loop corrections must be smaller than tree graphs if perturbation theory is to be trusted. On the other hand, at the minimum of the effective potential the (derivative of the) tree graphs must be canceled by the (derivative of the) one-loop corrections, and in general this contradicts perturbation theory. However, in theories with two coupling constants, one coupling constant may be of the same order as the square of another, and if the second coupling constant only appears in loops and the first in tree graphs, a consistent perturbative scheme for quantum induced spontaneous symmetry breaking is possible. As we shall see, $\lambda\varphi^4$ theory is an example of a theory in which the one-loop corrections to the effective potential are larger than the tree graph contributions, but complex scalars coupled to electromagnetism and with $\lambda(\varphi^*\varphi)^2$ self-interactions leads to consistent perturbation theory if λ is of order e^4. (If one thinks of λ as being induced by one-loop Coulomb scattering, the assumption of equality of λ and e^4 has even some physical motivation). In these models, "dimensional transmutation" occurs: a massless coupling constant in the classical theory is exchanged for the massive constant $<\varphi>$ at the quantum level. The reason a mass parameter is generated by quantum corrections is that renormalization of a massless theory requires that one defines renormalized coupling constants at nonzero values for φ, say $\varphi = M$, and this mass parameter M can then be exchanged for the nonzero $< \varphi >$. A change in $< \varphi >$ leads to a change in the value of the renormalized coupling ¿constant, and this shows that the number of independent parameters has not changed.

Consider first $\lambda\varphi^4$ theory. We shall begin with a mass m^2, and only later set m^2 to zero. We could have started without mass term but this is algebraically more complicated.

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu \varphi)^2 - \frac{1}{2}m^2\varphi^2 - \frac{1}{4!}\lambda\varphi^4 \tag{17.1.1}$$

To compute the effective potential at the one-loop level, we must evaluate the sum of φ-loops with 1,2,3,... vertices. Each vertex has two external φ fields, and if one takes the external φ fields constant, one obtains the effective potential. In the path integral one finds the exponent $\exp \frac{i}{\hbar} S = \exp \frac{i}{\hbar} \int (T - V) dt$, and by expanding $\exp \frac{i}{\hbar} S_{int}$ and using the Wick contraction rules, one finds the one-loop corrections to the potential

$$\frac{-i}{\hbar} V^{(1)}(\bar{\varphi}) = \sum_{n=1}^{\infty} \left(\frac{i}{\hbar}\right)^n \frac{1}{n!} \Big(\frac{-\lambda}{4!} \ \varphi^4\Big)\Big(\frac{-\lambda}{4!} \ \varphi^4\Big) \cdots \Big(\frac{-\lambda}{4!}\varphi^4\Big) \ \frac{(n-1)!}{2} \tag{17.1.2}$$

The constant external fields φ are denoted by $\bar{\varphi}$. We have removed the volume of spacetime on the left-hand side, and a factor $\delta^4(0)$ for energy-momentum conservation on the right-hand side. (These factors are of course equal). The first contraction yields a combinatorial factor $4 \times 4 \times (n-1)$ because at each vertex one can choose a field φ in 4 different ways, while from a given vertex one can still go to $(n-1)$ other vertices. The second contraction yields then a factor $3 \times 4 \times (n-2)$, because there are only 3 fields left at the second vertex but still 4 fields at the third vertex. The contraction from the third to the fourth vertex yields similarly a factor $3 \times 4 \times (n-3)$, etc. Finally, the last contraction from the last to the first vertex closes the loop and yields a combinatorial factor 3×3; this contraction is indicated on top of equation (17.1.2). However, in this way the contraction indicated on top of (17.1.2) is counted twice because it was already taken into account by the very first contraction we performed, so one must divide the final result by a factor 2. (An easy check on these combinatorial factors is to consider two vertices: at each vertex one can extract two external fields in $\binom{4}{2}$ ways, and then only 2 contractions between the remaining fields are left. This yields a factor $(\frac{4\times 3}{2})^2 \times 2$ which is indeed equal to $(4\times 4)\times(3\times 3)\times\frac{1}{2})$. Substituting the propagators for the scalar fields we find

$$\begin{aligned} -\frac{i}{\hbar}V^{(1)}(\bar{\varphi}) &= \sum_{n=1}^{\infty}\left(\frac{i}{\hbar}\right)^n \frac{1}{2n}\left(\frac{-\lambda}{2}\bar{\varphi}^2\right)^n \int\left(\frac{-i\hbar}{k^2+m^2-i\epsilon}\right)^n \frac{d^4k}{(2\pi)^4} \\ &= -\frac{1}{2}\int \ln\left(1+\frac{V''}{k^2+m^2-i\epsilon}\right)\frac{d^4k}{(2\pi)^4} \end{aligned} \tag{17.1.3}$$

where the symbol $V''=\frac{1}{2}\lambda\bar{\varphi}^2$ denotes the vertices with $\bar{\varphi}$ constant fields. Note that all terms in this sum are infrared finite due to the mass m^2. If we had started with a massless theory, we would at this point have found a sum of increasingly infrared singular terms.

The terms with $n=1$ and $n=2$ are divergent, but the divergences are eliminated by renormalization. We choose the renormalization conditions temporarily at $\bar{\varphi}=0$[1]

$$\frac{d^2}{d\bar{\varphi}^2}V^{\text{ren}}(\bar{\varphi})\big|_{\bar{\varphi}=0} = m^2_{\text{ren}};\ \frac{d^4}{d\bar{\varphi}^4}V^{\text{ren}}(\bar{\varphi})\big|_{\bar{\varphi}=0} = \lambda_{\text{ren}} \tag{17.1.4}$$

and renormalize $\mathcal{L}$ as

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu\varphi_{\text{ren}})^2 - \frac{1}{2}m^2_{\text{ren}}\varphi^2_{\text{ren}} - \frac{1}{4!}\lambda_{\text{ren}}\varphi^4_{\text{ren}} + \text{ counter terms} \tag{17.1.5}$$

Since $V^{(0)}=\frac{1}{2}{m_{\text{ren}}}^2{\varphi_{\text{ren}}}^2+\frac{1}{4!}\lambda_{\text{ren}}\varphi^4_{\text{ren}}$ already satisfies the renormalization conditions, $V^{(1),\text{ren}}$ should contain no $\bar{\varphi}^2$ and $\bar{\varphi}^4$ terms, and renormalization consists in removing the $\bar{\varphi}^2$ and $\bar{\varphi}^4$ terms from $V^{(1)}$. The simplicity of this renormalization procedure was the reason for choosing (17.1.4) as renormalization conditions. From the graphs without counterterms one obtains

$$-\frac{i}{\hbar}V^{(1),\text{ren}}(\bar{\varphi}) =$$

[1]Note that $\varphi=0$ is a solution of the classical field equations, and $<\varphi>=\bar{\varphi}=0$ is maintained at the quantum level because all terms in $V(\bar{\varphi})$ are quadratic in $\bar{\varphi}$. Hence, tadpole graphs vanish. The renormalization conditions in (17.1.5) are then the usual renormalization conditions for Green's functions at vanishing external momenta. (Vanishing external momenta correspond to constant external fields).

$$-\frac{1}{2}\int\left[\ln\left(1+\frac{V''}{k^2+m_{\rm ren}^2-i\epsilon}\right)-\frac{V''}{k^2+m_{\rm ren}^2-i\epsilon}+\frac{1}{2}\left(\frac{V''}{k^2+m_{\rm ren}^2-i\epsilon}\right)^2\right]\frac{d^4k}{(2\pi)^4} \tag{17.1.6}$$

Rather than evaluate the contributions from the counter terms separately, we shall directly determine the renormalized effective potential from the renormalization conditions. To evaluate the momentum integral we make a Wick rotation $dk_0 = id\kappa$,

$$V^{(1),\rm ren} = \frac{\hbar}{2}\int\left[\ln\left(1+\frac{V''}{\kappa^2+m_{\rm ren}^2}\right)-\frac{V''}{\kappa^2+m_{\rm ren}^2}+\frac{1}{2}\left(\frac{V''}{\kappa^2+m_{\rm ren}^2}\right)^2\right]\frac{d^4\kappa}{(2\pi)^4} \tag{17.1.7}$$

Using polar coordinates, $d^4\kappa = \pi^2\kappa^2 d\kappa^2$, the integration over $\kappa^2 \equiv x$ yields

$$\begin{aligned}
&\Big[\frac{1}{2}x^2\ln(x+A)-\frac{1}{4}(x+A)^2+Ax-\frac{1}{2}A^2\ln(x+A)\\
&-\frac{1}{2}x^2\ln(x+m_{\rm ren}^2)+\frac{1}{4}(x+m_{\rm ren}^2)^2-m_{\rm ren}^2x+\frac{1}{2}m_{\rm ren}^4\ln(x+m_{\rm ren}^2)\\
&+m_{\rm ren}^2V''\ln(x+m_{\rm ren}^2)-V''x+\frac{1}{2}(V'')^2\left(\ln(x+m_{\rm ren}^2)+\frac{m_{\rm ren}^2}{x+m_{\rm ren}^2}\right)\Big]_0^\infty\\
&=\frac{1}{2}A^2\ln\frac{A}{m_{\rm ren}^2}-\frac{3}{4}(V'')^2-\frac{1}{2}V''m_{\rm ren}^2
\end{aligned} \tag{17.1.8}$$

where $A = V'' + m_{\rm ren}^2$. In this way we obtain

$$\begin{aligned}
V^{(1),\rm ren}(\bar\varphi) &= \frac{\hbar}{(8\pi)^2}\Bigg[\left(\frac{1}{2}\lambda_{\rm ren}\bar\varphi^2+m_{\rm ren}^2\right)^2\ln\left(1+\frac{\lambda_{\rm ren}\bar\varphi^2}{2m_{\rm ren}^2}\right)\\
&- \frac{\lambda_{\rm ren}\bar\varphi^2}{2}\left(\frac{3}{4}\lambda_{\rm ren}\bar\varphi^2+m_{\rm ren}^2\right)\Bigg]
\end{aligned} \tag{17.1.9}$$

As it stands, we cannot take the limit $m_{\rm ren}\to 0$ in $V^{(1),\rm ren}$, and since we want to consider massless fields ($m_{\rm ren}=0$), we switch at this point to another renormalization condition for the coupling constant, now at $\bar\varphi = M$

$$\lambda_M = \frac{d^4}{d\bar\varphi^4}V^{\rm ren}(\bar\varphi)\Big|_{\bar\varphi=M} \tag{17.1.10}$$

The relation between λ_M and $\lambda_{\rm ren}$ can be found by differentiating $\frac{1}{4!}\lambda_{\rm ren}\bar\varphi^4+V^{(1),\rm ren}(\bar\varphi)$ four times

$$\lambda_M = \lambda_{\rm ren}+\frac{\hbar}{(8\pi)^2}\lambda_{\rm ren}^2\left[6\ln\left(1+\frac{\lambda_{\rm ren}M^2}{2m_{\rm ren}^2}\right)+16+\mathcal{O}\left(\frac{m^2}{M^2}\right)\right] \tag{17.1.11}$$

(The factor 16 comes from differentiating the logarithm once, twice, 3 times and 4 times and, of course, from the last term in $V^{(1),\mathrm{ren}}$).

Replacing λ_{ren} by λ_M in $V^{(1),\mathrm{ren}}$, the tree-graph and one-loop contributions to the effective potential read

$$V^{\mathrm{ren}}(\bar{\varphi}) = \frac{1}{4!}\lambda_M \bar{\varphi}^4 + \hbar \frac{\lambda_M^2 \bar{\varphi}^4}{(16\pi)^2}\left(\ln \frac{\bar{\varphi}^2}{M^2} - \frac{25}{6}\right) + \mathcal{O}\left(\frac{m_{\mathrm{ren}}^2}{M^2}\right) + \mathcal{O}(\hbar^2) \tag{17.1.12}$$

This is the Coleman-Weinberg formula. As a check on the factor $-\frac{25}{6}$ one may verify that the fourth derivative of the one-loop correction at $\bar{\varphi} = M$ vanishes (the renormalization condition in (17.1.10)). At this point we can take the limit $m_{\mathrm{ren}}^2 \to 0$, keeping M fixed. As we shall discuss later in more detail, we have removed the infrared divergence for $m_{\mathrm{ren}} \to 0$ in (17.1.9) by the "recalibration" in (17.1.11) (which is a finite renormalization for finite m_{ren}^2 but an infinite infrared renormalization for $m_{\mathrm{ren}}^2 \to 0$. We recall that also in QED infrared divergences appear in Z_1 and Z_2 if one uses on-shell renormalization). From (17.1.12) it is clear that one cannot define the coupling constant by a renormalization condition at $M = 0$.

The result seems to indicate that spontaneous symmetry breaking may have been induced by quantum corrections, because the minimum of $V^{(0)} + V^{(1),\mathrm{ren}}$ may occur at nonzero $\bar{\varphi}$. Evaluating $dV/d\bar{\varphi} = 0$ one finds that the minimum occurs at $< \bar{\varphi} >$ where

$$\frac{32\pi^2}{3} + \hbar\lambda_M \left(\ln \frac{< \bar{\varphi} >^2}{M^2} - \frac{11}{3}\right) = 0 \tag{17.1.13}$$

However, since higher-order corrections will be proportional to powers of $\hbar\lambda_M \ln \frac{<\varphi>^2}{M^2}$, perturbation theory is only valid if these factors are much smaller than one, in flat contradiction to (17.1.13). Hence the perturbative conclusions reached in this model cannot be trusted.

One can find models with more than one coupling constant where quantum induced spontaneous symmetry breaking also occurs but where perturbation theory now is a consistent approximation. As an example consider a charged scalar with minimal electromagnetic coupling and a $\lambda(\varphi^*\varphi)^2$ coupling

$$\mathcal{L} = -(\partial_\mu + ieA_\mu)\varphi^*(\partial^\mu - ieA^\mu)\varphi - \frac{1}{4}F_{\mu\nu}^2 - \frac{\lambda}{3!}(\varphi^*\varphi)^2 \tag{17.1.14}$$

To avoid the complications that one-loop diagrams with external scalars contribute in which both photons and scalars propagate in the loop, we choose the Landau gauge

$$D_{\mu\nu} = -i(\eta_{\mu\nu} - k_\mu k_\nu/k^2)(k^2 + M^2 - i\epsilon)^{-1} \tag{17.1.15}$$

(This gauge corresponds to a gauge fixing term $-\frac{1}{2\alpha}(\partial^\mu A_\mu)^2$ with gauge field propagator $\{\eta_{\mu\nu} + (\alpha-1)k_\mu k_\nu/(k^2+\alpha M^2)\}/(k^2+M^2)$ where $M = e < \varphi_1 >$. Taking the limit $\alpha \to 0$ yields (17.1.15)). Since the scalar-photon vertex $-ieA^\mu(\varphi^* \overleftrightarrow{\partial}_\mu \varphi)$ yields a factor $k_\mu A^\mu$ where k is the loop momentum (since the external $\bar{\varphi}$ carry no momentum), the contractions of k_μ with $D_{\mu\nu}(k)$ vanish. Hence, only graphs with photons in the loop contribute and thus only the vertices $\mathcal{L}^{int} = -e^2\varphi^*\varphi {A_\mu}^2$ need be taken into account. (There are 3 modes contributing to $\eta_{\mu\nu} - k_\mu k_\nu/k^2$, namely two transversal modes and one longitudinal mode. Of course, photons have only two degrees of freedom, but in the Landau gauge the vertices $-ieA_\mu\varphi^*\overleftrightarrow{\partial}_\mu\varphi$ do not contribute, and their contributions are now supplied by the third polarization of the photon in the Landau propagator). Since the action has the $U(1)$ rigid symmetry $\varphi \to e^{i\alpha}\varphi, \varphi^* \to e^{-i\alpha}\varphi^*$, we can always rotate a nonvanishing expectation value of $< \varphi >$ such that only $< \varphi_1 >$ is nonzero but $< \varphi_2 >= 0$, where $\varphi = (\varphi_1 + i\varphi_2)/\sqrt{2}$. Then only graphs with external φ_1 need be considered and one finds

$$V^{(1),\text{ren}}(\bar{\varphi}_1) = \frac{\lambda_M}{4}{\varphi_1}^4 + \frac{\hbar {\varphi_1}^4}{(8\pi)^2}\left(\ln\frac{{\varphi_1}^2}{M^2} - \frac{25}{6}\right)C \tag{17.1.16}$$

where

$$C = \left(\frac{1}{2}\lambda_M\right)^2 + \frac{1}{9}\left(\frac{1}{2}\lambda_M\right)^2 + 3e^4 \tag{17.1.17}$$

(The trace over the Landau propagator gives the factor 3 in $3e^4$, and the factor $\frac{1}{9}$ comes from the fact that one can extract two external φ_1 fields in 6 ways from the first term in the vertex ${\varphi_1}^4 + 2{\varphi_1}^2\varphi_2^2$ but only in one way from the second term. Together with the factor 2 in the second term this yields a relative factor 1/3 at each vertex.)

Assume now that for some reason λ_M is numerically of the order of $\hbar e^4$. Then the terms with λ_M^2 can be dropped

$$V^{(1),\text{ren}} = \frac{\lambda_M}{4!}{\bar{\varphi}_1}^4 + 3\hbar\frac{{\bar{\varphi}_1}^4}{64\pi^2}e^4\left(\ln\frac{{\bar{\varphi}_1}^2}{M^2} - \frac{25}{6}\right) \tag{17.1.18}$$

Up to this point the parameter M was still free, but we now choose the value of M to be the value of $\bar{\varphi}$ at the minimum of the potential

$$M = \bar{\varphi}_{1,\text{min}} \tag{17.1.19}$$

At the minimum of the potential, $\ln(\bar{\varphi}_1^2/M^2)$ vanishes, and we determine $< \varphi_1 >$ by evaluating $dV/\bar{\varphi}_1 = 0$. Denoting the value of λ_M for $M = \bar{\varphi}_{1,\text{min}}$ by $\bar{\lambda}$, we find

$$dV/d\bar{\varphi}_1 = \left(\frac{\bar{\lambda}}{6} - \hbar\frac{11e^4}{16\pi^2}\right){\bar{\varphi}_1}^3 = 0 \text{ at } \bar{\varphi}_1 = \bar{\varphi}_{1,\text{min}} \tag{17.1.20}$$

Hence[2]

$$\bar{\lambda} = \hbar\frac{33}{8\pi^2}e^4 \tag{17.1.21}$$

where $\bar{\lambda}$ is thus defined by $\bar{\lambda} = \frac{d^4}{d\bar{\varphi}_1}V(\bar{\varphi}_1)$ at $\bar{\varphi}_1 = \bar{\varphi}_{1,\text{min}}$. Substituting this value of $\bar{\lambda}$ back into the effective potential, we arrive at a very simple formula

$$V^{(1),\text{ren}} = \hbar\frac{3e^4}{64\pi^2}{\bar{\varphi}_1}^4\left(\ln\frac{\bar{\varphi}_1^2}{\bar{\varphi}_{\text{min}}^{(1),2}} - \frac{1}{2}\right) \tag{17.1.22}$$

In this form it is clear that V has a minimum at $\bar{\varphi}_1 = \bar{\varphi}_{\text{min}}^{(1)}$ which is away from the origin and this result is consistent with our assumption that $\bar{\lambda}$ is of order e^4.

[2]The dimension of λ is $(\hbar c)^{-1}$ while the parameter e in (17.1.14) has dimension $(\hbar c)^{-1/2}$. From $d^4k = d^3kcdk_0$ one gets a factor c. Altogether the dimensions match. For notational simplicity we have suppressed the factors c in the text.

However, it is equally clear from (17.1.20) that the value of $\bar{\varphi}_{1,\text{min}}$ cannot be fixed; rather, it is a new free parameter, while $\bar{\lambda}$ is completely eliminated as a free parameter. Since all masses in the spontaneously broken gauge theory are proportional to $\bar{\varphi}_1$, we can still determine mass ratios. (One finds then from (17.1.22) that both the photon and the Higgs scalar φ_1 become massive and that their mass ratio is given by $m^2(\varphi_1)/m^2(A_\mu) = V''/e^2 = \frac{3}{2\pi}\frac{e^2}{4\pi}$. This is an amusing result, but since it is model-dependent one should probably not take it too seriously.) So we have traded the massless coupling constants $\bar{\lambda}$ for the dimensionful constant $\bar{\varphi}_{1,\text{min}}$. This transition from a dimensionless constant to a dimensionful one is sometimes called "dimensional transmutation". It is possible because we defined the coupling constant at some point $\bar{\varphi}_1 = \bar{\varphi}_{1,\text{min}} = M$, which is a renormalization condition for λ. We could not define the coupling constant by a renormalization condition at $\bar{\varphi}_1 = 0$ because there the one-loop graphs in (17.1.3) become singular for $m^2_{\text{ren}} \to 0$. Setting $m^2_{\text{ren}} = 0$ leads to an infrared divergence on top of the ultraviolet divergence. By defining the coupling constant at $\bar{\varphi} = M$ instead of $\bar{\varphi} = 0$, we subtracted this mass-divergence, and absorbed it into the definition (renormalization) of λ_M, see (17.1.11). Hence, in order to be able to take the limit $m^2_{\text{ren}} \to 0$, one must introduce a new dimensionful parameter in the theory, and the most natural choice of this dimensionful parameter is the expectation value $\bar{\varphi}_1$ of the Higgs field itself at the minimum of the potential, rather than some arbitrary parameter M.

The CW mechanism raises the possibility that the minima of the effective potential at the quantum level may be different, even qualitatively, from the minima at the classical level. As application one may consider grand unified theories, or the standard model. Usually one chooses extra Higgs multiplets whose interactions are carefully chosen such that spontaneous symmetry breaking $SU(5) \to SU(3) \times SU(2) \times U(1)$, or $SU(2) \times U(1) \to U(1)_{em}$, occurs at the classical level. A drawback is the fine-tuning of the Higgs potentials. We already mentioned that it would be nicer if quantum effects could replace the classical fine-tuning procedure, but two remarks dampen this prospect. First, although we did not add a mass term classically, we still needed a $\lambda\varphi^4$ term, hence in more complicated models there still are arbitrary constants in such potentials. Second, spontaneous symmetry breaking is thought to occur at very early times in the universe when the universe cooled down. Hence, one should take temperature effects into account and these seem far more important than the CW effect by itself. However, a combination of both effects is not ruled out.

2 One-loop contributions from fermions

In the standard model, Higgs fields couple also to fermions (which receive a mass due to this coupling), hence we are also interested in fermion loops with external scalars. Consider complex (Dirac) fermions ψ coupled to real scalars φ

$$\mathcal{L} = -\bar{\psi}^a \gamma^\mu \partial_\mu \psi^a - \bar{\psi}\left[m + g(\alpha + i\gamma_5\beta)\varphi\right]\psi \tag{17.2.1}$$

where α and β are hermitian matrices acting on the indices of the ψ^a. The one-loop contributions now read

$$\begin{aligned}\frac{-i}{\hbar}V &= \sum_{n=1}^{\infty}\left(\frac{i}{\hbar}\right)^n \frac{1}{n!}(\bar{\psi}(-Y)\psi)(\bar{\psi}\ (-Y)\psi)..,\ (\bar{\psi}(-Y)\psi) \\ &= (-)\sum_{n=1}^{\infty}\left(\frac{i}{\hbar}\right)^n \frac{1}{n}(-i\hbar)^n tr\int Y\frac{(-\not{k}-im)}{k^2+m^2-i\epsilon}Y\ldots Y\frac{(-\not{k}-im)}{k^2+m^2-i\epsilon}\,\frac{d^4k}{(2\pi)^4}\end{aligned} \tag{17.2.2}$$

where $Y = g(\alpha+i\gamma_5\beta)\bar{\varphi}$. There is no factor $\frac{1}{2}$ because the fields ψ and $\bar{\psi}$ are different, and there is an overall minus sign for the closed loop of fermions. In the numerators we set $m = 0$, and then only terms with an even number of vertices contribute to the trace over Dirac matrices. This reintroduces an overall factor 1/2, just as in the case of scalars. Using

$$Y\frac{-\not{k}}{(k^2+m^2)}Y\frac{-\not{k}}{(k^2+m^2)} = YY^\dagger\frac{k^2}{(k^2+m^2)^2} = g^2(\alpha^2+\beta^2)\frac{k^2}{(k^2+m^2)^2} \tag{17.2.3}$$

and replacing $k^2(k^2+m^2)^{-2}$ by $(k^2+m^2)^{-1}$ for small m^2 we obtain

$$V = -i\hbar \sum_{p=1}^{\infty} \frac{1}{2p} \int tr[g^2(\alpha^2+\beta^2)\bar{\varphi}^2/(k^2+m^2)]^p \frac{d^4k}{(2\pi)^4} \tag{17.2.4}$$

The net result is then obtained from the result for $\lambda\varphi^4$ theory by (i) adding an overall minus sign, (ii) replacing $-\frac{\lambda}{2}\bar{\varphi}^2$ by $g^2(\alpha^2+\beta^2)\bar{\varphi}^2$ and (iii) tracing over the indices a of ψ^a. This leads to

$$\begin{aligned} V^{(1),\mathrm{ren}}_{\mathrm{fermions}}(\bar{\varphi}) &= -\hbar\frac{1}{(8\pi)^2} tr\left[(YY^\dagger)^2 \ln \frac{YY^\dagger}{M^2}\right] + \text{ renorm. terms} \\ &= -\hbar \frac{g^4}{(8\pi)^2} tr(\alpha^2+\beta^2)^2 \left[\bar{\varphi}^4 \ln\frac{\bar{\varphi}^2}{M^2} - \frac{25}{6}\right] \end{aligned} \tag{17.2.5}$$

As an application consider a particular susy system, the Wess-Zumino model. The action reads

$$\begin{aligned} \mathcal{L} &= -\frac{1}{2}(\partial_\mu A)^2 - \frac{1}{2}(\partial_\mu B)^2 - \frac{1}{2}\bar{\psi}\partial\!\!\!/\psi + \frac{1}{2}F^2 + \frac{1}{2}G^2 \\ &+ m\left(FA + GB - \frac{1}{2}\bar{\psi}\psi\right) + g[F(A^2-B^2) + 2GAB - \bar{\psi}(A - i\gamma_5 B)\psi] \end{aligned} \tag{17.2.6}$$

It is easier to work with complex fields $\varphi = (A+iB)/\sqrt{2}$ and $\mathcal{F} = (F-iG)\sqrt{2}$. Then

$$\begin{aligned} \mathcal{L} &= -\partial_\mu\varphi^*\partial^\mu\varphi - \frac{1}{2}\bar{\psi}\partial\!\!\!/\psi + \mathcal{F}^*\mathcal{F} + m\left(\mathcal{F}\varphi + \mathcal{F}^*\varphi^* - \frac{1}{2}\bar{\psi}\psi\right) \\ &+ g\sqrt{2}\left[\mathcal{F}\varphi^2 + \mathcal{F}^*\varphi^{*2} - \bar{\psi}\left\{\varphi^*\left(\frac{1+\gamma_5}{2}\right) + \varphi\left(\frac{1-\gamma_5}{2}\right)\right\}\psi\right] \end{aligned} \tag{17.2.7}$$

Eliminating the auxiliary fields by their classical field equation $\mathcal{F}^* = -m\varphi - g\sqrt{2}\varphi^2$ and setting $m = 0$ we obtain[3]

$$\mathcal{L}^{int} = -2g^2(\varphi^*\varphi)^2 - g\sqrt{2}\bar{\psi}\left\{\varphi^*\left(\frac{1+\gamma_5}{2}\right) + \varphi\left(\frac{1-\gamma_5}{2}\right)\right\}\psi \tag{17.2.8}$$

Auxiliary fields such as $\mathcal{F}$ and $\mathcal{F}^*$ play an important role in supersymmetry, but for the reader who is not familiar with them, their role at the quantum level is often

[3]One might wonder whether using the quantum field equations of $\mathcal{F}$ would lead to a different result. Since the classical potential can be written as $-|\mathcal{F} + m\varphi^* + g\sqrt{2}\varphi^{*2}|^2 + |m\varphi^* + g\sqrt{2}(\varphi^*)^2|^2$, it is clear that eliminating $\mathcal{F}$ by using the quantum field equation will only lead to corrections of order $\hbar^2$.

confusing. It should make no difference whether one eliminates the auxiliary fields at the classical level using the classical field equations, or at the quantum level using the field equations of the effective action. We shall explicitly perform calculations with and without auxiliary fields, and check that the results agree.

With auxiliary fields present, we decompose $\mathcal{F}$ and φ into constant background parts and quantum fields

$$\mathcal{F} = < \mathcal{F} > + \mathcal{F}_{qu}, \varphi = < \varphi > \ + \varphi_{qu} \tag{17.2.9}$$

We write the terms quadratic in the bosonic quantum fields φ_{qu} and F_{qu} in matrix form as follows $(\varphi_{qu}, \varphi^*_{qu}, \mathcal{F}_{qu}, \mathcal{F}^*_{qu}) M (\varphi^*_{qu}, \varphi_{qu}, \mathcal{F}^*_{qu}, \mathcal{F}_{qu})^T$. Then the one-loop contribution from the bosonic loops fields is given by $(\det M)^{-1}$

$$\det{}^{-1} \begin{bmatrix} -\frac{1}{2}k^2 & g\sqrt{2} < \mathcal{F} > & 0 & \sqrt{2}g < \varphi > \\ g\sqrt{2} < \mathcal{F}^* > & -\frac{1}{2}k^2 & \sqrt{2}g < \varphi^* > & 0 \\ 0 & \sqrt{2}g < \varphi > & \frac{1}{2} & 0 \\ \sqrt{2}g < \varphi^* > & 0 & 0 & \frac{1}{2} \end{bmatrix} \tag{17.2.10}$$

Using $\det \binom{AB}{CD} = \det(A - BD^{-1}C) \det D$, this becomes (dropping from now on the expectation brackets $< >$)

$$\det{}^{-1} \begin{pmatrix} \frac{1}{2}k^2 + 4g^2\varphi^*\varphi & -g\sqrt{2}\mathcal{F} \\ -g\sqrt{2}\mathcal{F}^* & \frac{1}{2}k^2 + 4g^2\varphi^*\varphi \end{pmatrix} \tag{17.2.11}$$

On the other hand, without auxiliary fields the one-loop determinant reads

$$\det{}^{-1} \begin{pmatrix} \frac{1}{2}k^2 + 4g^2\varphi^*\varphi & 2g^2\varphi^{*2} \\ 2g^2\varphi^2 & \frac{1}{2}k^2 + 4g^2\varphi^*\varphi \end{pmatrix} \tag{17.2.12}$$

Clearly, substituting the classical field equation $< \mathcal{F} >^* = -g\sqrt{2} < \varphi^2 >$ into the first determinant yields the second one. Writing in both cases the determinant as

$$\det{}^{-1}(k^2 + \mathcal{M}^2) \tag{17.2.13}$$

the one-loop correction due to bosonic loops with auxiliary fields still present are proportional to

$$tr\mathcal{M}^4 \ln \frac{\mathcal{M}^2}{M^2}, \mathcal{M}^2 = \begin{pmatrix} 8g^2\varphi^*\varphi & -2g\sqrt{2}\mathcal{F} \\ -2g\sqrt{2}\mathcal{F}^* & 8g^2\varphi^*\varphi \end{pmatrix} \tag{17.2.14}$$

Without auxiliary fields one must replace $\mathcal{F}^*$ by $-g\sqrt{2}\varphi^2$ (and $\mathcal{F}$ by $-g\sqrt{2}\varphi^{*2}$).

Diagonalizing the system without auxiliary fields yields

$$\mathcal{M}^2_{\text{diag}} = \begin{pmatrix} 12g^2\varphi^*\varphi & 0 \\ 0 & 4g^2\varphi^*\varphi \end{pmatrix} \tag{17.2.15}$$

and one clearly recovers the factor 3 between A-loops and B loops which we already encountered in the example of complex scalars coupled to electromagnetism. With auxiliary fields still present, the nonrenormalization theorems of $N = 1$ susy tell us that no terms which only depend on $\mathcal{F}$ and φ (or only on $\mathcal{F}^*$ and φ^*) can be generated. This is clearly the case. With auxiliary fields present, we determine the eigenvalues of $\mathcal{M}^2$ and then take the trace. This yields

$$\begin{aligned} V^{(1)} &= tr(8g^2\varphi^*\varphi + \sqrt{8g^2\mathcal{F}^*\mathcal{F}})^2 \ln(8g^2\varphi^*\varphi + \sqrt{8g^2\mathcal{F}^*\mathcal{F}}) \\ &+ tr(8g^2\varphi^*\varphi - \sqrt{8g^2\mathcal{F}^*\mathcal{F}})^2 \ln(8g^2\varphi^*\varphi - \sqrt{8g^2V^*\mathcal{F}}) \end{aligned} \tag{17.2.16}$$

After inserting the field equation $\mathcal{F}^* = -2\sqrt{g}\varphi^2$ we find the same one-loop potential as for the theory without auxiliary fields.

The final result is

$$V^{(1),\text{ren}} = \frac{\hbar}{(8\pi)^2}(trX^4 \ln\frac{X^2}{M^2} - trY^4 \ln\frac{Y^2}{M^2}) \tag{17.2.17}$$

where $X^2 = \begin{pmatrix} MM^\dagger & <-\mathcal{F}> \\ <-\mathcal{F}^*> & M^\dagger M \end{pmatrix}$ is due to bosonic loops and $Y^2 = \begin{pmatrix} MM^\dagger & 0 \\ 0 & M^\dagger M \end{pmatrix}$ is due to fermionic loops. Clearly, if susy is unbroken (which implies $<\mathcal{F}>= 0$), there is no one-loop correction to the potential.

3 The mass of the Higgs boson

At the tree graph level, the mass of the Higgs boson follows from the classical potential $V = \frac{1}{2}m^2\varphi_1^2 + \frac{\lambda}{24}\varphi_1^4$, and is given by ${m_H}^2 = -2m^2 = \frac{1}{3}\lambda v^2$. Experimentally, $v \sim 250\ GeV$, but λ is undetermined, and by letting λ tend to zero, the Higgs mass can be made arbitrarily small. Adding the one-loop correction to the effective potential, if λ becomes too small, the one-loop corrections take over, and this leads to a lower bound on the Higgs mass as we now show.

The relevant interactions of the Higgs field are given by

$$\begin{aligned}\mathcal{L}^{int} &= -\left(g_2^2 W_\mu{}^+ W^{\mu,-} + \frac{1}{2}(g_1^2+g_2^2) Z_\mu Z^\mu\right)\varphi_1{}^2 \\ &- \sum_f \left(\frac{m_f}{v}\right)\bar{\psi}_f \psi_f \varphi_1 - m^2\phi^\dagger\phi - \lambda(\phi^\dagger\phi)^2 \end{aligned} \tag{17.3.1}$$

where g_2 and g_1 are the SU(2) and $U(1)$ coupling constants and $\phi = \{\frac{(\varphi_1+i\varphi_2)}{\sqrt{2}}, \varphi^-\}$. The one-loop corrections to the effective potential are then given by

$$\begin{aligned} V^{(1)}(\varphi_1) &= B\varphi_1^4\left(\ln\frac{\varphi_1^2}{M^2} - \frac{25}{6}\right) \\ B &= \frac{\hbar}{(8\pi)^2}\left[3\left\{g_2{}^4 + g_2{}^4 + \frac{1}{4}(g_1{}^2+g_2{}^2)^2\right\} - 2\sum_f\left(\frac{m_f}{v}\right)^4 + \left(\frac{1}{2}\lambda\right)^2\left\{1+\frac{1}{3}\right\}\right] \end{aligned} \tag{17.3.2}$$

The factor 3 is due to the Landau gauge for the gauge bosons W^+, W^- and Z, the factor -2 is due to taking all fermions chiral, and in the scalar sector the 3 real bosons in φ_2 and φ^- yield each 1/9 of the contribution of φ_1.

The minimum of the potential is at v, where $\frac{\partial}{\partial\varphi_1}V(v) = 0$ and the Higgs mass is $V''(v)$. However, in order that the minimum at $\varphi_1 = v$ be lower than the value of the effective potential at the origin, $V(\varphi_1 = 0) = 0$, we have a stability requirement

$$V(v) < 0 \tag{17.3.3}$$

Solving $Bv^2 \ln\frac{v^2}{M^2}$ from $\frac{\partial}{\partial v}V(v) = 0$ yields

$$Bv^2\ln\frac{v^2}{M^2} = -\frac{1}{4}m^2 - \frac{1}{4!}\lambda v^2 + \frac{11}{3}Bv^2 \tag{17.3.4}$$

and solving v^2 from $m_H{}^2 = V''(v^2)$ yields

$$m_H{}^2 = -2m^2 + 8Bv^2 \tag{17.3.5}$$

Using these results in the constraint $V(v) < 0$ leads to $m^2 - 2Bv^2 < 0$, which can be rewritten as

$$
{m_H}^2 \geq 4Bv^2 \tag{17.3.6}
$$

How interesting is this bound? In terms of masses it reads

$$
\begin{aligned}
v^2 {m_H}^2 \geq & \quad \frac{1}{16\pi^2}\left[6{m_W}^4 + 3{m_Z}^4 - 2\sum_f {m_f}^4 + 3\left(\frac{1}{3}\lambda v^2\right)^2\right] \\
& \geq \frac{1}{16\pi^2}\left[6{m_W}^4 + 3{m_Z}^4 - 2\sum_f {m_f}^4\right]
\end{aligned} \tag{17.3.7}
$$

Before the top quark was discovered, this inequality only involved the W and Z bosons and yielded $m_H \gtrsim 7\ GeV$. The top quark with a mass 170 GeV makes the bound on the Higgs mass uninteresting.

4 Gauge-choice dependence of the effective potential

The effective potential is the generator of proper Green's functions with vanishing external momenta (constant fields), and because it is a Green's function, it is in general dependent on the choice of gauge. The minimum value of the effective action has the meaning of the energy of the vacuum, and should clearly be gauge-choice independent. Another quantity of great interest is the expectation value of the Higgs field in the Standard Model which leads to the W and Z masses. Is it gauge-choice independent? We shall determine which parts of the effective action are gauge-choice independent.

Consider as an example the abelian Higgs model with 't Hooft gauge fixing term

$$
\begin{aligned}
\mathcal{L} = & - \frac{1}{4}F_{\mu\nu}^2 - D_\mu\varphi^* D^\mu\varphi + m^2\varphi^*\varphi - \frac{\lambda}{6}(\varphi^*\varphi)^2 \\
& - \frac{1}{2\xi}(\partial^\mu A_\mu - \xi e v\chi)^2 + b(\partial_\mu\partial^\mu c - \xi e v\sigma c)
\end{aligned} \tag{17.4.1}
$$

where $D_\mu\varphi = \partial_\mu\varphi - ieA_\mu\varphi$ and $\varphi = (\sigma + i\chi)/\sqrt{2}$. The classical gauge invariance is $\delta\varphi = ie\alpha(x)\varphi$, or $\delta\sigma = -e\alpha(x)\chi, \delta\chi = e\alpha(x)\sigma$ and $\delta A_\mu = \partial_\mu\alpha(x)$. The parameter v in the gauge fixing and ghost terms is at this point an arbitrary parameter, but later we shall set it equal to $<\sigma>= v$. (The classical interaction

$$\mathcal{L}^{int} = ie(\partial_\mu\varphi^*\varphi - \varphi^*\partial_\mu\varphi)A^\mu = e(\sigma\partial_\mu\chi - \chi\partial_\mu\sigma)A^\mu \tag{17.4.2}$$

leads after shifting $\sigma = \sigma' + v$ to an off-diagonal kinetic term which is canceled by a similar term from the gauge fixing term). The effective potential is

$$V = -\frac{1}{2}m^2(\sigma^2+\chi^2) + \frac{\lambda}{4!}(\sigma^2+\chi^2)^2 + \frac{1}{2\xi}(ev)^2\chi^2 + b\xi ev\sigma c \tag{17.4.3}$$

and is clearly gauge-choice (ξ) dependent. The minimum is reached when the field equations are satisfied. They read

$$\sigma\left\{-m^2 + \frac{\lambda}{6}(\sigma^2+\chi^2)\right\} = 0; \chi\left\{-m^2 + \frac{\lambda}{6}(\sigma^2+\chi^2) + \frac{1}{\xi}(ev)^2\right\} = 0 \tag{17.4.4}$$

One solution (the usual one) is

$$<\chi> = 0, <\sigma>^2 = 6mn^2/\lambda \tag{17.4.5}$$

This solution is gauge-choice independent. Another solution (a spurious one as we shall see) is

$$<\sigma> = 0, <\chi>^2 = 6m^2/\lambda - \frac{1}{\xi}(ev)^2 \tag{17.4.6}$$

This solution is clearly ξ-dependent.

BRST invariance of the quantum theory requires that gauge-fixing terms should vanish at infinity as we now show. If $\partial^\mu A_\mu - \xi ev\chi$ is to vanish at infinity for all field configurations, χ should vanish in the vacuum, and this rules out the spurious solution. To show that the gauge-fixing terms should vanish at infinity we use the auxiliary field of BRST symmetry. Then

$$\begin{aligned} \mathcal{L}(\text{fix}) &= \frac{1}{2}\xi d^2 - d(\partial^\mu A_\mu - \xi ev\chi) \\ \delta_{BRST}b &= d, \delta_{BRST}d = 0 \end{aligned} \tag{17.4.7}$$

In operator form, $\{Q_{BRST}, b\} = d, \{Q_{BRST}, d\} = 0$, and hence

$$< 0 \mid \{Q_{BRST}, b\} \mid 0 >= 0 =< 0 \mid F \mid 0 > \tag{17.4.8}$$

where F is the gauge fixing term. Hence, BRST invariance of the vacuum, $Q_{BRST} \mid 0 >= 0$, implies that the gauge fixing terms have vanishing vacuum expectation value, and since $< A_\mu >= 0$, this implies that $< \chi> = 0$.

Some gauges are "bad gauges". An example is $\mathcal{L}(\text{fix}) = \frac{1}{2}\xi d^2 - d(\partial^\mu A_\mu + \beta\sigma)$. Then $< d >= 0$ requires

$$< \partial^\mu A_\mu + \beta\sigma >= 0 \tag{17.4.9}$$

and if $< \sigma >\neq 0$, this is only possible for x-dependent $< A_\mu >$. This bad gauge then cannot preserve both BRST symmetry and translational invariance.

Since the sum of gauge-fixing and ghost terms is BRST exact,

$$S(\text{fix}) + S(\text{ghost}) = \int \delta_{BRST}[b(F + \xi d)]d^4x \tag{17.4.10}$$

while physical states are BRST inert (in particular, the vacuum is BRST invariant, since it is by definition the physical state with lowest energy) it follows that $< 0 \mid S(\text{fix}) + S(\text{ghost}) \mid 0 >= 0$. Hence at tree graph level (the classical theory) the gauge-fixing and ghost terms do not contribute to the vacuum energy. (Heisenberg fields satisfy the field equations.)

The action

$$\mathcal{L} = \mathcal{L}(\text{class}) + \frac{1}{2}\xi d^2 + d(\partial^\mu A_\mu - \xi ev\chi) + b(\Box c - \xi e^2 v\sigma c) \tag{17.4.11}$$

satisfies

$$\frac{\partial}{\partial\xi}\mathcal{L} = \frac{1}{2}d^2 - dev\chi - be^2 v\sigma c = \delta_B b(\frac{1}{2}d - ev\chi) \tag{17.4.12}$$

Hence

$$< 0 \mid \frac{\partial}{\partial\xi}\mathcal{L} \mid 0 >=< 0 \mid \frac{\partial}{\partial\xi}\left\{Q_B, \frac{1}{2}d - ev\chi\right\} \mid 0 >= 0 \tag{17.4.13}$$

Hence, the gauge-artefacts do not contribute to the vacuum energy at the classical level. So, the classical potential is gauge-choice independent **at its minimum** (the Heisenberg fields satisfy the field equations).

To prove the same statement at the quantum level we use path integrals. We couple not only external sources to the fields and to the nonlinear parts of the BRST transformations, but also to $b(\frac{1}{2}d - ev\chi)$

$$\begin{aligned} \mathcal{L} &= \mathcal{L}(\text{class}) + \mathcal{L}(\text{fix, ghost}) + J^\mu A_\mu + j_\sigma \sigma + j_\chi \chi \\ &+ j_d d + i\bar{\eta} c + b\eta + K_\sigma(-e\chi c) + K_\chi(e\sigma c) + Mb\left(\frac{1}{2}d - ev\chi\right) \quad (17.4.14) \end{aligned}$$

All fields are hermitian, except the antighost b which is antihermitian, and all terms in the action are hermitian. Then we find in the usual way the Ward identity for connected graphs

$$\begin{aligned} 0 &= J^\mu < \delta A_\mu > + j_\sigma < \delta\sigma > + j_\chi < \delta\chi > + i\bar{\eta} < \delta c > - < \delta b > \eta + M\delta_B b\left(\frac{1}{2}d - ev\chi\right) \\ &= J^\mu \partial_\mu \frac{1}{i}\frac{\partial}{\partial\bar{\eta}}W + j_\sigma \frac{\delta}{\delta K_\sigma}W + j_\chi \frac{\delta}{\delta K_\chi}W + i\frac{\delta}{\delta j_d}W\eta + M\delta_B b\left(\frac{1}{2}d - e\chi\right) = 0 \quad (17.4.15) \end{aligned}$$

Since $\frac{\delta S}{\delta \xi} = \delta_B\{b(\frac{1}{2}d - e\chi)\}$, we can rewrite this Ward identity as follows

$$\begin{aligned} \int \left[M(x) < \frac{\delta\mathcal{L}}{\delta\xi} >\right] \delta^4 x = \int \Big[&-J^\mu \partial_\mu \frac{\partial}{\partial\bar{\eta}}W \\ &- j_\sigma \frac{\delta}{\delta K_\sigma}W - j_\chi \frac{\delta}{\delta K_\chi}W + \frac{\delta}{\delta j_d}W\eta \Big] d^4 x \quad (17.4.16) \end{aligned}$$

Taking the $\frac{\partial}{\partial M(y)}$ derivative, and then integrating over y leads to

$$\begin{aligned} < \frac{\partial S}{\partial\xi} > = \frac{\partial}{\partial\xi}W = \int \Big[&-J^\mu(x)\frac{\partial}{\partial x^\mu}\frac{\partial^2 W}{\partial\bar{\eta}(x)\partial M(y)} - j_\sigma(x)\frac{\delta^2 W}{\partial K_\sigma(x)\partial M(y)} \\ &- j_\chi(x)\frac{\delta^2 W}{\partial K_\chi(x)\delta M(y)} + \frac{\delta^2 W}{\delta j_d(x)\delta M(y)}\eta(x)\Big] d^4x d^4y \quad (17.4.17) \end{aligned}$$

After a Legendre transformation from W to the effective action Γ, $\frac{\delta}{\delta\xi}W = \frac{\delta}{\delta\xi}\Gamma$ and we obtain

$$\begin{aligned} \frac{\partial\Gamma}{\partial\xi} = \int \Big[&\frac{\delta\Gamma}{\delta A_\mu(x)}\frac{\partial}{\partial x^\mu}\frac{\delta c(x)}{\delta M(y)} - \frac{\delta\Gamma}{\delta\sigma(x)}\frac{\delta^2\Gamma}{\delta K_\sigma(x)\delta M(y)} \\ &- \frac{\delta\Gamma}{\delta\chi(x)}\frac{\delta^2\Gamma}{\delta K_\chi(x)\delta M(y)} + \frac{\delta d(x)}{\partial M(y)}\frac{\delta\Gamma}{\delta b(x)}\Big] d^4x d^4y \quad (17.4.18) \end{aligned}$$

Considering now constant external σ and vanishing external other fields, Γ becomes the effective potential, and we arrive at

$$\frac{\partial}{\partial\xi}V^{eff} = \frac{\delta V^{eff}}{\delta\sigma}\int\frac{\delta^2\Gamma}{\delta K_\sigma(0)\delta M(y)}d^4y \equiv \frac{\delta V^{eff}}{\delta\sigma}C(\sigma,\xi) \tag{17.4.19}$$

So clearly, at the minimum of the effective potential where $\frac{\delta}{\delta\sigma}V^{eff} = 0$, we have gauge-choice independence:

$$\frac{\partial}{\partial\xi}V^{eff}(\min) = 0 \tag{17.4.20}$$

Furthermore, if we vary ξ and σ simultaneously according to

$$\frac{\partial\sigma}{\partial\xi} = C(\sigma,\xi) \tag{17.4.21}$$

then V^{eff} is everywhere invariant

$$\frac{d}{d\xi}V_{eff} = 0 \tag{17.4.22}$$

The expectation value of $\sigma(x)$ was ξ independent at tree level

$$<\sigma^2>= gm^2/\lambda \tag{17.4.23}$$

However, quantum corrections can bring in ξ dependence, because one must modify $\sigma = \sigma' + v$ at each loop level in order that tadpoles remain zero

$$\frac{\partial}{\partial\sigma'}\Gamma(\sigma')\mid_{\sigma'=0}= 0 \tag{17.4.24}$$

This ξ-dependence of v can also be derived from the Ward identity for the effective potential

$$\begin{aligned}\frac{\delta W}{\delta j_\sigma(x)} &= <\sigma(x)>= v\\ \frac{\partial v}{\partial\xi} &= \frac{\delta}{\delta j_\sigma(x)}\left(\frac{\delta W}{\delta\xi}\right)\Big|_{J=j=\eta=0} = \int\frac{\delta^2\Gamma}{\delta M(x)\delta K_\sigma(0)}d^4x = C(\sigma,\xi)\end{aligned} \tag{17.4.25}$$

This is precisely (17.4.21), hence the effective potential is gauge-choice independent, see (17.4.22). However, $<\sigma(x)>= v$ itself is in general gauge dependent! Clearly, we cannot define the mass of the W boson by ev at the quantum level after taking loop corrections to v into account such that tadpoles vanish. Nor should one use such a definition: the physical mass of the Z boson is determined not by ev, but by the pole in the propagator, and this definition of the physical mass is gauge-choice independent. What happens is this: the propagator $\Pi_{\mu\nu}(k)$ decomposes into a transversal part (proportional to $\eta_{\mu\nu} - k_\mu k_\nu / m_Z^2$). The physical mass is given by the pole in the former and is gauge-choice independent, whereas the longitudinal part of the propagator is gauge-choice dependent but not used to define the physical mass.

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Chapter 18

Finite temperature field theory

The early universe contained matter at a very high density and radiation at a very high temperature. As it expanded and cooled down, phase transitions probably occurred which led to spontaneous symmetry breaking of certain gauge field theories, for example from $SU(5)$ to $SU(3) \times SU(2) \times U(1)$, and later from $SU(2) \times U(1)$ to $U(1)_{em}$. To describe these situations a field theory at finite temperature and finite density is needed. We shall compute the temperature at which phase transitions occur; in a simple toy model we obtain $T_c = 6.10^{15}$ degrees Kelvin (corresponding[1] to 500 GeV). Lattice QCD has estimated that the quark deconfining temperature lies around 150-200 MeV. Also in relativistic heavy ion collisions, many nucleons interact with each other and are squeezed together, and again a finite temperature field theory which also takes density of matter into account (more precisely, density of matter minus density of antimatter) may be used to describe this quark-gluon plasma. Finally, in astrophysics there are situations with high density (white dwarfs, neutron stars) and high temperature (supernovas), and also in these cases finite temperature and finite density field theory is used. Some textbooks which discuss finite temperature physics are given in [1–8].

[1] The Boltzmann constant k equals 1.39 $\times 10^{-16}$ erg/K $\simeq 10^{-4}$ eV/K so that room temperature corresponds to 1/40 eV.

The simplest situation to study is a system in thermodynamical equilibrium; then the occupation number of states is given by the Boltzmann factor $\exp -\beta E$ with $\beta = (kT)^{-1}$. We shall call β the inverse temperature. To keep the discussion as simple as possible, we shall not consider curved space, although gravity played a big role in the early universe. We shall first be interested in high temperature systems for which canonical ensembles with $\exp -\beta\hat{H}$ are sufficient.[2] To describe systems with a nonvanishing density of matter one uses grand canonical ensembles, with $-\beta(\hat{H} - \mu\hat{N})$ in the exponent where $\hat{N}$ is a number operator for particles minus the number operator for antiparticles, and μ is the Gibbs (chemical) potential. This may be used for example, in relativistic heavy ion collisions where the density of nucleons is much higher than that given by blackbody radiation. Since the baryon current is conserved, one can fix the density of nucleons by choosing a suitable value for μ (the density of nucleons is the number of nucleons minus the number of antinucleons; each is, of course, constant and nonvanishing at a given temperature).

Adding the factor $\exp -\beta\hat{H}$, or $\exp -\beta(\hat{H} - \mu\hat{N})$, in the usual Green's functions and taking the average over a canonical, or grand canonical ensemble, the resulting theory is a combination of relativistic field theory and statistical mechanics, sometimes called statistical field theory, or thermal field theory, or field theory at finite temperature. We shall not discuss nonrelativistic field theory at finite temperature. Nor shall we discuss nonequilibrium phenomena.

Finite temperature (and finite density) field theory (FTF) started out in the 1950's as a nonrelativistic program based on quantum mechanics under the name "the many-body problem" (it was called "the many-body problem" or "many-particle systems" because it was mainly used in condensed matter physics and in nuclear physics). An extension to relativistic field theory was achieved by Fradkin in 1965 [9].

[2] Blackbody radiation by itself leads to a definite density of matter and an equal density of antimatter. For systems with different densities of matter and antimatter one needs the formalism with a chemical potential. The integrated density of matter minus the integrated density of antimatter should be the charge corresponding to a conserved current in order that it be time independent.

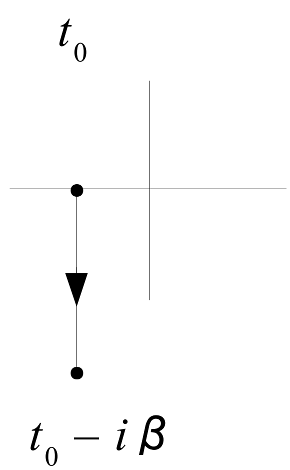
In 1972, Kirzhnitz and Linde [10] applied FTF to electroweak phase transitions in the context of the Big-Bang (thus at high T), see also S. Weinberg [11], and Dolan and Jackiw [12]. Somewhat later, Collins and Perry [13] applied it to neutron stars using QCD at high T and high density; the high temperature ($\mu, kT >> \Lambda_{QCD}$) allowed them to use asymptotic freedom to do perturbative calculations. In 1981, Gross, Pisarski and L. Jaffe [14], and earlier Kalashnikov and Klimov in the USSR (1980), and later Weldon [15] and others applied FTF more extensively to QCD. However, by the end of the 80's, it was realized that there are serious infrared divergencies in FTF and gauge dependence in supposedly physical quantities. A major break through was made by Braaten and Pisarski [16] who reorganized the perturbation theory such that apparent inconsistencies were avoided, and introduced the concept of hard thermal loops (HTL), subsequently extended to hard dense loops (HDL). Hard thermal loops and heard dense loops are concepts in equilibrium physics, but also in non-equilibrium physics hard loops have recently been considered [17].

By the mid 1990's, FTF came back into the limelight because one studied the electroweak phase transition with "sphalerons" (a kind of unstable solitons), which were used to describe anomalous baryon-violation processes in Grand Unified Theories. It was hoped that these studies would explain that all observed baryon excess in our world is due to these CP violating processes. Many numerical studies were made, assuming that a first-order phase transition occurred; however, as the predicted value of the Higgs mass increased near the end of the 1990's, the likelihood of such phase transitions declined, and at present the only hope in this direction are susy models where more Higgs particles are present. The main interest at present in FTF lies in QCD, namely in the relativistic heavy ion collisions (RHIC) observed at Brookhaven National Laboratory (BNL) on Long Island and to be observed at the Large Hadron Collider (LHC) at CERN near Geneva, and further in cosmology, for example the matter-antimatter problem and neutron stars [18].

Over time three approaches to finite temperature field theory have emerged.

These are: the imaginary-time formalism, the real-time formalism and thermal field dynamics (also called thermo field dynamics).

In the imaginary-time formalism one makes a Wick rotation of the time coordinate and interprets the Wick-rotated time as the temperature: $t \to -i\beta$ (where $\beta = 1/kT$). This formalism is very well suited to compute physical quantities which are the same in Minkowski space as in Euclidean space, so static quantities, in particular the effective potential, but also equilibrium Green functions can be computed although in this case one must make an analytical continuation to Minkowski space [19] which is nontrivial [20]. (The external fields of the effective potential carry vanishing four-momenta, hence they are independent of the space and time coordinates and thus a Wick rotation does not change the effective potential.) To compute thermal Green's functions $Tre^{-\beta\hat{H}}T\varphi(x_1)\ldots\varphi(x_n)/Tre^{-\beta\hat{H}}$ one converts this operator expression into a Euclidean path integral on the Euclidean time interval $[0, \beta]$ by inserting as usual complete sets of states at intermediate times. The trace leads then to periodic boundary conditions for bosons (or antiperiodic boundary conditions for fermions, as we shall show). The action in this path integral is $\int_0^\beta \mathcal{L} dt$ where t is the Euclidean time. In terms of the original Minkowski time coordinate this implies that one considers a straight contour from $t = t_0$ to $t = t_0 - i\beta$ in the complex t plane, the Matsubara contour. On this contour fields are still time-ordered, or rather "contour-ordered".

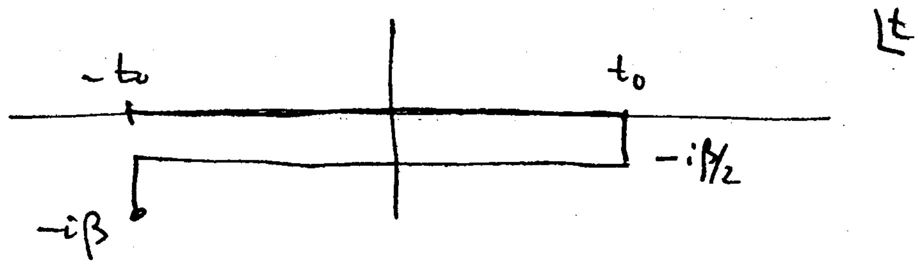


(18.0.1)

Figure caption: the Matsubara contour.

The imaginary-time formalism by itself corresponds to statistical mechanics, and hence the integrands of the Euclidean path integrals are to be interpreted as probabilities. If, on the other hand, one starts with a field theory in Minkowski spacetime, one may still use the imaginary-time formalism to compute Wick-rotated matrix elements, but these are then amplitudes and not probabilities.

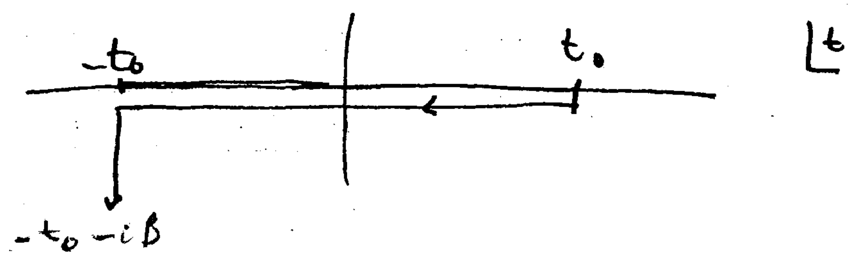
The real-time formalism is a path integral formalism with both an ordinary Minkowski time and a temperature. In this formalism one can calculate the same quantities as in the imaginary time formalism, but in addition one can study non-equilibrium phenomena. The Minkowski time still begins at $t = t_0$ and ends at $t = t_0 - i\beta$ but translational invariance of the thermal Green's functions allows us to choose any real initial point t_0 for C, and it is convenient to choose a negative time $-t_0$ as initial point. Instead of a straight contour from t_0 to $t_0 - i\beta$ one may choose any contour with these points as endpoints. The contour is then a curve in the complex $t-$ plane but the fields on this contour are still real. So the fact that the amplitudes do not depend on the choice of contour is not a kind of Cauchy theorem, but it is due to analytic continuation of the time coordinates, and at the end one should still continue back. In the imaginary time formalism all points have the same real part, and analytic continuation to real Minkowski times which are different is much more difficult. For example there are ambiguities to be fixed due to factors $\exp i\beta\omega_n$ which are unity before the analytic continuation [31]). A contour C in the complex t-plane starts at $t = -t_0$, moves to $+t_0$, then goes down to $t_0 - i\beta_1$, turns left to $-t_0 - i\beta_1$, and finally goes down again to end at $-t_0 - i\beta$. It can be shown that in the limit of $t_0 \to \infty$ only the two horizontal segments of the contour contribute and the propagator can then be written as a 2×2 matrix. An elegant choice (mostly used in thermal field dynamics, see below) is $\beta_1 = \frac{1}{2}\beta$; in this case the propagator matrix is symmetric.



(18.0.2)

Figure caption: the contour of thermal field dynamics.

However, for nonequilibrium processes the choice $\beta_1 = 0$ is much simpler; this corresponds to the original Schwinger-Keldysh contour [21] (although these authors did not take the vertical pieces of the contour into account).



(18.0.3)

Figure caption: the Schwinger-Keldysh contour.

It is sometimes stated that the first horizontal segment (on which increasing time moves to the right) contains the time coordinates of the physical fields whose thermal Green's function are computed, while the second segment corresponds to fields which describe the heat bath. Hence there is field-doubling. There are then four Feynman propagators: $D_F^{11}(x-y), D_F^{12}(x-y), D_F^{21}(x-y)$ and $D_F^{22}(x-y)$, depending on whether x^0 and/or y^0 lie on segment 1 or segment 2. The vertices for fields from segment 2 are the complex-conjugated of the original vertices (with factors $-ig$ instead of ig). Obviously, this situation resembles the cutting rules for unitarity, and there is indeed a close connection, as becomes clear in discussions of unitarity at finite temperature.

The third approach to thermal field theory is called thermo field dynamics. It is based on operator methods instead of path integrals. It provides a canonical formalism with a thermal Hilbert space, thermal vacuum and thermal creation and absorption operators. It is really the operator formulation of the real-time path integral formalism, and, as expected, its results coincide with those obtained from the real-time formalism. However, it clarifies some issues of the real-time formalism which are less clear in the path integral approach, such as the correct vacuum and field doubling: for example, as we shall show, field doubling is needed (in the real-time formalism but not in the Matsubara formalism) if one wants to define inner products in the Hilbert space of thermal states.

We shall first derive the propagators of the imaginary-time formalism and compute the thermal mass of a real scalar field in $\lambda\varphi^4$ theory. Next we calculate the critical temperature for the phase transition in a model with scalar fields and spontaneous symmetry breaking. This is followed by a discussion of gauge theories and fermions at finite temperature. In particular the treatment of ghosts at finite temperature is studied, and using BRST methods we prove that ghosts should be periodic in time, like gauge fields but unlike physical fermions. Then we consider supersymmetry at finite temperature and compute the contribution of fermions to the critical temperature. To this purpose we evaluate in each case the one-loop corrections to the effective potential. Finally we briefly discuss the real-time formalism. To clarify the problems about Hilbert spaces, inner products and Goldstone fermions, we work through explicit examples with bosonic and fermionic harmonic oscillators. We begin this chapter with an elementary discussion of the relation between field theory and thermodynamics where we show that the effective potential is equal to the Gibbs free energy.

1 Elements of thermodynamics

Since thermodynamics [22] is for many particle physicists only a discipline they studied as students, we rederive in this section those few results we will need. In thermodynamics the probability that a quantum system in thermal equilibrium be in an eigenstate $|n>$ of the Hamiltonian is given by the Boltzmann factor

$$p_n = \frac{1}{Z(\beta)} e^{-\beta E_n}, \ \beta = (kT)^{-1} \tag{18.1.1}$$

Since the sum of all probabilities should be one, the normalization factor $Z(\beta)$ is given by

$$Z(\beta) = \sum_n \exp -\beta E_n = \sum_n < n| \exp -\beta \hat{H} |n >= Tr \exp -\beta \hat{H} \tag{18.1.2}$$

The function $Z(\beta)$ is called the partition function. For an observable A we then define the expectation value at a temperature β by

$$< A >_\beta = \sum p_n < n|\hat{A}|n >= \frac{Tr \hat{A} e^{-\beta \hat{H}}}{Tr e^{-\beta \hat{H}}} \tag{18.1.3}$$

The averaging in $< n|\hat{A}|n >$ is the quantum mechanical averaging of expectation values which is also present at zero temperature, but the averaging in $< A >_\beta$ contains also the averaging over ensembles in statistical mechanics. To indicate this double averaging, one sometimes uses the notation $<< A >>$ for $< A >_\beta$. One can introduce a density matrix $\hat{p} = \sum_n p_n |n >< n|$ and write $< A >_\beta =< \hat{\rho} A > / < \hat{\rho} >$.

We first consider canonical ensembles for which the average energy is fixed. Later we discuss grand canonical ensembles for which also the average number of particles (for example nucleons minus antinucleons) is fixed. The average energy, usually denoted by U, is given by

$$U(\beta) \equiv < E >_\beta = Tr \hat{H} e^{-\beta \hat{H}} / Tr e^{-\beta \hat{H}} = -\frac{\partial}{\partial \beta} \ln Z(\beta) \tag{18.1.4}$$

We shall from now on omit the hats on operators when no confusion should arise.

The lack of order of a quantum mechanical system is given by the entropy, which is defined by

$$S = -\sum_n p_n \ln p_n = - < \ln p > \tag{18.1.5}$$

From this definition it follows that S is dimensionless. Clearly $S = 0$ if the system is in only one state, and $S > 0$ if it is in more than one state. For a micro canonical ensemble with N states, each with the same probability, one finds from (18.1.5)

$$S = -\sum p_n \ln p_n = -\sum \frac{1}{N} \ln \frac{1}{N} = -\ln \frac{1}{N} = \ln N \tag{18.1.6}$$

This agrees with the usual definition of entropy in the micro canonical ensemble as the logarith of the number of states available at a given fixed energy E and number of particles N.

For a system in thermodynamical equilibrium, the entropy can be expressed in terms of the partition function by substituting (18.1.1).

$$\begin{aligned} S(\beta) &= -\sum_n \frac{e^{-\beta E_n}}{Z(\beta)}(-\beta E_n - \ln Z(\beta)) \\ &= \beta < E >_\beta + \ln Z(\beta) \\ &= -\beta^2 \frac{\partial}{\partial \beta}\left(\frac{1}{\beta} \ln Z(\beta)\right) = \frac{\partial}{\partial T}(T \ln Z) \end{aligned} \tag{18.1.7}$$

The Helmholtz free energy $F(\beta)$ is defined by

$$\begin{aligned} F(\beta) &= U(\beta) - \frac{S(\beta)}{\beta} \\ &= -\frac{\partial}{\partial \beta} \ln Z(\beta) + \beta \frac{\partial}{\partial \beta} \frac{1}{\beta} \ln Z(\beta) \\ &= -\frac{1}{\beta} \ln Z(\beta) \end{aligned} \tag{18.1.8}$$

or

$$Z(\beta) = e^{-\beta F(\beta)} \tag{18.1.9}$$

Clearly, U and S are derivation of F w.r.t. β

$$\begin{aligned} U(\beta) &= -\frac{\partial}{\partial \beta} \ln Z(\beta) = F(\beta) + \beta \frac{\partial F}{\partial \beta} \\ S(\beta) &= -\beta^2 \frac{\partial}{\partial \beta}\left(\frac{1}{\beta} \ln Z(\beta)\right) = \beta^2 \frac{\partial F}{\partial \beta} \end{aligned} \tag{18.1.10}$$

Let us now discuss grand canonical ensembles. These are the ensembles to be used for quantum field theories because in these theories particles can be created or annihilated. In this case both the average energy and the average number of particles is fixed. We implement these constraints by Lagrange multipliers. In equilibrium, the entropy is maximal if the following expression with Lagrange multipliers is extremal

$$\sum p_n \ln p_n + \lambda(\sum p_n - 1) + \alpha \sum (p_n N_n - N) + \beta(\sum p_n E_n - U) \quad (18.1.11)$$

Assuming that $\hat{H}$ and $\hat{N}$ commute, one can diagonalize them simultaneously. The eigenstates $|n>$ of $\hat{H}$ and $\hat{N}$ have eigenvalues E_n and N_n. At the minimum, the derivative $\partial/\partial p_k$ of this expression must vanish

$$\ln p_k + 1 + \lambda + \alpha N_k + \beta E_k = 0 \quad (18.1.12)$$

from which we conclude that

$$p_k = e^{-1-\lambda-\alpha N_k - \beta E_k} \quad (18.1.13)$$

The normalization $\sum p_k = 1$ fixes the constant λ

$$p_k = \frac{e^{-\alpha N_k - \beta E_k}}{\sum_k e^{-\alpha N_k - \beta E_k}} \quad (18.1.14)$$

The corresponding density matrix is given by

$$\hat{\rho} = \frac{e^{-\alpha \hat{N} - \beta \hat{H}}}{Tr(e^{-\alpha \hat{N} - \beta \hat{H}})}; \hat{\rho} = \sum p_n |n><n| \quad (18.1.15)$$

The denominator is denoted by $Z(\alpha, \beta)$

$$Z(\alpha, \beta) = Tr e^{-\alpha \hat{N} - \beta \hat{H}} \quad (18.1.16)$$

We then obtain an extension of the formula in (18.1.4) for the case of the grand canonical ensemble

$$\frac{\partial}{\partial \beta} \ln Z(\alpha, \beta) = -<\hat{\rho}\hat{H}> = -U; \frac{\partial}{\partial \alpha} \ln Z(\alpha, \beta) = -<\hat{\rho}\hat{N}> = -N \quad (18.1.17)$$

The functional Z is an extensive quantity (because U and N are extensive quantities), hence we can introduce a corresponding intrinsic quantity P by extracting the volume of the system

$$\ln Z = \beta V P \quad (18.1.18)$$

For reasons to be explained, one calls P the thermodynamical pressure, and $-\frac{1}{\beta}\ln Z$ is called the grand canonical thermodynamic potential Ω. It corresponds to the free energy F in the canonical formalism.

It is useful to extract a factor β from α

$$\alpha = -\mu\beta \tag{18.1.19}$$

where μ is called the chemical potential. Then

$$\hat{\rho} = \frac{1}{Z(\beta,\mu)} e^{-\beta(\hat{H}-\mu\hat{N})} \tag{18.1.20}$$

So μ measures the energy which is due to the presence of particles. The entropy S was defined by $S = -\sum p_k \ln p_k$, and $S =< -\ln\hat{\rho} >$ since $\hat{\rho} = \sum p_n|n >< n|$. Using (18.1.20)

$$S = \langle -\ln\hat{\rho}\rangle = \langle -\ln \frac{e^{-\beta(\hat{H}-\mu\hat{N})}}{Z}\rangle = \ln Z + \beta U - \beta\mu N = \beta V P + \beta U - \beta\mu N \tag{18.1.21}$$

we obtain the well-known relation[3] (using $\beta = (kT)^{-1}$ and setting $k = 1$)

$$TS = VP + U - \mu N \tag{18.1.22}$$

This justifies calling P in (18.1.18) the pressure. Note that in terms of $Z(\beta,\mu)$ instead of $Z(\beta,\alpha)$, the average energy is given by

$$U = -\frac{\partial}{\partial\beta}\ln Z(\beta,\mu) + \mu N \tag{18.1.23}$$

because the derivative $-\partial/\partial\beta$ brings down $\hat{H} - \mu\hat{N}$.

[3]By writing $S(\beta) = \beta U + \ln Z$, or $S(\beta,\alpha) = \beta U + \ln Z + \alpha N$ in the grand canonical ensemble with $Z = Tr\exp(-\beta\hat{H} - \alpha\hat{N})$, it is clear that the entropy is the Legendre transform of the logarithm of the partition function. Using $\delta\ln Z(\beta,\alpha,V) = U\delta\beta - N\delta\alpha + (\frac{d}{\delta V}\ln Z)\delta V$ and substituting the Legendre transform, one finds the usual relation ("Gibbs relation") $\delta S = \alpha\delta N + \beta\delta U + (\frac{\delta}{\delta V}\ln Z)\delta V$. Comparing with classical thermodynamics one identifies $\alpha = -\beta\mu$ and $\frac{\delta}{\delta V}\ln Z = \frac{1}{V}\ln Z = -\beta P$, and $\beta = (kT)^{-1}$.

We can again define the Helmholz free energy F by $F = U - TS$. Then we see that the Gibbs free energy $G = -VP$ is the Legendre transform of the free energy in the grand canonical ensemble

$$G = U - TS - \mu N = F - \mu N \ . \tag{18.1.24}$$

We shall now show that the Gibbs free energy corresponds in field theory to the effective action.

In field theory one considers the partition function

$$Z(\beta, \mu, J) = Tr \exp[-\beta(\hat{H} + \mu\hat{N}) + J\hat{A}] \tag{18.1.25}$$

(usually without $\mu\hat{N}$ term), where $J \cdot \hat{A}$ is given by $\int J(\vec{x})\hat{A}(x)d^3x$ and $\hat{A}(x)$ are the fundamental fields. All operators and sources are time independent since we consider thermodynamical equilibrium. In addition, if one wants to compute the effective potential, one takes external fields as constant in space, and then both J and $\hat{A}$ are constants. (One can think of $\hat{A}$ as $\hat{A} = \int A(\vec{x})d^3x$.) The extra term $J\hat{A}$ is similar to adding an interaction term to the exponent in (18.1.1), for example an external magnetic field $\vec{B}$ coupled to the spin $\vec{\sigma}$ on a lattice. Then one is still in the canonical ensemble. Or one can add a term $\mu\hat{N}$ with $\hat{N}$ a number operator, for example the number of charged particles or the number of baryons. Then one is in the grand canonical ensemble. If one makes a Legendre transform from $F(\beta, \vec{B})$ or $F(\beta, \vec{B}, \mu)$ to $G(\beta, \vec{\sigma})$ or $G(\beta, \vec{\sigma}, \mu)$, one obtains the Gibbs free energy of thermodynamics

$$G\left(\beta, \vec{\sigma}, \mu\right) = F\left(\beta, \vec{B}, \mu\right) + \frac{1}{\beta}\vec{\sigma} \cdot \vec{B} \tag{18.1.26}$$

In (18.1.24) the interaction $\frac{1}{\beta}\vec{\sigma} \cdot \vec{B}$ is replaced by $-\mu\mathcal{N}$. Hence $J\hat{A}$ has the same form as $\vec{B} \cdot \vec{\sigma}$.

Interpreting the exponent in (18.1.26) divided by Z again as a probability p_n, we find, repeating the steps in (18.1.7) and (18.1.21),

$$S(\beta, \mu, J) \ = \ \frac{-1}{Z} Tr \left\{ e^{-\beta(\hat{H}+\mu\hat{N})+J\hat{A}} \left(-\beta(\hat{H} + \mu\hat{N}) + J\hat{A} - \ln Z \right) \right\}$$

$$\begin{aligned} &= \beta < \hat{H} > +\beta\mu < \hat{N} > -J < \hat{A} > +\ln Z \\ &= -\beta^2 \frac{\partial}{\partial\beta}\left(\frac{1}{\beta}\ln Z\right) - J < \hat{A} > \end{aligned} \tag{18.1.27}$$

We define the Helmholtz free energy F again as[4]

$$Z = e^{-\beta F} \tag{18.1.28}$$

Clearly, $F(\beta, \mu, J)$ corresponds to the generating functional for connected Feynman graphs. Using (18.1.27) we obtain

$$\begin{aligned} F(\beta,\mu,J) &= -\frac{1}{\beta}\ln Z = -\frac{\partial}{\partial\beta}\ln Z + \beta\frac{\partial}{\partial\beta}\frac{1}{\beta}\ln Z \\ &= E - \frac{1}{\beta}(S + JA) + \mu N \end{aligned} \tag{18.1.29}$$

All quantities E, N and A depend on β, μ and J because they are expectation values. Making a Legendre transformation from J to A,

$$F(\beta,\mu,J) + \frac{1}{\beta}JA = G(\beta,\mu,A) \tag{18.1.30}$$

we see that $\beta G(\beta, \mu, A)$ is equal to the sum of all one-particle irreducible Feynman diagrams $\Gamma(\beta, \mu, A)$. (The factor β in front of G is needed in order that $\Gamma(\beta, \alpha, A) = \beta G(T, \mu, A)$ be dimensionless.) It follows that **the Gibbs free energy in statistical mechanics corresponds to the effective potential in field theory**. It is a well-known property of the Legendre transformation that $\frac{\partial}{\partial\beta}(\beta F) = \frac{\partial}{\partial\beta}(\beta G)$. (In classical mechanics this corresponds to $\frac{\partial}{\partial q}L = -\frac{\partial}{\partial q}H$). Consequently, it follows from (18.1.30) and (18.1.10) that the total energy is given by[5]

$$U(\beta,\mu,A) = G(\beta,\mu,A) + \beta\frac{\partial}{\partial\beta}G(\beta,\mu,A) + \mu N - \frac{1}{\beta}JA \tag{18.1.31}$$

The identification of βG with Γ and the relation between U and G in (18.1.31) are the only results from thermodynamics which we need below.

[4] Strictly speaking the Helmholtz free energy is only defined in the canonical ensemble (so without μ). In the grand canonical ensemble one has instead $P(T,\mu) = -\frac{T}{V}\ln Z(\mu, \beta; V)$.

[5] In thermodynamics, one views the entropy S as a function $S(E, N, A, V)$ and G as a function $G(T, \mu, A, V)$.

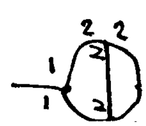
2 Propagators at finite temperature

We begin by evaluating the propagators of finite temperature field theory. The main difference between zero-temperature and finite-temperature field theory lies in the form of the propagators, hence they require a detailed discussion. The vertices, Wick contraction rules, etc., are all as in the zero-temperature case. (The other difference is that the vacuum is a thermal heat bath in equilibrium, so that matrix elements of creation or annihilation operators are not equal to unity but involve factors $\sqrt{n_B}$ or $\sqrt{n_B+1}$ where the Boltzmann factor n_B gives the number of particles in a given state.)

One might expect that the two-point Green function in thermal field theory for a real scalar field is defined by

$$\begin{aligned} D_F(x-y)_\beta &= Tre^{-\beta H} T\varphi(x)\varphi(y)/Tre^{-\beta H} \\ &= \theta(x^0-y^0)D_+(x-y)_\beta + \theta(y^0-x^0)D_-(x-y)_\beta \end{aligned} \quad (18.2.1)$$

where the symbol T denotes time-ordering. This will only yield the $D_F^{11}(x-y)_\beta$ propagator of the real-time formalism but, as we shall see, that is sufficient if one is interested in static quantities at the one-loop level. At higher loops one needs the other propagators also, for example in a tadpole graph



(18.2.2)

For $\beta \to \infty$ (zero-temperature) only the ground state contributes, and one falls back on the Feynman propagator of ordinary quantum field theory. However, for finite β, all eigenstates $|n>$ of the Hamiltonian contribute, weighted by the Boltzmann factor $e^{-\beta E_n}$ as one would expect for a system in thermodynamical equilibrium.

We shall begin by deriving the propagator for an arbitrary contour in the complex t plane (the so-called Mills propagator [23]). The imaginary-time and real-time propagators are special cases. However, later we shall also give alternative derivations of the imaginary-time and real-time propagators separately. By comparing these derivations one gets a better understanding. The evaluation of the propagator can be performed in two ways:

(i) by using operator methods using second quantization to expand $\varphi(x_1)$ and $\varphi(x_2)$ into annihilation and creation operators. By using the usual vacuum of field theory, one obtains then the propagator in the same way as in zero temperature field theory.

(ii) by first analytically continuing the time coordinates x_1^0 and x_2^0 to $x_1^0 = -i\tau_1^0$ and $x_2^0 = -i\tau_2^0$, respectively, with real τ_1 and τ_2. Then, by inserting complete sets of "p and q" eigenstates (where q really means the Schrödinger field operators $\hat{\varphi}(\vec{x}, 0)$ and p are their canonically conjugate momenta) we obtain a Euclidean path integral. The propagators corresponding to this path integral are then simply obtained by inverting the kinetic operator. Since the trace Tr leads to periodic boundary conditions for the bosons (or antiperiodic boundary conditions for the fermions), the fields $\varphi(\vec{x}, \tau)$ in the path integral are expanded into a Fourier sum, using $\exp i(2\pi n/\beta)\tau$ for the bosons and $\exp i2\pi(n + 1/2)/\beta\tau$ for the fermions. The energies $\omega_n = 2\pi n/\beta$ or $\omega_n = 2\pi(n + 1/2)/\beta$ are called Matsubara frequencies [24].

We shall end this section by showing that both forms of the propagators are equivalent. The definition of the thermal propagators in (18.2.1) leads to the so-called Kubo-Martin-Schwinger (KMS) periodicity condition [25]

$$D_+(\vec{x} - \vec{y}, x^0 - y^0)_\beta = D_-(\vec{x} - \vec{y}, x^0 - y^0 + i\beta)_\beta \tag{18.2.3}$$

and we shall check that both forms of the propagator have this periodicity. Then we shall show that they are, in fact, equal after analytic of the time coordinate.

The propagator for gauge fields can be derived in a similar manner. In the real-time approach one starts in the Lorentz gauge from $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 + \frac{1}{2}(\partial^\mu A_\mu^a)^2 = -\frac{1}{2}(\partial_\mu A_\nu^a)^2$ and defines

$$D^{ab}_{F,\mu\nu}(x-y)_\beta = Tre^{-\beta\hat{H}}T\hat{A}^a_\mu(x)\hat{A}_\nu(y) \tag{18.2.4}$$

where

$$A^a_\mu(x) = \int \frac{d^3k}{\sqrt{2w_k}}\left(\epsilon^m_\mu(\vec{k})a_m(\vec{k})e^{ikx} + (\epsilon^m_\mu(\vec{k}))a_m(\vec{k})^\dagger e^{-ikx}\right) \tag{18.2.5}$$

The real $\epsilon^m_\mu(\vec{k})$ are the four polarization vectors and $[a_m(\vec{k}), a_n(\vec{\ell})^\dagger] = \eta_{mn}\delta^3(\vec{k} - \vec{\ell})$. The Hamiltonian has the same form as for scalars.

The thermal propagator for a free real scalar field $\varphi(x)$ is defined by

$$\begin{aligned} D_F(x-y)_\beta &= Tre^{-\beta\hat{H}}T\hat{\varphi}(x)\hat{\varphi}(y)/Z_\beta \\ &= \{\theta(x^0-y^0)Tre^{-\beta\hat{H}}\hat{\varphi}(2)\hat{\varphi}(y) + \theta(y^0-x^0)Tre^{-\beta\hat{H}}\hat{\varphi}(y)\hat{\varphi}(x)\}/Z_\beta \end{aligned} \tag{18.2.6}$$

where $\hat{H} = \int d^3x\left(\frac{1}{2}\dot{\varphi}^2 + \frac{1}{2}(\partial_k\varphi) + \frac{1}{2}m^2\varphi^2\right)$ is given by[6] $\hat{H} = \int d^3k(2\pi)^3\omega_k(a^\dagger_{\vec{k}}a_{\vec{k}} + \frac{1}{2})$. Note that $e^{-\beta\hat{H}}$ stands in front of the time-ordered product; one cannot move it to other places because $\hat{H}$ does not commute with $\hat{\varphi}$.

We shall set $\hbar = c = 1$ in most derivations. Using second quantization one obtains

$$\begin{aligned} D_F(x-y)_\beta &= \int \frac{d^3k}{(2\pi)^{3/2}}\frac{1}{\sqrt{2\omega_k}}\int\frac{d^3p}{(2\pi)^{3/2}}\frac{1}{\sqrt{2\omega_p}} \\ &\quad [\,\theta(x^0-y^0)\{< a(\vec{k})a(\vec{p})^\dagger >_\beta e^{ikx-ipy} + < a(\vec{k})^\dagger a(\vec{p}) >_\beta e^{-ikx+ipy}\} \\ &+ \theta(y^0-x^0)\{< a(\vec{p})a(\vec{k})^\dagger >_\beta e^{-ikx+ipy} + < a(\vec{p})^\dagger a(\vec{k}) >_\beta e^{ikx-ipy}\}\,] \end{aligned} \tag{18.2.7}$$

There are no terms with two a's or two $(a^\dagger)$'s because their expectation values in eigenstates of the Hamiltonian vanish.

[6]Recall that the terms with two a's or two $a^\dagger$'s from $\frac{1}{2}\dot{\varphi}^2$ cancel those from $\frac{1}{2}(\partial_k\varphi)^2$.

To evaluate $< a(\vec{k})a(\vec{p})^\dagger >_\beta$ and other such matrix elements, one may use cyclicity of the trace, and the commutation relations between $a(\vec{k}), a(\vec{k})^\dagger$ and the Hamiltonian

$$[H, a(\vec{k})] = -\omega_k a(\vec{k}) \tag{18.2.8}$$

Then

$$\begin{aligned} & Tre^{-\beta H} a(\vec{k})a(\vec{p})^\dagger / Tre^{-\beta H} = Tra(\vec{p})^\dagger e^{-\beta H} a(\vec{k})/Tre^{-\beta H} \\ = \; & Tre^{-\beta H} e^{\beta \omega_p} a(\vec{p})^\dagger a(\vec{k})/Tre^{-\beta H} \\ = \; & e^{\beta\omega_p} Tre^{-\beta H}\{a(\vec{k})a(\vec{p})^\dagger - \delta^3(\vec{k}-\vec{p})\}/Tre^{-\beta H} \end{aligned} \tag{18.2.9}$$

Hence

$$< a(\vec{k})a(\vec{p})^\dagger >_\beta = [1 + n_B(\omega_p)]\delta^3(\vec{k}-\vec{p}) \; ; n_B(\omega) = \frac{1}{e^{\beta\omega}-1} \tag{18.2.10}$$

Similarly

$$< a(\vec{k})^\dagger a(\vec{p}) >_\beta = n_B(\omega)\delta^3(\vec{k}-\vec{p}) \tag{18.2.11}$$

The terms with n_B give the temperature corrections to the propagator

$$\begin{aligned} D_F(x-y)_\beta \; &= \; \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega} [\, \theta(x^0 - y^0) e^{ik(x-y)} \\ &+ \; \theta(y^0 - x^0)e^{-ik(x-y)} + n_B(\omega)(e^{ik(x-y)} + e^{-ik(x-y)}) \,] \end{aligned} \tag{18.2.12}$$

where $kx = \vec{k}\cdot\vec{x} + k_0 x^0$ with $-k_0 = k^0 = E = (\vec{k}^2 + m^2)^{1/2}$. If x and y have both real time components we can rewrite this as usual as a four-dimensional integral

$$D_F(x-y)_\beta = \int \frac{d^4k}{(2\pi)^4} \left[\frac{-i\hbar}{k^2 + m^2 - i\epsilon} + \frac{2\pi\hbar}{e^{\beta E}-1}\delta(k^2+m^2) \right] e^{ik(x-y)} \tag{18.2.13}$$

where $d^4k = d^3k dk_0, k^2 = \vec{k}^2 - k_0^2$ and $kx = \vec{k}\vec{x} + k_0 x^0$. The second term gives the temperature corrections, and the whole expression is the $D_F^{11}(x-y)_\beta$ part of the real-time propagator. At no time in this derivation did we go from Minkowski spacetime to Euclidean space. The propagator in (18.2.13) was first obtained by Dolan and Jackiw [12].

We can also obtain the propagator in the imaginary-time formalism by making an analytic continuation of the time coordinate. We begin again with the definition in (18.2.1), but now we write

$$\hat{\varphi}(\vec{x}, x^0) = e^{\frac{i}{\hbar}\hat{H}x^0}\hat{\varphi}(\vec{x}, 0)e^{\frac{-i}{\hbar}\hat{H}x^0} = e^{\frac{1}{\hbar}\hat{H}\tau_x}\hat{\varphi}(\vec{x}, 0)e^{\frac{-1}{\hbar}\hat{H}\tau_x} \tag{18.2.14}$$

We treat $\tau_x = ix^0$ as real, which amounts to an analytic continuation of x^0. The propagator now contains terms of the form

$$Tre^{-(\beta-\tau_x)\hat{H}}\hat{\varphi}(\vec{x}, 0)e^{-(\tau_x-\tau_y)\hat{H}}\hat{\varphi}(\vec{y}, 0)e^{-\tau_y\hat{H}} \tag{18.2.15}$$

Inserting complete sets of $\hat{\varphi}(\vec{x})$ and $\hat{\pi}(\vec{x})$ eigenstates, one obtains the path-integral representations of the propagator. After integrating out the momenta, one is left with the usual configuration-space path integral and the time ordering of (18.2.6) is still present, but in Euclidean space and with fields $\varphi(\vec{x}, \tau)$ which are periodic in τ with period β due to the trace Tr.[7]

Hence, one can expand the field $\varphi(\vec{x}, \tau)$ into a Fourier sum and a Fourier integral

$$\varphi(\vec{x}, \tau) = \sum_{n=-\infty}^{\infty}\int \frac{d\vec{k}}{(2\pi)^{3/2}}e^{i(\vec{k}\cdot\vec{x}+\omega_n\tau)}\varphi(\vec{k}, n) \tag{18.2.16}$$

where the energies ω_n ("Matsubara frequencies") are given by

$$\omega_n = \frac{2\pi n}{\beta}, -\infty < n < +\infty \tag{18.2.17}$$

Because fields in path integrals are off-shell, the function into which φ is expanded are the discrete counterparts of 4-dimensional plane waves, so ω is not expressed in terms of $\vec{k}$. The propagator at temperature β is then obtained either by coupling the $\varphi(\vec{k}, n)$ to external currents $J(\vec{k}, n)$ and completing squares, or directly by inverting the kinetic operator. The result is the usual propagator, but with $\int dk_0/2\pi$ replaced

[7] Strictly speaking, the trace only implies that $\varphi(\vec{x}, t_i) = \varphi(\vec{x}, t_f)$ but not that also their derivatives are equal. Hence, the fields are even instead of periodic. However, the propagator of π must also be even, $\pi(\vec{x}, t_i) = \pi(\vec{x}, t_f)$, hence φ (and π) are periodic, after all.

by $\frac{1}{\beta}\sum_n$

$$D_F(\vec{x}, -i\tau)_\beta = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\beta} \sum_n \frac{\hbar e^{i(\vec{k}\cdot\vec{x}+\omega_n\tau)}}{\vec{k}^2 + m^2 + \omega_n^2} \tag{18.2.18}$$

Since we started from (18.2.1), and then used a series of identities, the final result for the Euclidean propagator in (18.2.18) should be the analytic continuation of the Minkowski propagator in (18.2.13). (In momentum space one gets a term $-i\epsilon k_0$ instead of $-i\epsilon$, so retarded or advanced propagators. See R. Mills [23]). This we shall show in (18.2.26).

Finally we derive the KMS periodicity condition, and afterwards check that both the imaginary-time propagator and the real-time propagator satisfy this condition. From its definition

$$\begin{aligned}
&Tre^{-\beta H}T\varphi(x)\varphi(y)/Z_\beta = [\theta(x^0-y^0)D_+(x-y)_\beta + \theta(y^0-x^0)D_-(x-y)_\beta)] \\
&= \theta(x^0-y^0)Tre^{-\beta H}e^{\frac{i}{\hbar}Hx^0}\varphi_S(\vec{x})e^{\frac{-i}{\hbar}H(x^0-y^0)}\varphi_S(\vec{y})e^{\frac{-i}{\hbar}Hy^0}/Z_\beta \\
&+\theta(y^0-x^0)Tre^{-\beta H}e^{\frac{i}{\hbar}Hy^0}\varphi_S(\vec{y})e^{\frac{-i}{\hbar}H(y^0-x^0)}\varphi_S(\vec{x})e^{\frac{-i}{\hbar}Hx^0}/Z_\beta
\end{aligned} \tag{18.2.19}$$

where $\varphi_A(\vec{x}) = \varphi(\vec{x}, t)$ at $t = 0$. Using cyclicity of the trace, we see that

$$\begin{aligned}
D_+(x-y)_\beta &= Tre^{-\beta H}e^{\frac{i}{\hbar}Hx^0}\varphi_S(\vec{x})e^{\frac{-i}{\hbar}H(x^0-y^0)}\varphi_S(\vec{y})e^{\frac{-i}{\hbar}Hy^0}/Z_\beta \\
&= Tre^{\frac{i}{\hbar}H(x^0+i\beta)}\varphi_S(\vec{x})e^{\frac{-i}{\hbar}H(x^0+i\beta)}e^{-\beta H}e^{\frac{i}{\hbar}Hy_0}\varphi_S(\vec{y})e^{\frac{-i}{\hbar}Hy_0}/Z_\beta
\end{aligned} \tag{18.2.20}$$

is equal to

$$\begin{aligned}
&Tre^{-\beta H}e^{\frac{i}{\hbar}Hy_0}\varphi_S(\vec{y})e^{\frac{-i}{\hbar}Hy_0}e^{\frac{i}{\hbar}H(x^0+i\beta)}\varphi_S(\vec{x})e^{-\frac{i}{\hbar}H(x^0+i\beta)}/Z_\beta \\
&= Tre^{-\beta H}\hat{\varphi}(y)\hat{\varphi}(\vec{x}, x^0+i\beta)/Z_\beta
\end{aligned} \tag{18.2.21}$$

Hence

$$D_+(\vec{x}-\vec{y}, x^0-y^0)_\beta = D_-(\vec{x}-\vec{y}, x^0+i\beta-y^0) \tag{18.2.22}$$

We claim that also (18.2.12) has this periodicity. To prove this, replace $n_B(\omega)$ by $\theta(x^0-y^0)n_B(\omega)+\theta(y^0-x^0)n_B(\omega)$ and identify

$$\begin{aligned} D_+(x-y)_\beta &\sim (1+n_B(\omega))e^{ik(x-y)}+n_B(\omega)e^{-ik(x-y)} \\ D_-(x-y)_\beta &\sim n_B(\omega)e^{ik(x-y)}+(1+n_B(\omega))e^{-ik(x-y)} \end{aligned} \tag{18.2.23}$$

Using then $1+n_B(\omega)=(\exp\beta\omega)n_B(\omega)$, we can extract an overall factor $n_B(\omega)$,

$$\begin{aligned} D_+(x-y)_\beta &\sim e^{\beta\omega+ik(x-y)}+e^{-ik(x-y)} \\ D_-(x-y)_\beta &\sim e^{ik(x-y)}+e^{\beta\omega-ik(x-y)} \end{aligned} \tag{18.2.24}$$

The KMS periodicity in (18.2.22) is now obvious. For the propagator in the imaginary-time formalism (18.2.18), the whole propagator and hence both D_+ and D_- are separately periodic in $\tau_x-\tau_y$ with period β, and thus also in this case the KMS condition holds.

Using the KMS periodicity, one can directly show that the propagators in Minkowski space and in Euclidean space are equal. The propagator in Minkowski space, after the analytic continuation $x^0\to -i\tau_x$ follows from (18.2.12) and (18.2.24)

$$\begin{aligned} D_F(x)_\beta &= \int \frac{d^3k}{(2\pi)^3}\,\frac{1}{2\omega}\,\frac{1}{e^{\beta\omega}-1}\Big[\theta(\tau_x)\left\{e^{i\vec{k}\cdot\vec{x}-\omega\tau_x+\beta\omega}+e^{-i\vec{k}\cdot\vec{x}+\omega\tau_x}\right\} \\ &\quad +\theta(-\tau_x)\left\{e^{-i\vec{k}\cdot\vec{x}+\omega\tau_x+\beta\omega}+e^{i\vec{k}\cdot\vec{x}-\omega\tau_x}\right\}\Big] \end{aligned} \tag{18.2.25}$$

The propagator in Euclidean space is given by (18.2.18). Converting all $e^{-i\vec{k}\cdot\vec{x}}$ to $e^{i\vec{k}\cdot\vec{x}}$, the equality to prove is

$$\frac{1}{\beta}\sum_n \frac{-ie^{i\omega_n\tau_x}}{\vec{k}^2+m^2+\omega_n^2}=\frac{1}{2\omega}\,\frac{1}{e^{\beta\omega}-1}\left[e^{-\omega|\tau_x|+\beta\omega}+e^{\omega|\tau_x|}\right] \tag{18.2.26}$$

It is clear that the right-hand side is a periodic function in τ_x on the interval $[-\beta/2,\beta/2]$. (The function and its derivative at $\tau=\beta/2$ are the same as at $\tau=-\beta/2$.) We can therefore expand it into a Fourier series, and this produces indeed the left-hand side. This directly proves that the real-time and imaginary-time propagators are indeed equal after analytic continuation in the time parameter.

The preceding remarks were all based on canonical methods, but let us now give a path integral treatment. We begin with quantum mechanics.

$$\begin{aligned} Z &= Tre^{-\beta\hat{H}}, \hat{H} = H - \mu_j N_j \\ &= \int dq \langle q|e^{-\beta\hat{H}}|q\rangle \end{aligned} \tag{18.2.27}$$

By inserting complete sets of p's and q's, we obtain a path integral in Euclidean space. Comparing with the families "Feynman" path integral in Minkowski spacetime

$$\begin{aligned} Z &= \langle q''|e^{-\frac{i}{\hbar}H(t''-t)}|q'\rangle \\ &= N\int Dq(t)e^{\frac{i}{\hbar}\int_{t'}^{t''} L(q,q')dt} \\ & \quad q(t') = q'; q(t'') = q'' \end{aligned} \tag{18.2.28}$$

we see that $-\frac{1}{\hbar}(t''-t')$ has been replaced by $-i\beta$, and to obtain the trace, we have set $q' = q'' = q$ and integrated over q.

The path integral in Euclidean space can likewise be written as

$$\begin{aligned} Z = \mathcal{N}\int dq \int Dq(t)e^{\frac{i}{\hbar}\int_{t_0}^{t=-i\beta\hbar} dt L_M(t)} \\ q(t') = q(t'') = q \end{aligned} \tag{18.2.29}$$

Setting $t = -i\tau$ we obtain

$$Z = \mathcal{N}\int dq \int Dq(t)e^{\frac{i}{\hbar}\int_0^{\beta\hbar} L_M(t\to -ir)}d\tau \tag{18.2.30}$$

where

$$L_M(t \to -i\tau) \equiv L_E = -\frac{1}{2}(\partial_\tau\phi)^2 - \frac{1}{2}(\vec{\nabla}\phi)^2 - V \tag{18.2.31}$$

Let us now make the transition from quantum mechanics to field theory by replacing q by $\phi(\vec{x},\tau)$. Because the fields satisfy $\phi(\vec{x},\tau') = \phi(\vec{x},\tau'')$, the bosonic fields have periodic boundary conditions. Fermionic fields have anti-periodic boundary conditions as we shall derive.

From here on one uses standard manipulations of quantum field theory. One adds external sources

$$Z[j] = \mathcal{N} \int_C D\phi e^{i \int_c [d^4x \mathcal{L}(x) + j(x)\phi(x)]} \tag{18.2.32}$$

where the contour C runs from t_0 to $t_0 - i\beta h$. Then

$$T\phi(x_1)\dots\phi(x_N) = (-i)^n \frac{\delta}{\delta j(x_1)} \cdots \frac{\delta}{\delta j(x_N)} Z\Big|_{j=0} \tag{18.2.33}$$

and the fields are "contour-ordered".

3 Thermal masses

As a first applications of the imaginary time formalism we study a seagull graph, and show that it gives rise to a mass correction proportional to T^2. Consider a real scalar field φ whose action is given by $\mathcal{L} = -\frac{1}{2}(\partial_\mu \varphi)^2 - \frac{1}{2}m^2\varphi^2 - \frac{\lambda}{4!}\varphi^4$. Then one obtains a mass correction

$$\Delta m^2 = \frac{1}{2}\lambda \left(\frac{1}{\beta}\sum_n\right) \int \frac{d^3k}{(2\pi)^3} \frac{1}{\left(\frac{2\pi n}{\beta}\right)^2 + \vec{k}^2 + m^2} \tag{18.3.34}$$

One can perform the sum over n by contour integration[8], and then one obtains

$$\begin{aligned} \Delta m^2 &= \frac{\lambda}{4} \int \frac{d^3k}{(2\pi)^3} \frac{1}{\omega_k} \coth\left(\frac{\beta\omega_k}{2}\right); \omega_k = \sqrt{\vec{k}^2 + m^2} \\ &= \frac{\lambda}{2} \int \frac{k^2 dk}{(2\pi)^2} \frac{1}{\omega_k} \left(1 + 2\frac{1}{e^{\beta\omega_k} - 1}\right) \end{aligned} \tag{18.3.35}$$

The first term gives the result at $t = 0$, but the second term is the T-dependent mass correction we are interested in.

For large T we can neglect m if $m << T$, and find then

$$\Delta m^2(T) = \frac{\lambda}{4\pi^2} \int_0^\infty k dk \frac{1}{e^{\beta k} - 1} = \frac{\lambda(kT)^2}{24} + \mathcal{O}\left(\frac{m}{T}\right). \tag{18.3.36}$$

[8] Use $\sum_n \frac{1}{n^2 + y^2} = \frac{\pi \coth \pi y}{y}$.

where we used $\int_0^\infty x(e^x-1)^{-1}dx = \sum_{n=1}^\infty \frac{1}{n^2} = \zeta(2) = \pi^2/6$.

Let us now try to calculate Δm^2 for arbitrary m. As we shall see, the result is nonanalytic in m^2 and divergent, namely of the form $\Delta m^2 = aT^2 + b\sqrt{m^2}T + cm^2 + \cdots$ where c diverges. To regulate, we use dimensional regularization. We separate off the terms with $n=0$ and calculate the sum over $|n| \geq 1$ separately. For the $n=0$ term we get from (18.3.34)

$$\Delta m^2 = \frac{1}{2}\frac{\lambda}{\beta}\int \frac{d^{3-2\epsilon}k}{(2\pi)^{3-2\epsilon}}\frac{1}{k^2+m^2} = \frac{1}{2}\lambda kT\sqrt{m^2}\frac{\Gamma(-1/2)}{(4\pi)^{3/2}} \tag{18.3.37}$$

where we used dimensional regularization for Euclidean momenta

$$\int \frac{d^n k}{(2\pi)^n}\frac{1}{(k^2+m^2)^\alpha} = \frac{(m^2)^{\frac{n}{2}-\alpha}}{2^n\pi^{n/2}}\frac{\Gamma\left(\alpha-\frac{n}{2}\right)}{\Gamma(\alpha)} \tag{18.3.38}$$

On the other hand, we obtain for the terms with $n \neq 0$

$$\begin{aligned}
&\sum_{n\neq 0}\int \frac{d^d k}{(2\pi)^d}\frac{1}{\omega_n^2+k^2+m^2} = 2\sum_{n=1}^\infty \int \frac{d^d k}{(2\pi)^d}\sum_{l=0}^\infty \frac{(-m^2)^l}{(\omega_n^2+k^2)^{l+1}} \\
&= 2\sum_{n=1}^\infty\sum_{l=0}^\infty (-m^2)^l \cdot \frac{1}{(4\pi)^{d/2}}\frac{\Gamma(l+1-d/2)}{\Gamma(l+1)}\left(\frac{2\pi n}{\beta}\right)^{d-2l-2} \\
&= 2\left(\frac{2\pi}{\beta}\right)^{d-2}\frac{1}{(4\pi)^{d/2}}\sum_{l=0}^\infty (-)^l\left(\frac{m\beta}{2\pi}\right)^{2l}\frac{\Gamma(l+1-d/2)}{\Gamma(l+1)}\zeta(2l+2-d)\mu^{2\epsilon}
\end{aligned} \tag{18.3.39}$$

where $d = 3-2\epsilon$. The factor $\mu^{2\epsilon}$ contains the renormalization mass which is needed to keep the dimension of all terms in the sum equal to that of Δm^2. For the first few terms with a zeta function we use

$$\zeta(-1) = -\frac{1}{12}; \zeta(0) = -\frac{1}{2}; \zeta(1+2\epsilon) = \frac{1}{2\epsilon}+\gamma_E, \zeta(2) = \frac{\pi^2}{6}\ . \tag{18.3.40}$$

where γ_E is the Euler-Mascharoni constant.

We find then

$$\Delta m^2 = \frac{\lambda}{2}\left[\frac{T^2}{12} - \frac{mT}{4\pi} - \frac{m^2}{8\pi^2}\left(\frac{1}{2\epsilon}+\ln\frac{\mu}{\pi T}+\gamma_E\right)+\cdots\right] \tag{18.3.41}$$

The second term contains $\sqrt{m^2}$, and this term is thus nonanalytic in m^2. The T-dependence of Δm^2 has the physical meaning that the heat bath is a sticky medium which gives an extra mass to the scalar particle. The sign of the leading (T^2) term is positive, and this will be of importance when we study symmetry restoration of spontaneously broken field theories at high temperature.

4 Phase transitions at high temperature

To compute the effective potential, one would expect that one should begin by writing down expressions for one-particle irreducible Feynman graphs with constant external fields $\bar{\varphi}$ in Minkowski spacetime at finite temperature, and then make a Wick rotation on k_0 (or some suitably generalized variable) to Euclidean space. This is the procedure at zero temperature, but it does not work at finite temperature because the real-time propagator cannot be written as the usual expectation value of two field operators between a pure "thermal vacuum state" in the Hilbert space. One could still try to construct amplitudes for Feynman graphs using thermal propagators, but these are quite complicated (the contour for each real-time propagator contains four segments). Rather, what one does is to convert $Tr(\varphi(x_1)\dots\varphi(x_n)e^{-\beta H})/Z_\beta$ directly to a Euclidean path integral in Euclidean space with Euclidean Feynman rules. For zero temperature, one finds the same result for the effective potential, whether one starts in Minkowski or Euclidean space, and it seems very likely that also at finite temperature both approaches give the same result. In any case, we shall start with the Euclidean field theory and imaginary-time propagators.[9]

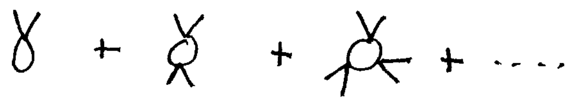
First we consider spontaneously symmetry breaking in $\lambda\varphi^4$ theory with $V =$

[9] In [1] it has been shown for the two-point function that the Euclidean Green's function $G(\omega_n, \vec{k})$ where ω_n are the Matsubara frequencies, satisfies a dispension relation $G(i\omega_n, \vec{k}) = \int d\omega \rho(\omega, \vec{k})/i\omega_n - \omega$. The Green's function obtained from the real-time formalism can be written the same way, and one can show that the corresponding spectral densities ρ agree (27). From this it follows that both expressions for the effective potential are the same.

$-\frac{1}{2}\mu^2\phi^2 + \frac{1}{4!}\lambda\phi^4$ in Minkowski space at zero temperature, and discuss the Coleman-Weinberg potential. Shifting $\phi \to \phi + v$ with $v = \pm\sqrt{6\mu^2/\lambda}$, one finds

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu\phi)^2 - \mu^2\phi^2 - \frac{1}{6}\lambda v\phi^3 - \frac{1}{4!}\lambda\phi^4 \tag{18.4.1}$$

The sum of the one-loop diagrams with $1, 2, 3\ldots$ vertices



(18.4.2)

for the unshifted $\mathcal{L} = -(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{1}{4!}\lambda\phi^4$ yields the effective potential

$$\Gamma^{(2n)}(p=0, p=0\ldots, p=0) = i\frac{(2n)!}{2^n\dot{2}n}\int\frac{d^4k}{(2\pi)^4}\left(\frac{-i\lambda}{k^2+m^2-i\epsilon}\right)^n\phi^n \tag{18.4.3}$$

Then the effective potential becomes

$$\begin{aligned}
V(\phi) &= -i\int\frac{d^4k}{(2n)^4}\sum_{n=1}^{\infty}\frac{1}{2n}\left(\frac{\frac{1}{2}\lambda}{k^2+m^2-i\epsilon}\right)^n \\
&= -\frac{i}{2}\int\frac{d^4k}{(2n)^4}\ln\left(1-\frac{\lambda\phi^2/2}{k^2+m^2-i\epsilon}\right) \\
&= \frac{\wedge^2}{32\pi^2}(m^2+\lambda\phi^2/2) + \frac{1}{64\pi^2}\left(m^2+\lambda\frac{\phi^2}{2}\right)^2\ln\left[\frac{m^2+\lambda\phi^2/2+i\epsilon}{\wedge^2}-\frac{1}{2}\right]
\end{aligned} \tag{18.4.4}$$

We renormalize by adding counter terms with ϕ^2 and ϕ^4, setting $V \to V + \frac{A}{2}\phi^2 + \frac{B}{4!}\phi^4$, and requiring $m^2 = \frac{d^2V}{d\phi^2}\Big|_{\phi=M}$ and $\lambda = \frac{d^4\phi}{d\phi^4}\Big|_{\phi=M}$.

The result is

$$\begin{aligned}
V &= \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4 + \frac{1}{64\pi^2}\left[\left(m^2+\frac{\lambda}{2}\phi^2\right)^2\ln\left(\frac{m^2+\lambda\phi^2/2}{m^2}\right)\right. \\
&-\frac{1}{2}\lambda m^2\phi^2 - \frac{25}{24}\lambda^2\phi^4 + \frac{1}{2}\lambda^2\phi^4\ln\frac{2m^2}{\lambda M^2}
\end{aligned} \tag{18.4.5}$$

For $m \to 0$ we get

$$V_{eff} = \frac{1}{4!}\lambda\phi^4 + \frac{\lambda^2\phi^4}{256\pi^2}\left(\ln\frac{\phi^2}{M^2} - \frac{25}{6}\right) \tag{18.4.6}$$

The minimum is at

$$\lambda \log \frac{<\phi^2>}{M^2} = -\frac{32\pi^2}{3} + \mathcal{O}(\lambda) \tag{18.4.7}$$

Let us now repeat this calculation at finite temperature. Coupling an external source to the quantum field, we see that the Wick contraction rules are unmodified at finite temperature because finite temperature now just means using Euclidean field theory. The one-loop corrections to the effective unrenormalized potential are then given by

$$V^{(1)} = \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\frac{1}{\beta}\sum_{n=-\infty}^{\infty} \ln\left(1 + \frac{V''}{\vec{k}^2 + \left(\frac{2\pi n}{\beta}\right)^2 + m^2}\right) \tag{18.4.8}$$

where $V'' = \frac{1}{2}\lambda\bar{\varphi}^2$ for$\lambda\varphi^4$ theory. The only difference with the zero-temperature case is that the integral $\int dk_0/2\pi$ has been replaced by the sum $\frac{1}{\beta}\sum_n$, and $-k_0^2$ has been replaced by $(2\pi n/\beta)^2$.

We shall now first evaluate the sum over $\ln[(\frac{2\pi n}{\beta})^2 + \vec{k}^2 + m^2 + V'']$, and then subtract its divergence by using the same renormalization conditions as at zero temperature. As we shall see, the extra contributions due to a nonvanishing temperature are finite by themselves. To evaluate the divergent sum

$$\sum_n \ln\left[\left(\frac{2\pi n}{\beta}\right)^2 + \vec{k}^2 + m^2 + V''\right] \equiv \sum_n \ln\left[\left(\frac{2\pi n}{\beta}\right)^2 + E^2\right] \equiv v(E) \tag{18.4.9}$$

where $E^2 = \vec{k}^2 + m^2 + V''$, we first differentiate w.r.t. E

$$\frac{\partial}{\partial E}v(E) = \sum_{n=-\infty}^{\infty} \frac{2E}{\left(\frac{2\pi n}{\beta}\right)^2 + E^2} = \left(\frac{\beta}{\pi}\right)\sum_{n=-\infty}^{\infty}\frac{(\beta E/2\pi)}{n^2 + \left(\frac{\beta E}{2\pi}\right)^2} \tag{18.4.10}$$

The result is a finite sum. It can be evaluated by using[10]

$$\sum_{n=1}^{\infty}\frac{2y}{n^2+y^2} = -\frac{1}{y} + \pi\coth\pi y \tag{18.4.11}$$

[10]To derive this result, consider the contour integral $\oint \frac{\cot(\pi z)}{z-y}dz$ and choose as contour a square in the complex z-plane which passes between the point $z = N$ and $z = N+1$, and also between $z = -N$ and $z = -N-1$. For $N \to \infty$, the contribution of the contour vanishes, and since the poles of $\cot\pi z$ at $z = 0, \pm 1, \pm 2, \ldots$ have residue π^{-1}, one finds $\sum_{n=1}^{\infty}\left(\frac{1}{n-y} + \frac{1}{-n-y}\right) + \frac{1}{(-y)} + \pi\cot\pi y = 0$. Analytic continuation of y to iy yields then the result.

and we obtain

$$\begin{aligned}\frac{\partial v}{\partial E}(E) &= \frac{\beta}{\pi}\left[\pi\left(\frac{e^{\beta E/2}+e^{-\beta E/2}}{e^{\beta E/2}-e^{-\beta E/2}}\right)\right] \\ &= \beta\left[1+\frac{2}{e^{\beta E}-1}\right] \qquad (18.4.12)\end{aligned}$$

Integration over E yields

$$v(E) = \beta E + 2\ln(1-e^{-\beta E}) + E{-}\text{independent constant} \qquad (18.4.13)$$

where the E-independent constant is still to be fixed.

The one-loop effective action can then be written as follows

$$\begin{aligned}V^{(1)} &= \frac{1}{2}\int\frac{d^3k}{(2\pi)^3}E + \frac{1}{\beta}\int\frac{d^3k}{(2\pi)^3}\ln(1-e^{-\beta E}) \\ E^2 &= \vec{k}^2 + V'' + m^2; V'' = \frac{1}{2}\lambda\bar{\varphi}^2 \qquad (18.4.14)\end{aligned}$$

In the limit of zero temperature ($\beta \to \infty$), the second term vanishes. There remains the sum over the zero-point energies, and since this sum is equal to the expectation value of the free part of the Hamiltonian (which is a sum of harmonic oscillators, each oscillator yielding a term $\frac{1}{2}\hbar\omega$), (18.4.14) yields the difference of the sum of zero-point energies of the interacting theory minus the sum of zero-point energies of the free theory. This fixes the free constant in (18.4.13).[11] The second term contains the temperature-dependent corrections to the effective potential. It is clearly ultraviolet and infrared finite, and also finite for m tending to zero. Hence, renormalization at zero temperature also removes the divergences in the theory at finite temperature.

To evaluate these temperature dependent corrections for large temperature (small β), we must compute the following integral

$$I_B(y) = \int_0^\infty (drr^2)\ln[1-e^{-(r^2+y^2)^{1/2}}]; y^2 = \beta^2(V''+m^2) \qquad (18.4.15)$$

[11]The usual form of the one-loop corrections to the effective action at zero temperature is

$$V^{(1)} = \frac{1}{2}\frac{1}{(2\pi)^4}\int d^4k\ln[1+V''/(\vec{k}^2+k_0^2+m^2)]$$

Since $\frac{1}{2\pi}\int_{-\infty}^{\infty}dk_0\ln[\vec{k}+k_0^2-m^2+V''] = E$, both expressions are correct. To prove the last equation one may first differentiate w.r.t. m^2.

In the section where we discuss supersymmetry at finite temperature we shall find an analogous integral $I_F(y)$ for fermions with $\ln\{1+\exp -(r^2+y^2)^{1/2}\}$. Expanding $I_B(y) = I_0 + y^2 I_2 + \dots$, we find[12]

$$\begin{aligned} I_0 &= -\sum_{n=1}^{\infty}\frac{1}{n}\int_0^\infty drr^2 e^{-nr} = -2\sum_{n=1}^{\infty}\frac{1}{n^4} = -\frac{\pi^4}{45} \\ I_2 &= \frac{1}{2}I''(0) = \frac{1}{2}\int_0^\infty drre^{-r}[1-e^{-r}]^{-1} \\ &= \frac{1}{2}\int_0^\infty drr\sum_{n=1}^{\infty}e^{-nr} = \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{12} \end{aligned} \tag{18.4.16}$$

We briefly discuss the next term in the expansion since there is a subtlety: the function $I_B(y)$ is not analytic at $y=0$, as follows from taking its second derivative w.r.t. y^2. Since the logarithm depends only on r^2+y^2, it is simpler to take the second derivative w.r.t. r^2 instead y^2, and to partially integrate. This yields

$$\frac{\partial^2}{\partial y^4}I_B(y) = \int_0^\infty dr\frac{r^{-\epsilon}}{(r^2+y^2)^{1/2}[e^{\sqrt{r^2+y^2}}-1]} \tag{18.4.17}$$

where a factor $r^{-\epsilon}$ has been inserted to regulate the singularity at $r=y=0$. We must thus evaluate the integral

$$\int_0^\infty r^{-\epsilon}\frac{1}{\sqrt{r^2+y^2}[(\exp\sqrt{r^2-y^2})-1]}dr \tag{18.4.18}$$

Multiplying numerator and denominator by a factor $\exp -\frac{1}{2}\sqrt{r^2+y^2}$, we can express this integral into a term with $\coth(\frac{1}{2}(r^2+y^2)^{1/2}$ and a term with only a factor $(r^2+y^2)^{-1/2}$. The coth can be written as an infinite sum by using (18.4.11), yielding

$$\begin{aligned} \frac{\partial^2}{\partial y^4}I_B(y) &= I_\epsilon^{(1)}(y) + I_\epsilon^{(2)}(y) \\ I_\epsilon^{(1)}(y) &= \int_0^\infty drr^{-\epsilon}\sum_{n=-\infty}^{\infty}\frac{1}{r^2+y^2+4\pi^2n^2} \\ I_\epsilon^{(2)}(y) &= -\frac{1}{2}\int_0^\infty drr^{-\epsilon}(r^2+y^2)^{-1/2} \end{aligned} \tag{18.4.19}$$

[12]Expand $\ln(1-e^{-r})$. To evaluate $\sum n^{-4}$, consider $\oint \frac{\cot\pi z}{z^4}dz$ and choose the contour as a circle with large radius and use the expansion $\cot\pi z = (\pi z)^{-1}[1-\frac{1}{3}(\pi z)^2 - \frac{1}{45}(\pi z)^4 + \dots]$.

For $y \neq 0$, both integrals exist. In the first integral we write each term as an integral $\int_0^\infty dx x^{-\epsilon}(x^2+1)^{-1}$ times a factor $(y^2+4\pi^2 n^2)^{-\frac{1}{2}(1+\epsilon)}$. Because the latter factor has a pole $\frac{1}{\epsilon}$, we cannot set $\epsilon = 0$ in the integrand. Splitting off the term with $n = 0$, we write the remaining factors $(y^2+4\pi^2n^2)^{-1/2(1+\epsilon)}$ for $n \neq 0$ as a sum of their value at $y = 0$ (which gives a Riemann zeta function $\sum_{n=1}^\infty \frac{1}{n^{1+\epsilon}} = \frac{1}{\epsilon} + \gamma + \ldots$) plus their difference. In this difference we can easily take the limit ϵ to zero. The function $I_\epsilon^{(2)}(y)$ is a beta function, so here the limit ϵ to zero can also easily be taken. T he ϵ poles in both integrals cancel each other. The final result is

$$\frac{\partial^2}{\partial y^4} I_B(y) = \frac{\pi}{2y} + \frac{1}{2}\ln\frac{y}{4\pi} + \frac{1}{2}\gamma + \frac{1}{2}\sum_{n=1}^\infty \frac{1}{n}\left[\left(1+\frac{y^2}{4\pi^2n^2}\right)^{-1/2} - 1\right] \tag{18.4.20}$$

This result clearly shows that a Taylor series around $y = 0$ does not exist.

Inserting the values for I_0 and I_2 into $V^{(1)}$ we find for the effective potential

$$\begin{aligned} V(\bar\varphi,\beta) \;=\;& \frac{1}{2}m^2\bar\varphi^2 + \frac{\lambda}{4!}\bar\varphi^4 + \frac{1}{\beta^4}\frac{4\pi}{(2\pi)^3}\left(-\frac{\pi^4}{45} + \frac{\pi^2\beta^2}{12}\left(\frac{1}{2}\lambda\bar\varphi^2 + m^2\right)\right) \\ & - \frac{1}{\beta}\frac{1}{12\pi}\left(\frac{1}{2}\lambda\bar\varphi^2+m^2\right)^{3/2} - \frac{1}{64\pi^2}\left(\frac{1}{2}\lambda\bar\varphi^2+m^2\right)^4 \ln\left[\left(\frac{1}{2}\lambda\bar\varphi^2+m^2\right)\beta^2\right] \\ & + \frac{\left\{\frac{3}{2} + 2\ln 4\pi - 2\gamma\right\}}{64\pi^2}\left(\frac{1}{2}\lambda\bar\varphi^2+m^2\right)^2 + \mathcal{O}\left[\left(\frac{1}{2}\lambda\bar\varphi^2+m^2\right)^3\beta^2\right] \end{aligned} \tag{18.4.21}$$

The term with β^{-4} is, of course, the Stefan-Boltzmann term. It reads $V = -\frac{\pi^2}{90}T^4$, and the negative sign may seem startling, but recall that $V^{(1)}$ gives the free energy, not the energy. According to (18.1.31), the energy is given by $F + \beta\partial/\partial\beta F$ and this yields a positive Stefan Boltzmann energy $E_{SB} = \frac{\pi^2}{30}T^4$. The term with the factor 3/2 in the last line comes from the Coleman-Weinberg zero-temperature 1-loop correction

$$V_{\beta=0}^{(1)} = \frac{\hbar}{64\pi^2}\left[(V''+m^2)^2\ln\left(1+\frac{V''}{m^2}\right) - V''\left(\frac{3}{2}V''+4m^2\right)\right] \tag{18.4.22}$$

to the effective potential, i.e., first term in (18.4.14).

Anticipating that at sufficiently high temperature the symmetry is unbroken ($\bar\varphi = 0$), we study what happens when a phase transition occurs ($\bar\varphi \neq 0$, but small).

For sufficiently small $\bar{\varphi}$ the logarithm in (18.4.21) becomes complex (we take m^2 is negative because we want at zero temperature spontaneous symmetry breaking to occur). The one-loop approximation breaks down, and one should really perform another approximation than the loop expansion. However, we are only interested in the case of small β (high temperature), and in that case we retain only the first few terms which are nonsingular. Even this is strictly speaking inconsistent, as for small β the loop approximation breaks down.

For $\bar{\varphi} = 0$ and $T = 0$ the loop corrections are normalized to zero, see (18.4.8). Let us consider the $\bar{\varphi}$ dependent terms to study whether the symmetry of the $T = 0$ spontaneously broken theory is restored at high T

$$\begin{aligned} V &= \left(\frac{1}{2}m^2 + \frac{\lambda}{48\beta^2}\right)\bar{\varphi}^2 + \frac{\lambda}{4!}\bar{\varphi}^4 + \dots \\ &= \frac{1}{2}m^2\left(1 - \frac{T^2}{T_c^2}\right)\bar{\varphi}^2 + \frac{\lambda}{4!}\bar{\varphi}^4 + \dots \end{aligned} \qquad (18.4.23)$$

We see that for $kT \geq kT_c = (-24\frac{m^2}{\lambda})^{1/2}$ the minimum of the potential has moved to the origin. Recalling that m^2 is negative, and that $-2m^2$ is the physical mass of the Higgs scalar, while $<\varphi>^2 = -6m^2/\lambda$ is the order parameter at tree graph level, we find the value of the critical temperature above which spontaneous symmetry breaking disappears

$$kT_c = 2 <\varphi> \qquad (18.4.24)$$

With $<\varphi> = v = 250\ GeV$ in the electroweak sector of the Standard Model, and room temperature corresponding to $\frac{1}{40}\ eV$, the critical temperature T_c in this toy model is about 6.10^{15} degrees Kelvin. Hence we get symmetry restoration of the electroweak interactions at 6.10^{15} degrees Kelvin GeV.[13]

The phase transition at $T = T_c$ is a second-order phase transition, because $<\varphi>$ moves continuously to zero as T approaches T_c. (For a first-order phase transition $<\varphi>$ jumps discontinuously to zero.)

[13] The chiral symmetry restoration in QCD at 170 MeV is due to an entirely different mechanism, involving deconfinement.

5 Gauge theories, fermions and ghosts at finite T

As we shall show in this section and use in the next section, fermionic fields satisfy antiperiodic boundary conditions in the time direction of the imaginary-time (Euclidean) path integral formulation, whereas bosonic fields satisfy periodic boundary conditions. But this raises an interesting problem for gauge theories: Faddeev-Popov ghosts and antighosts are anticommuting, so on this basis one would expect them to satisfy antiperiodic boundary conditions. On the other hand, the BRST variation of a gauge field ${A_\mu}^a$ is given by $\delta A_\mu = D_\mu c\Lambda$, and if A_μ has periodic boundary conditions, also $A_\mu + \delta A_\mu$ and hence δA_μ should satisfy periodic boundary conditions, but then the equality $\delta A_\mu = D_\mu c\Lambda$ shows that also $D_\mu c$ and hence c must satisfy periodic boundary conditions. Which boundary conditions do the ghosts satisfy? On physical grounds, one would expect periodic boundary conditions because the role of ghosts is in general to cancel the effects of the longitudinal and timelike modes of gauge fields. We have run into a dilemma.

The resolution of this dilemma follows from a detailed analysis of the BRST charge and the ghost charge (to be defined), but the main idea can be explained easily. The correct partition function is not the trace $Tr \exp -\beta H$ over the Fock space of all, physical and unphysical (ghost, antighosts, longitudinal, timelike) modes, but rather only over the physical (transversal) modes. Rewriting this restricted trace $\bar{T}r \exp -\beta H$ as an unrestricted trace one needs to insert inside the trace a projection operator which projects to the subsector of physical states. As we shall prove, this projection operator is given by $P_0 \exp i\pi Q_{gh}$ where P_0 is the projection operator onto physical states. The operator $\exp i\pi Q_{gh}$ is the ghost number operator ($\exp i\pi Q_{gh} = (-)^{n_{gh}}$), and acts on the initial bra or the final ket state such that it switches the boundary conditions from antiperiodic to periodic for ghosts as we shall demonstrate.

This mechanism is the same as what happens when one calculates the chiral anomaly of quantum field theories as the quantum mechanical trace $An(\text{chiral}) =$

$Tr(-)^F \exp -\beta H$. In that case the fermion number operator $(-)^F$ (which is used as a quantum mechanical representation of the chiral matrix γ_5) switches the boundary conditions of the anticommuting quantum mechanical functions $\psi^a(\tau)$ from antiperiodic to periodic. In both cases one can rewrite the trace over operators as a quantum-mechanical path integral by inserting complete sets of x- and p- eigenstates for the bosons, and coherent states for the fermions. [31]

As already mentioned, the partition function $\bar{Tr} \exp -\beta H$ contains by definition only a trace over the physical states in the whole Fock space. A physical state $| \, phys >$ is by definition a state which is annihilated by the BRST charge Q_B but which is itself not BRST exact.[14] One can prove that physical states necessarily have ghost number zero and are transversal in the Lorentz gauge. Let us denote the projection operator onto the representatives of physical states by P_0. Then the partition function we have to evaluate can be written as

$$\bar{Tr} e^{-\beta H} = Tr P_0 e^{-\beta H} = Tr P_0 e^{i\pi Q_{gh}} e^{-\beta H} \tag{18.5.1}$$

The operator $\exp i\pi Q_{gh}$ has been added for free in the last term but it will play a central role.

The operator P_0 commutes with the BRST charge Q_B because the commutator $[P_0, Q_B]$ vanishes both on all physical and on all unphysical states

$$\begin{aligned} [P_0, Q_B] \mid \text{phys} > &= Q_B P_0 \mid \text{phys} >= Q_B \mid \text{phys} >= 0 \\ [P_0, Q_B] \mid \text{unphys} > &= P_0 Q_B \mid \text{unphys} >= 0 \end{aligned} \tag{18.5.2}$$

Note that we used in the last step that BRST exact states are unphysical.

[14] We call a state $Q_B|\chi>$ unphysical because by itself it is not a physical state (it has vanishing norm since $Q_B^2 = 0$). Often one considers the whole equivalence class of states $|\text{phys} > +Q_B|\chi>$ for all $|\chi>$ as one physical state. We pick a representative $|\text{phys}>$ from this equivalence class and restrict the summation in the trace to these representatives. This amounts to gauge fixing in the partition function. One does not need extra ghosts associated with this gauge fixing, because the symmetry $|\text{phys} >\to |\text{phys} > +Q_B|\chi>$ is abelian.

The ghost charge Q_{gh} is the Noether charge for the rigid ghost symmetry $\delta c = \alpha c$ and $\delta b = -\alpha b$ of the Lagrangian $L = \int(\dot{b}\dot{c} + \ldots)d^3x$. A charge is the integral over space of the time component of the Noether current, hence $Q_{gh} = i\int(c\pi_c - b\pi_b)d^3x$ where $\pi_c = \frac{\partial}{\partial \dot{c}}L = -\dot{b}$ and $\pi_b = \frac{\partial}{\partial \dot{b}}L = \dot{c}$. Since c is hermitian and b antihermitian (in order that L be hermitian), Q_{gh} is antihermitian (use that $\{c, \pi_c\} - \{b, \pi_b\} = 0$), and $[\alpha Q_{gh}, c] = \alpha c$ and $[\alpha Q_{gh}, b] = -\alpha b$ as follows from the equal-time anticommutation relations $\{\pi_c(x), c(y)\} = -i\delta(\vec{x} - \vec{y})$ and $\{\pi_b(x), b(y)\} = -i\delta(\vec{x} - \vec{y}$. Since the BRST charge Q_B has in general the form $\int(c(x)\varphi(x) + \ldots)d^3x$ where $\varphi(x)$ are first-class constraints, it is hermitian and has ghost-number $+1$. Hence

$$[Q_{gh}, Q_B] = Q_B \tag{18.5.3}$$

Next consider the operator $e^{i\pi Qgh}$. It has eigenvalues ± 1, namely $+1(-1)$ on states with an even (odd) number of ghosts and antighosts. We call this operator the ghost number operator. Consider now the expression $(\exp i\pi Q_{gh})Q_B = Q_B + i\pi Q_{gh}Q_B + \ldots$. Using (18.5.3), it can be rewritten as $Q_B + i\pi Q_B(Q_{gh} + 1) + \ldots = Q_B \exp i\pi(Q_{gh} + 1) = -Q_B \exp i\pi Q_{gh}$. Hence the BRST charge **anti**commutes with the ghost number charge

$$\left\{e^{i\pi Q_{gh}}, Q_B\right\} = 0 \tag{18.5.4}$$

Next we consider in more detail the projection operator P_0 onto the physical states. Unphysical states correspond to unphysical excitations (ghost, antighost, longitudinal and timelike modes). We can decompose the unphysical state sector of the whole Fock space into a sector with one unphysical mode, two unphysical modes, etc. Denoting the corresponding projection operators by P_1, P_2, etc., we have $P_0 + \sum_{i=1}^n P_i = 1$. Since $[P_0, Q_B] = 0$, also $[\sum_{i=1}^n P_i, Q_B] = 0$. Hence, the operator $\sum_{i=1}^n P_i$ is BRST-closed. It can be shown that it is actually BRST exact,

$$\sum_{i=1}^{n} P_i = \{Q_B, R\} \tag{18.5.5}$$

This is not immediately obvious; for example also P_0 is BRST closed but not BRST exact. (If P_0 would be equal to $\{Q_B, R'\}$ there would be no physical states.)

We now return to the physical partition function, and using the properties derived above, we rewrite it as follows

$$\begin{aligned}\bar{T}re^{-\beta H} &= TrP_0 e^{i\pi Q_{gh}} e^{-\beta H} = Tr\left(1 - \sum_{i=1}^{n} P_i\right) e^{i\pi Q_{gh}} e^{-\beta H} \\ &= Tr(1 - \{Q_B, R\}) e^{i\pi Q_{gh}} e^{-\beta H} = Tre^{i\pi Q_{gh}} e^{-\beta H} \\ &- Tr(Re^{i\pi Q_{gh}} e^{-\beta H} Q_B + RQ_B e^{i\pi Q_{gh}} e^{-\beta H}) \end{aligned} \tag{18.5.6}$$

We used the cyclicity of the trace[15].

Using that the ghost number and the BRST charge are both conserved

$$[Q_B, H] = 0 \ ; \ [Q_{gh}, H] = 0 \tag{18.5.7}$$

we find that the last two terms combine into $-TrR\{\exp i\pi Q_{gh}, Q_B\} \exp -\beta H$ and hence vanish. Hence

$$\bar{T}re^{-\beta H} = Tre^{i\pi Q_{gh}} e^{-\beta H} = Tre^{-\beta H} e^{i\pi Q_{gh}} \tag{18.5.8}$$

Similarly we find for any physical operator A (satisfying by definition $[A, Q_B] = 0$) that

$$< A >_\beta = Z^{-1}(\beta)\, \overline{Tr}(e^{-\beta H} A) = Z^{-1}(\beta)\, Tr(e^{i\pi Q_{\rm gh}} e^{-\beta H} A) \tag{18.5.9}$$

The reason we introduced the factor $\exp i\pi Q_{gh}$ is now clear: it allows us to eliminate P_0 and to transform the restricted trace $\bar{T}r$ into an unrestricted trace Tr. (For an operator which is not physical, we find an extra term in the (thermal) expectation value

$$< A >_\beta = Tre^{i\pi Q_{gh}} \{e^{-\beta H} A - [A, Q_B] R\} \tag{18.5.10}$$

[15]The equality $TrQ_BA = TrAQ_B$ holds both for bosonic and fermionic A because the trace over fermionic indices has a minus sign. Thus for fermionic A one has

$$trQ_BA = (Q_B)^\alpha{}_n (A)^n{}_\alpha (-)^\alpha = A^n{}_\alpha (Q_B)^\alpha{}_n = trAQ_B$$

We shall not pursue nonphysical operators further).

We shall now show that the factor $\exp i\pi Q_{gh}$ switches the boundary conditions for b and c in the path integral from antiperiodic to periodic.

To convert the trace to a path integral, we must insert complete sets of $\varphi(x)$ and $\pi(x)$ eigenstates in the bosonic sector, $\psi(x)$ and $\psi^\dagger(x)$ eigenstates in the fermionic sector, and $c(x)$ and $\pi_c(x)$, and $b(x)$ and $\pi_b(x)$ eigenstates in the ghost sector. First we show that fermions have antiperiodic boundary conditions. Then we focus on the ghost-antighost sector.

For simplicity we consider only one degree of freedom instead of a whole field. So we divide space at fixed time t_n into little cubes with volume ΔV, and we consider only one such cube. We consider complex fermions[16]. Let the average $\int \psi(x) d^3x/\sqrt{\Delta V}$ in such a cube be denoted by ψ, and similarly $\psi^\dagger$. From $\{\psi(x), \psi^\dagger(y)\} = \delta(x-y)$ it follows that $\{\psi, \psi^\dagger\} = 1$.

For these complex fermions we use complete sets of coherent states

$$| \chi >= e^{\psi^\dagger \chi} | 0 >, \; < \bar{\eta} |=< 0 | e^{\bar{\eta}\psi} \tag{18.5.11}$$

where $\psi \mid 0 >= 0$. The following completeness relation and inner products hold

$$\begin{aligned} I_f^{(n)} &= \int d\bar{\eta}_n d\chi_n \mid \chi_n > e^{-\bar{\eta}_n \chi_n} < \bar{\eta}_n \mid \\ & < \bar{\eta}_n \mid \chi_n >= e^{\bar{\eta}_n \chi_n} \end{aligned} \tag{18.5.12}$$

where n refers to t_n and $I_f =| 0 >< 0 | + \psi^\dagger | 0 >< 0 | \psi$ is the unit operator in the Fock space of the fermions. For any bosonic operator A the trace over the fermionic Fock space is given by

$$TrA = \int d\chi d\bar{\eta} e^{\bar{\eta}\chi} < \bar{\eta} \mid A \mid \chi > \; = \; < \; 0 \mid A \mid 0 > + < 1 \mid A \mid 1 > \tag{18.5.13}$$

[16]If one is dealing with real (Majorana) fermions one can construct complex (Dirac) fermions either by adding a second set of free (noninteracting) Majorana fermions, or by combining the components of the real fermion. This is explained in detail in [31].

where $\mid 1 >$ denotes $\psi^\dagger \mid 0 >$. Note that we need $d\chi d\bar{\eta}$ in (18.5.13) but $d\bar{\eta}d\chi$ in (18.5.12). Then

$$Tre^{-\beta H} = \int d\chi d\bar{\eta} e^{\bar{\eta}\chi} < \bar{\eta} \mid e^{-\epsilon H} I_f^{(n-1)} e^{-\epsilon H} I_f^{(n-2)} \dots e^{-\epsilon H} I_f^{(1)} e^{-\epsilon H} \mid \chi > \quad (18.5.14)$$

where $\epsilon = \beta/n$. Substituting the expression for I_f we find matrix elements $< \bar{\eta}_k \mid e^{-\epsilon H} \mid \chi_k >= e^{-\epsilon H} e^{\bar{\eta}_k \chi_k}$. The product of the factors $\exp -\epsilon H$ gives the usual $\exp - \int_{-\beta}^0 H d\tau$ and it plays no role in our considerations. Instead we focus on the remaining exponentials

$$e^{\bar{\eta}\chi} e^{\bar{\eta}\chi_{n-1}} e^{-\bar{\eta}_{n-1}\chi_{n-1}} e^{\bar{\eta}_{n-1}\chi_{n-2}} \dots e^{-\bar{\eta}_1\chi_1} e^{\bar{\eta}_1\chi} \quad (18.5.15)$$

The first exponential comes from the trace, while the others come from the inner products and completeness relations. Defining $\chi \equiv \chi_0$ and $\bar{\eta} \equiv \bar{\eta}_n$, it is clear that all terms form pairs which become $-\bar{\eta}\dot{\chi}d\tau$ in the continuum limit, except that the first two terms read

$$e^{\bar{\eta}\chi} e^{\bar{\eta}\chi_{n-1}} = e^{\bar{\eta}_n(\chi_0 + \chi_{n-1})} \quad (18.5.16)$$

In order that also this combination becomes of the form $-\bar{\eta}\dot{\chi}$, we must introduce a new variable χ_n and set it equal to $\chi_0 = -\chi_n$. To avoid confusion, we note that this new variable χ_n is not present in the discretized path integral, but in the continuum limit we must introduce it and continuity (actually, differentiability) of $\chi(\tau)$ requires then that $\chi_n = \chi(\tau = 0) = -\chi_0 = \chi(\tau = -\beta)$. But this means that $\chi(\tau)$ must be an antiperiodic function of τ. To show that also $\bar{\eta}(\tau)$ is antiperiodic, we move the factor $e^{\bar{\eta}\chi}$ to the far right, and find then that $\bar{\eta} = \bar{\eta}_n$ must be equal to $-\bar{\eta}_0$ in order that $\bar{\eta}_1\chi + \bar{\eta}\chi = (\bar{\eta}_1 + \bar{\eta}_n)\chi_0$ limits to $\dot{\bar{\eta}}\chi$. Hence, in order that the trace an be written as a path integral with differentiable functions, we must impose antiperiodic boundary conditions on the fermions.[17]

[17]Note that this antiperiodicity is w.r.t. **time**. Often one puts field theories in a box with volume V to discretize the continuum spectrum. Then one usually imposes periodic boundary conditions in space (although one could also here consider antiperiodic boundary conditions, as for example in string theory).

Consider finally the path integral in the ghost sector. We denote again the average of the fields $c(x)$ and $b(x)$ in a little cube at time t_k by c_k and b_k, and $\{c_k, b_k\} = -i$. Then the path integral becomes a product of the following kernels

$$< c_{k+1} \mid e^{-\epsilon H} \mid \pi(c)_{k+1} >< \pi(c)_{k+1} \mid c_k > \tag{18.5.17}$$

where $\epsilon = \beta/N$ and one should integrate overall c_k and $\pi(c)_k$. Similar expressions with $\mid b >$ and $\mid \pi(b) >$ are present in the antighost sector. Hence, this approach is similar to the treatment of bosons with complete sets of x and p eigenstates, even though in this case c and $\pi(c)$ are anticommuting.

The complete sets of eigenstates should satisfy

$$\begin{aligned} \hat{c}_k \mid c_k > &= c_k \mid c_k > ; \ \hat{\pi}(c)_k \mid \pi(c)_k >= \pi(c)_k \mid \pi(c)_k > \\ < c_k \mid \hat{c}_k &= < c_k \mid c_k ; \ < \pi(c)_k \mid \hat{\pi}(c)_k =< \pi(c)_k \mid \pi(c)_k \end{aligned} \tag{18.5.18}$$

These states are constructed from vacua $\mid 0 >$ and $< 0 \mid$ satisfying $\hat{c} \mid 0 > = 0$ and $< 0 \mid \hat{c} = 0$ as follows

$$\begin{aligned} \mid c > &= e^{i\hat{\pi}(c)c} \mid 0 >; < c \mid =< 0 \mid e^{ic\hat{\pi}(c)} \\ \mid \pi(c) > &= e^{i\hat{c}\pi(c)}\hat{\pi}(c) \mid 0 > ; \ < \pi(c) \mid =< 0 \mid \hat{\pi}(c)e^{i\pi(c)\hat{c}} \end{aligned} \tag{18.5.19}$$

These are clearly eigenfunctions with the correct eigenvalues. The operators $\hat{\pi}(c)$ are needed, otherwise one would find zero in the second line. Since $< 0 \mid \{\hat{c}, \hat{\pi}(c)\} \mid 0 >= -i < 0 \mid 0 >= 0$, we see that $< 0 \mid 0 >= 0$. We normalize the vacua such that $-i \ < 0 \mid \hat{\pi}(c) \mid 0 >= 1$. Then $< c \mid c' >=< 0 \mid ic\hat{\pi}(c) + i\hat{\pi}(c)c' \mid 0 >= c - c'$. (Note that $\hat{\pi}(c)$ denotes the conjugate momentum of c but does not depend on the Grassman variables c and c'. Hence, we need $\hat{\pi}(c)$ in the second term and an operator $\hat{\pi}(c')$ does not make sense. Grasmann variables $\pi(c')$ make sense, however). Further $\int dc \mid c >< c \mid$ equals $\int dc(1 + i\hat{\pi}(c)c) \mid 0 >< 0 \mid (1 + ic\hat{\pi}(c))$ which yields $-i\hat{\pi}(c) \mid 0 >< 0 \mid +(-i) \mid 0 >< 0 \mid \hat{\pi}(c) = I$ where we used that $\mid 0 >< 0 \mid$ is anticommuting.

To prove that this is indeed the unit operator one may check all its matrix elements. Similar results hold for the conjugate momenta.

The following completeness relations and inner products thus hold

$$\begin{aligned}&\int dc_k \mid c_k >< c_k \mid = I \;;\; < c_k \mid c'_k >= c_k - c'_k\\&\int d\pi(c)_k \mid \pi(c)_k >< \pi(c)_k \mid = -iI \;;\; < \pi(c)_k \mid \pi'(c)_k >= i\pi_k(c) - i\pi'_k(c)\end{aligned} \tag{18.5.20}$$

Furthermore

$$< c \mid \pi(c) >= ie^{ic\pi(c)}, < \pi(c) \mid c >= ie^{i\pi(c)c} \tag{18.5.21}$$

In the antighost sector one finds the same relations.

Now comes the crucial moment. The trace formula for a bosonic operator A reads

$$TrA = \int dc < c \mid A \mid -c > \tag{18.5.22}$$

with $\mid -c >$ and not $\mid c >$ as ket. (This corresponds to $TrA = \int dx < x \mid A \mid x >$ in ordinary quantum mechanics.) One may check this for A equal to the unit operator. We convert the trace $tre^{-\beta H} \exp i\pi Q_{gh}$ into a path integral by dividing $e^{-\beta H}$ into N factors $e^{-\epsilon H}$ and inserting complete sets of c and $\pi(c)$ eigenstates. One finds then products of the kernels in (18.5.17), and using the relations for inner products one finds combinations

$$\begin{aligned}&< c_{k+1} \mid \pi(c)_{k+1} >< \pi(c)_{k+1} \mid c_k >\\&= - \exp[-i\pi(c)_{k+1}\{c_{k+1} - c_k\}]\end{aligned} \tag{18.5.23}$$

which limit to $-i\pi\dot{c}$ in the exponent. However, the last kernel contains the operator $(\exp -\epsilon H)(\exp i\pi Q_{gh})$. Using that

$$e^{i\pi Q_{gh}} \mid 0 >=\mid 0 >, e^{i\pi Q_{gh}}\hat{\pi}(c)e^{-i\pi Q_{gh}} = -\hat{\pi}(c) \tag{18.5.24}$$

as follows from $\hat{c} \mid 0 >= \hat{b} \mid 0 >= 0$ and $\hat{c}\hat{\pi}(c) - \hat{b}\hat{\pi}(b) = -\hat{\pi}(c)\hat{c} + \hat{\pi}(b)\hat{b}$, we see that $\exp i\pi Q_{gh} \mid -c >=\mid c >$. Hence, the boundary conditions for the ghost fields are **periodic**.

The analysis for the antighosts is the same. For completeness we mention that one obtains the complete ghost action by adding a gauge-fermion operator $\{\hat{Q}_B, \hat{\chi}\}$ to the expression $\bar{T}r \exp(-\beta\hat{H} + \{\hat{Q}_B, \hat{\chi}\})$. For infinitesimal $\hat{\chi}$, all terms linear in $\hat{\chi}$ cancel in the trace, since there is always an operator $\hat{Q}_B$ which one can pull either to the left or to the right past factors of H such that it reaches the physical states where it vanishes. This shows that the partion function is gauge-choice independent. Finite $\hat{\chi}$ can be built up from successive infinitesimal steps.

The fields c and b satisfy the same anti-commutation relations as ψ and $\psi^\dagger$. Hence we can introduce a vacuum $|0\rangle$ with $c|0\rangle$, and another vacuum $\langle 0|$ with $\langle 0|b = 0$. As coherent states we consider

$$|\chi\rangle = e^{b\chi}|0\rangle; \langle\bar{\eta}| =< 0|e^{\bar{\eta}c} \tag{18.5.25}$$

and we can insert again completeness relations, calculate inner products, and define the trace operation. All this is similar to what we did for fermions. Now comes the crucial point: the factor $\exp i\pi Q_{\rm gh}$ in the trace acts on the final ket $|\chi\rangle$ in the trace as follows

$$\begin{aligned}
&e^{i\pi Q_{\rm gh}}|\chi\rangle = e^{i\pi Q_{\rm gh}}e^{b\chi}|0\rangle = \left(e^{i\pi Q_{\rm gh}}e^{b\chi}e^{-i\pi Q_{\rm gh}}\right)e^{i\pi Q_{\rm gh}}|0\rangle \\
&= \exp\left[e^{i\pi Q_{\rm gh}}b\chi e^{-i\pi Q_{\rm gh}}\right]|0\rangle = \exp\left[e^{i\pi}b\chi|0\rangle\right. \\
&= e^{-b\chi}|0\rangle = |-\chi\rangle
\end{aligned} \tag{18.5.26}$$

Thus the ghost number operator switches the sign of χ in $|\chi\rangle$, and as we have seen for the fermions, this switches the boundary conditions in the continuum path integral, from antiperiodic as they would for ordinary fermions, to **periodic** for ghosts.

The periodicity of ghosts yields a result for the path integral which confirms one's physical expectations. In the Lorentz gauge the path integral over A_μ (for free gauge fields, but one can extend the reasoning to interacting field theories) yields the product of four Gaussian integrals. The path integral over the ghosts is also a Gaussian integral, but because the ghosts are anticommuting, one gets the determinant of the field operator of the ghosts in the numerator, and that of the four fields A_μ in the denominator. The determinant is the product of the eigenvalues, and for the ghosts as well as for A_μ one needs the eigenvalues which belong to eigenfunctions with periodic boundary conditions in time. The final formula is $Z(\beta) = [(\det -\Box)^{-1/2}]^4 \det -\Box$ and one observes that the determinant of the ghosts cancels the determinant of the longitudinal and timelike photons. Thus the periodicity of the ghosts is needed for the gauge invariance of the partition function.

6 Supersymmetry violation at nonzero temperature

Ordinary symmetries are restored at high temperature. However, at finite temperature the boundary conditions of bosons (and ghosts) and fermions in the path integral become different (periodic and antiperiodic, respectively) hence finite temperature breaks susy if it was present at zero temperature. As we shall see, if susy is spontaneously broken at zero temperature, it cannot be restored at some finite temperature. We shall demonstrate this at the one-loop level, by considering a model with a so-called Fayet-Iliopoulos term, and evaluating the one-loop effective potential.

Recall that rigid supersymmetry is broken if and only if the vacuum expectation value of the auxiliary fields is nonzero. Since the energy of the vacuum $< E >$ is equal to the square of the auxiliary fields, susy is broken if and only if $< E >$ is positive. If susy is unbroken, $< E >= 0$; never can $< E >$ be negative.

For example, for an $N = 1$ abelian vector multiplet coupled to a scalar multiplet with Fayet-Iliopoulos term ξD, the action reads

$$\begin{aligned} \mathcal{L} &= -\frac{1}{4}F_{\mu\nu}{}^2 - \frac{1}{2}\bar{\lambda}\not{D}\lambda + \frac{1}{2}D^2 - (D_\mu\varphi^*)D^\mu\varphi - \bar{\psi}_L\not{D}\psi_L \\ &+ \mathcal{F}^*\mathcal{F} - g\sqrt{2}(\bar{\lambda}\psi_L\varphi^* + \bar{\psi}_L\lambda\varphi) + gD\varphi^*\varphi + \xi D \end{aligned} \tag{18.6.1}$$

where $D_\mu\varphi = (\partial_\mu + igA_\mu)\varphi$ and

$$\begin{aligned}
\delta A_\mu &= \bar{\epsilon}\gamma_\mu i\gamma_5\lambda\ ;\ \delta D = \bar{\epsilon}\not{D}\lambda; \delta\lambda = \frac{i}{2}\gamma^\mu\gamma^\nu F_{\mu\nu}\gamma_5\epsilon + D\epsilon \\
\delta\varphi &= \sqrt{2}\bar{\epsilon}_R\psi_L; \delta\varphi^* = \sqrt{2}\bar{\psi}_L\epsilon_R \\
\delta\psi_L &= \sqrt{2}(\not{D}\varphi\epsilon_R + \mathcal{F}\epsilon_L); \delta\bar{\psi}_L = -\sqrt{2}(\bar{\epsilon}_R\not{D}\varphi^* - \bar{\epsilon}_L\mathcal{F}^*) \\
\delta\mathcal{F} &= \sqrt{2}\bar{\epsilon}_L\not{D}\psi_L; \delta\mathcal{F}^* = -\sqrt{2}D_\mu\bar{\psi}_L\gamma^\mu\epsilon_L
\end{aligned} \tag{18.6.2}$$

One can construct nonabelian extensions with $D_\mu\varphi^i = \partial_\mu\varphi^i + gA_\mu{}^a(T_a)^i{}_j\varphi^j$ but we shall not consider them here. The potential reads after eliminating $\mathcal{F}$ and D

$$V = \frac{1}{2}(\xi + g\varphi^*\varphi)^2 \tag{18.6.3}$$

and for $\xi < 0$ there is no susy breaking ($< E >= 0$) and gauge invariance is broken ($< \varphi >$ is nonzero), while for $\xi > 0$ susy is broken ($< E >$ is positive) but gauge invariance is unbroken ($< \varphi> = 0$). One reaches the same conclusions by studying the supersymmetry transformation laws of the fermions: for $\xi < 0$ there is no constant term in $\delta\lambda$ (because $D = -g\varphi^*\varphi - \xi$ and $\varphi^*\varphi + \xi = 0$ at the minimum of V), while for $\xi > 0$ there is a constant term (at the minimum of V, $\varphi^*\varphi = 0$ so $D = -\xi$). Hence for $\xi < 0$ there is no Goldstone fermion, indicating that susy is unbroken, while for $\xi > 0$ there is a Goldstone fermion and susy is broken. We now study the one-loop corrections in this model to the effective potential at finite temperature.

At finite temperature, quantum corrections are added to the effective potential, and one may ask whether at high enough energies susy can be restored. One might at first believe that one should compute whether the zero-loop plus one-loop effective potential vanishes or is positive at its minimum. This is incorrect: the vacuum energy is given by

$$< E >= \left(V_{eff} + \beta\frac{\partial}{\partial\beta}V_{eff}\right)_{min} \tag{18.6.4}$$

as we have discussed in section 1. To evalue V_{eff}, we must first quantize the system. To fix the local U(1) symmetry we choose the unitary gauge $\varphi_2 = 0$. The gauge propagator becomes then of the same form as in the Landau gauge, namely proportional to $\eta_{\mu\nu} - k_\mu k_\nu/k^2$, and the photon loop will give an extra factor 3, just as in the Coleman-Weinberg calculation. The ghosts become nonpropagating, $\mathcal{L} = b\varphi_1 c$, and do not contribute. Hence, we are left with 4 bosonic degrees of freedom (A_μ and φ_1) and 4 fermionic degrees of freedom (ψ_L and λ).[18]

To compute the effective potential to one-loop order, we write down the linearized action (all terms in the action which are quadratic in quantum fields). Then we evaluate all one-loop determinants (equivalently: the sum of all one-loop graphs with any number of external v's). At tree graph level, $v = -2\xi/g$ if $\xi < 0$ and $v = 0$ if $\xi > 0$, but at the one-loop level v will acquire an order $\hbar$ correction which we shall determine. The linearized action in the gauge $\varphi_2 = 0$ reads

$$\begin{aligned} \mathcal{L} &= \left(-\frac{1}{4}F_{\mu\nu}^2 - \frac{1}{2}\partial_\mu{\varphi_1}^2 - \frac{1}{2}\bar{\lambda}\partial\!\!\!/\lambda - \bar{\psi}_L\partial\!\!\!/\psi_L\right) \\ &- \frac{1}{2}(g^2v^2){A_\mu}^2 - \frac{1}{2}(\xi g + \frac{3}{2}g^2v^2){\varphi_1}^2 - \bar{\lambda}(gv)\psi_L - \bar{\psi}_L(gv)\lambda \end{aligned} \tag{18.6.5}$$

The one-loop temperature-dependent corrections due to the bosons are obtained by combining (18.4.8), (18.4.14) and (18.4.21)

$$\begin{aligned} V^{(1)}_{B,\beta} &= \sum_j \frac{\hbar}{2}\epsilon_j \frac{1}{\beta}\sum_n \int \frac{d^3k}{(2\pi)^3} \ln\left(\left(\frac{2\pi n}{\beta}\right)^2 + E_j^2\right) \\ &= \sum_j \frac{\hbar}{\beta}\epsilon_j \int \frac{d^3k}{(2\pi)^3}\ln(1 - e^{-\beta E_j}) + \frac{\hbar}{2}\int \frac{d^3k}{(2\pi)^3}\left(\sum_j \epsilon_j E^j\right) \\ &= \sum_j \epsilon_j \frac{\hbar}{\beta^4}\frac{1}{2\pi^2}\left(-\frac{\pi^4}{45} + \frac{\pi^2\beta^2}{12}{M_j}^2 + \dots\right) \end{aligned} \tag{18.6.6}$$

where $E_j^2 = \vec{k}^2 + M_j^2$, and $\epsilon_j = 1$ and ${M_j}^2 = \xi g + \frac{3}{2}g^2v^2$ for φ_1, and $\epsilon_j = 3$ and

[18] One might also consider the 't Hooft gauge fixing term $\mathcal{L}(fix) = -\frac{1}{\alpha}(\partial^\mu A_\mu - \lambda e v \varphi_2)^2$ which cancels the off-diagonal kinetic term of φ_2 and A_μ after shifting $\varphi_1 = \sigma + v$. The ghost action then becomes $\mathcal{L}$ (ghost) $= b(\Box c - \lambda e^2 v\sigma c)$, and taking the limit $\alpha \to 0$, one recovers the same photon propagator, and the ghosts decouple. The would-be Goldstone boson φ_2 acquires a mass $\xi g + \frac{1}{2}g^2v^2$ which vanishes in the spontaneously broken case ($\xi < 0$) if one chooses v such that the classical potential is minimalized ($(\xi + \frac{1}{2}gv^2)^2 = 0$). Then φ_2 does not contribute any β-dependent terms. For $\xi > 0$, one has $v = 0$, and all fields (including the ghosts) are massless.

${M_j}^2 = g^2v^2$ for A_μ (using again the Landau gauge). Note that this function is everywhere negative, and monotonically increasing as a function of M_j^2.

For the fermions it is easiest first to rewrite the Majorana fermion λ as a right-handed Weyl fermion λ_R. Then $\mathcal{L} = -\bar{\psi}_L \partial\!\!\!/ \psi_L - \bar{\lambda}_R \partial\!\!\!/ \lambda_R - \bar{\lambda}_R (gv)\psi_L - \bar{\psi}_L(gv)\lambda_R$ becomes equal to $-\bar{\chi}\partial\!\!\!/\chi - \bar{\chi}gv\chi$ with the Dirac spinor $\chi = \{\psi_L, \lambda_R\}$, and

$$V^{(1)}_{F,\beta} = \frac{\hbar}{2}(-4)\frac{1}{\beta}\sum_n \int \frac{d^3k}{(2\pi)^3} \ln\left(\left(\frac{2\pi(n+1/2)}{\beta}\right)^2 + \vec{k}^2 + (gv)^2\right) \tag{18.6.7}$$

where $E^2 = \vec{k}^2 + (gv)^2$. The factor -4 is for the fermion loop with a Dirac spinor, and the summation over half-integers is due to the antiperiodic boundary conditions for fermions. Using

$$\sum_{n=-\infty}^{\infty} \frac{y}{(n+\frac{1}{2})^2 + y^2} = \sum_{n=-\infty}^{\infty} \frac{4y}{(2n+1)^2 + 4y^2} = \sum_{n=-\infty}^{\infty} \frac{2(2y)}{n^2 + (2y)^2} - \sum_{n=-\infty}^{\infty} \frac{y}{n^2 + y^2} \tag{18.6.8}$$

we obtain from (18.4.12)

$$\begin{aligned} &\frac{\partial}{\partial E}\sum_{n=-\infty}^{\infty} \ln\left[\left(\frac{2\pi(n+1/2)^2}{\beta}\right) + E^2\right] \equiv \frac{\partial}{\partial E}u(E) \\ &= 2\beta \coth \beta E - \beta \coth \frac{1}{2}\beta E \end{aligned} \tag{18.6.9}$$

Integrating w.r.t. E we obtain the fermionic counterpart of (18.4.13)

$$\begin{aligned} u(E) &= \left\{2\beta E + 2\ln(1 - e^{-2\beta E})\right\} - \left\{\beta E + 2\ln(1 - e^{-\beta E})\right\} \\ &= \beta E + 2\ln(1 + e^{-\beta E}) \end{aligned} \tag{18.6.10}$$

Note the appearance of the expected + sign of a Fermi-Dirac distribution. Then the temperature corrections due to fermions become

$$V^{(1)}_{F,\beta} = \frac{\hbar}{\beta}(-4)\int \frac{d^3k}{(2\pi)^3} \ln(1 + e^{-\beta E_f}) - 4\frac{\hbar}{2}\int \frac{d^3k}{(2\pi)^3} E_f \tag{18.6.11}$$

Again, we find at zero temperature only the zero-point energies, and again the temperature corrections are negative and and monotonically increasing as a function of M^2.

The temperature dependent corrections due to fermions are given by the first integral in (18.6.11). For high temperature we expand to first order in y^2

$$\begin{aligned} -I_F(y) &= -\int_0^\infty drr^2 \ln[1+e^{-(r^2+y^2)^{1/2}}]; y^2 = \beta^2 M^2 = (\beta g v)^2 \\ &= \left(2\sum_{n=1}^{\infty}(-)^n \frac{1}{n^4}\right) + y^2\left(-\frac{1}{2}\sum_{n=1}^{\infty}(-)^n\frac{1}{n^2}\right) + \dots \\ &= \frac{7}{8}\left(\frac{-\pi^4}{45}\right) + \frac{1}{2}y^2\left(\frac{\pi^2}{12}\right) + \dots \end{aligned} \quad (18.6.12)$$

Hence, compared to a real scalar field each complex fermionic degree of freedom gives 7/8 the amount of black-body radiation and 1/2 the $1/\beta^2$ term

$$V^{(1)}_{\beta,F} = \frac{\hbar}{\beta^4}\,\frac{1}{2\pi^2}(-4)\left(\frac{7}{8}\frac{\pi^4}{45} - \frac{1}{2}\frac{\pi^2\beta^2}{12}M^2\right) + \mathcal{O}\left(\frac{\hbar}{\beta}\right) + \mathcal{O}(\hbar^2) \quad (18.6.13)$$

The sum of all β-dependent corrections to the effective potential reads

$$\begin{aligned} V^{(1)}_{(\beta)} &= \frac{\hbar}{\beta^4 2\pi^2}\sum_\alpha \epsilon_\alpha I_\alpha(\beta M_\alpha) = \frac{\hbar}{\beta^4}\frac{1}{2\pi^2}\left[\left\{1+3+\frac{7}{2}\right\}\left(\frac{-\pi^4}{45}\right)\right. \\ &\left. + \frac{\pi^2\beta^2}{12}\left\{\left(\xi g + \frac{3}{2}g^2v^2\right) + 3(gv)^2 + 2(gv)^2\right\} + \dots\right] \end{aligned} \quad (18.6.14)$$

The Kugo-Ojima quartet of unphysical degrees of freedom (which consists of the ghost and antighost, the unphysical polarization $\epsilon^\mu = \left(\vec{k}\frac{k_0}{k}, -k\right)$, and the would-be Goldstone boson) all are massive, but their contributions to the effective action cancel as in the $T=0$ case.

The total effective potential at the one-loop level is given in closed form by

$$\begin{aligned} V_\beta &= V^{(0)}_\beta + V^{(1)}_\beta = \frac{1}{2}\left(\xi + \frac{1}{2}gv^2\right)^2 + \frac{\hbar}{2}\int\frac{d^3k}{(2\pi)^3}(E_{\varphi_1} + 3E_{A_\mu} - 4E_f) \\ &+ \frac{\hbar}{\beta}\int\frac{d^3k}{(2\pi)^3}\left[\ln(1-e^{-\beta E_{\varphi_1}}) + 3\ln(1-e^{-\beta E_{A_\mu}}) - 4\ln(1+e^{-\beta E f})\right] \end{aligned} \quad (18.6.15)$$

where $E^2_{\varphi_1} = \vec{k}^2 + \xi g + \frac{3}{2}(gv)^2$, $E^2_{A_\mu} = \vec{k}^2 + (gv)^2$ and $E^2_f = \vec{k}^2 + (gv)^2$. If the theory was classically susy ($\xi + \frac{1}{2}gv^2 = 0$), all boson and fermion masses are the same and the sum of the zero-point energies vanishes. This agrees with the well-known "nonrenormalization theorem" that in a susy theory the value and the position of the minimum of the effective action do not receive radiative corrections. Hence, the zero-temperature one-loop corrections preserve susy ($V(\beta \to \infty) = 0$). Conversely, if susy is spontaneously broken at the classical level ($\xi > 0, v = 0$), zero-temperature one-loop corrections break it even ¿more (the sum of the zero point energies is positive, namely $\frac{\hbar}{2} \int d^3k/(2\pi)^3 \{(\vec{k}^2 + \xi g)^{1/2} - (\vec{k}^2)^{1/2}\}$). To decide whether susy is broken at finite temperature, we must find out whether the vacuum energy is nonzero, in which case it should be positive. The energy is given by $E = V + \beta \frac{\partial}{\partial\beta} V$, and this is clearly positive for very high temperatures since the coefficient of β^{-4} is negative. This confirms that E is not simply V. For arbitrary values of β the finite temperature corrections to the energy of the vacuum are given by

$$E(\beta) = \frac{\hbar}{\beta} \int \frac{d^3k}{(2\pi)^3} \left[\frac{\beta E_{\varphi_1}}{1 - e^{-\beta E_{\varphi_1}}} + 3 \frac{\beta E_{A_\mu}}{1 - e^{-\beta E_{A_\mu}}} - 4 \frac{\beta E_f}{1 + e^{-\beta E_f}} \right] \tag{18.6.16}$$

The reader may verify that these corrections are positive when susy is classically unbroken ($\xi < 0$, all masses equal). (If the plus sign in the denominator of the last term would have been a minus sign all corrections would have canceled.) On the other hand, if susy is classically spontaneously broken ($\xi > 0, v = 0$ classically), the corrections are also positive.

Hence, if susy is spontaneously broken at zero temperature, it becomes even more broken at nonzero temperatures and when it was unbroken at zero temperature, it becomes broken at nonzero temperature. High temperature does not restore susy. One might have expected this from the fact that fermions and bosons have different boundary conditions in the path integral at finite temperature.

7 The real-time formulation

To evaluate $Tre^{-\beta H} T\varphi(x_1) \dots \varphi(x_n)$ for Minkovski field theories, we must take into account both a real time coordinate t, and the temperature β which we can again treat as an imaginary time coordinate. Taking the trace the factor $e^{-\beta H}$ propagates then from a time t_0 to a time $t_0 - i\beta$, and by inserting complete sets of "p's and q's", one constructs a path integral containing both t and β. The fields $\varphi(x_1), \ldots \varphi(x_n)$ all have real time coordinates, but since one ends up at $t_0 - i\beta$,[19] there must be a contour in the complex t plane which contains the real time axis and the point $t_0 - i\beta$.

We can say more about this contour C. Consider the two-point function, and insert complete sets $\sum_m |m >< m|$ and $\sum_n (n >< n)$ of eigenstates of the Hamiltonian. We use a decomposition which is similar to the one used in the zero temperature case and denote the finite temperature two-point function on C by $\bar{D}_F$. Hence

$$\bar{D}_F(x_1 - x_2)_\beta = \theta_C(t_1 - t_2)\bar{D}_+(x_1 - x_2)_\beta + \theta_C(t_2 - t_1)\bar{D}_-(x_1 - x_2)_\beta \qquad (18.7.1)$$

where $\theta_C(t_1 - t_2)$ is $+1$ if t_1 lies past t_2 on the contour C, and $\theta_C(t_1 - t_2) = 0$ if t_1 is earlier on C than t_2. The distribution $\bar{D}_+(x - y)_\beta$ is then given by

$$\begin{aligned} \bar{D}_+(x_1 - x_2)_\beta &= Tr e^{-\beta H}\varphi(x_1)\varphi(x_2) \\ &= \sum_{m,n} e^{-\beta E_m} < m|\varphi(x_1)|n >< n|\varphi(x_2)|m > /Z_\beta \\ &= \sum_{m,n} e^{\frac{i}{\hbar}E_m(t_1 - t_2 + i\beta)} e^{-\frac{i}{\hbar}E_n(t_1 - t_2)} < m|\varphi_S(\vec{x}_1)|n >< n|\varphi_S(\vec{x}_2)|m > \end{aligned} \qquad (18.7.2)$$

Assuming that the exponentials determine the convergence of these series, we find that $\bar{D}_+$ is an analytic function in $t_1 - t_2$ for $-\beta < Im(t_1 - t_2) < 0$. Similarly, $\bar{D}_-$

[19] For zero temperature, Green's functions are usually defined by reducing the transition element $< Q, T|Q, -T >_J$ to the vacuum-to-vacuum persistence amplitude $< 0, T|0, -T >_J$ where $|0 >$ denotes the vacuum and T is a large time, while $|Q, t >\equiv e^{+\frac{i}{\hbar}Ht}|Q >$. The corresponding path-integral has then a time contour from $t = -T$ to $t = +T$ and one takes the limit $T \to \infty$. Note that in $Tr e^{-\beta H} T\varphi(x_1) \ldots \varphi(x_n) = \int < Q, -t_0|e^{-\beta H} T\varphi(x_1) \ldots \varphi(x_n)|Q, -t_0) > dQ$ one starts at time $-t_0$, moves to t_n by the action of $\exp -\frac{i}{\hbar}H(t_n + t_0)$, then moves to t_{n-1} by the action of $e^{-\frac{i}{\hbar}H(t_{n-1} - t_n)}$, etc., until one is left with the factor $< Q, -t_0|e^{-\beta H} e^{\frac{i}{\hbar}Ht_1} =< Q|e^{\frac{i}{\hbar}H(t_0)} e^{-\beta H} e^{\frac{i}{\hbar}Ht_1}$ at the point t_1. Without the factor $e^{-\beta H}$ one would then move back to $-t_0$, but $e^{-\beta H}$ moves the contour first to the point $-i\beta$ and $e^{\frac{i}{\hbar}Ht_0}$ moves one then to the point $-t_0 - i\beta$. Interchanging $\exp \frac{i}{\hbar}Ht_0$ and $\exp -\beta H$, one finds the contour considered in the text. Clearly mathematically there are infinitely many equivalent contours possible, but the one in the text is the simplest and, more importantly, it has a physical interpretation.

converges and hence is analytic in $t_1 - t_2$ for $0 < Im(t_1 - t_2) < \beta$. This means that the contour C must always go down or stay horizontal in the complex $t_1 - t_2$ plane as one moves along it from its beginning to its endpoint. At the boundaries $Im(t_1 - t_2) = 0$ and $Im(t_1 - t_2) = \pm\beta$, these analytic functions limit to continuous distributions. Hence, the 2-point function is defined in the strip

$$-\beta \leq Im(t_1 - t_2) \leq \beta \tag{18.7.3}$$

There are still many contours which start at t_0, contain the real t-axis, and end at $t_0 - i\beta$, and never go up in the complex plane, but an obvious choice is the following contour

Given this contour, the path integral for a scalar field becomes

$$Z_\beta(J) = \int DAD\pi e^{\frac{i}{\hbar}\int_C (\pi\partial_C A - \mathcal{H} + JA) d^3x d\tau_C} \tag{18.7.4}$$

where τ_C is a real coordinate along C and $\partial_C A$ is the derivative of $A(\vec{x}, \tau_C)$ w.r.t. τ_C. More explicitly, at the discretized level one has factors $\exp(-\Delta z H)$ and $\exp ip(A)[(A)(\tau + \Delta\tau) - A(\tau)]$. Integrating out the momenta $p(A)$, one obtains a complex contour integral which is parametrized by the real τ_C. Note that **$A(\tau_C)$ and τ_C are real but $dz = \frac{dz}{d\tau^C} d\tau^C$ is complex**. Integrating out the momenta, we obtain $\mathcal{L}(A) = \frac{1}{2}(\partial_C A)^2 - \frac{1}{2}(\partial_k A)^2 - V(A)$. Taking the quadratic part of $\mathcal{L}(A)$, and completing squares, one finds

$$Z_\beta(J) = \int DA \exp \frac{i}{\hbar} S^{int}\left(\frac{\hbar}{i}\frac{\delta}{\delta J}\right) \exp -\frac{i}{2\hbar}\int_C J(x)\left(\frac{1}{i\hbar}\bar{D}_F(x-y)_\beta\right) J(y) d^3x d\tau_x d^3y d\tau_y \tag{18.7.5}$$

where

$$(-\partial_C^2 + \partial_k^2 - m^2)\bar{D}_F(x-y)_\beta = \delta_C(x-y) \tag{18.7.6}$$

and $\delta_C(x-y) = \delta(\vec{x} - \vec{y})\delta_C(\tau_x - \tau_y)$. The contour-delta function δ_C is related to the contour theta function θ_C by

$$\begin{aligned} &\int_C \delta_C(\tau - \tau') f(\tau') d\tau' = f(\tau), \tau' \epsilon C \\ &\theta_C(\tau - \tau') = \int_C^\tau \delta_C(\tau - \tau') d\tau' \end{aligned} \tag{18.7.7}$$

The propagator is then obtained by using $\delta J(x)/\delta J(y) = \delta_C(x-y)$ and is equal to $\bar{D}_F(x-y)_\beta$. Because the trace is cyclic no matter what contour is used, the propagators $\bar{D}_\pm(x-y)_\beta$ must satisfy the KMS periodicity condition

$$\bar{D}_+(\vec{x}-\vec{y},\tau_{C,x}-\tau_{C,y})_\beta = \bar{D}_-(\vec{x}-\vec{y},\tau_{C,x}-\tau_{C,y}+i\beta)_\beta \tag{18.7.8}$$

This, together with a hermiticity property, fixes the propagator as we now show. Fourier transforming $\vec{x}$, we define

$$\bar{D}_F(\vec{x},\tau_C)_\beta = \int d\vec{x} e^{-i\vec{k}\cdot\vec{x}} \bar{D}_F(\vec{k},\tau_C)_\beta \tag{18.7.9}$$

and decomposing $\bar{D}_F(\vec{k},\tau_C)_\beta$ into parts $\bar{D}_\pm(\vec{k},\tau_C)_\beta$

$$\bar{D}_F(\vec{k},\tau_C)_\beta = \theta(\tau_C)\bar{D}_+(\vec{k},\tau_C)_\beta + \theta(-\tau_C)\bar{D}_-(\vec{k},\tau_C)_\beta \tag{18.7.10}$$

we find that the KMS condition leads to

$$\begin{aligned}
\bar{D}_+(\vec{k},\tau_C)_\beta &= f(E)\{e^{-iE\tau_C} + \alpha e^{iE(\tau_C+i\beta)}\} \\
\bar{D}_-(\vec{k},\tau_C)_\beta &= f(E)\{\alpha e^{iE\tau_C} + e^{-iE(\tau_C-i\beta)}\}
\end{aligned} \tag{18.7.11}$$

where $E^2 = \vec{k}^2 + m^2$. This yields a solution which satisfies both the differential equation and the periodicity condition, but for zero temperature ($\beta \to \infty$) we should find the usual Feynman propagator for which $\alpha = 1$. The value $\alpha = 1$ follows from hermiticity: taking the hermitian conjugate of $Tre^{-\beta H} T\varphi(x_1)\varphi(x_2)$ yields $(\bar{D}_+(\vec{k},\tau_C)_\beta)^* = \bar{D}_-(\vec{k},\tau_C)_\beta$, which implies $\alpha = 1$. It is now straightforward to verify that

$$\{-\partial_C^2 + E(\vec{k})^2\}\bar{D}_F(\vec{k},\tau)_\beta = \delta_C(\tau) \tag{18.7.12}$$

by using $\partial_C\theta_C(\tau) = \delta_C(\tau)$. One finds

$$f(E) = \frac{1}{2}\frac{1}{E(1-e^{-\beta E})} \tag{18.7.13}$$

Hence

$$\bar{D}_F(\vec{x},\tau)_\beta = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} \left[\theta_C(\tau)\frac{1}{2E}\frac{(e^{-iE\tau}+e^{iE\tau-\beta E})}{1-e^{-\beta E}} + \theta_C(-\tau)\frac{1}{2E}\frac{(e^{iE\tau}+e^{-iE\tau-\beta E})}{1-e^{-\beta E}}\right] \tag{18.7.14}$$

The propagator $\bar{D}_F(\vec{k}, \tau_1 - \tau_2)_\beta$ tends to zero for $t_0 \to \infty$ if τ_1 lies on C_1 or C_2 and τ_2 lies on C_3 or C_4 (or vice-versa), because then the factors $E^{\pm iEt_0}$ oscillate rapidly and yield no contribution (the Riemann-Lebesgue theorem[20]). Hence, for large t_0 we can neglect the vertical contours C_3 and C_4, and the path integral factorizes

$$Z_{0,\beta}[J] = Z_{0,\beta}[J; C_1 C_2] Z_{0,\beta}[J, C_3 C_4] \tag{18.7.15}$$

where

$$Z_{0,\beta}[J, C_1 C_2] = \exp -\frac{i}{2\hbar} \int_{C_1 C_2} dx dy J(x) \frac{\bar{D}_F(x-y)_\beta}{i\hbar} J(y) \tag{18.7.16}$$

with τ_x on either C_1 or C_2, and τ_y also on either C_1 or C_2. If we require that the external sources $J(\vec{x}, t)$ vanish for $t \to \pm\infty$, we can omit $Z_{0,\beta}[J, C_3 C_4]$ as it is canceled by the normalization factor of path integrals. Then

$$Z_{0,\beta}[J] = Z_{0,\beta}[J, C_1 C_2]. \tag{18.7.17}$$

Next we rewrite this path integral as a path integral on the usual time interval $-\infty < \tau < \infty$ by defining

$$J_1(\vec{x}, t) \equiv J(\vec{x}, t); J_2\left(\vec{x}, t) \equiv J(\vec{x}, t - \frac{i}{2}\beta\right) \tag{18.7.18}$$

This reduces the propagator on the complex time contour to a matrix-propagator on the usual real time doman,

$$Z_{0,\beta}[J] = \exp \frac{-i}{2\hbar} \int_{C_1} dx dy J_a(x) \frac{\bar{D}_F^{ab}(x-y)_\beta}{i\hbar} J_b(y) \tag{18.7.19}$$

where one is to sum over $a, b = 1, 2$. From these definitions we find

$$\begin{aligned}
\bar{D}_F^{11}(x-y)_\beta &= D_F(x-y)_\beta \\
\bar{D}_F^{22}(x-y)_\beta &= D_F(y-x)_\beta = D_F(x-y)^*_\beta, \text{ see (18.2.12).} \\
\bar{D}_F^{12}(x-y)_\beta &= D_-(\vec{x}-\vec{y}, x^0 - y^0 + \frac{i}{2}\beta)_\beta \\
\bar{D}_F^{21}(x-y)_\beta &= D_+(\vec{x}-\vec{y}, x^0 - y^0 - \frac{i}{2}\beta)_\beta
\end{aligned} \tag{18.7.20}$$

[20]This theorem can only be applied to functions in L^1. One should therefore keep the usual $i\epsilon$ as a Gaussian $\rho(\vec{k})$which in the limit $\epsilon \to 0$ yields the usual $\delta(k^2 + m^2)$ In fact, all higher loop calculations should be performed with these $\rho(k)$ present. This avoids ill-defined products of distributions. Only at the end one should take the limit $\epsilon \to 0$ [2].

Interactions are dealt with as in the zero-temperature case. We begin again with the complex contour

$$Z_\beta[J] = \exp \frac{i}{\hbar} \int_C L^{int} \left(\frac{\hbar}{i} \frac{\delta}{\delta J} \right) Z_{0,\beta}[J] \tag{18.7.21}$$

Since the contour C consists of the two contours C_1 and C_2, this can be written as

$$Z_\beta[J_1, J_2] = \exp \frac{i}{\hbar} \int_{C_1} \left\{ L^{int} \left(\frac{\hbar}{i} \frac{\delta}{\delta J_1} \right) - L^{int} \left(\frac{\hbar}{i} \frac{\delta}{\delta J_2} \right) \right\} Z_{0,\beta}[J_1, J_2] \tag{18.7.22}$$

There is a minus sign in the second term because the τ integration on C_1 runs from $t = -\infty$ to $t = +\infty$.

We can formally construct a path integral which leads to $Z_\beta[J_1, J_2]$

$$\begin{aligned} Z_\beta[J_1, J_2] &= \int DA^1 DA^2 \exp \left[\frac{i}{\hbar} \int A^a(x) D^{-1}_{F,ab}(x-y) A^b(y) dx dy \right. \\ & \quad + \frac{i}{\hbar} \int [\mathcal{L}^{int}(\varphi_1) - \mathcal{L}^{int}(\varphi_2) + J_a A^a \,] \, d^4x \end{aligned} \tag{18.7.23}$$

We thus find **field-doubling**: there are the original fields which appear in the Green's functions and which are denoted by $A^1(x)$, and then there are new fields $A^2(x)$ which come from the contour C_2, and which we shall identify with fields describing the heat bath. From the original definition $Tre^{-\beta H} T\varphi(x_1) \ldots \varphi(x_n)$ of thermal Green's functions, we found a local path integral on the complex time contour C, but now we have an ordinary path integral on the usual real-time contour C_1, but the number of fields has doubled. For perturbation calculations, however, one does not need to work out the inverse of D_F^{ab}, but one only needs the propagators D_F^{ab} and vertices $\mathcal{L}^{int}(\varphi^a)$. (Is the action in (18.7.23) a sum of two ordinary local actions asked van Weert.)

The complete matrix propagator follows from (18.2.13), (18.7.14) and (18.7.20) and reads

$$\begin{aligned} D_F^{11}(k)_\beta &= \left(\frac{-i}{k^2 + m^2 - i\epsilon} + 2\pi \frac{\delta(k^2 + m^2)}{e^{\beta E} - 1} \right), E > 0 \\ D_F^{12}(x-y)_\beta &= \int \frac{1}{2E} \frac{1}{1 - e^{-\beta E}} \left\{ e^{iE(t + \frac{1}{2} i\beta)} + e^{-iE((t + \frac{1}{2} i\beta) - i\beta)} \right\} \frac{e^{i\vec{k}\vec{x}}}{(2\pi)^3} d^3k \end{aligned}$$

$$= \int 2\pi\delta(k^2+m^2)\frac{e^{\frac{1}{2}\beta E}}{e^{\beta E}-1}\frac{e^{ikx}}{(2\pi)^4}d^4k \equiv \int D_F^{12}(k)_\beta \frac{e^{ikx}}{(2\pi)^4}d^4k \tag{18.7.24}$$

and $D_F^{21}(k)_\beta = D_F^{12}(k)_\beta$. In matrix notation

$$\bar{D}_F(k)_\beta = \begin{pmatrix} D_F(k) & 0 \\ 0 & D_F(k)^* \end{pmatrix} + \frac{2\pi\delta(k^2+m^2)}{e^{\beta E}-1}\begin{pmatrix} 1 & e^{\frac{1}{2}\beta E} \\ e^{\frac{1}{2}\beta E} & 1 \end{pmatrix} \tag{18.7.25}$$

Note that $\bar{D}_F$ is symmetric only because we choose the contour C_2 at $t-\frac{1}{2}i\beta$; other values (for example, $t-i\epsilon$ or $t-i\beta$) are also used in the literature but do not lead to a symmetric propagator.

For photons, one replaces E by $|\vec{k}|$. For electrons one finds in a similar manner

$$\bar{S}_F(k)_\beta = \begin{pmatrix} S_F(k) & 0 \\ 0 & S_F(k)^* \end{pmatrix} + \frac{2\pi\delta(k^2+m^2)}{e^{\beta E}+1}\begin{pmatrix} 1 & \epsilon(k_0)e^{\frac{1}{2}\beta E} \\ -\epsilon(k_0)e^{\frac{1}{2}\beta E} & 1 \end{pmatrix} \tag{18.7.26}$$

The propagator can be diagonalized by a nonunitary transformation

$$\begin{aligned} \bar{D}_F(k)_\beta &= S(E)^\dagger \bar{D}_F(k)_\infty S(E) \\ S(E) &= \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix}; \cosh^2\theta = \frac{1}{1-e^{-\beta E}} \end{aligned} \tag{18.7.27}$$

where $\bar{D}_F(k)_\infty$ is the diagonal matrix on the right-hand side of (18.7.25) with the Feynman propagator ($D_F(k) = \frac{-i}{k^2+m^2-i\epsilon}$) and its complex conjugate. The right-hand side of (18.7.27) yields

$$\begin{pmatrix} D_F + (D_F + D_F^*)\sinh^2\theta & (D_F + D_F^*)\sinh\theta\cosh\theta \\ (D_F + D_F^*)\sinh\theta\cosh\theta & D_F^* + (D_F + D_F^*)\sinh^2\theta \end{pmatrix} \tag{18.7.28}$$

and using $D_F + D_F^* = 2\epsilon/[(k^2+m^2)^2+\epsilon^2] = 2\pi\delta(k^2+m^2)$ and $\sinh^2\theta = (e^{\beta E}-1)^{-1}$, hence $\sinh\theta\cosh\theta = (e^{\frac{1}{2}\beta E} - e^{-\frac{1}{2}\beta E})^{-1}$, (18.7.27) is proven. A similar transformation of the fields A_1 and A_2 in the path integral (18.7.23) diagonalizes the kinetic terms (at the expense of making the interactions and the term JA more complicated)

$$\begin{aligned} B_1 &= \cosh\theta A_1 + \sinh A_2 \\ B_2 &= \sinh\theta A_1 + \cosh A_2 \end{aligned} \tag{18.7.29}$$

8 The canonical approach to thermal field theory

Up to this point we have rewritten an operator expression, $Tr\{(\exp -\beta\hat{H})T\varphi(x_1)\ldots\varphi(x_n)\}$, as a path integral, and deduced the Feynman rules which yield the amplitudes. These thermal Green's functions are, however, not vacuum expectation values; rather, they are sums of expectation values over all states in the Hilbert space, weighted with the Boltzmann factor. It is, however, possible to introduce a thermal vacuum $|0,\beta>$, and thermal creation and annihilation operators. The thermal creation operators, when acting on the thermal vacuum, create states in a thermal Hilbert space, and the usual formulas of quantum field theory such as the Wick contraction formula, go through. In fact, as we shall see, one can go from the vacuum and operators of ordinary (temperature zero) field theory to those of thermal field theory by a Bogoliubov transformation, which involves a thermal angle $\theta_F(\beta)$ for fermions and $\theta_B(\beta)$ for bosons. The main advantage of this canonical approach is that it puts the results of the real-time formalism in a more familiar framework, in particular it makes it clear why one needs field-doubling at nonvanishing temperature: a thermal annihilation operator can either annihilate a creation operator in the fields in the correlation functions (as in $T=0$ field theory), or it can remove a particle from the heat bath.

Hints of such formulation with field doubling came from the path integral, where we found a second contour running from right to left, and from the possibility that one could diagonalize the 2×2 propagators of the real-time formalism as in (18.7.4). In fact, let us write the 2×2 matrix of propagators in (18.7.4) as

$$D_{ab} = M_a{}^{a'}(k_0)M_b{}^{b'}(k_0)\langle 0|T_C\phi'_{a'}\phi'_{b'}|0\rangle \tag{18.8.1}$$

where ϕ_a for $a=1$ is one normal quantum field with ordinary time ordering, and ϕ_b for $b=2$ has anti-time ordering. Then it would be natural to introduce fields

$$M_a{}^c\phi_c \equiv \phi_a^{TFD} = \begin{pmatrix} \phi(\theta) \\ \tilde{\phi}^\dagger(\theta) \end{pmatrix} \tag{18.8.2}$$

where TFD stands for thermal field dynamics. One can write this as a similarity transformation

$$\phi^{TFD} = U\phi U^{-1} \tag{18.8.3}$$

where U is hermitian only if $\delta = \beta/2$. Thus a Bogoliubov transformation leads one from $a, a^\dagger$ and $\tilde{a}, \tilde{a}^\dagger$ operators to corresponding operators in thermal field dynamics which are dressed by the heat bath. We now work these ideas out.

We shall begin with a canonical approach to thermal quantum mechanics, and study the fermionic, bosonic and supersymmetric harmonic oscillator, first in the free case and then with a simple type of interaction. First we give a short summary of the partition function for the harmonic oscillator in field theory.

For real massless bosonic field theory with $\mathcal{L} = -\frac{1}{2}(\partial_\mu\varphi)(\partial^\mu\varphi)$, the Hamiltonian is given by

$$H = \int \left(\dot{\varphi}\frac{\partial}{\partial\dot{\varphi}}\mathcal{L} - \mathcal{L}\right) d^3x = \int \left[\frac{1}{2}(\partial_k\varphi)^2 + \frac{1}{2}(\partial_0\varphi)^2\right] d^3x \tag{18.8.4}$$

Using second quantization

$$\varphi(\vec{x}, t) = \sum_{\vec{k}} \sqrt{\frac{\hbar}{2\omega V}} c \left[a(\vec{k})e^{i(\vec{k}\cdot\vec{x}-\omega t)} + a(\vec{k})^\dagger e^{-i(\vec{k}\cdot\vec{x}-\omega t)}\right] \tag{18.8.5}$$

we find

$$\begin{aligned} H &= \sum_{\vec{k}} \frac{\hbar}{2\omega}\frac{1}{2}\Big[\vec{k}^2c^2\left(a(-\vec{k})e^{-i\omega t} + a(\vec{k})^\dagger e^{i\omega t})(a(\vec{k})e^{-i\omega t} + a(-\vec{k})^\dagger e^{i\omega t}\right) \\ &\quad +(a(-\vec{k})(-i\omega)e^{-i\omega t} + a(\vec{k})^\dagger(i\omega)e^{i\omega t})(a(\vec{k})(-i\omega)e^{-i\omega t} + a(-\vec{k})^\dagger(i\omega)e^{i\omega t})\ \Big] \end{aligned} \tag{18.8.6}$$

The terms with two creation or two annihilation operators are multiplied by $\vec{k}^2c^2 - \omega^2$ and vanish. (In the massive case one finds the combination $\vec{k}^2c^2 + m^2 - \omega^2$ which again vanishes.) The rest yields

$$\begin{aligned} H &= \frac{\hbar}{2\omega}\frac{1}{2}\left[(\vec{k}^2c^2 + \omega^2)(a(\vec{k})a(\vec{k})^\dagger + a(\vec{k})^\dagger a(\vec{k})\right] \\ &= \hbar\omega\left[a(\vec{k})^\dagger a(\vec{k}) + \frac{1}{2}\right] \end{aligned} \tag{18.8.7}$$

Hence, bosonic harmonic oscillators have positive zero point energy $\frac{1}{2}\hbar\omega$.

The partition function for one bosonic oscillator is given by

$$Z(\beta) = \sum_{n=0}^{\infty} e^{-\beta(n+\frac{1}{2})\hbar\omega} = \frac{1}{e^{\frac{1}{2}\beta\hbar\omega} - e^{-\frac{1}{2}\beta\hbar\omega}} \tag{18.8.8}$$

The average energy is then given by

$$\begin{aligned} < E >_\beta &= -\frac{\partial}{\partial\beta}\ln Z(\beta) = \frac{1}{2}\hbar\omega\left(\frac{e^{\frac{1}{2}\beta\hbar\omega} + e^{-\frac{1}{2}\beta\hbar\omega}}{e^{\frac{1}{2}\beta\hbar\omega} - e^{-\frac{1}{2}\beta\hbar\omega}}\right) \\ &= \frac{1}{2}\hbar\omega + \frac{\hbar\omega}{e^{\beta\hbar\omega} - 1} \end{aligned} \tag{18.8.9}$$

This is Planck's law. For low temperatures, $< E >_\beta = \frac{1}{2}\hbar\omega$ but for high temperatures $< E >_\beta = kT$, in agreement with equipartition of energy.

For a fermionic field theory, the Hamiltonian is obtained from the Dirac action $\mathcal{L} = -\hbar c\bar{\psi}\gamma^\mu\partial_\mu\psi = -\hbar c\psi^\dagger i\gamma^0\gamma^\mu\partial_\mu\psi$ in the same way

$$H = \int\left[\dot{\psi}\left(\frac{\partial}{\partial\dot{\psi}}\mathcal{L}\right) - \mathcal{L}\right]d^3x = \hbar c\int\psi^\dagger i\gamma^0\gamma^k\partial_k\psi d^3x \tag{18.8.10}$$

Since Heisenberg fields satisfy the field equations, $\gamma^k\partial_k\psi$ can be replaced by $-\gamma^0\partial_0\psi$, and one finds

$$H = i\hbar\int\psi^\dagger\dot{\psi}d^3x \tag{18.8.11}$$

In second quantization

$$\psi^\alpha(\vec{x},t) = \sum_{\vec{k},r=1,2}\frac{1}{\sqrt{V}}\left[b^r(\vec{k})u_r^{+,\alpha}(\vec{k})e^{i(\vec{k}\cdot\vec{x}-\omega t)} + d^r(\vec{k})^\dagger u_r^{-,\alpha}(-\vec{k})e^{-i(\vec{k}\cdot\vec{x}-\omega t)}\right] \tag{18.8.12}$$

where $r = 1,2$ denotes the helicities and the four spinors $u_r^\pm$ are orthonormal, $\sum_\alpha u_r^{+,\alpha}(\vec{k})^* u_s^{+,\alpha}(\vec{k}) = \delta_{rs}$ and $\sum_\alpha u_r^{+,\alpha}(\vec{k})^* u_s^{-,\alpha}(\vec{k}) = 0$. Then

$$\begin{aligned} H &= \sum_{\vec{k},r} i\hbar\left[b^r(\vec{k})^\dagger u_r^{+,\alpha}(\vec{k})^* e^{+i\omega t} + d^r(-\vec{k})u_r^{-,\alpha}(\vec{k})^* e^{-i\omega t}\right] \\ &\quad \left[b^{r'}(\vec{k})u_{r'}^{+,\alpha}(\vec{k})(-i\omega)e^{-i\omega t} + d^{r'}(-\vec{k})^\dagger u_{r'}^{-,\alpha}(\vec{k})(i\omega)e^{-i\omega t}\right] \end{aligned} \tag{18.8.13}$$

Using the orthogonality of the u's, the terms with two creations or two annihilation operators cancel, and one finds

$$\begin{aligned} H &= \sum_{\vec{k},r} \hbar\omega \left[b^r(\vec{k})^\dagger b^r(\vec{k}) - d^r(-\vec{k}) d^r(-\vec{k})^\dagger \right] \\ &= \sum_{\vec{k},r} \hbar\omega (b^r(\vec{k})^\dagger b^r(\vec{k}) - 1/2) + \hbar\omega \left(d^r(\vec{k})^\dagger d^r(\vec{k}) - \frac{1}{2} \right) \end{aligned} \tag{18.8.14}$$

Thus a fermionic harmonic oscillator has $-\frac{1}{2}\hbar\omega$ as zero point energy.

The partition function for a fermionic harmonic oscillator becomes then

$$Z(\beta) = \sum_{n=0}^{1} e^{-\beta\hbar\omega(n-\frac{1}{2})} = e^{\frac{1}{2}\beta\hbar\omega} + e^{-\frac{1}{2}\beta\hbar\omega} \tag{18.8.15}$$

and the average energy is given by

$$< E >_\beta = -\frac{\partial}{\partial\beta} \ln Z(\beta) = -\frac{\frac{1}{2}\hbar\omega(e^{\frac{1}{2}\beta\hbar\omega} - e^{-\frac{1}{2}\beta\hbar\omega})}{e^{\frac{1}{2}\beta\hbar\omega} + e^{-\frac{1}{2}\beta\hbar\omega}} = -\frac{1}{2}\hbar\omega + \frac{\hbar\omega}{e^{\beta\hbar\omega}+1} \tag{18.8.16}$$

The zero point energy is negative, and $< E >_\beta$ tends to zero for large temperatures as one would expect since both levels are then equally populated. In supersymmetric theories the sum of all zero point energies cancels, leading to a vanishing cosmological constant.

The thermal average of an operator $A(a, a^\dagger)$ depending on bosonic and/or fermionic creation and absorption operators a and $a^\dagger$ is

$$< A >_\beta = \left(\sum_n e^{-\beta E_n} < n|A|n > \right) \left(\sum_n e^{-\beta E_n} \right)^{-1} \tag{18.8.17}$$

where $|n>$ are the eigenstates of the Hamiltonian. We claim that we can rewrite this expression as a vacuum expectation value in a thermal vacuum state $|0,\beta>$. This thermal vacuum is defined by

$$|0,\beta> = \left(\sum_{n=\tilde{n}} e^{-\frac{1}{2}\beta E_n} |n> \otimes |\tilde{n}> \right) \left(\sum_n e^{-\beta E_n} \right)^{-1/2} \tag{18.8.18}$$

where the thermal Hilbert space is replaced by the direct product of two copies: the original Hilbert space H with states $|n>$ and another Hilbert space with states $|\tilde{n}>$.

The analogy with the field-doubling of the path integral approach is clear. Using that $<\tilde{m}|\tilde{n}>=\delta_{\tilde{m},\tilde{n}}$ and that $A(a,a^\dagger)$ only acts in the first but not in the second Hilbert space, we find

$$<A>_\beta=<0,\beta|A|0,\beta> \tag{18.8.19}$$

The reason for Hilbert space doubling is clear: the second Hilbert space provides the Kronecker deltas which project on the diagonal matrix elements of A in the first Hilbert space. If there were no second Hilbert space, one would find a sum over m and n involving arbitrary matrix elements $<m|A|n>$. Of course, one could add two, three, or even more, extra Hilbert spaces and still achieve the same goal. This would correspond to contours for the path integral approach which move a few times from left to right and back in the complex t plane, but nothing would be gained by this complication and we shall restrict our discussion to the minimum extension in (18.8.18).

Consider now first **the fermionic harmonic oscillator** (we begin with the fermionic oscillator because in that case one only needs to sum over two states in the Hilbert space). The Hamiltonian, and creation and annihilation operators acting in the Hilbert space are (setting $tr=1$)

$$\begin{aligned} H &= \omega(a_F^\dagger a_F-1/2);\tilde{H}=\omega(\tilde{a}_F^\dagger\tilde{a}_F-1/2) \\ \{a_F,a_F^\dagger\} &= \{\tilde{a}_F,\tilde{a}_F^\dagger\}=1;\{a_F \text{ or } a_F^\dagger,\tilde{a}_F \text{ or } \tilde{a}_F^\dagger\}=0 \end{aligned} \tag{18.8.20}$$

There are clearly four states in the Hilbert space: $|\,0>\otimes\,|\,\tilde{0}>,|\,0>\otimes\,|\,\tilde{1}>,|\,1>\otimes\,|\,\tilde{0}>$ and $|\,1>\otimes\,|\,\tilde{1}>$. We shall denote these states by $|\,0,0>,|\,0,1>$ etc. to simplify the notation. Then

$$|\,0,\beta>=\frac{1}{\sqrt{1+e^{-\beta\omega}}}(|\,0,0>+e^{-\frac{1}{2}\beta\omega}\,|\,1,1>) \tag{18.8.21}$$

The thermal vacuum $|\,0,\beta>$ can be obtained from the zero-temperature vacuum $|\,0>\otimes\,|\,\tilde{0}>$ by a "unitary" transformation

$$|\,0,\beta>_F \;=\; U_F(\beta)\,|\,0,0>_F$$

$$U_F(\beta) \;=\; \exp\{-\theta_F(\beta)(\tilde{a}_F a_F - a_F^\dagger \tilde{a}_F^\dagger)\} \tag{18.8.22}$$

The proof is easy: expanding the exponent, one finds monomials

$$\begin{aligned}(\tilde{a}_F a_F - a_F^\dagger \tilde{a}_F^\dagger)^{2n} \mid 0,0>_F &= (-)^n \mid 0,0>_F \\ (\tilde{a}_F a_F - a_F^\dagger \tilde{a}_F^\dagger)^{2n+1} \mid 0,0>_F &= (-)^{n+1} \mid 1,1>_F\end{aligned} \tag{18.8.23}$$

because each factor $\tilde{a}_F a_F$ maps $\mid 1,1>_F$ to $\mid 0,0>_F$ and each factor $a_F^\dagger \tilde{a}_F^\dagger$ maps $\mid 0,0>_F$ to $\mid 1,1>_F$. (Note the ordering of the operator.) It follows that

$$\mid 0,\beta>_F = \cos\theta_F(\beta) \mid 0,0>_F + \sin\theta_F(\beta) \mid 1,1>_F \tag{18.8.24}$$

Comparing with our earlier expression for $\mid 0,\beta>_F$ in (18.8.21) fixes the thermal angle $\theta_F(\beta)$

$$\cos\theta_F(\beta) = \frac{1}{\sqrt{1+e^{-\beta\omega}}}, \quad \sin\theta_F(\beta) = \frac{e^{-\frac{1}{2}\beta\omega}}{\sqrt{1+e^{-\beta\omega}}}. \tag{18.8.25}$$

It is then natural to consider also thermal annihilation and creation operators defined by

$$a_F(\beta) = U_F(\beta) a_F U_F^{-1}(\beta), \text{ idem for } \tilde{a}_F, a_F^\dagger \text{ and } \tilde{a}_F^\dagger \tag{18.8.26}$$

Writing $U = \exp u$ with $u = -\tilde{a}_F a_F + a_F^\dagger \tilde{a}_F^\dagger$, and using $UaU^{-1} = a + [u,a] + \frac{1}{2!}[u,[u,a]] + \ldots$, one finds easily

$$\begin{aligned}a_F(\beta) &= a_F \cos\theta_F(\beta) - \tilde{a}_F^\dagger \sin\theta_F(\beta) \\ \tilde{a}_F^\dagger(\beta) &= \tilde{a}_F^\dagger \cos\theta_F(\beta) + a_F \sin\theta_F(\beta) \\ a_F^\dagger(\beta) &= a_F^\dagger \cos\theta_F(\beta) - \tilde{a}_F \sin\theta_F(\beta) \\ \tilde{a}_F(\beta) &= \tilde{a}_F \cos\theta_F(\beta) + a_F^\dagger \sin\theta_F(\beta)\end{aligned} \tag{18.8.27}$$

It follows that

$$\begin{aligned}a_F(\beta) \mid 0,\beta> &= (U_F(\beta) a_F U_F^{-1}(\beta)) U_F \mid 0,0>_F = 0 \\ \tilde{a}_F(\beta) \mid 0,\beta> &= 0\end{aligned} \tag{18.8.28}$$

Hence, the thermal annihilation operators $a_F(\beta)$ and $\tilde{a}_F(\beta)$ annihilate the thermal vacuum. The thermal Hilbert space is spanned by the states

$$\mid 0,\beta>_F; a_F^\dagger(\beta)\mid 0,\beta>_F; \tilde{a}_F^\dagger(\beta)\mid 0,\beta>_F; a_F^\dagger(\beta)\tilde{a}_F^\dagger(\beta)\mid 0,\beta>_F \tag{18.8.29}$$

The thermal annihilation operators are linear combinations of nonthermal annihilation operators in one Hilbert space and creation operators in the other space. We may view the operators $a^\dagger(\beta)$ (and $\tilde{a}^\dagger(\beta)$) as creation operators of dressed ordinary particles (and dressed particles in the heat bath). In a physical experiment such as electron-photon scattering in a heat bath, the states in the first Hilbert space are also present at a zero-temperature, but the states in the second Hilbert space form a heat bath. Consider

$$\begin{aligned} & a_F(\beta)\mid 0,\beta>_F= \\ & (a_F\cos\theta_F(\beta)-\tilde{a}_F^\dagger\sin\theta_F(\beta))(\cos\theta_F(\beta)\mid 0,0>_F+\sin\theta_F(\beta)\mid 1,1>_F) \\ & =\cos\theta_F(\beta)\sin\theta_F(\beta)(a_F\mid 1,1>_F-\tilde{a}_F^\dagger\mid 0,0>_F)=0 \end{aligned} \tag{18.8.30}$$

We see that in this example annihilation of an ordinary particle (a_F) is equivalent to creation of a particle in the heat bath ($\tilde{a}_F^\dagger$). In other words, adding a quantum to the heat bath, a thermal particle ($a_F^\dagger(\beta)\mid 0,\cdot>$) loses energy with respect to the average energy of the heat bath. Similarly

$$\tilde{a}_F(\beta)\mid 0,\beta>_F=\cos\theta_F(\beta)\sin\theta_F(\beta)(\tilde{a}_F\mid 1,1>_F+a_F^\dagger\mid 0,0>_F)=0 \tag{18.8.31}$$

shows that a dressed particle in the heat bath may lose energy ($\tilde{a}_F(\beta)$) either by losing one of its constituent heath-bath particles ($\tilde{a}_F$) or by adding a quantum in the non-heat-bath sector ($a_F^\dagger$).

The thermal states $(a_F^\dagger(\beta))^n(\tilde{a}_F^\dagger(\beta))^{\tilde{n}}\mid 0,\beta>$ are not eigenstates of the Hamiltonian $H=\omega(a_F^\dagger a_F-1/2)$, but they are eigenstates of $H-\tilde{H}$. This can be checked explicitly

$$\begin{aligned} H-\hat{H} &= \sum\omega(a_F^\dagger a_F-\tilde{a}_F^\dagger\tilde{a}_F) \\ &= \sum\omega(a_F^\dagger(\beta)a_F(\beta)-\tilde{a}_F^\dagger(\beta)\tilde{a}_F(\beta)) \end{aligned} \tag{18.8.32}$$

A more elegant way to understand this β independence of the total Hamiltonian $H - \tilde{H}$ is to note that the operators a and $\tilde{a}^\dagger$ form doublets under the action of U

$$A_F = \begin{pmatrix} a_F \\ \tilde{a}_F^\dagger \end{pmatrix}; A_F(\beta) = U A_F U^{-1} = \begin{pmatrix} a_F(\beta) \\ \tilde{a}_F^\dagger(\beta) \end{pmatrix} = \begin{pmatrix} \cos\theta_F(\beta) & -\sin\theta_F(\beta) \\ \sin\theta_F(\beta) & \cos\theta_F(\beta) \end{pmatrix} A_F \tag{18.8.33}$$

Similarly $a_F^\dagger$ and $\tilde{a}_F$ form a doublet

$$\tilde{A}_F = \begin{pmatrix} a_F^\dagger \\ \tilde{a}_F \end{pmatrix} \to \tilde{A}_F(\beta) = \Big(\text{same matrix}\Big) \tilde{A}_F(\beta) \tag{18.8.34}$$

The inner product $\tilde{A}_F \cdot A_F - 1 = a_F^\dagger a_F + \tilde{a}_F \tilde{a}_F^\dagger - 1 = a_F^\dagger a_F - \tilde{a}_F^\dagger \tilde{a}_F$ is proportional to the total Hamiltonian in (18.8.32), and is clearly rotationally invariant. The minus sign in front of $\tilde{H}$ in $H - \tilde{H}$ can better be understood from the path integral point of view; it is due to changing the orientation of the second contour such that it runs in the same direction as the first contour.

Next we consider **the bosonic harmonic oscillator.** We start from

$$\begin{aligned} H &= \sum \omega(a_B^\dagger a_B + 1/2), [a, a^\dagger] = [\tilde{a}_B, \tilde{a}_B^\dagger] = 1 \\ & \quad [a_B \text{ or } a_B^\dagger, \tilde{a}_B \text{ or } \tilde{a}_B^\dagger] = 0 \end{aligned} \tag{18.8.35}$$

The bosonic thermal vacuum is according to (18.8.17) given by

$$| \, 0, \beta >_B = \sqrt{1 - e^{-\beta\omega}} \sum_{n=0}^{\infty} e^{-\frac{1}{2} n\beta\omega} \, | \, n, \tilde{n} >_B \tag{18.8.36}$$

The unitary (Bogoliubov) transformation which maps the zero-temperature vacuum $| \, 0, 0 >$ onto the thermal vacuum is

$$\begin{aligned} | \, 0, \beta >_B &= U_B(\beta) \, | \, 0, 0 >_B \\ U_B(\beta) &= \exp\{-\theta_B(\beta) u_B\}; u_B = \tilde{a}_B a_B - a_B^\dagger \tilde{a}_B^\dagger \end{aligned} \tag{18.8.37}$$

Before proving this and fixing $\theta_B(\beta)$, we first construct the thermal creation and annihilation operators

$$\begin{aligned} a_B(\beta) &= U_B(\beta) a_B U_B^{-1} = a_B - \theta_B(\beta)[u_B, a_B] + \frac{1}{2}\theta_B(\beta)^2 [u_B, [u_B, a_B]] + \ldots \\ &= a_B - \theta_B(\beta) \tilde{a}_B^\dagger + \frac{1}{2}\theta_B(\beta)^2 a_B + \ldots \\ &= \cosh\theta_B(\beta) a_B - \sinh\theta_B(\beta) \tilde{a}_B^\dagger \end{aligned} \tag{18.8.38}$$

Similarly

$$\begin{aligned}\tilde{a}_B^\dagger(\beta) &= U_B(\beta)\tilde{a}_B^\dagger U_B(\beta)^{-1} = \tilde{a}_B^\dagger - \theta_B(\beta)[u_B, \tilde{a}_B^\dagger] + \dots \\ &= \cosh\theta_B(\beta)\tilde{a}_B^\dagger - \sinh\theta_B(\beta) a_B \end{aligned} \tag{18.8.39}$$

Hence, a_B and $\tilde{a}_B^\dagger$ form again a doublet

$$A(\beta) = \begin{pmatrix} a_B(\beta) \\ \tilde{a}_B^\dagger(\beta) \end{pmatrix} = \begin{pmatrix} \cosh\theta(\beta) & -\sinh\theta(\beta) \\ -\sinh\theta(\beta) & \cosh\theta(\beta) \end{pmatrix} \begin{pmatrix} a_B \\ \tilde{a}_B^\dagger \end{pmatrix} \tag{18.8.40}$$

Similarly

$$\begin{aligned}\tilde{a}_B(\beta) &= \tilde{a}_B - \theta_B(\beta)[u_B, \tilde{a}_B] + \dots = \tilde{a}_B - \theta_B(\beta) a_B^\dagger + \dots \\ &= \cosh\theta_B(\beta)\tilde{a}_B - \sinh\theta_B(\beta) a_B^\dagger \\ a_B^\dagger(\beta) &= a_B^\dagger - \theta_B(\beta)[u_B, a_B^\dagger] + \dots = a_B^\dagger - \theta_B(\beta)\tilde{a}_B \\ &= \cosh\theta_B(\beta) a_B^\dagger - \sinh\theta_B(\beta)\tilde{a}_B \\ \tilde{A}(\beta) &= \begin{pmatrix} a_B^\dagger(\beta) \\ \tilde{a}_B(\beta) \end{pmatrix} = \begin{pmatrix} \cosh\theta_B(\beta) & -\sinh\theta_B(\beta) \\ -\sinh\theta_B(\beta) & \cosh\theta_B(\beta) \end{pmatrix} \begin{pmatrix} a_B^\dagger \\ \tilde{a}_B \end{pmatrix}\end{aligned} \tag{18.8.41}$$

It is obvious from (18.8.37) and (18.8.40) that $a_B(\beta)$ and $\tilde{a}_B(\beta)$ annihilate the thermal vacuum. Furthermore, since the transformation matrix from A to $A(\beta)$ is a Lorentz matrix, we have

$$\tilde{A}_\mu(\beta)\eta^{\mu\nu}A_\nu(\beta) \equiv a_B^\dagger(\beta)a_B(\beta) - \tilde{a}_B(\beta)\tilde{a}_B^\dagger(\beta) = a_B^\dagger a_B - \tilde{a}_B\tilde{a}_B^\dagger \tag{18.8.42}$$

Hence, $H - \hat{H}$ is invariant under the Bogoliubov transformations

$$H - \tilde{H} = H(\beta) - \tilde{H}(\beta) = U_B(\beta)(H - \tilde{H})U_B(\beta)^{-1} \tag{18.8.43}$$

This shows that the thermal states obtained by acting with the thermal creation operators $a_B^\dagger(\beta)$ and $\tilde{a}_B^\dagger(\beta)$ on the thermal vacuum $\mid 0, \beta >$ are again eigenstates of the total Hamiltonian $H - \hat{H}$.

The only task left is to determine $\theta(\beta)$, and to prove that $\mid 0,\beta >_B = U_B(\beta) \mid 0,0 >_B$. Expanding $U_B(\beta)$, we find monomials

$$(\tilde{a}_B a_B - a_B^\dagger \tilde{a}_B^\dagger)^n \mid 0,0 > \tag{18.8.44}$$

but now this series is not so easily summed because for fixed n there are several terms which are proportional to a state $\mid k, \vec{k} >$ for $k < n$. Rather, we employ a trick and evaluate the thermal average of the number operator $a_B a_B^\dagger$

$$\begin{aligned} < a_B a_B^\dagger >_\beta &= \left(\sum_{n=0}^{\infty} n e^{-\beta\omega n}\right) \Big/ \left(\sum_{n=0}^{\infty} e^{-\beta\omega n}\right) \\ &= (1-e^{-\beta\omega})\left(-\frac{\partial}{\partial(\beta\omega)}\right)\left(\frac{1}{1-e^{-\beta\omega}}\right) = \frac{1}{1-e^{-\beta\omega}} \end{aligned} \tag{18.8.45}$$

On the other hand, from the inverse relations between A and $A(\beta)$ we find

$$A = \begin{pmatrix} \cosh\theta_B(\beta) & \sinh\theta_B(\beta) \\ \sinh\theta_B(\beta) & \cosh\theta_B(\beta) \end{pmatrix} A(\beta), \text{idem } \tilde{A}(\beta) \tag{18.8.46}$$

we obtain

$$a_B a_B^\dagger = \{\cosh\theta_B(\beta) a_B(\beta) + \sinh\theta_B(\beta)\tilde{a}_B^\dagger(\beta)\}\{\cosh\theta_B(\beta) a_B^\dagger(\beta(+\sinh\theta_B(\beta)\tilde{a}_B(\beta)\} \tag{18.8.47}$$

Hence

$$_B < 0,\beta \mid a_B a_B^\dagger \mid 0,\beta >_B = \cosh\theta(\beta)^2 \tag{18.8.48}$$

By comparing both expressions for $< aa^\dagger >_\beta$ we find

$$\cosh\theta_B(\beta) = \frac{1}{\sqrt{1-e^{-\beta\omega}}}, \sinh\theta(\beta) = \frac{e^{-\frac{1}{2}\beta\omega}}{\sqrt{1-e^{-\beta\omega}}}. \tag{18.8.49}$$

As expected, the bosonic angle θ_B is related to the Bose-Einstein factor.

To prove that $\mid 0,\beta >_B$ is equal to $U_B(\beta) \mid 0,0 >_B$ for this value of $\theta_B(\beta)$, it is sufficient to show that they have the same inner products with all thermal states $a_B^\dagger(\beta)^k \mid 0,\beta >_B, k = 0,1,2,\ldots$. This is immediately obvious if one uses that $a_B(\beta) \mid 0,\beta >_B = 0$ and $a_B^\dagger(\beta)^k \mid 0,\beta >_B = U_B(\beta) a_B^\dagger k \mid 0,0 >_B$.

We consider now **the supersymmetric harmonic** oscillator. The action is given by

$$\mathcal{L} = \frac{1}{2}\dot{q}^2 + i\psi^\dagger\dot{\psi} + \frac{1}{2}F^2 - FW'(q) - W''(q)\psi^\dagger\psi \tag{18.8.50}$$

We have chosen a system with $N = 2$ susy in order to obtain Dirac (complex) instead of Majorana (real) fermions; this avoids Dirac brackets for the quantization. It is easy to check that the action is invariant under

$$\begin{array}{ll} \delta q = \frac{1}{\sqrt{2}}\psi^\dagger\epsilon & \bar{\delta} q = \frac{1}{\sqrt{2}}\epsilon^\dagger\psi \\ \delta\psi = -\frac{i}{\sqrt{2}}\dot{q}\epsilon - \frac{1}{\sqrt{2}}F\epsilon & \bar{\delta}\psi = 0 \\ \delta\psi^\dagger = 0 & \bar{\delta}\psi^\dagger = \frac{i}{\sqrt{2}}\dot{q}\epsilon^\dagger - \frac{1}{\sqrt{2}}F\epsilon^\dagger \\ \delta F = \frac{-i}{\sqrt{2}}\dot{\psi}^\dagger\epsilon & \delta F = \frac{i}{\sqrt{2}}\epsilon^\dagger\dot{\psi} \end{array} \tag{18.8.51}$$

The susy algebra is $[\delta(\epsilon_1), \delta(\epsilon_2)] = [\bar{\delta}(\epsilon_1^\dagger), \bar{\delta}(\epsilon_2^\dagger)] = 0$ and $[\delta(\epsilon_1), \bar{\delta}(\epsilon_2^\dagger)] = (-i\epsilon_2^\dagger\epsilon_1)\frac{\partial}{\partial\tau}$. The classical Hamiltonian is

$$H = \frac{1}{2}p^2 + \left(FW'(q) - \frac{1}{2}F^2\right) + W''(q)\psi^\dagger\psi \tag{18.8.52}$$

and the nonvanishing quantum (anti)commutators are

$$[q, p] = i, \{\psi, \psi^\dagger\} = 1 \tag{18.8.53}$$

For local ϵ and $\epsilon^\dagger$ the action varies into the Noether current, $\delta\mathcal{L} = Q^\dagger\dot{\epsilon} + \dot{\epsilon}^\dagger Q$, where

$$\begin{aligned} Q^\dagger &= \frac{1}{\sqrt{2}}\left(p - iW'\right)\psi^\dagger; Q = \frac{1}{\sqrt{2}}(p + iW')\psi \\ &\{Q, Q\} = \{Q^\dagger, Q^\dagger\} = 0; \{Q^\dagger, Q\} = H = \frac{1}{2}p^2 + \frac{1}{2}W'^2 + \frac{1}{2}W''[\psi^\dagger, \psi] \end{aligned} \tag{18.8.54}$$

As always, the auxiliary fields have disappeared in the Hamiltonian formalism and the Hamiltonian comes out Weyl-ordered.

The free harmonic oscillator corresponds to $W = \frac{1}{2}\omega q^2$. Then $W' = \omega q$ and we define

$$\begin{aligned} \frac{1}{\sqrt{2\omega}}(p - i\omega q) &= a_B \; ; \; \frac{1}{\sqrt{2\omega}}(p + i\omega q) = a_B^\dagger \\ \psi &= a_F \; ; \; \psi^\dagger = a_F^\dagger \end{aligned} \tag{18.8.55}$$

The energy is bounded from below. The susy generators then read

$$Q = \sqrt{\omega} a_B^\dagger a_F, Q^\dagger = \sqrt{\omega} a_F^\dagger a_B \tag{18.8.56}$$

The zero-temperature Hilbert space consists of states

$$\mid n_B, n_F >= (a_B^\dagger)^{n_B} / \sqrt{n_B!} (a_F^\dagger)^{n_F} \mid 0, 0 > \tag{18.8.57}$$

with energy

$$H \mid n_B, n_F >= \omega(a_B^\dagger a_B + a_F^\dagger a_F) \mid n_B, n_F >= \omega(n_B + n_F) \mid n_B, n_F > \tag{18.8.58}$$

The zero-temperature vacuum is supersymmetric

$$Q \mid 0, 0 >= Q^\dagger \mid 0, 0 >= 0 \tag{18.8.59}$$

Since Q maps the state $\mid n_B, n_F >$ to $\mid n_B + 1, n_F - 1 >$, and $Q^\dagger$ maps $\mid n_B, n_F >$ to $\mid n_B - 1, n_F + 1 >$, all states can be grouped in bose-fermi pairs with equal energy, $\mid n_B + 1, n_F >$ and $\mid n_B, n_F + 1 >$, except the ground state which is bosonic (by definition) and has no fermionic partner. Hence the Witten index is nonzero

$$tr(-)^F = tr(1 - 2a_F^\dagger a_F) = +1 \tag{18.8.60}$$

Consider now this system at finite temperature. We double all operators as before, and consider states $\mid n_B, n_F > \otimes \mid \tilde{n}_B, \tilde{n}_F >$ which we write as $\mid n_B, n_F; \tilde{n}_B, \tilde{n}_F >$. The Bogoliubov operator is now the product of the Bogoliubov operators in the fermionic and bosonic Hilbert spaces, $U = U_B(\theta_B(\beta))U_F(\theta_F(\beta))$, and all four thermal annihilation operators $a_B(\beta), a_F(\beta), \tilde{a}_B(\beta)$ and $\tilde{a}_F(\beta)$ annihilate the thermal vacuum $\mid 0, \beta >_B \otimes \mid 0, \beta >_F \equiv \mid 0, \beta >$.

We shall now study whether susy is broken at finite temperature. From our study of the effective potential computed using the imaginary-time formalism we certainly expect that susy will be broken at nonzero temperature, but the issue in the canonical formalism is rather which operators should be used to study susy breaking. Let us first collect some facts.

(i) The energy (by which we mean the expectation value of H, not $H-\tilde{H}$) of the thermal vacuum is nonvanishing

$$< 0,\beta \mid H \mid 0,\beta >=_B< 0,\beta \mid \omega a_B^\dagger a_B \mid 0,\beta >_B +_F < 0,\beta \mid \omega a_F^\dagger a_F \mid 0,\beta >_F$$
$$= \omega \left[\cosh^2\theta_B(\beta) + \cos^2\theta_F(\beta)\right] = \omega \left(\frac{e^{-\beta\omega}}{1-e^{-\beta\omega}} + \frac{e^{-\beta\omega}}{1+e^{-\beta\omega}} \right) \qquad (18.8.61)$$

(ii) The Witten index (by which we mean the trace of $(-)^F$, not $(-)^{F+\tilde{F}}$) in the thermal Hilbert space is nonvanishing and fractional

$$Tr(-)^F = Tr(1-2a_F^\dagger a_F) = Tr\left[1 - 2\cos\theta_F(\beta)a_F^\dagger(\beta) + \sin\theta_F(\beta)\tilde{a}_F(\beta)\}\right.$$
$$\left.\{\cos\theta_F(\beta)a_F(\beta) + \sin\theta_F(\beta)\tilde{a}_F^\dagger(\beta)\}\ \right]$$
$$= Tr\left[1 - 2\cos^2\theta_F(\beta)a_F^\dagger(\beta)a_F(\beta) - 2\sin^2\theta_F(\beta)\tilde{a}_F(\beta)\tilde{a}_F^\dagger(\beta)\right]$$
$$=< 0,\beta \mid (-)^F \mid 0,\beta >= 1 - 2\sin^2\theta_F(\beta) = \frac{1-e^{-\beta\omega}}{1+e^{\beta\omega}} \qquad (18.8.62)$$

(iii) The thermal vacuum is not supersymmetric (not invariant under the zero-temperature susy charges)

$$Q \mid 0,\beta >= \sqrt{\omega}a_B^\dagger a_F \mid 0,\beta >=$$
$$= \sqrt{\omega}\left\{\cosh\theta_B(\beta)a_B^\dagger(\beta) + \sinh\theta_B(\beta)\tilde{a}_B(\beta)\right\}\{\cos\theta_F(\beta)a_F(\beta) +$$
$$\left.\sin\theta_F(\beta)\tilde{a}_F^\dagger(\beta)\right\}$$
$$= \sqrt{\omega}\cosh\theta_B(\beta)\sin\theta_F(\beta) \mid n_B(\beta) = 1, \tilde{n}_B(\beta) = 0, n_F(\beta) = 0, \tilde{n}_F^{(\beta)} = 1 >$$
$$= \frac{\sqrt{\omega}e^{-\frac{1}{2}\beta\omega}}{[(1-e^{-\beta\omega})(1+e^{-\beta\omega})]^{1/2}} \mid 1,0,0,1 > \qquad (18.8.63)$$

Similarly

$$Q^\dagger \mid 0,\beta >= \sqrt{\omega}a_F^\dagger a_B \mid 0,\beta >= \sqrt{\omega}\{\cos\theta_F(\beta)a_F^\dagger(\beta) + \sin\theta_F(\beta)\tilde{a}_F(\beta)\}$$
$$\{\cosh\theta_B(\beta)a_B(\theta) + \sinh\theta_B(\beta)\tilde{a}_B^\dagger(\beta)\}$$
$$= \frac{\sqrt{\omega}e^{-\frac{1}{2}\beta\omega}}{[(1-e^{-\beta\omega})(1+e^{-\beta\omega}]^{1/2}} \mid 0,1,1,0 >_\beta \qquad (18.8.64)$$

(iv) The Goldstone fermions (Goldstinos) are the states $\mid \chi >_\beta$ which are generated by fermionic field operators $\hat{\chi}$ for which the vacuum expectation value of the susy transformation is nonvanishing:

$$< 0,\beta \mid \{Q,\chi\} \mid 0,\beta > = < 0,\beta \mid Q \mid \chi >_\beta + _\beta < \chi \mid Q \mid 0,\beta > \neq 0 \tag{18.8.65}$$

and idem for $\{Q^\dagger,\chi\}$. It is clear that the states $Q \mid 0,\beta >$ and $Q^\dagger \mid 0,\beta >$ are the Goldstinos because $< 0,\beta \mid Q^\dagger Q \mid 0,\beta > = > \|Q \mid 0,\beta > \|^2 \neq 0$.

Finally we consider an interacting susy quantum mechanical system, the susy anharmonic oscillator. We take $W(q) = \xi q + \frac{1}{2}\omega q^2 + \frac{1}{3}gq^2$, where ξ is the Fayet-Iliopoulos term, and ω and g are taken positive. Then the quantum Hamiltonian is

$$H = \frac{1}{2}p^2 + \frac{1}{2}(\xi + \omega q + gg^2)^2 + \frac{1}{2}(\omega + 2gq)[\psi^\dagger,\psi] \tag{18.8.66}$$

The classical potential at zero temperature is

$$V^{(0)}(\beta = 0) = \frac{1}{2}\left[g\left(q + \frac{\omega}{2g}\right)^2 + \left(\xi - \frac{\omega^2}{4g}\right)\right]^2 \tag{18.8.67}$$

For $\xi > \omega^2/(4g)$, its minimum is positive and at $v = -\omega/(2g)$. Decomposing $q = \sigma + v$, we find a zero-temperature Goldstino, as expected since susy is spontaneously broken

$$H = \frac{1}{2}p^2 + \frac{1}{2}\left(g\sigma^2 + \xi - \frac{\omega^2}{4g}\right)^2 + g\sigma[\psi^\dagger,\psi] \tag{18.8.68}$$

For $\xi < \omega^2/(4g)$, the minimum of the potential is zero and occurs at $v = -\omega/(2g) \pm (\omega^2/(4g) - \xi)^{1/2}$. Then

$$H = \frac{1}{2}p^2 + \frac{1}{2}\left(\pm 2g\sigma\left(\frac{\omega^2}{4g} - \xi\right)^{1/2} + g\sigma^2\right)^2 + g\left\{\sigma \pm \left(\frac{\omega^2}{4g} - \xi\right)^{1/2}\right\}[\psi^\dagger,\psi] \tag{18.8.69}$$

Now susy is classically unbroken, and the fermion is massive. Expressing H^{int} in terms of thermal creation and annihilation operators, just as in the case of the free system, we find

$$\begin{aligned} H^{int} &= \frac{1}{2}\left[\xi - \frac{g}{2\omega}(a_B - a_B^\dagger)^2\right]^2 \\ &+ i\sqrt{\frac{\omega}{2}}(a_B - a_B^\dagger)\left[\xi - \frac{g}{2\omega}(a_B - a_B^\dagger)^2\right] \\ &+ \frac{ig}{\sqrt{2\omega}}(a_B - a_B^\dagger)[a_F, a_F^\dagger] \end{aligned} \tag{18.8.70}$$

Expressing the zero-temperature creation and annihilation operators by their thermal equivalents, we can evaluate the thermal expectation value of the interaction Hamiltonian

$$\begin{aligned} &< 0,\beta \mid H^{int} \mid 0,\beta >= \frac{1}{2}\xi^2 - \frac{\xi g}{2\omega}\{-2\sinh\theta_B(\beta)\cosh\theta_B(\beta) \\ - &\cosh^2\theta_B(\beta) - \sinh\theta_B^2(\beta)\ \} \\ = &\frac{1}{2}\xi^2 + \frac{\xi g}{2\omega}\frac{1+e^{-1/2\beta\omega}}{1-e^{-1/2\beta\omega}} + \frac{3g^2}{8\omega^2}\left(\frac{1+e^{-\frac{1}{2}\beta\omega}}{1-e^{-1/2\beta\omega}}\right)^2 \\ = &\frac{1}{2}(\xi + \frac{g}{2\omega}\frac{1+e^{-\frac{1}{2}\beta\omega}}{1-e^{-1/2\beta\omega}})^2 + \frac{g^2}{4\omega^2}\left(\frac{1+e^{-1/2\beta\omega}}{1-e^{-1/2\beta\omega}}\right) \end{aligned} \tag{18.8.71}$$

Adding the expectation value of H^0 obtained before

$$< 0,\beta \mid H^0 \mid 0,\beta >= \omega\left(\frac{e^{-\beta\omega}}{1-e^{-\beta\pi}} + \frac{e^{-\beta\omega}}{1+e^{-\beta\omega}}\right) \tag{18.8.72}$$

it is clear that the total energy is positive, indicating that susy is broken at the quantum level, no matter what the sign of ξ is.

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Chapter 19

Quantum Chern-Simons theory in 3 dimensions

1 Quantum Chern-Simons theory

The Chern-Simons action for Yang-Mills theory in three Euclidean dimensions reads

$$S(CS) = \frac{-ik}{4\pi} \int \epsilon^{\mu\nu\rho} \left\{ \frac{1}{2} A_\mu{}^a \partial_\nu A_\rho{}^a + \frac{1}{6} f_{abc} A_\mu{}^a A_\nu{}^b A_\rho{}^c \right\} d^3x \tag{19.1.1}$$

The factor i is absent in Minkovski spacetime where the action is real, and is due to the Wick rotation of ∂_ν and A_ν. Under a gauge transformation $\delta A_\mu{}^a = D_\mu \lambda^a$, the Lagrangian transforms into a total derivative

$$\begin{aligned}
\frac{\delta \mathcal{L}(CS)}{(-ik/4\pi)} &= \epsilon^{\mu\nu\rho} \left\{ (D_\mu \lambda^a) \partial_\nu A_\rho{}^a + \frac{1}{2} f_{abc} (D_\mu \lambda^a) A_\nu{}^b A_\rho{}^c + \frac{1}{2} \partial_\nu (A_\mu{}^a D_\rho \lambda^a) \right\} \\
&= \frac{1}{2} \epsilon^{\mu\nu\rho} \left\{ (D_\mu \lambda^a)(G_{\nu\rho}{}^a) + \partial_\nu (A_\mu{}^a D_\rho \lambda^a) \right\} \\
&= \frac{1}{2} \epsilon^{\mu\nu\rho} \partial_\mu \left[\lambda^a G_{\nu\rho}{}^a - A_\nu{}^a D_\rho \lambda^a \right] \\
&= \frac{1}{2} \epsilon^{\mu\nu\rho} \partial_\mu (\lambda^a \partial_\nu A^a_\rho)
\end{aligned} \tag{19.1.2}$$

(In the one but last step we used the Bianchi identity). For a large gauge transformation one finds then a surface integral

$$\oiint \frac{1}{2}\epsilon^{\mu\nu\rho}\lambda^a \partial_\nu A_\rho{}^a d\Omega_\mu = A \tag{19.1.3}$$

and gauge invariance would seem to require that $\exp \frac{-ik}{4\pi} A = 1$, or $kA = 8\pi^2 n$ with n an integer. However, this argument is incomplete since no gauge fixing terms have been included.

It is useful to define $\frac{k}{4\pi} = g^{-2}$ and $A_\mu{}^a = gA'^a_\mu$; then the action takes on the form

$$S = -i \int \epsilon^{\mu\nu\rho} \left\{ \frac{1}{2} A'^a_\mu \partial_\nu A'^a_\rho + \frac{1}{6} g f^{abc} A'^a_\mu A'^b_\nu A'^c_\rho \right\} d^3x \tag{19.1.4}$$

We shall use this latter form, but drop the primes for notational simplicity. The ϵ tensor requires a subtle treatment in dimensional regularization, as we shall see. Adding the following Landau gauge-fixing and ghost terms to the action

$$S(\text{fix} + \text{ghost}) = \int d^3x \left\{ -d^a \partial^\mu A^a_\mu + b^a \partial^\mu D_\mu c^a \right\} \tag{19.1.5}$$

one can write down, at least in 3 dimensions, the propagators and vertices, and study the divergences by power counting. The kinetic matrix of the gauge field and auxiliary field is

$$\frac{1}{2} \left(A^a_\mu, d^a \right) \delta_{ab} \begin{pmatrix} \epsilon^{\mu\rho\nu} p_\rho & ip^\mu \\ -ip^\nu & 0 \end{pmatrix} \begin{pmatrix} A^b_\nu \\ d^b \end{pmatrix} \tag{19.1.6}$$

and the naive, 3-dimensional, propagators are

$$\begin{aligned} < A^a_\mu(p) A^b_\nu(-p) > &= \epsilon_{\mu\rho\nu} \frac{p^\rho}{p^2} \delta^{ab} \\ < A^a_\mu(p) d^b(-p) > &= -i \frac{p_\mu}{p^2} \delta^{ab} = - < d^a(p) A_\mu(-p) > \\ < d^a(p) d^a(-p) > &= 0; < c^a(p) b^b(-p) >= \frac{\delta^{ab}}{p^2} \end{aligned} \tag{19.1.7}$$

For a 1PI graph with E_A external A legs, E_d external d legs, $(E_b)E_c$ external (anti)ghost legs, and I_{AA}, I_{Ad}, I_{bc} propagators, and V_{AAA} CS vertices and V_{bAc} ghost vertices, one finds the following degree of divergence of an L-loop graph.

$$D = -I_{AA} - I_{Ad} - 2I_{bc} + V_{bAc} + 3L - E_b \tag{19.1.8}$$

Since each A line (either external or internal) has to end at a vertex, one has

$$2I_{AA} + I_{Ad} + E_A = 3V_{AAA} + V_{bAc} \tag{19.1.9}$$

The same argument gives for the d, b and c lines

$$\begin{aligned} I_{Ad} + E_d &= 0 \\ 2I_{bc} + E_b + E_c &= 2V_{bAc} \end{aligned} \tag{19.1.10}$$

Since there are (at least in this action) no vertices with d lines, $I_{Ad} = 0$ and $E_d = 0$, and since ghost number is conserved, $E_b = E_c$. Since at each vertex energy-momentum conservation eliminates one independent 3-momentum, one finds the usual topological relation

$$L = I_{AA} + I_{bc} - V_{AAA} - V_{bAc} + 1 \tag{19.1.11}$$

Substituting back into D, one finds

$$D = 3 - E_A - E_b - E_c \tag{19.1.12}$$

It follows that the only divergent 1PI graphs are:

(i) the vacuum selfenergies (cubically divergent). These we will no further consider.

(ii) the AA and bc selfenergies (linearly divergent)

(iii) the AAA and bAc vertices (logarithmically divergent)

The theory has thus a chance to be renormalizable. As we shall see, the theory is actually <u>finite</u>.

To compute loops, one has to define a regularization scheme. A very simple scheme would be to replace in each propagator p^{-2} by $p^{-2} - (p^2 + M^2)^{-1}$, i.e. the $\frac{1}{p^2}$ are replaced by $\frac{M^2}{p^2(p^2+M^2)}$. Each propagator acquires an extra power p^{-2},and power counting yields now $D = 3 - 3V_{AAA} - 3V_{bAc}$, showing that all graphs are finite (there are no tadpole graphs since all fields carry colour indices). For the AA selfenergy graph due to an A loop one finds then an expression of the form

$$S\,(A \text{ loop }) = \int \frac{\epsilon\epsilon\epsilon\epsilon\; q(q+p)\; d^3q}{q^2(q^2+M^2)(q+p)^2[(q+p)^2+M^2]} \tag{19.1.13}$$

The contribution to the AA selfenergy due to a ghost loop is of the same form, but without ϵ tensors in the numerator. Evaluating the contractions of the ϵ tensors in 3 dimensions, one finds that the sum of these two integrands cancels! The same holds for the one-loop corrections to the AAA and bAc vertices: again in each case there are two diagrams whose integrands sum up to zero (with 6 and no ϵ tensors in the first case, and with one and three ϵ tensors in the second case). At the two-loop level, the same results are found! One might therefore be tempted to conclude that "the theory" has no quantum corrections at all [1].

Actually, this regularization scheme is not gauge-invariant, and we shall later see that there exist gauge-invariant regularization schemes which yield finite, local but nonzero contributions at the one-and two-loop level. To see that this scheme is not gauge-invariant it suffices to write down the corresponding action which gives these propagators and vertices. It clearly reads

$$\begin{aligned} S \quad &\sim \quad \frac{1}{2}A_\mu{}^a\left(1-\frac{\partial^2}{M^2}\right)\partial_\nu A_\rho{}^a + \text{ same } A^3 \text{ term as before} \\ &+ \quad b_a\left(1-\frac{\partial^2}{M^2}\right)\partial^2 c^a + \text{ same } bAc \text{ term } - d_a(1-\frac{\partial^2}{M^2})\partial^\mu A_\mu{}^a \end{aligned} \tag{19.1.14}$$

The bare derivatives in ∂^2 destroy the gauge invariance of the classical action.

The reduced theory in the gauge $A_0{}^a = 0$ gives the same results for the gauge-invariant parts of the one-loop effective action[8] (the parameter α, see below) as the gauge-invariant schemes, but disagrees with the "naive-Pauli-Villars scheme" described above. Also general arguments by Witten [2] state that $< 0|\mathcal{O}|0 >$ with

$\mathcal{O}$ given by products of $tr P \exp \oint \vec{A}\vec{dx}$ (Wilson loops) are functions $f(q)$ with $q = \exp \frac{2\pi i}{\frac{1}{\hbar}k+N}$. (Since these correspond to connected graphs, there are also two- and higher-loop corrections. These are obtained by expanding $f(q)$ in terms of N. The 1PI graphs are the ones which only receive one-loop corrections[1]). Again this seems to disagree with the vanishing of the 2-and 3-point functions.

One might try to construct a gauge-invariant regularization scheme by adding a gauge invariant higher derivative term to the action

$$S(reg) = \frac{1}{4m^{1+2n}} \int tr(D^n F_{\nu\rho})(D^n F^{\nu\rho}) d^3x \tag{19.1.15}$$

For $n = 0$ the term $\frac{1}{m} tr F_{\mu\nu} F^{\mu\nu}$ regulates the 3-and higher-loop graphs but not the one-and two-loop graphs, whereas for $n = 1$ the term $\frac{1}{m^3}(D_\mu F_{\nu\rho})^2$ regulates also the two-loop graphs, but not the one-loop graphs. In fact, for no n are the one-loop graphs regulated by this higher-derivative scheme since in a one-loop graph there are as many vertices as propagators and what each propagator gains in convergence each vertex loses in a nonabelian theory. For the one-loop graphs one may then use Pauli-Villars regularization, as studied in detail by Lee and Slavnov. There exist, actually, two PV schemes for gauge-theories, a gauge noncovariant one and a gauge-covariant one. As discussed in the section on PV regularization, the latter does not regulate all one-loop graphs. (For example, there are extra PV ghost loops due to the extra vertices introduced by covariantization of the PV ghost actions.)

Using the covariant PV scheme, Alvarez-Gaumé et al. [3] found for the one-loop corrections to the two- and three-point function

$$\begin{aligned} \Pi^{ab}_{\mu\nu} &= N\epsilon_{\mu\rho\nu} p^\rho \delta^{ab} \\ V^{abc}_{\mu\nu\rho} &= N\epsilon_{\mu\nu\rho} f^{abc} \end{aligned} \tag{19.1.16}$$

[1]The two-loop corrections C to 1PI graphs, if they would have been nonzero, would enter as $\exp \frac{2\pi i}{\frac{1}{\hbar}k+N+\hbar C}$. In the Wilson action, all corrections are only to gauge-invariant objecs (to the parameter α, see below). In field theory, the BRST trivial parts can, and, in fact, do receive higher-loop corrections.

while the two-point function for the ghosts vanished! These results seemed to prove that at one loop k is renormalized into $k + \hbar N$. However, their calculations have been criticized for having taken the limit $m \to \infty, m_j \to \infty$ too soon (namely in the integrands) and for having omitted some diagrams. A more careful calculation by Ruiz et al. [4] found different results for $\Pi^{ab}_{\mu\nu}(AA)$ and Π^{ab} (ghost). A renormalization $\frac{1}{\hbar}k \to \frac{1}{\hbar}k + N$ looks very convincing, however. As we now show, $\Pi^{ab}_{\mu\nu}(AA)$ contributes both to a gauge-invariant part of the effective action and to a BRST exact part. It is only after subtracting the latter that one finds again $k \to k + \hbar N$. For the one-loop and two-loop calculations we shall use dimensional regularization, and so we must now first come to grips with $\epsilon^{\mu\nu\rho}$.

We begin by showing that the CS action plus gauge fixing term $-d^a \partial^\mu A_\mu{}^a$ does not fix the gauge in n dimensions (by which we mean that the kinetic matrix is singular). Adding the Yang Mills action $\frac{1}{m} F_{\mu\nu}{}^2$ does fix the gauge and leads to well-defined propagators. In the end one may then take the limit $m \to \infty$. Power counting shows that in this theory the one and two loop graphs are still divergent

$$D = 4 - L + E_c - V^{CS}_{AAA} - E_A \tag{19.1.17}$$

We shall use dimensional regularization to regulate these divergences. However, this immediately revives the question how to treat $\epsilon^{\mu\nu\rho}$ in n dimensions. It is easy to show that treating $\epsilon^{\mu\nu\rho}$ as an n-dimensional object is inconsistent. Namely, defining

$$\epsilon^{\mu\nu\rho}\epsilon_{\alpha\beta\gamma} = f(n)\left(\delta^\mu_\alpha \delta^\nu_\beta \delta^\rho_\gamma + 5 \text{ more terms}\right) \tag{19.1.18}$$

one finds that contractions of 3 or more ϵ symbols are inconsistent: they depend on the order in which one performs the contractions. As a simple example we will show that

$$\left(\epsilon^{\mu\nu\rho}\epsilon_{\alpha\nu\rho}\right)\epsilon_{\mu\beta\gamma} \neq \epsilon^{\mu\nu\rho}\left(\epsilon_{\alpha\nu\rho}\epsilon_{\mu\beta\gamma}\right) \tag{19.1.19}$$

Using n-dimensional Kronecker deltas δ^μ_ν, one finds on the left-hand side

$$(n-1)(n-2)\delta^\mu_\alpha \epsilon_{\mu\beta\gamma} \tag{19.1.20}$$

while the right hand side yields only two terms because μ can only be contracted with α

$$\epsilon^{\alpha\beta\gamma} - \epsilon^{\alpha\gamma\beta} = 2\epsilon^{\alpha\beta\gamma} \tag{19.1.21}$$

Hence at $n = 3$ (or $n = 0$) the contractions are order-independent, but in n dimensions they are inconsistent.

Let us now study the propagator. In n-dimensions, without $\frac{1}{m}F^2$ term, the kinetic matrix reads

$$K_{\mu\nu} = \begin{pmatrix} \epsilon_{\mu\alpha\nu}\bar{p}^\alpha & ip_\mu \\ -ip_\nu & 0 \end{pmatrix} \begin{array}{l} \bar{p}^\alpha \text{ is } 3-\text{dimensional}, \\ p_\mu, p_\nu \text{ are } n-\text{dimensional}, n > 3. \\ \hat{p}_\rho \text{ is } (n-3)-\text{dimensional} \end{array} \tag{19.1.22}$$

We first show that $K_{\mu\nu}$ is not invertible. The propagator (if it were to exist) would have the form

$$D_{\nu\rho} = \begin{pmatrix} f_1\epsilon_{\nu\beta\rho}\bar{p}^\beta + f_2\left(\bar{p}_\nu\hat{p}_\rho - \hat{p}_\nu\hat{p}_\rho\right) & g_1\bar{p}_\nu + g_2\hat{p}_\nu \\ -g_1\bar{p}_\rho - g_2\hat{p}_\rho & h \end{pmatrix} \tag{19.1.23}$$

Requiring $K_{\mu\nu}D_{\nu\rho} = \delta_{\mu\rho}$, one finds from the off-diagonal entries $h = 0$ and $f_2 = 0$. From the right lower corner one finds $g_1 = g_2 = \frac{i}{p^2}$ and the left upper corner then yields to the requirement

$$f_1\left(\bar{p}^\mu\bar{p}^\rho - \delta^{\mu\rho}\bar{p}^2\right) + \frac{p^\mu p^\rho}{p^2} \sim \delta^{\mu\rho} \tag{19.1.24}$$

Only in $d = 3$ does one find a solution (with $f_1 = -1$).

Adding the $\frac{1}{m}F^2$ term to the action, the kinetic term in the $A^a_\mu A^b_\nu$ sector acquires a part which is symmetric in μ, ν and the same calculation of the propagator now yields

$$\begin{aligned} D^{ab}_{\mu\nu} &= \frac{\delta^{ab}m}{p^4 + m^2\bar{p}^2}\Big[m\epsilon_{\mu\rho\nu}p^\rho + (p^2\delta_{\mu\nu} - p_\mu p_\nu) \\ &+ \frac{m^2}{p^2}\left\{\frac{\hat{p}^2}{p^2}p_\mu p_\nu + \bar{p}^2\hat{\delta}_{\mu\nu} + \hat{p}_\mu\hat{p}_\nu - p_\mu\hat{p}_\nu - \hat{p}_\mu p_\nu\right\}\Big] \end{aligned} \tag{19.1.25}$$

The reader who wants to check this result is invited to contract it with the kinetic matrix $K_{\mu\nu}$ (now including the contribution from $F^2_{\mu\nu}$ to $K_{\mu\nu}$. The off-diagonal parts of $D_{\mu\nu}$ and $D_{\mu\nu}$ are the same). It will be advantageous to rewrite $D^{ab}_{\mu\nu}$ as a sum of a purely n-dimensional part $\Delta^{ab}_{\mu\nu}$ and a remainder $R^{ab}_{\mu\nu}$, such that at $d=3$ one has $R^{ab}_{\mu\nu}=0$,

$$D^{ab}_{\mu\nu} = \Delta^{ab}_{\mu\nu} + R^{ab}_{\mu\nu} \tag{19.1.26}$$

Since

$$\frac{1}{p^4+m^2\bar{p}^2} - \frac{1}{p^4+m^2p^2} = \frac{m^2\hat{p}^2}{(p^4+m^2\bar{p}^2)p^2(p^2+m^2)} \tag{19.1.27}$$

one finds easily

$$\begin{aligned} \Delta^{ab}_{\mu\nu} &= \frac{\delta^{ab}m}{p^2(p^2+m^2)}\left[m\epsilon_{\mu\rho\nu}p^\rho + (p^2\delta_{\mu\nu} - p_\mu p_\nu)\right] \\ R^{ab}_{\mu\nu} &= \frac{\delta^{ab}m^3}{p^2(p^4+m^2\bar{p}^2)}\left[\frac{\hat{p}^2}{p^2+m^2}\left\{m\epsilon_{\mu\rho\nu}p^\rho + p^2\delta_{\mu\nu}\right.\right. \\ &+ \left.\left.\frac{p_\mu p_\nu m^2}{p^2}\right\} + \bar{p}^2\hat{\delta}_{\mu\nu} - \hat{p}_\mu\hat{p}_\nu + p_\mu\hat{p}_\nu + \hat{p}_\mu p_\nu\right] \end{aligned} \tag{19.1.28}$$

Clearly $R \to 0$ as $d \to 3$. Note that $\Delta \to \frac{1}{p^2}$ for large p, while $R \to \frac{1}{p^4}$. This fact will be crucial in the coming analysis of ultraviolet divergences. To avoid confusion we reemphasize that all p^μ in Δ are n dimensional but $\epsilon^{\mu\nu\rho}$ is 3 dimensional. Clearly, Δ is a nice object to work with, so our strategy will be to show that $R_{\mu\nu}$ may be omitted from loop calculations.[2]

The quantum action we consider is given by

$$\mathcal{L} = -i\epsilon^{\mu\nu\rho}\left(\frac{1}{2}A_\mu{}^a\partial_\nu A_\rho{}^a + \frac{1}{6}gf^{abc}A_\mu{}^aA_\nu{}^bA_\rho{}^c\right)$$

[2]Note that there are no infrared divergences in the Green's functions since $\frac{d^3q}{q^2}$ is infrared finite. In field theories with massless particles the S matrix contains infrared divergences which often cancel in the cross section. Here there are no particles as follows for example from the fact that the hamiltonian corresponding to the classical CS action vanishes. Indeed, the gravitational stress tensor is zero, since $\epsilon^{\mu\nu\rho}$ does not allow any metric fields in $\mathcal{L}(CS)$ in curved space, while also the canonical stress tensor is trivial (as one may further study using Dirac formalism). Hence there are no physical degrees of freedom in CS theory. The theory with an extra $\frac{1}{m}F^2$ term has one degree of freedom and is unitary. Do here IR divergences cancel? A one-loop AA selfenergy with a selfenergy insertion goes like $\int d^3k k^{-4}\prod(k)$ where $\prod_{\mu\nu}(k) \sim \int d^3q q^{-2}(q+k)^{-2}k_\mu k_\nu$ goes like $(k^2)^{-1/2}$. This seems to give an infrared divergence. However, there are also $k_\mu q_\nu$ and $q_\mu q_\nu$ terms in $\prod(k)$ and they might cancel in the sum.

$$\begin{aligned} &- \quad d^a \partial^\mu A_\mu{}^a + b_a \partial^\mu D_\mu c^a + \frac{1}{4m} F_{\mu\nu}{}^a F_{\mu\nu}{}^a \\ &+ \quad K_a{}^\mu D_\mu c^a - L_a \left(\frac{1}{2} g f^{abc} c^b c^c \right) \end{aligned} \tag{19.1.29}$$

The sources $K_a{}^\mu(x)$ and $L_a(x)$ couple to the BRS variations of $A_\mu{}^a$ and c^a. We shall now first make a cohomological study of the effective action to extract the gauge-invariant parts, and then come back to loop calculations.

The one-loop effective action $\Gamma^{(1)}$, after taking the limit $m \to \infty$, satisfies $\Delta\Gamma^{(1)} = 0$ where Δ is the BRST operator of the quantum action without $\frac{1}{mF^2}$ term. Defining $\bar{\Gamma}$ as the part of Γ without the tree level term $\mathcal{L}(\text{fix})$

$$\Gamma = \bar{\Gamma} - d^a \partial^\mu A_\mu{}^a \tag{19.1.30}$$

it follows that $\bar{\Gamma}$ is independent of d^a (since there are not vertices with d^a) and depends on $K^\mu{}_a$ and b_a only through the combination $G^\mu{}_a = K^\mu{}_a - \partial^\mu b_a$. Then

$$\Delta\bar{\Gamma}^{(1)} = 0, \Delta = \int \left(\frac{\delta\bar{\Gamma}^{(0)}}{\delta A_\mu{}^a} \frac{\delta}{\delta G^\mu{}_a} + \frac{\delta\bar{\Gamma}^{(0)}}{\delta G^\mu{}_a} \frac{\partial}{\partial A_\mu{}^a} + \frac{\delta\bar{\Gamma}^{(0)}}{\delta L_a} \frac{\delta}{\delta c^a} + \frac{\delta\bar{\Gamma}^{(0)}}{\delta c^a} \frac{\delta}{\delta L_a} \right) d^3x \tag{19.1.31}$$

where $\bar{\Gamma} = \bar{\Gamma}(A_\mu{}^a, c^a, G^\mu{}_a, L_a)$. We decompose Δ into $\Delta_0 + \Delta_1$ where Δ_i contains $i+2$ fields, and decompose $\bar{\Gamma} = \sum_{n=0}^\infty \bar{\Gamma}_n$ where $\bar{\Gamma}_n$ contains n fields. Of course, $\Gamma_0^{(1)} = 0, \Gamma_1^{(1)} = 0$. From $\Delta^2 = 0$ it follows that

$$\Delta_0{}^2 = \Delta_1{}^2 = 0, \Delta_0\Delta_1 + \Delta_1\Delta_0 = 0 \tag{19.1.32}$$

Then we find a hierarchy of relation (a "filtration")

$$\begin{aligned} \Delta_0\Gamma_2^{(1)} &= 0 \\ \Delta_0\Gamma_3^{(1)} &+ \Delta_1\Gamma_2^{(1)} = 0 \\ \Delta_0\Gamma_4^{(1)} &+ \Delta_1\Gamma_3^{(1)} = 0 \\ \Delta_0\Gamma_5^{(1)} &+ \Delta_1\Gamma_4^{(1)} = 0 \text{ etc.} \end{aligned} \tag{19.1.33}$$

We now show that $\Gamma_2^{(1)}$ and $\Gamma_3^{(1)}$ are local. This follows from the fact that with the regulators with $p^2[p^2 + M^2]^{-1}$ discussed before, the 2 and 3 point functions vanished.

Therefore, any other regularization scheme will produce at most local finite terms. The most general parametrization is then

$$\begin{aligned}\Gamma_2^{(1)} &= \alpha_1 \partial A^2 + \alpha_2 G \partial c \\ \Gamma_3^{(1)} &= \beta_1 A^3 + \beta_2 G A c + \beta_3 L c c\end{aligned} \tag{19.1.34}$$

and the BRS identities fix some of the β's. Substitution into $\Delta_0 \Gamma_4^{(1)} + \Delta_1 \Gamma_3^{(1)} = 0$ shows that $\Delta_1 \Gamma_3^{(1)} = 0$, hence

$$\Delta_1 \Gamma_3^{(1)} = \Delta_0 \Gamma_4^{(1)} = 0 \tag{19.1.35}$$

One finds then

$$\Gamma_2^{(1)} + \Gamma_{(3)}^{(1)} = \text{ local } = \alpha \mathcal{L}_{CS} + (\Delta_0 + \Delta_1) \int (\beta G A + \gamma L c)\, d^3x \tag{19.1.36}$$

It is easy to evaluate the terms with β and γ since Δ acts as a counting operator for GA and Lc

$$\begin{aligned}\Delta \int \beta G A &= \beta\,(-N_G - N_A)\,\bar{\Gamma}_0 \\ \Delta \int \gamma L c &= \gamma\,(+N_L - N_c)\,\bar{\Gamma}_0\end{aligned} \tag{19.1.37}$$

The result is

$$\begin{aligned}\Gamma^{(0)} &+ \Gamma_2^{(1)} + \Gamma_3^{(1)} = -i \int \epsilon^{\mu\nu\rho} \Big[\frac{1}{2}(1+\alpha+2\beta) A_\mu{}^a \partial_\nu A_\rho{}^a \\ &+ \frac{1}{6}(1+\alpha+3\beta) A_\mu{}^a A_\nu{}^b A_\rho{}^c f_{abc} \,\Big] \\ &+ \int \Big[G^\mu{}_a \partial_\mu c^a (1-\beta) + f^{abc} G^\mu{}_a A_\mu{}^b c^c (1-\gamma) \\ &- \frac{1}{2} g f_{abc} L_a c^b c^c (1-\gamma) \,\Big]\, d^3x\end{aligned} \tag{19.1.38}$$

To compute α, β, γ, one considers the AA two-point function, (which gives $(1+\alpha+2\beta)$) the Gc two point function (which gives β) and the Lcc vertex (which gives γ). A suitable linear combination then determines α.

The claim is that all gauge invariant regularization schemes give the same α, but in general different β and γ. It has been shown by direct calculation that the one-loop corrections to α amount to $\frac{1}{\hbar}k \to \frac{1}{\hbar}k + N$ while at the 2-loop level there are no corrections to α. [4] The correct way to compute loops is to take limits as follows

$$m \overset{\lim}{\to} \infty \left[d \overset{\lim}{\to} 3 \,(\ldots) \right] \tag{19.1.39}$$

In the divergent terms one must keep $d \neq 3$, but in the convergent terms one may already put $d = 3$.

We shall now show that loops for $\frac{1}{m}F^2 + \mathcal{L}(CS)$ are finite to all loop orders. Hence, the β function for this model vanishes even for finite m. Furthermore, $\mathcal{L}(CS)$ is the infrared regulator of $\mathcal{L}(YM) = F^2$, and $\mathcal{L}(YM)$ is the ultraviolet regulator of $\mathcal{L}(CS)$ beyond two loops. (As we discussed, F^2 theory is probably infrared divergent but the topological mass from $\mathcal{L}(CS)$ cures that.)

There are two useful "decoupling theorems" which say what terms one may drop in advance if in the end one is going to take the limit $m \to \infty$. As we now discuss, one may completely drop all $R^{ab}_{\mu\nu}$ propagators. This is very useful since one-loop calculations with $\hat{q}$ and $\bar{q}$ in the propagators are as complicated as two loop calculations with $d\hat{q}d\bar{q}$.

Power counting for the quantum action with $\frac{1}{m}F^2_{\mu\nu}$ term yields [3]

$$D = 3 - \frac{1}{2}\left(E_A + 2E_{b,c} + V_3 + V_{gh} + 2V_4 + 3V_K + 2V_L\right) \tag{19.1.40}$$

It follows that the only divergent graphs are

(i) one-loop AA graphs (with an A loop or an A seagull or a ghost loop): linearly divergent

(ii) two-loop AA graphs (16 diagrams): logarithmically divergent. (They have either 2 V_4 vertices or 3 or more other vertices).

[3] One finds actually by direct calculation $D = 3 - \frac{1}{2}E_A - \frac{3}{2}E_b - \frac{1}{2}E_C - \frac{3}{2}V_{CS} - \frac{1}{2}V_3 - \frac{1}{2}V_4 - \frac{1}{2}V_{gh} - 2V_K - 2V_L$, but using $-E_b + E_c = V_K + 2V_L$, this result is obtained.

(iii) one-loop ghost selfenergy: logarithmically divergent

(iv) one-loop AAA vertex logarithmically divergent

We shall show that only $\Delta_{\mu\nu}$ but not $R_{\mu\nu}$ contributes to the divergences in one-loop graphs. In the convergent one-loop graphs, $R_{\mu\nu}$ clearly does not contribute, but also in the divergent graphs $R_{\mu\nu}$ will not contribute. This is actually a trivial consequence from the fact that at one loop there are no divergences at all because integrals like$\int \frac{d^n p}{(p^2)^\alpha (p^2+m^2)^\beta}$ have not poles at odd n.

It similarly follows that also the double poles at 2 loop level cancel. To show that in the finite parts of one-and two-loop graphs the $R_{\mu\nu}$ do not contribute, we first construct the quantum action which yields as propagator $\Delta_{\mu\nu}$.

The action which yields the $\Delta^{ab}_{\mu\nu}$ propagator instead of the $D^{ab}_{\mu\nu}$ propagator differs from the $CS + \frac{1}{m}F^2$ action by an extra term involving an operator which we call $\mathcal{O}$

$$\begin{aligned}
S'\left(\text{for}\Delta^{ab}_{\mu\nu}\right) &= S(CS + F^2 + \text{rest}) \\
&+ \frac{1}{2}\int dx \int dy A_\mu{}^a(x)\mathcal{O}^{\mu\nu}_{ab}(x-y)A_\nu{}^b(y) \\
\mathcal{O}_{\mu\nu}(p) &= \frac{m^2}{p^4 + m^2\hat{p}^2}\left[\hat{p}^2\left\{-m\epsilon_{\mu\rho\nu}p^\rho + p^2\delta_{\mu\nu} - p_\mu p_\nu\right\}\right. \\
&+ (p^2+m^2)\left\{\hat{p}^2 p_\mu p_\nu/p^2 + \bar{p}^2\hat{\delta}_{\mu\nu} + \hat{p}_\mu\hat{p}_\nu - p_\mu\hat{p}_\nu - \hat{p}_\mu p_\nu\right\}\,]
\end{aligned} \tag{19.1.41}$$

One may verify that $p^\mu\mathcal{O}_{\mu\nu}(p) = 0$ and $\Delta_{\mu\rho}K_{\rho\nu} + \frac{p_\mu p_\nu}{p^2} = \delta_{\mu\nu}$. It follows that under BRST variations

$$\delta S' = \int dx \int dy \left(A_\mu{}^a(x)\mathcal{O}^{\mu\nu}_{ab}(x-y)f^b{}_{pq}A^p_\nu c^q\right) dxdy \tag{19.1.42}$$

We now show that $\Pi^{ab}_{\mu\nu}$ (with D) $= \Pi^{ab}_{\mu\nu}$ (with Δ) by using that

(i) $\Pi^{ab}_{\mu\nu}$(with D) is finite and transversal. This follows from the gauge invariance of the action.

(ii) $\Pi^{ab}_{\mu\nu}$(with D)$-\Pi^{ab}_{\mu\nu}$ (with Δ) = finite and local $= cm\delta_{\mu\nu}\delta^{ab}$. At one-loop, all Green's functions computed with Δ are equal to those computed with D, since at one loop there are no poles while R vanishes at $d = 3$. (One does not need to use that there are no poles in ϵ, one only needs to use that R goes like p^{-4} while Δ goes like p^{-2}, so graphs with an R are finite, and vanish since R vanishes in $d = 3$). At two loops, the difference between calculations with D and with Δ must be local and finite, since $\mathcal{O}$ vanishes at $d = 3$, and using $\mathcal{O}$ is like using a different regularization scheme.

(iii) Also $\Pi^{ab}_{\mu\nu}$ (with Δ) is transversal at $d = 3$ because the breaking of the Ward identity disappears at $d = 3$. More precisely, only the terms with $\frac{1}{\epsilon}$ can contribute to $\prod_D - \prod_\Delta = cm\delta_{\mu\nu}$. These terms are each transversal: $\prod_{\mu\nu}(D)$ is finite and therefore $\prod_{\mu\nu}(\Delta)$ is finite (using $\mathcal{O}$ is like using a regulator).

(iv) From $k^\mu\Pi^{ab}_{\mu\nu}$ (with D) $-k^\mu\Pi^{ab}_{\mu\nu}$ (with Δ) $= 0$ it then follows that $c = 0$. QED.

So we now analyze the Ward identity in the theory with only Δ propagators.

We begin with

$$(\Gamma, \Gamma) = \int \delta\mathcal{L}'(\mathcal{O}) \bullet \Gamma \quad d^3x \tag{19.1.43}$$

All graphs with precisely one $\mathcal{O}$ insertion (and of course only Δ propagators) have the following divergences according to power counting

$$D(\mathcal{O}) = -\frac{1}{2}\left[\, E_A + E_d + 2\left\{E_c + E_b + E_H\right\} + 3E_j + V_{A^3} + V_{bAc} + 2V_{A^4}\,\right] \tag{19.1.44}$$

We now show that

(i) all 1-loop, and later 2 loop, graphs with one $\mathcal{O}$ insertion are finite (using the power counting rules given by (19.1.44), and also using the 1 and 2 loop finiteness of the original graphs (the graphs without $\mathcal{O}$ insertion)

(ii) therefore all 1-loop contributions with one $\delta\mathcal{L}(\mathcal{O})$ go to zero at $d = 3$ (because $\mathcal{O} \to 0$ for $d \to 3$)

(iii) for 2 loops we use that in CS theory the 1 and 2 loop subgraphs were finite, and we show that also subgraphs with one $\delta\mathcal{L}(\mathcal{O})$ insertion are finite.

1 loops: all 1 loop graphs, even with an insertion, are 1 loop finite because $\int d^n p$ gives no poles in $d = 3$. So, BRST breaking disappears at $d = 3$.

2 loops: a finite result at 1-loop might still give a pole at 2 loops and then for $d \to 3$ one might end up with a nonvanishing result. However, $D(\mathcal{O})$ in (19.1.44) shows that all 2 loops with one insertion are finite. So, one may drop the BRST breaking term.

3 and higher loops: are power counting finite. Also 1- and 2 loop subgraphs are finite, so here one may omit BRS breaking.

Conclusion: in $(\Gamma, \Gamma) = \delta S(\mathcal{O}) \bullet \Gamma$ one may drop the right-hand side for $d \to 3$. So, at $d = 3$, the Ward identities hold, and the AA graphs are transversal even when one uses Δ instead of D propagators. Since this showed that the AA graphs with Δ and with D are the same, we have shown that $R_{\mu\nu}$ does not contribute in loop calculations.

It has been shown that the (one-loop) effective action for CS theory can also be obtained from fermion loops in $d = 3$ coupled to YM fields. The mass terms in the fermionic $d = 3$ action violates parity, just like the CS action itself. [Korchemsky,......].

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Chapter 20

Pauli Villars regularization of gauge theories

Dimensional regularization has become the preferred regularization scheme, but it requires careful treatment of theories with γ_5 or epsilon tensors, to say the least. The higher derivative regularization schemes which we consider below are purely four-dimensional, and for that reason they have been advocated for theories with γ_5 and/or epsilon tensors. However, higher derivative regularization schemes do not regulate one-loop graphs, and for the latter Pauli-Villars regularization has been advocated. One must then write down an action containing both higher-derivative terms and Pauli-Villars terms, and compute each graph using this entire action. As we shall see, the Pauli-Villars regularization scheme for gauge theories is not gauge-invariant (or better: it violates the BRST Ward-identities). It is believed, and can be proven in some cases, that all gauge-invariant regularization schemes give the same answer for local physical quantities such as β functions (or the renormalization of the coupling constant in front of the Chern-Simons action). On the other hand, a non-gauge-invariant regularization scheme will give results which differ by local finite counter terms (at the regularized level), and hence local quantities cannot be calculated in a scheme-independent way using a gauge-noninvariant scheme. For these reasons, Slavnov has tried to construct a gauge-invariant modification of Pauli-Villars regularization which we will discuss. It requires extra vertices to achieve gauge invariance, but, as we shall see, the resulting theory is no longer merely a regularization of the original theory; rather, the net effect of adding the extra vertices is to introduce a new complex scalar ghost in the action. As a result, the β function is modified and also unitarily is violated in this modified Pauli-Villars scheme of Slavnov, even in the limit of regulator masses tending to infinity. In the usual Pauli-Villars scheme unitarity is also violated at energies above the regulator masses, but by taking the regulator masses to infinity, unitarity is restored (because then the regulator fields cannot be intermediate states in the unitarity equation).

We consider pure Yang-Mills theory in 4 Euclidean dimensions, and add a gauge-invariant higher-derivative term to the action

$$\mathcal{L} = \frac{1}{4} {F_{\mu\nu}}^2 + \frac{1}{4\Lambda^2} \left(D^n F_{\mu\nu}\right)^2 \tag{20.0.1}$$

where $(D^n F_{\mu\nu})^2$ has $2n$ more derivatives than ${F_{\mu\nu}}^2$. We shall shortly fix the precise form of this term. As gauge fixing and ghost terms we take

$$\mathcal{L} = \frac{-\alpha}{2} d^a \frac{1}{f^2\left(\frac{-\partial^2}{\Lambda^2}\right)} d^a - d^a \partial^\mu {A_\mu}^a - b_a \partial^\mu D_\mu c^a \tag{20.0.2}$$

Also the function f will be fixed shortly. The gauge propagator then comes out as

$$\begin{aligned} < {A_\mu}^a {A_\nu}^b > &= \Lambda^{2n} \left(1 + p^{2n}/\Lambda^{2n}\right)^{-1} \left(\frac{\delta_{\mu\nu}}{p^2} - \frac{p_\mu p_\nu}{p^4}\right) \\ &+ \frac{\alpha p_\mu p_\nu}{p^4 f^2(p^2/\Lambda^2)} \sim \frac{1}{p^{2+2n}} \text{ for } \alpha = 0 \end{aligned} \tag{20.0.3}$$

Since there are no vertices involving d^a fields, we do not need the dd or Ad propagators at this point. (In fact, the dd propagator vanishes and the Ad propagator equals ip_μ/p^2). In the limit $\alpha \to 0$, power counting gives for the degree of divergence of a 1PI graph

$$D = 4 - 2n(L-1) - E_A - (\frac{3}{2} + n) E_{gh} \leq 4 \tag{20.0.4}$$

All diagrams with external ghosts are clearly finite if $n \geq 1$. For $n = 1$, there are still two-loop divergent graphs (for example the AA selfenergy), but for $n \geq 2$, there are only one-loop divergences (we do not consider one-loop and higher-loop selfenergy diagrams since we consider connected diagrams). Hence we fix $n = 2$, and take as higher-derivative term

$$\frac{1}{4\Lambda^4}(D^2 F_{\mu\nu})^2 \quad , \quad (D^2 \equiv D^\lambda D_\lambda) \tag{20.0.5}$$

Next we fix $f(-\partial^2/\Lambda^2)$ such that all α-dependent terms in 1PI graphs are finite. We choose

$$f = 1 + p^4/\Lambda^4 \tag{20.0.6}$$

because then the α-dependent term in the AA propagator behave like p^{-10} for large p, which is an improvement over the first term with a factor p^{-4}. Clearly, all α-dependent terms are then convergent, (because $D < 4$ for $\alpha = 0$, see above). For that reason we shall from now on only use the propagators at $\alpha = 0$ (The choice $f = 1 + p^2/\Lambda^2$ would not have led to any improved convergence for the α term in the propagator. Note that we do not have to specify $i\epsilon$ prescriptions for the p^4 terms since we work in Euclidean space. The unitarity in the corresponding Minkovski theory is not obvious.)

To regulate the one-loop graphs with only external A^a_μ legs and with α-independent propagators, we construct an action for Pauli-Villars fields $A^a_{\mu,j}(j = 1, 2...)$ which have the same propagators (except that they are massive) and the same one-loop vertices. This action is

$$\begin{aligned} \mathcal{L}(PV) &= \sum_{J=1}^{N} [\sum_{j=1}^{\alpha_J} \frac{1}{2} A^a_{\mu j} \frac{\delta^2 \mathcal{L}(\text{total})}{\delta A^a_\mu \delta A_\nu{}^b} A_{\nu j}{}^b + \frac{1}{2} M_J{}^2 A_{j\mu}{}^a A_{j\mu}{}^a] \\ &- d_j{}^a \partial^\mu A_{\mu j}{}^a + b_{ja} \left(\partial^\mu D_\mu + M_j{}^2 \right) c_j{}^a \\ \mathcal{L}(\text{total}) &= \frac{1}{4}(F_{\mu\nu}{}^a)^2 + \frac{1}{4\Lambda^4} \left(D^2 F_{\mu\nu} \right)^2 \end{aligned} \tag{20.0.7}$$

(Since graphs with external ghosts were finite we do not need vertices involving $b_j A_{\mu j}{}^a$

or $A_{\mu j}{}^a c_j$). In path integral notation, we have

$$Z = \int (DA_\mu{}^a Db_a Dc^a Dd^a) \left(DA_{\mu j}{}^a Db_{aj} Dc_j{}^a Dd_j^a\right)$$
$$e^{-\frac{1}{\hbar}S(A_\mu^a, d^a, b_a, c^a)} \prod_{J=1}^{N} (\det A_J)^{-\frac{1}{2}\alpha_j} (\det C_J)^{\alpha_J} \tag{20.0.8}$$

where A_J is the complete kinetic operator in $A_{\mu j}{}^a, d_j{}^a$ space and C_J the kinetic operator in $b_{aj}, c_j{}^a$ space. Thus, for fixed j, there are α_j *copies* of the $A_{\mu j}{}^a$ action and also α_j copies of the b_{aj}, c_j^a ghost action, all with the same mass M_J. (One gets a *sum* of α_j actions for the PV fields and PV ghosts, not a coefficient α_j in front of one action. Such a coefficient would cancel in one-loop graphs). We take the α_j equal to integers, and if α_j is positive, one has commuting PV fields, while for negative α_j, one has anticommuting PV fields.[1]

Since the complete quantum action in the PV sector is gauge invariant, we can view it as a kind of gauge-invariant "matter action" which has been added to the BRST invariant YM quantum action. Hence this action will preserve its own BRST identities at the loop level.

For every loop in the original theory there are now $\sum \alpha_j$ exact PV copies. For suitable α_j and M_j, this scheme regulates all one-loop graphs as we shall later show. However, it is not gauge invariant as we now show. Under a gauge transformation $A_\mu \to g^{-1}(\partial_\mu + A_\mu)g$, one has

$$\frac{\delta^2 S}{\delta A_\mu{}^a(x)\delta A_\nu{}^b(y)} \to g^{-1}(x)\frac{\delta^2 S}{\delta A_\mu{}^a(x)\delta A_\nu{}^b(y)}g(y) \tag{20.0.9}$$

(For example, for S equal to the YM action, $\delta^2 S/\delta A_\mu{}^a(x)\delta A_\nu{}^b(y)$ equals $\frac{\delta}{\delta A_\nu{}^b(y)}$ $(D^\mu G_{\mu\nu}(A(x))$ which is covariant since $\delta/\delta A_\nu{}^b(y) G_{\mu\nu}(x)$ is covariant and also $D^\mu(A(x)$ is covariant). Redefining $A_{\mu j} \to g A_{\mu j} g^{-1}$ in the path integral (which is allowed as no $A_{\mu j}{}^a$ come out of graphs, while the Jacobian is unity), all terms are gauge invariant except the term $d^a \partial^\mu A_\mu{}^j$ and the ghost action. As we already explained this gauge noninvariance makes the scheme not very useful.

[1] More precisely $2\alpha_j$ complex anticommuting and α_j real commuting PV fields.

To make the scheme gauge-invariant, *Slavnov adds by hand extra vertices* which replace $d^a \partial^\mu A_{j\mu}{}^a$ by $d^a D^\mu A_{j\mu}{}^a$ and $b_{ja}\partial^\mu D_\mu c_j^a$ by $b_{ja} D^\mu D_\mu C_j{}^a$. The underlying justification of this modification is that physics "should not depend on the details in the unphysical sector." However, we must now first investigate whether this gauge-invariant PV regularization scheme still regulates all diagrams.

Since there are now vertices involving $d_j{}^a$, we quote its propagator (at $\alpha = 0$ as we explained)

$$< d_j{}^a \, d_j{}^b >= \delta^{ab} \frac{M_j{}^2}{p^2} \; ; \; < d_j{}^a A_{j\mu}{}^b >= \delta^{ab} \, \frac{p_\mu}{p^2} \tag{20.0.10}$$

(The $< d^a d^a >$ propagator vanishes, but the PV masses give a nonvanishing propagator for the PV partners $d_j{}^a$). Let us consider the $A_\mu{}^a A_\nu{}^b$ one-loop selfenergy. The complete set of diagrams consists of the original set of diagrams which are due to the original, noncovariant, PV scheme, and extra diagrams due to the new couplings.

We observe that all new graphs except one are quadratically divergent.

The original selfenergy and seagull graphs are regulated by their PV counterparts. The PV $A_{\mu j}{}^a$ propagator reads

$$< A_{\mu j}{}^a A_{\nu j}{}^b >= \frac{\Lambda^4 (p^2 \delta_{\mu\nu} - p_\mu p_\nu) \delta^{ab}}{p^2 (p^6 + p^2 \Lambda^4 + M_j{}^2 \Lambda^4)} \tag{20.0.11}$$

hence for ${M_j}^2 = 0$ it is equal to the $< {A_\mu}^a {A_\nu}^b >$ propagator at $\alpha = 0$. Expanding in ${M_j}^2$, we see that the terms proportional to ${M_j}^2$ get an extra factor $\sim p^{-6}$, hence these contributions are finite. Thus the AA selfenergies due to an AA loop or A tadpole are regulated as long as

$$1 + \sum \alpha_j = 0. \tag{20.0.12}$$

No condition $\sum \alpha_j M_j^2$ is needed. However, the original ghost loop together with its PV partner yields a contribution

$$\int \left[\frac{(q+p)^\mu q^\nu}{q^2 (q+p)^2} + \sum_j \alpha_j \frac{(q+p)^\mu q^\nu}{(q^2 + {M_j}^2)((q+p)^2 + {M_j}^2)} \right] d^4 q \tag{20.0.13}$$

and expanding in terms of ${M_j}^2$ we now must require

$$1 + \sum \alpha_j = 0, \quad \sum \alpha_j {M_j}^2 = 0 \tag{20.0.14}$$

Hence, all original graphs are properly regularized as long as (20.0.14) holds.

On the other hand, the extra vertices in the ghost sector which replace $b_j \partial^\mu D_\mu c_j$ by $b_j (D^\mu D_\mu + {M_j}^2) c_j$ lead, together with the old ghost vertices $b \partial D c$, to the ghost PV loop

$$\sum_j \alpha_j \frac{(2q+p)^\mu (2q+p)^\nu}{(q^2 + {M_j}^2)[(q+p)^2 + {M_j}^2]} \tag{20.0.15}$$

(compare with (20.0.13)) and clearly there are now divergences left, as we already mentioned. Since the topology of the new graphs is not uniformly the same, one has to introduce a second regularization scheme for the set of 6 new graphs.

At this point a "miracle" occurs: using dimensional regularization to compute the remaining divergent graphs, all divergences cancel provided $1 + \sum \alpha_j = 0$. One can directly show that the remaining set of (quadratically) divergent graphs is finite by expanding in ${M_j}^2$. Terms proportional to ${M_j}^2$ are more convergent by a factor p^{-6} and are finite. Putting ${M_j}^2 = 0$ in the graphs, one decomposes denominators like $[q^4 (q^4 + \Lambda^4)]^{-1} = \{ q^{-4} - (q^4 + \Lambda^4)^{-1} \} \Lambda^{-4}$. In the end one obtains integrals like $\int d^4 q \; q^\mu q^\nu (q^2)^\alpha \; (q+p)^2)^\beta$ where α, β are negative integers, and these integrals can easily be estimated, proving that the divergences in the extra one-loop graphs in the covariant PV scheme indeed cancel. An "explanation" of this finiteness at finite values of Λ and M_j is given by a formal analysis of the path integral in the textbook by Faddeev and Slavnov. The original functional is transformed into another functional which is finite and which differs from the original functional by terms which are finite but nonlocal. Hence, there was, after all, <u>some</u> truth to the claim that physics does not depend on gauge artefacts. However, the finite parts are modified in an essential way as we now show.

In Slavnov's formal proof, only graphs with external ${A_\mu}^a$ are considered. However, in higher loop graphs, one should also make sure that one-loop subgraphs with external ${A_{\mu j}}^a$ and ${d_j}^a$ are finite. One may check that the extra one loop graphs with external ${d_j}^a$ are finite by themselves. Defining a $S_{\Lambda'}$ with (say) 6 extra D derivatives to yield the ${A_{\mu j}}^a$ action, graphs with two external ${A_{\mu j}}^a$ are finite. Thus using $S_\Lambda \neq S_{\Lambda'}$. Furthermore, the graphs with d^a fields are not regularized, since the ${d_j}^a$ couple to $D_\mu {A_{\mu j}}^a$ and not to $\partial_\mu {A_{\mu j}}^a$ and so there are extra vertices.

The complete AA two-point function is found to be

$$\begin{aligned} \Pi^{ab}_{\mu\nu} &= \frac{\delta^{ab} g^2 C_A}{16\pi^2} \left(p^2\delta_{\mu\nu} - p_\mu p_\nu\right) \\ & \left[A_2 \ln p^2/\Lambda^2 + B_2 \sum_j \alpha_j \ln p^2/{M_j}^2 + \text{ finite terms}\right] \end{aligned} \qquad (20.0.16)$$

(For $SU(N), C_A = N$). The divergent part is then (putting ${M_j}^2 = \Lambda^2$ and using $\sum \alpha_j = -1$)

$$\Pi^{ab,div}_{\mu\nu} = \frac{\delta^{ab} g^2 C_A}{16\pi^2} (p^2\delta_{\mu\nu} - p_\mu p_\nu) \left[-(-A_2 + B_2) \ln \Lambda^2/\mu^2\right] \qquad (20.0.17)$$

where μ^2 is the subtraction point. One finds

$$-A_2 + B_2 = \frac{\alpha}{2} - 7/3 \qquad (20.0.18)$$

where α is the gauge-fixing parameter, which of course is present in the finite parts of the graphs (finite before taking the limit $\Lambda \to \infty, M_j \to \infty$). (There are no quadratic divergences $\prod^{ab}_{\mu\nu} \sim \Lambda^2 g_{\mu\nu}\delta^{ab}$ since the scheme is gauge invariant. A general non-gauge invariant scheme would give

$$\begin{aligned}\Pi^{ab}_{\mu\nu} &= \delta^{ab}\,[\,a_1\Lambda^2 g_{\mu\nu} + a_2(p^2 g_{\mu\nu} - p_\mu p_\nu)\ln p^2/\Lambda^2 \\ &+ \;\; a_3 p^2 g_{\mu\nu} + a_4 p_\mu p_\nu\,]\end{aligned} \tag{20.0.19}$$

Since the difference between two regularization schemes is *local*, the terms with $\ln p^2/\Lambda^2$ is always and the same and hence transversal).

Similarly, the two-point $< c^a b_b >$ function of the original ghost system (which is finite for finite Λ since $< AA > \sim p^{-6}$) yields

$$\Pi^{ab}_{\text{ghost}}\,(\Lambda) = \frac{\delta^{ab} g^2 C_V}{16\pi^2}\left[\frac{(3-\alpha)}{4}\ln\frac{p^2}{\Lambda^2} + \text{ const.}\right]p^2 \tag{20.0.20}$$

It yields a divergence for large Λ

$$\Pi^{ab,div}_{\text{ghost}}\,(\Lambda\to\infty) = \frac{\delta^{ab} g^2 C_V}{16\pi^2}\left[-\frac{(3-\alpha)}{4}\ln\Lambda^2/\mu^2\right]p^2 \tag{20.0.21}$$

Finally, we conpute the 1-loop ghost-ghost-gauge vertex. Also this vertex is finite and the usual two graphs yield

$$V^{abc}{}_{\mu} = -igf^{abc}\frac{g^2 C_A}{16\pi^2}\left[-\frac{\alpha}{2}\ln p^2/\Lambda^2\right]p_\mu + \text{finite} \tag{20.0.22}$$

For large Λ one finds then

$$V_\mu{}^{abc,div} = -igf^{abc}\frac{g^2 C_A}{16\pi^2}\left[\frac{\alpha}{2}\ln\Lambda^2/\mu^2\right]p_\mu \tag{20.0.23}$$

From these three divergences ("Z-factors") one can then extract the 1-loop β function as usual. One finds

$$\beta = \frac{-23}{6}\,\frac{g^3 C_V}{16\pi^2} \tag{20.0.24}$$

Note that in the sum α cancels: $(\frac{\alpha}{2} - \frac{7}{3}) - 2(\frac{3-\alpha}{4}) + 2(-\frac{1}{2}\alpha) = \frac{-23}{6}$. However, in pure YM theory

$$\beta = \frac{-11}{3}\,\frac{g^3 C_V}{16\pi^2} \tag{20.0.25}$$

because in the AA propagator one finds $\frac{13}{6} - \frac{\alpha}{2}$ instead of $\frac{7}{3} - \frac{\alpha}{2}$. (Only in the AA propagator the divergences depend on ${M_j}^2$, and thus only in the AA sector is there a possibility that the contribution to the β function changes). The discrepancy is due to those diagrams which for $\Lambda \to \infty$ and $M_j \to \infty$ have nonvanishing integrands. There are two such diagrams

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Taking the $\Lambda \to \infty, M_j \to \infty$ parts of their integrands, dimensional regularization can be used to calculate their contributions to the β function. We shall not reproduce this calculation but instead take the limit $\Lambda \to \infty, M_j \to \infty$ inside the path integral (this should really be justified).

Since $\frac{-23}{6} - \frac{11}{3} = \frac{1}{6}$, it suggests that Slavnov introduced a new scalar particle into the theory. This is indeed the case. To see this, consider the path integral with action $\frac{1}{2}A_j\left(\frac{\delta^2 S}{\delta A^2} + {M_j}^2\right)A_j$ and redefine $A_j = B_j/M_j$. Rescale also d_j as $d_j = M_j d'$ such that the $d_j D^\mu A_{\mu j}$ terms remains M independent. The extra powers of M_j in the measure are neglected (they should cancel). Then, taking the limit $\Lambda \to \infty, M_j \to \infty$, the terms with $\frac{1}{\Lambda}(D^2F^2)$ go away for $\Lambda \to \infty$ and one finds in the ${d_j}', B_j$ sector the following contribution

$$\int dd_j DB_j \; e^{d'jD\cdot B_j} \; e^{\frac{-1}{2}B_j(1)B_j} \tag{20.0.26}$$

Completing squares and integrating over B_j, one arrives at

$$\int dd_j' e^{\frac{1}{2}(Dd')^2} \tag{20.0.27}$$

This is the new degree of freedom. It has an unphysical sign, hence it corresponds to an unphysical scalar. Thus Slavnov's scheme is not a good regularization scheme since unitarity is violated due to the unphysical scalar.

We conclude that the extra vertices which were added by hand to make the PV scheme gauge invariant, still regulate the theory (if one uses dimensional regularization to show that all one-loop divergences cancel), but taking the regulating parameters Λ and M_j to infinity, the divergences thus obtained are not the same as the divergences obtained with another scheme (for example, ordinary dimensional regularization without PV regularization). We conclude that gauge-invariant regularization gives incorrect answers for physical quantities because the vertices one has added, truly change the theory. Since anyhow dimensional regularization was needed at the one loop level, one can better forget about PV regularization for gauge fields.

Incidentally, even a gauge-invariant regularization scheme will not fix local physical terms in the effective action of a generic renormalizable theory, since one can always make finite additional renormalizations. However, in a finite theory like CS theory in 3 dimensions, the local physical quantities ("observables"), namely Wilson loops and all objects obtained from them (like the coefficient of the CS action) are unique and independent of the particular gauge invariant scheme chosen to demonstrate that the theory is finite.

Incidentally, the Yang-Mills result $-11/3$ follow from the general rule

$$\begin{aligned}
&\beta \sim C_2 \frac{1}{\epsilon}(-)^{2S} \sum_z \left(\frac{1}{3} - (2S_z)^2\right) \\
&S = 0 : \frac{1}{3} \\
&S = 1/2 : \frac{2}{3} \\
&S = 1 : \frac{-11}{3}
\end{aligned} \tag{20.0.28}$$

The current is decomposed into a convection and a spin part

$$\begin{aligned}
\text{spin } 1/2 : A^\mu \bar{\psi}\gamma_\mu \psi &= \bar{\psi} \overleftrightarrow{\partial}_\mu \psi + \overline{\psi}\sigma_{\mu\nu}\psi F^{\mu\nu} \\
\text{spin } 1 : A^\mu j_\mu(A) &= A_a{}^\mu \left\{ A_b^\nu \overleftrightarrow{\partial}_\mu A_{\nu c} f^{abc} \right\} + A_a^\mu S_{\mu\nu}{}^{\alpha\beta} A_b{}^\nu F_{\alpha\beta}{}^c f^{abc}
\end{aligned} \tag{20.0.29}$$

(For spin 1, the ghost only contributes to the convection current since it has spin 0). See R. Hughes, *Nucl. Phys. B* 1980/1981. His corresponding *Phys. Lett.* has less detail).

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Theor. Math. Phys. **10**, 99 (1972) (higher deriv. in YM theory)
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Chapter 21

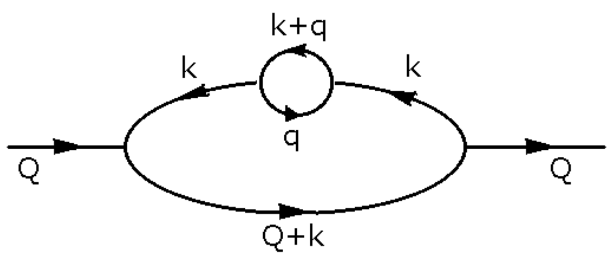
The infrared R^* operation

Massless particles can introduce infrared divergences (IRD) in Feynman graphs. In Minkowski space there are IRD in the S matrix for QED due to the emission of soft real photons, and also in graphs with virtual photons. These IRD cancel in the cross section [1]. Furthermore, in QCD and also in QED in the approximation that quarks are massless, one has the situation that massless particles couple to themselves or to other massless particles, and this leads to further IRD in the S matrix in Minkowski space, the so-called collinear divergences. These cancel also if one averages over the color of the incoming particles and sums over the momenta of the initial states with soft collinear particles [2], or if one uses factorization methods. In this note we discuss IRD in Green's functions in Euclidean space; they are unrelated to the IRD in Minkowski space which occur in the S matrix.

Four-dimensional quantum field theories with only dimensionless coupling constants contain in Euclidean space for generic external momenta no IRD. Thus in the proof of renormalizability of proper graphs in QED, all divergences which one encounters (in Euclidean space) are ultraviolet divergences (UVD). The Z factors contain thus only information about the small-distance behavior, and for this reason they can be used to construct running coupling constants. The same applies to QCD and massive quarks. Even if one sets the masses of all particles to zero, it remains true that for generic Euclidean external momenta the proper Green function in four dimensions are free from IRD[1].

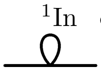
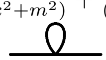
Spontaneously broken field theories have in general dimensionful coupling constants. For example, the $O(2)$ Goldstone model and the $SU(2)$ Higgs model contain an interaction term $\lambda v \sigma^3$ with λv a dimensionful coupling constant. Nevertheless these four-dimensional Goldstone and Higgs models do not contain any IRD in the Euclidean proper graphs; this is due to the Goldstone theorem which states that proper selfenergies $\Pi(p)$ for Goldstone bosons vanish at $p^2 = 0$ even when loops with massive scalars σ contribute to the Goldstone boson selfenergy.

However for such theories as $\lambda\varphi^4 + h\varphi^3$ theory in four dimensions with a superrenormalizable dimensionful coupling constant h, IRD do occur. A simple graph which shows this explicitly is as follows

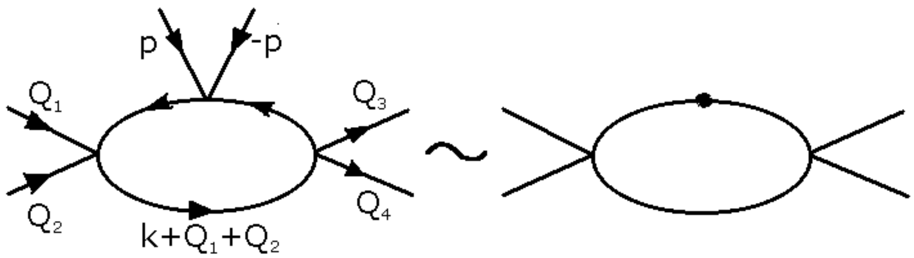


(21.0.1)

We have a massless boson in the larger loop with loop momentum k and a selfenergy insertion with loop momentum q due to a massive scalar. The two massless propagators k^{-2} lead to a logarithmic IRD $\int d^4k/k^4$ because now the proper selfenergy of the massive scalar $\Pi(k) = \int \frac{1}{q^2+m^2} \frac{1}{(k-q)^2+m^2} d^4q$ does not vanish for small k.

[1]In dimensional regularization one sets the following gluon selfenergy graph  to zero. This graph has an UVD but not an IRD, and its vanishing should be considered as the result of a computation, not as an independent rule. (One may for example replace $\frac{1}{k^2}$ by $\frac{m^2}{k^2(k^2+m^2)} + \frac{1}{(k^2+m^2)}$, and one finds then that the sum of both integrals vanishes). Similarly  $\sim \int d^4k/k^4$ vanishes, but now the IRD (evaluated at $n > 4$) cancels the UVD (evaluated at $n < 4$). Again this cancellation follows from the rules of dimensional regularization.

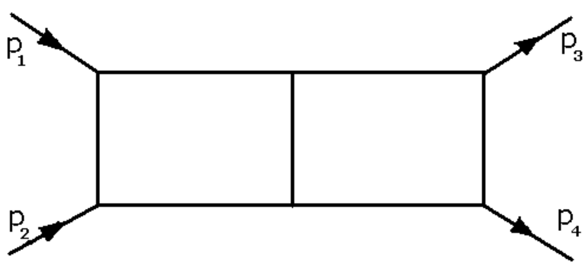
Also at exceptional momenta IRD in Euclidean space can occur as the following example shows



(21.0.2)

The two external lines in the middle carry momenta p and $-p$ and one can also write them as a φ^2 -insertion. The loop integral yields $\int d^4k \frac{1}{k^4}\frac{1}{(k+Q)^2}$ with $Q = Q_1 + Q_2$, and contains clearly an IRD at $k = 0$.

Another area where IRD create problems is in the calculations of higher-loop β functions. Suppose we want to compute the divergences in the following 4-point graph



(21.0.3)

It simplifies the calculation a great deal if one sets $p_3 = p_4 = 0$. When we set some external momenta to zero, we shall call this operation "nullification of momenta". Nullification of some of the external momenta makes loop calculations a lot easier but it creates spurious IRD. (By spurious we mean that for generic momenta there are no such infrared divergences). To compute the UVD in this graph after having put $p_3 = p_4 = 0$, one must first remove the spurious IRD [3].

The conclusion is that in four-dimensional Euclidean space Feynman graphs may contain IRD in the following cases
(i) if there are superrenormalizable coupling constants such as $\lambda\varphi^3$ interactions in $D=4$.
(ii) at special external momenta. More precisely, when the sum of some external momenta is equal to the sum of some (or none) of the other external momenta; in particular when some external momenta are nullified. (In the example with external momenta $+p$ and $-p$, the sum of these two external momenta vanishes).

When there are both massless and massive particles in the theory, the situation is much more complicated, and one must proceed by studying each theory separately. For example the interaction $g\varphi\chi^2$ in $D=6$ with a massless field φ and a massive field χ leads to IRD in the φ selfenergy with two or more closed χ loops,but if one first renormalizes and imposes the renormalization condition that the φ self-energy vanishes like k^2, the IRD disappears. This example shows that in general **one should first renormalize and then study IRD**.

In higher spacetime dimensions the degree of IRD in general decreases because of the measure d^Dk, but in lower spacetime dimensions one encounters more IRD. In particular, in $D=2$ the nonlinear sigma models with action $g_{ij}\partial_\mu\varphi^i\partial^\mu\varphi^j$ have dimensionless coupling constants (for example $g_{ij}=\delta_{ij}(1+g\varphi^2)$ has the dimensionless coupling constant g) but tadpoles and selfenergies are IRD. On the other hand, $\lambda\varphi^3$ theory in $D=6$ dimensions is renormalizable but not superrenormalizable (because λ is now dimensionless) and has no IRD.[2]

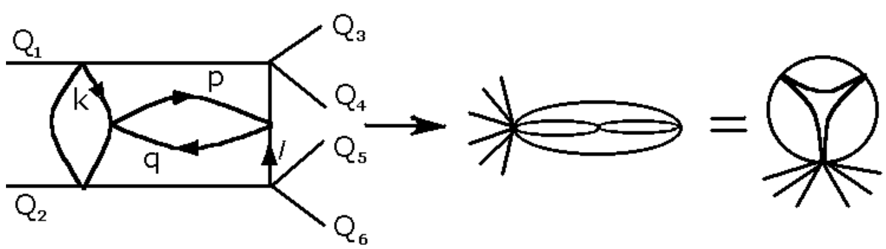
To explain these various results on IRD one can use some simple IR power counting. Consider a proper graph with nonvanishing external momenta p_i in D dimensions.

[2]As an example, consider in 6 dimensions a selfenergy graph with a massless scalar in the loop, and insert into this loop a string of M selfenergies with massive scalars in the loops. Then the propagators yield a factor $(\frac{1}{k^2})^{M+1}$ and the measure yields $\int d^6k$, but now each massive renormalized selfenergy yields a factor $\int d^6q\frac{1}{(q^2+m^2)(k-q)^2+m^2}\sim k^2$, and there is indeed no infrared divergence.

We shall assume that the external momenta are nonexceptional by which we mean that there does not exist a relation

$$\sum_{j=1}^{M} p_j = \sum_{k=1}^{N} p_k \tag{21.0.4}$$

for any $M \geq 0$ and $N > 0$ other than overall energy-momentum conservation. The propagators contain loop momenta and external momenta. Choosing a particular momentum flow through the diagram, there are "soft propagators" with only loop momenta and "hard propagators" with a combination of loop momenta and external momenta. (There are no propagators with only a combination of external momenta since the graph is proper). For vanishing loop momenta, the hard propagators do not become singular if we do not have exceptional external momenta. Hence, for nonexceptional values, the external momenta provide an infrared cut-off for the hard propagators. We can then determine the IRD which occur if one or more loop momenta tend to zero by shrinking all hard propagators to a point. The following example in $\lambda\varphi^4$ theory illustrates this procedure



(21.0.5)

The contracted graphs are still proper when the original graph was proper. Because the hard propagators form a connected graph, there is only one contracted vertex for a given proper graph when the external momenta are nonexceptional.

Just as one can count the degree of UVD of a proper graph by UV counting rules one can also develop IR counting rules. To perform IR counting, consider a contracted proper graph with N external lines, L loops, I internal propagators and vertices V_j with j lines, in addition to the contracted vertex. Let N_i be the number of soft lines at the contracted vertex (the subscript i stands for internal). Because the contracted graph is still proper the number of soft lines connecting it to the rest of the graph is at least 2, hence $N_i \geq 2$. Then the usual counting rule for the number of loops and the relation which states that any internal (external) line ends at two (one) vertices, lead to

$$\begin{aligned} L &= I - \left(\sum_j V_j + 1\right) + 1 \\ N + 2I &= \sum_j jV_j + (N + N_i) \end{aligned} \tag{21.0.6}$$

(In the example $L = 4, N = 6, V_4 = 2, I = 6$ and $N_i = 4$. Then $L = 4 = 6 - 2$ and $N + 2I = 18 = 8 + 6 + 4$). If there are no superrenormalizable couplings, a vertex V_j in D dimensions carries $D - \frac{1}{2}j(D - 2)$ momenta attached to it. (For example, in gauge theory in $D = 4$, the $AA\partial A$ vertex carries one momentum and the $AAAA$ vertex carries no momentum). When all loop momenta tend to zero, minus the overall degree of IRD of a contracted graph is given by

$$\begin{aligned} \omega_{IR} &= DL - 2I + \sum V_j \left(D - \frac{1}{2}j(D-2)\right) \\ &= \frac{1}{2}(D-2)N_i \geq D - 2 \end{aligned} \tag{21.0.7}$$

Therefore in $D \geq 3$ there are in general no overall IRD, but in $D = 2$ all contracted graphs are logarithmically IRD. In particular the widely used WZWN models contain IRD. On the other hand, gauge theories in 4 dimensions have no IRD, as we already discussed.

What happens if only some of the loop momenta tend to zero, but others do not? If a number $I_H \leq I$ of the internal momenta are kept hard, one can further contract the original diagram such that only the soft lines ($I - I_H$ in number) remain. For example, one could make the loop momentum of the soft loop at the top in figure (21.0.5) hard. If one were to make the two propagators in the loop on top hard, contraction of this new hard lines would yield a second contracted vertex, but contraction of one of the loops on the side would still leave only one contracted vertex.

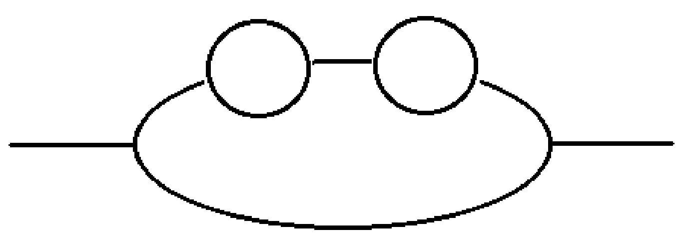
Let the new hard lines form a subgraph with L_H loops, I_H propagators and V_{jH} vertices. The remaining degree of IRD is in this case

$$\begin{aligned}\omega_{IR}\ (\text{subcase})\ &= \omega_{IR} - \Delta\omega_{IR}\\ \Delta\omega_{IR} &= DL_H - 2I_H + \sum jV_{jH}\left(D - \frac{1}{2}j(D-2)\right)\end{aligned} \tag{21.0.8}$$

The subgraph can be worse IR divergent than the original graph because ω_{IR} (subcase) is equal to ω_{IR} of the original graph minus the degree of infrared divergence $\Delta\omega_{IR}$ of the subgraph. The original set of hard lines together with the new set of hard lines form a new (possibly disconnected) set of hard lines, and we can again apply the IR counting rules to the new contracted graph. In the example in (21.0.5) with $I = 6$ soft lines we can make the two soft lines on the right hard. Then the soft part subdivides as follows

$$\begin{aligned}&\omega_{IR}\ (\text{subcase})\ = 3D - 8 + (-D+4) = 2D - 4\\ &\Delta\omega_{IR} = D - 4 + (D - 2(D-2)) = 0\end{aligned} \tag{21.0.9}$$

The subcase remains indeed IR finite in $D \geq 3$. As another example, one may make the loop on top of figure (21.0.5) hard. Than one finds that $\Delta\omega_{IR} = D-4+(-D+4) = -D+4$, and $\omega_{IR}(\text{subcase}) = 3D-8$, which is again IR for $D \geq 3$. However, an example of a subgraph which is divergent while the minimal graphis not is the following self energy in D=6:



(21.0.10)

The graph is overall IR finite (as we have proven generally) but contracting the two subloops, one finds an IRD proportional to $\int d^6k/k^6$. We must thus learn how to subtract IR subdivergences.

There exists a general scheme for subtracting IRD, which is the counter part of the so-called R-scheme of BPHZ for subtracting UVD [4]. The scheme which subtracts both UVD and IRD is called the R^* scheme [5], and one can formally write

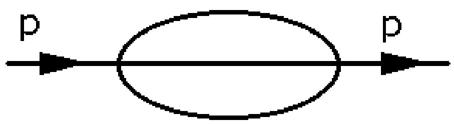
$$R^* = R_{UV} R_{IR} \tag{21.0.11}$$

According to this prescription, one first removes all IRD of given graph, and only afterwards subtracts all UVD. This is correct for most cases, but there are graphs where $R_{UV} R_{IR}$ is not equal to $R_{IR} R_{UV}$. The question then arises which order is the correct one, and the answer is that the correct order is $R_{IR} R_{UV}$, and not (21.0.11). We already argued before that in the case of theories with massive and massless particles one should first renormalize before extracting IRD. In most of our examples we shall follow the prescription in (21.0.11) because this is technically easier, but we shall also discuss a 5-loop graph where (21.0.11) is not correct.

The UVD can be canceled by the usual UV counter terms, but the IRD are discarded by hand. For the calculation of β functions this is no problem because the IR divergences due to nullification of external momenta were anyhow spurious, but for field theories such as $h\varphi^3$ in $D = 4$ discarding genuine IRD by hand seems a dubious procedure. One would prefer to have also an IR renormalization procedure similar to the UV renormalization procedure, but it does not seem to exist. Speculations have been made that the sum of infrared divergences vanishes in the two-dimensional WZWN model (when properly summed). [4] If this does not happen in this model, or in massless superrenormalizable theories, one would have to exclude such theories.

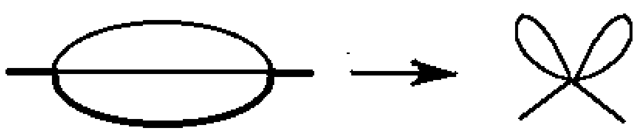
In the R subtraction scheme of BPHZ, graphs are expanded into a Taylor series in the external momenta. We shall instead use dimensional regularization to compute both the IRD and the UVD. Before going on, we make a comment on the relation $\int d^4k/k^4 = 0$ in dimensional regularization. The reason this integral vanishes is that it contains both an UVD and an IRD, whose sum cancels. (One may separately compute the UVD from $\int d^4k/(k^2 + m^2)^2$ and the IRD from $\int d^4k \frac{1}{k^4} \frac{1}{k^2+m^2}$ and show that their sum cancels). If one were to use $\int d^4k/k^4 = 0$ in higher-loop β function calculations, one would drop some UVD, and hence one would make an error. In fact, one never encounters the need for setting $\int d^4k/k^4 = 0$ in the computation of the β function at the one-loop and two-loop level, but at higher loops care is required not to discard UVD. The claim is that the R^* scheme does not loose UVD even though it sets tadpoles to zero according to the rules of dimensional regularization.

Let us now explain the infrared subtraction procedure by some simple examples. Consider massless $\lambda\varphi^4$ theory in $D = 4$. The proper graph



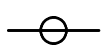
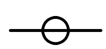
(21.0.12)

contains no IRD at generic p, hence there is nothing to subtract. This is clear after contracting the graph

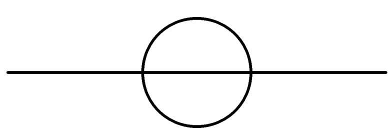


(21.0.13)

Let us introduce an operator R_{IR} which projects out the infrared finite part from a given graph. Then in this example we obtain

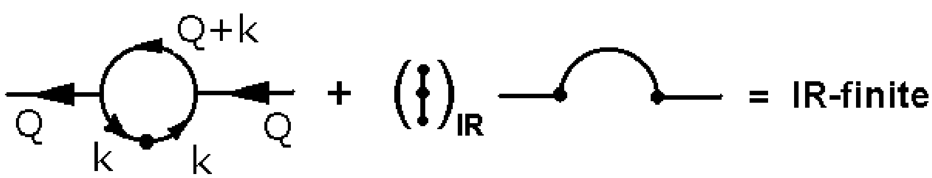
R_{IR}  =  = IR-finite (21.0.14)

However, consider in $D = 4$ the graph



(21.0.15)

where the dot indicates that two external momenta have been nullified. There is then clearly a logarithmic divergence $\int d^4k/k^4$. This IRD is due to the double propagator and we can determine the divergence it produces by inserting it into the simplest graph where it yields an IRD. So we work in two steps: we determine first the IR divergent part associated with and then we use it to subtract the IRD from the original graph. We indicate the IRD induced by the double propagator by $\left(\; \right)_{IR}$. Then $\left(\; \right)_{IR}$ should make the simplest graph with an insertion finite as far as IR divergences go



(21.0.16)

Note that we obtain the remainder of the graph after extracting the IRD by deleting the subgraph which yields the IRD, not by contracting it. Analytically the meaning of this graphic equation is as follows

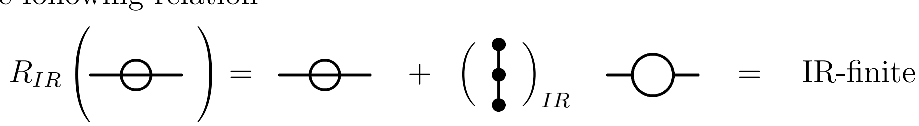
$$\int d^Dk \frac{1}{k^4(Q-k)^2} + \left(c_1 \frac{1}{\epsilon}\right) \frac{1}{Q^2} = \text{ IR-finite} \tag{21.0.17}$$

We can also formulate the subtraction procedure by replacing $\frac{1}{k^4}$ in the original graph by a term with $\delta^4(k)$

$$\frac{1}{k^4} \to \frac{1}{k^4} + c_1 \frac{1}{\epsilon} \delta^D(k) \mu^\epsilon \;, \; \epsilon = D - 4 \tag{21.0.18}$$

The factor μ^ϵ is needed in dimensional regularization to keep the dimension of the last term the same as that of k^{-4}. Given the rules of dimensional regularization, one can compute c_1.

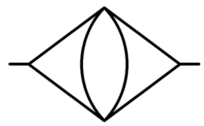
We now return to the original graph in (21.0.15). As before, the infrared finite part of the original graph is projected out by the operator R_{IR}, and pictorially one has the following relation



(21.0.19)

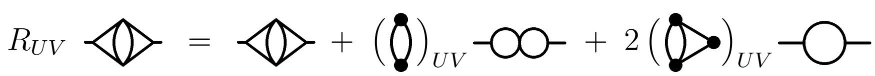
Note that one always begins with the original graph on the right-hand side, and then one adds terms which subtract the IRD of the various subgraphs of the original graph. By convention one always writes plus signs on the right-hand side.

The UV subtraction scheme can be formulated in the same way. Consider for example the graph in $D = 4$



(21.0.20)

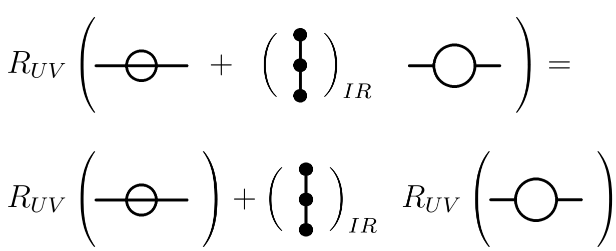
It has a logarithmic UVD due to the loop in the middle, and also two logarithmic UV divergences due to the two 2-loop subgraphs on the left and on the right. (These two subgraphs are overlapping but that does not modify the subtraction rules). Let us introduce an operator R_{UV} which projects out the UV finite part of a graph. One obtains then



(21.0.21)

Again by convention we always use plus signs for the terms to be subtracted.

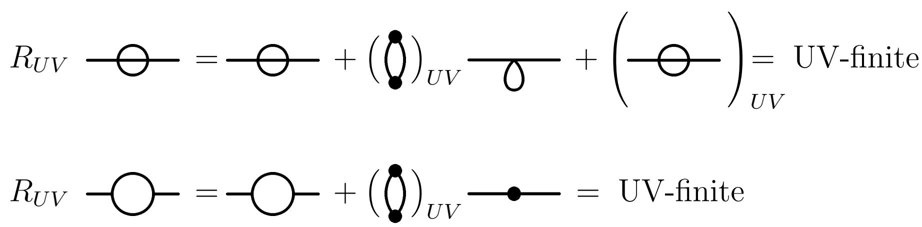
After the IRD have been subtracted, one may subtract the UVD. One does this for each term on the right-hand side of the R_{IR} equation separately. Consider for example (21.0.19). The subtraction of UVD proceeds as follows



(21.0.22)

We never subtract IRD or UVD from counter terms, so there is no "nesting" of the subtraction procedure.

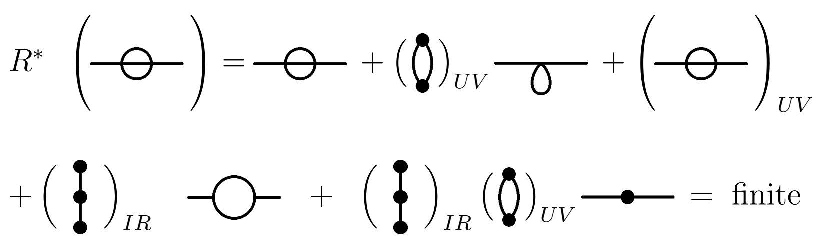
The UVD are easily located. Only the subgraph  is UVD. Hence



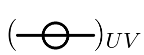
(21.0.23)

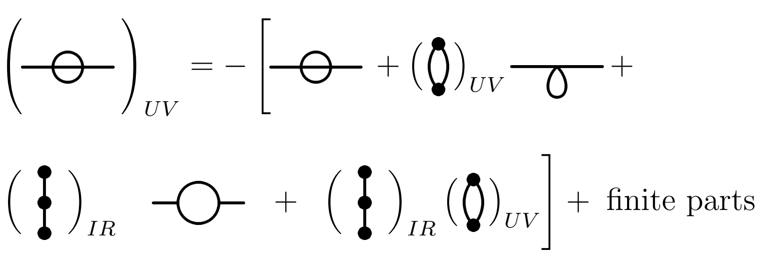
The tadpole graph in the first line vanishes according to the rules of dimensional regularization. (We set it to zero even though it contains an infrared and an ultraviolet divergence. The ultraviolet divergence in this diagram is accounted for by the whole R^* procedure). Note that again we always begin with the original graph on the right-hand side, and then subtract divergences by adding counter terms. (Again by convention we use plus signs for these subtractions).

Combining the IR and UV subtraction procedure, we obtain for the graph in (21.0.15)



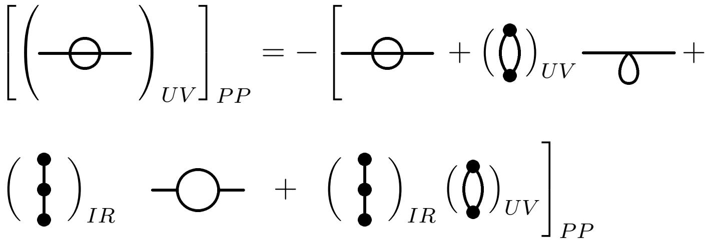
(21.0.24)

We can now determine the overall UV counter term which makes the graph finite after all subdivergences have been removed. This is the UV counter term one needs for the β function. It is given by  and can be computed by evaluating the r.h.s. of the following equation



(21.0.25)

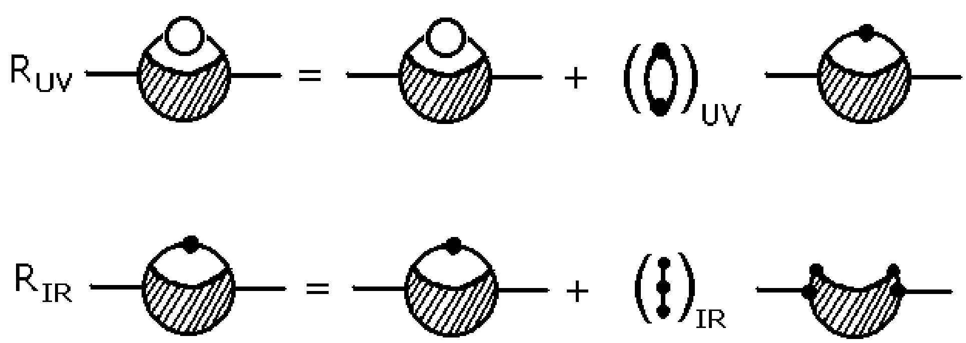
Taking the pole parts (PP), we can also write



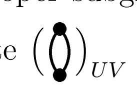
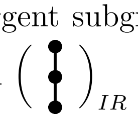
(21.0.26)

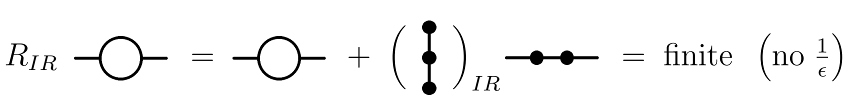
Each graph on the r.h.s. is computed with dimensional regularization, including the original graph (which is of course the most difficult to compute). The subtraction terms denoted by $()_{UV}$ and $()_{IR}$ are polynomials in $\frac{1}{\epsilon}$, so one must compute also some graphs on the right-hand side to order ϵ, ϵ^2 etc. More precisely when there is a higher order pole $\frac{1}{\epsilon^k}$ due to $()_{UV}$ and $()_{IR}$, one must compute the corresponding graph to order ϵ^{k-1}.

To illustrate the analogies and differences of R_{UV} and R_{IR} consider the following example



(21.0.27)

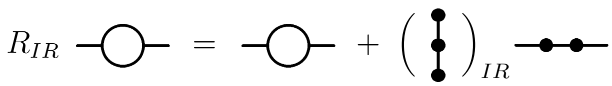
The hatched part of the graph denotes any nonsingular subgraph. As this example shows, to remove UVD one shrinks the divergent proper subgraphs to a point, but one deletes the IR divergent subgraphs. We compute  as the $\frac{1}{\epsilon}$ pole part of this one-loop graph, and  is computed from



(21.0.28)

In both cases one gets $\frac{1}{\epsilon}$ poles (if one uses dimensional regularization).

Recall that the graphical identity

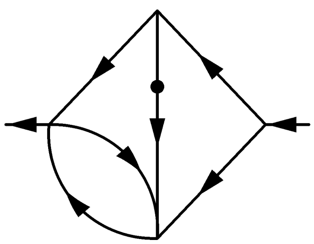


(21.0.29)

corresponds to the following analytical expression

$$R_{IR}\int d^4k \frac{1}{k^4}\frac{1}{(k-Q)^2} = \int d^4k \left(\frac{1}{k^4}+\frac{c}{\epsilon}\delta^4(k)\right)\frac{1}{(k-Q)^2} \tag{21.0.30}$$

The IRD occurs at $k=0$ and is taken care of by the $\frac{1}{\epsilon}\delta^4(k)$ insertion. One may find products of such IR factors $\delta^4(k)/\epsilon$ with different loop momenta, but never with the same loop momentum. As an example consider the following Feynman graph in $\varphi^4+\varphi^3$ theory



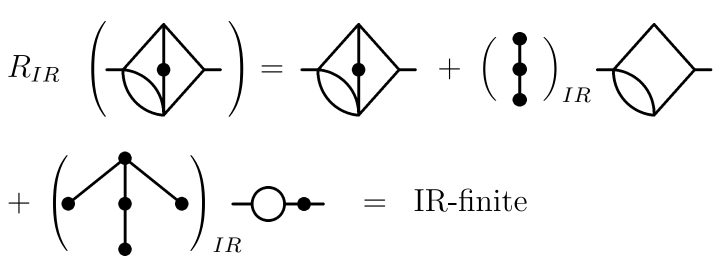
$$F = \int \frac{d^4k d^4q d^4p}{(k-p)^4 p^2 k^2}\frac{1}{(Q-k)^2}\frac{1}{(Q-q)^2(p-q)^2} \tag{21.0.31}$$

In this example there is an IRD at $k=p$, and an overall IRD at $k=p=0$. To subtract these IRD we replace some propagators by delta function in D dimensions

$$\begin{aligned} R_{IR}(F) &= F + \textstyle\int \left(\frac{1}{(k-p)^4} \to \mu^\epsilon C_1 \delta^D(k-p)\right)\frac{1}{p^2}\frac{1}{k^2}\frac{1}{(Q-k)^2}\frac{1}{(Q-q)^2}\frac{1}{(p-q)^2} \\ &+ \textstyle\int \left(\frac{1}{(p-k)^4}\frac{1}{p^2}\frac{1}{k^2} \to (\mu^{2\epsilon})^2 C_2 \delta^D(p)\delta^D(k)\right)\left(\frac{1}{(Q-k)^2(Q-q)^2}\frac{1}{(p-q)^2}\right) \end{aligned} \tag{21.0.32}$$

Because there are now two D-dimensional Dirac functions one needs the factor $\mu^{2\epsilon}$ with $\epsilon = D-4$, as usual in dimensional regularization, to make the dimensions come out correctly.

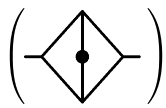
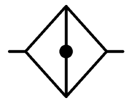
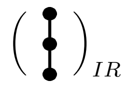
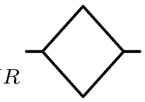
We can write this pictorially as follows

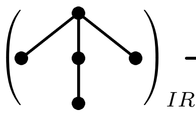
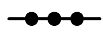


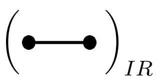
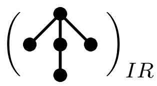
(21.0.33)

We recall that we must first compute C_1 and C_2, and then by substitution we should find that the r.h.s. is IR-finite.

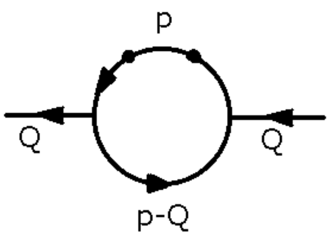
The computation of C_1 was discussed before, it follows from  . The computation of C_2 follows from requiring that the following expression be IR-finite

R_{IR}  $=$  $+$  

$+2$   $=$ IR-finite (21.0.34)

The factor 2 is due to the two overall IRD, one upstairs and one downstairs. Note that  $= \frac{a}{\epsilon}$ but  $= \frac{b}{\epsilon} + \frac{c}{\epsilon^2}$. (If the two lower propagators would have been massive, we would have needed a factor one). Note also that  $= 1/Q^4$. Having determined C_1 and C_2 we can determine R_{IR}  .

The IR counter terms are always polynomials in $\frac{1}{\epsilon}$ and $\delta^4(k_j)$. However, as our next example shows, one sometimes needs derivatives of $\delta^4(k)$. Consider the following massless graph



$$= \int \frac{d^D p}{p^6 (p-Q)^2} \tag{21.0.35}$$

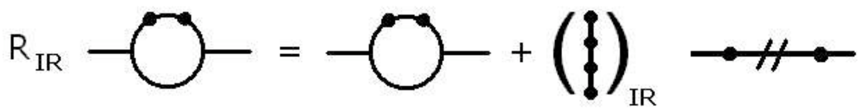
There is now a quadratic IRD proportional to $\int d^4p/p^6$. We want to extract it from the original graph by adding a term involving $\delta^D(p)$ but in order that dimensions match, we need a Dalembertian acting on $\delta^D(p)$

$$R_{IR} \int \frac{d^D p}{p^6 (p-Q)^2} = \int d^D p \left[\frac{1}{p^6} + \mu^\epsilon \bar{C}_1 \Box_p \delta^D(p) \right] \frac{1}{(p-Q)^2} \tag{21.0.36}$$

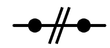
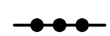
Partially integrating the operator $\Box_p = \frac{\partial}{\partial p^\mu} \frac{\partial}{\partial p_\mu}$ we obtain for the last term in the integrand

$$\int d^D p \delta^D(p) \Box_p \frac{1}{(p-Q)^2} = \left[\Box_p \frac{1}{(p-Q)^2} \right] \Bigg|_{p=0} \tag{21.0.37}$$

To fix $\bar{C}_1$ we consider the simplest graph with this divergence; this is unfortunately the graph itself but we nevertheless proceed (we could make the original graph more complicated to lift this degeneracy). Hence we fix $\bar{C}_1$ from



(21.0.38)

The two slashes in the last part of the equation indicate the action of $\Box_p$ on $1/(p-Q)^2$. However, $\Box_p(p-Q)^{-2}\Big|_{p=0} = 2\epsilon Q^{-4}$, so we find the equation  $= -2\epsilon$  and

$$\int d^D p \frac{1}{p^6} \frac{1}{(p-Q)^2} + \bar{C}_1 2\epsilon \frac{1}{Q^4} = \text{IR-finite} \tag{21.0.39}$$

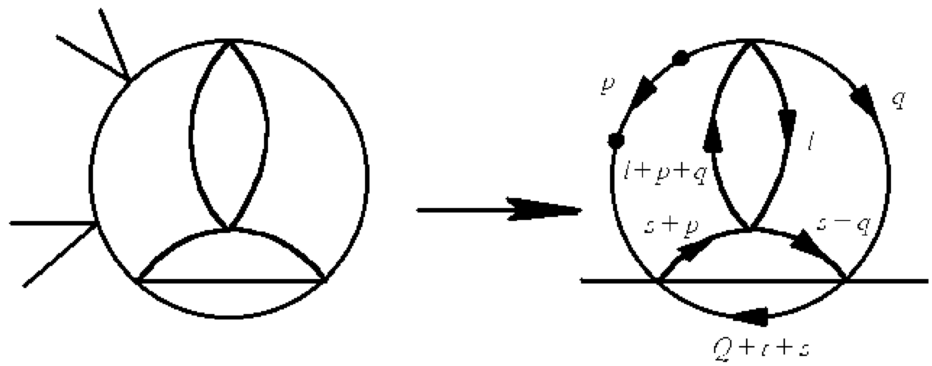
One would expect that $\bar{C}_1$ is proportional to $\frac{1}{\epsilon}$, so the original graph happens to be IR-finite (due to the peculiar properties of dimensional regularization).

To evaluate $\bar{C}_1$ we therefore need another graph. One could take a massive propagator with $(p-Q)^2$ in which case

$$\Box_p[(p-Q)^2 + m^2]^{-1} = \frac{(8-2D)(p-Q)^2 - 2Dm^2}{[(p-Q)^2 + m^2]^3} \tag{21.0.40}$$

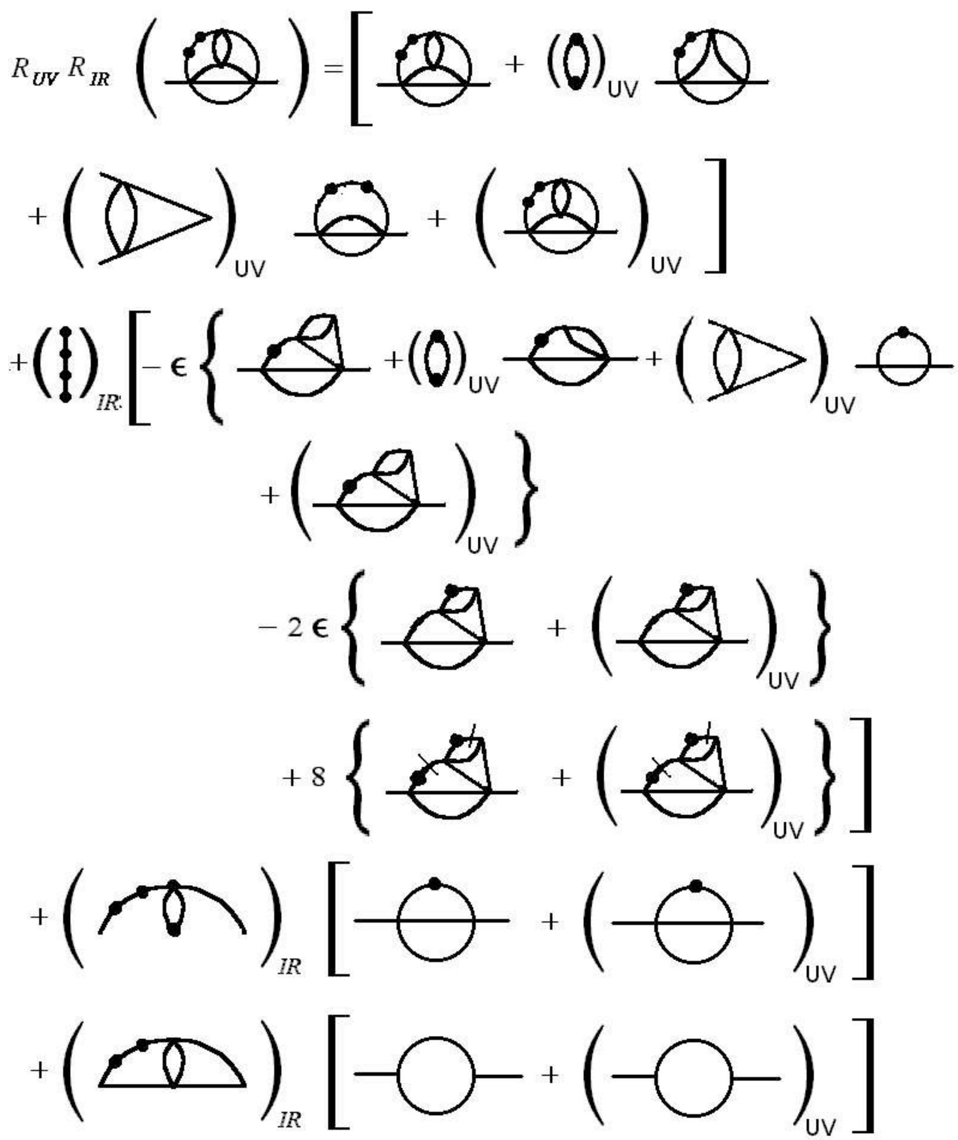
is no longer proportional to ϵ. One can then determine $\bar{C}_1$.

We now discuss a subtlety having to do with the order in which one applies R_{UV} and R_{IR}. The combined operation is denoted by R^*. A priori one might expect that $R_{IR}R_{UV}$ is equal to $R_{UV}R_{IR}$, but there exist counter examples at the five-loop level [7]. Consider the following 5-loop graph



(21.0.41)

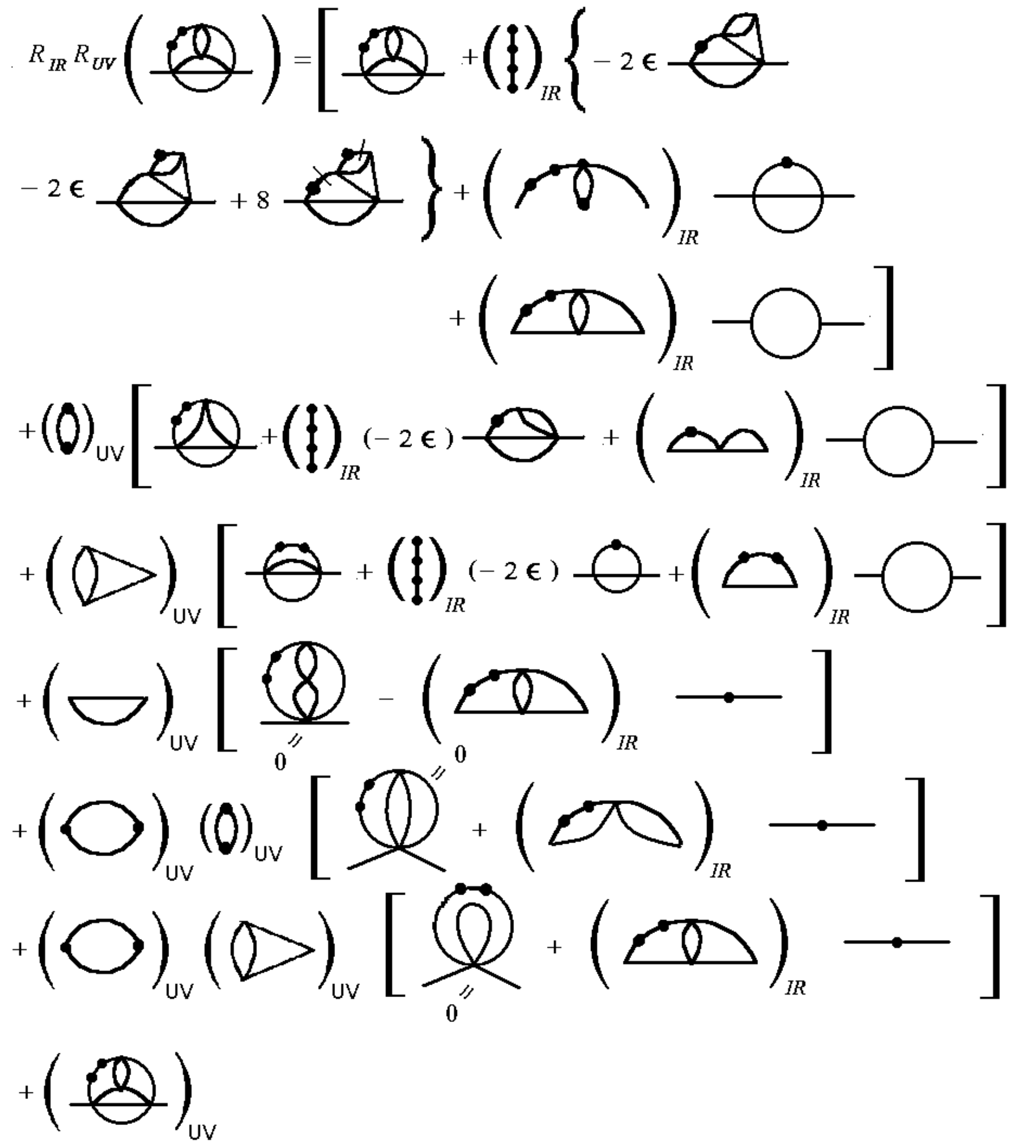
Operating with $R_{UV}R_{IR}$ gives the incorrect result



(21.0.42)

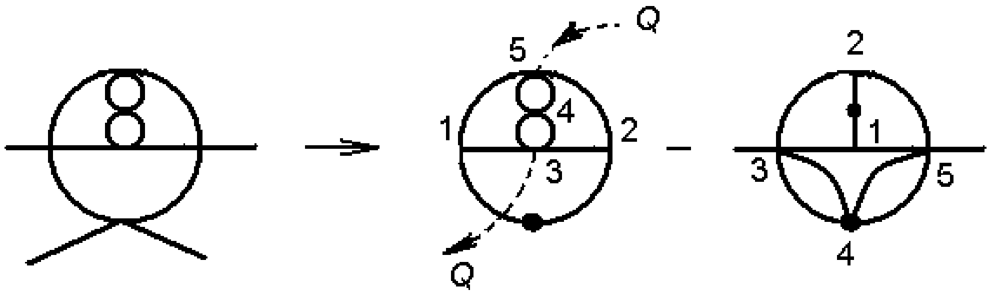
The IRD arise when the following loop momenta vanish: p, pql, and $pqls$. When pt vanish (or pl), one finds an IRD of the term $\int dpdtp^{-6}t^{-2}$, but this was already considered in the IRD with $\int dpp^{-6}$ so we do not count it separately. We repeatedly used that tadpoles vanish, no matter how many loops they contain. We also used (21.0.36)–(21.0.39), but because there are two propagators on which $\Box_p$ can act, we get also cross terms where each propagator carries one derivative. Operating with

$R_{IR}R_{UV}$ gives the correct result



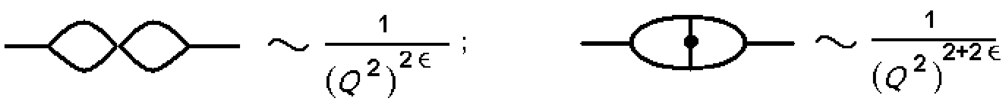
(21.0.43)

As another example, we consider a 5-loop graph which is needed to compute the β function at 5-loops. The original diagram gives a vertex correction for the $\lambda\varphi^4$ coupling, namely a graph with 4 external lines, but we nullify all 4 lines. Since vacuum graphs vanish in dimensional regularization we add two new external lines carrying a new external momentum Q.



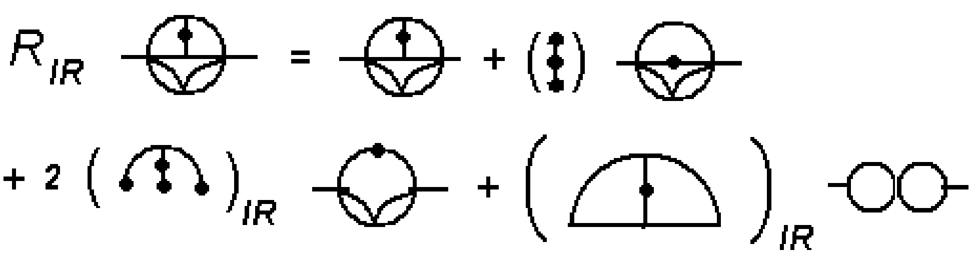
(21.0.44)

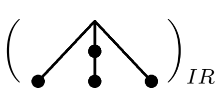
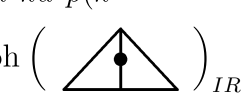
The reason we let momenta Q flow in and out the graph at these particular points is that this allows to compute the original graph easily. Indeed, to compute the graph itself (which is always needed), one may first compute the subgraphs



(21.0.45)

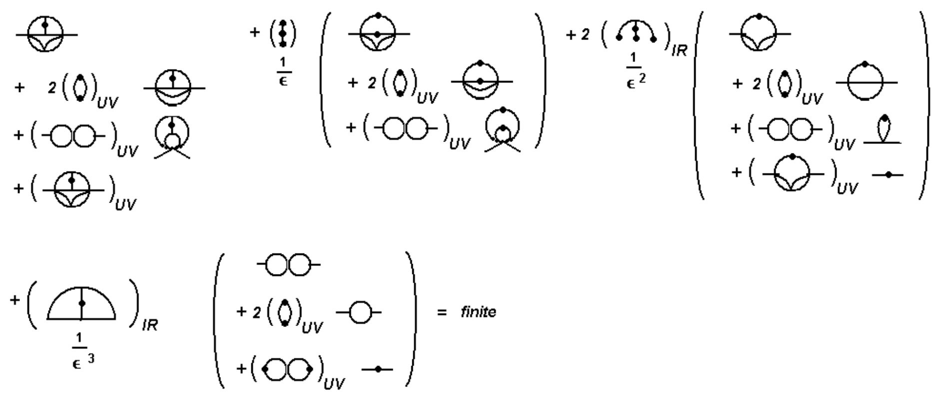
Then one evaluates $\int d^D q (Q-q)^2)^{-2\epsilon}(q^2)^{-2-2\epsilon}$. We now evaluate R^* on this diagram. Since there are no $\frac{1}{p^6}$ terms, there is no term with $\Box_p \delta^D(p)$ and thus no ambiguity whether one should choose $R_{UV}R_{IR}$ or $R_{IR}R_{UV}$. We choose the former. We first record the result for acting with R_{IR} on the graph, and then record the results due to acting with R_{UV} on each of the terms in the result for R_{IR}. We write the results such that each column in the result for R_{UV} corresponds to one term in the result for R_{IR}.



The subgraph  (which corresponds to the IR divergent integral $\int d^4k d^4p (k - p)^{-2} k^{-2} p^{-4}$ for small k and p) is written as $c\delta^D(k)\delta^D(p)$ whereas the subgraph 

contains only one delta function $\delta^D(k+q)$ (since there is only one external momentum, one can only use one $\delta^D(p)$). At the end no IRD are left ( is IR finite).

Now we perform R_{UV} on each of these terms. We write the result of each R_{UV} operation as a column.



(21.0.46)

Only the term at the bottom of the first column is needed for the β function. Since we need all $\frac{1}{\epsilon}$ poles, we must calculate each graph to some power in ϵ. In particular, since the IRD in the second column is proportional to $\frac{1}{\epsilon}$ we need the graphs in this column to order ϵ^0, but since the IRD in front of the third column is $\frac{a}{\epsilon^2}+\frac{b}{\epsilon}$, we need the graphs in this column to order ϵ, and we need the terms proportional to $1, \epsilon$ and ϵ^2 in the graphs of the last column since the IRD in front of the last column contains a leading term $1/\epsilon^3$. The three 1-loop, 2-loop and 3-loop tadpoles graphs in the first three columns in the third row all vanish according to the rules of dimensional regularization.

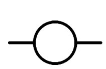
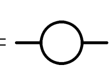
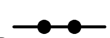
All these calculations were done using dimensional regularization, not dimensional reduction (the latter is inconsistent at higher loops [8]). For theories with γ_5 and $\epsilon_{\mu\nu\rho\sigma}$ the approach followed is to first compute graphs without ϵ tensors (substituting for γ_5 the product $\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma$), and only after the calculation is finished one contracts with four-dimensional ϵ tensors. In fact, to compute $R_{UV}F$ of a divergent graph with several ϵ tensors, one can use the fact that $R_{UV}F$ is UV finite, and write the product of two ϵ tensors in terms of D-dimensional δ functions. The error one makes is of order ϵ, so vanishes as $\epsilon \to 0$. These D-dimensional Kronecker delta function one can then insert inside the expression $R_{UV}F$ to obtain a scalar. If one has a single $\epsilon_{\mu\nu\rho\sigma}$, one can multiply by $k_1^\alpha k_2^\beta k_3^\gamma k_4^\delta \epsilon_{\alpha\beta\gamma\delta}$ and work out the product $\epsilon_{\alpha\beta\gamma\delta}\epsilon^{\mu\nu\rho\sigma}$ in terms of D-dimensional delta functions. Effectively this means contracting the open indices in $R_{UV}F$ with momenta k^μ to obtain a Lorentz scalar. So, in the end one never computes with open indices. One obtains the correct answer for the 3-loop chiral anomaly [9]. This approach works only for multiplicatively renormalizable quantities, and not diagram-by-diagram. The reason is that for multiplicatively renormalizable models

$$R_{UV}F = ZF \tag{21.0.47}$$

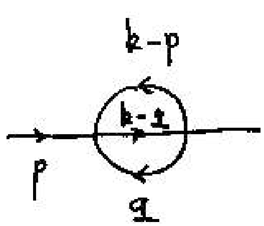
Then the error in using D-dimensional contractions is of order $D-4$.

Another example where the correct subtraction of IRD is crucial for determining the UVD is the massless WZWN model in $D = 2$ dimensions [6]. The simplest IR subtraction corresponds to

$$R_{IR}\int \frac{d^2k}{k^2} = \int d^2k \left(\frac{1}{k^2} + \frac{\pi}{\epsilon}\delta^2(k)\right), \quad \epsilon = n-2$$

R_{IR}  $=$  $+\,2\,\Big($  $\Big)_{IR}$  $=$ IR-finite (21.0.48)

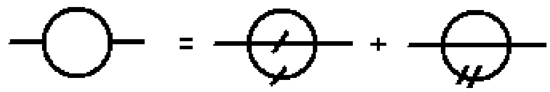
A more complicated example is



$$= \int \frac{q_\rho(k-2q)_\sigma}{(k-q)^2 q^2 (k-p)^2} d^2q d^2p \tag{21.0.49}$$

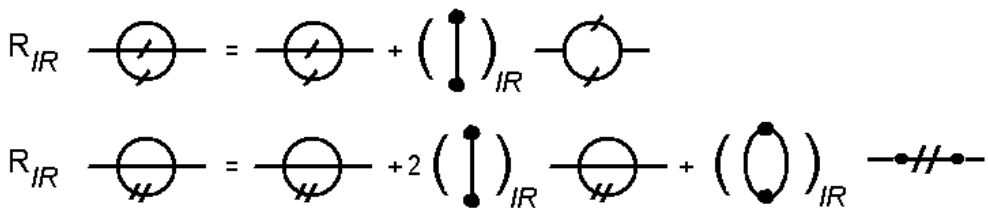
We write the numerator in terms of the momenta which appear in the propagators,

$q_\rho(k-2q)_\sigma = q_\rho(k-q)_\sigma - q_\rho q_\sigma$, and obtain then graphically



(21.0.50)

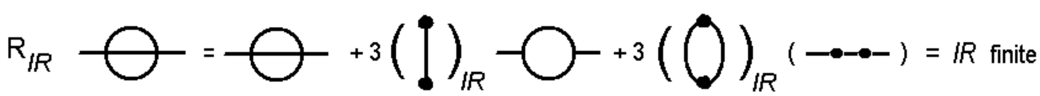
The slashes denote momenta, and reduce the IRD. We obtain then



(21.0.51)

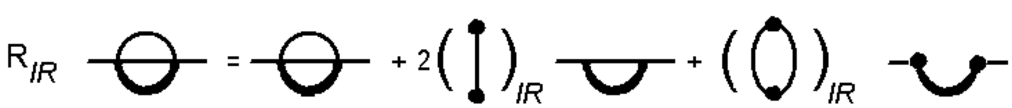
Having subtracted the IRD, one may then proceed to compute the UVD. These are supposed to cancel in the $D=2$ WZWN model at $L \geq 2$ loops, but one clearly needs to be careful with first subtracting the correct amount of IRD.

If there are no momenta in the numerator, one obtains



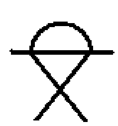
(21.0.52)

If one of the propagators is massive, we obtain (denoting the massive propagator by a solid line)



(21.0.53)

We close with two further examples. First consider in $D=4$ the following 2-loop graph in massless $\lambda\varphi^4$

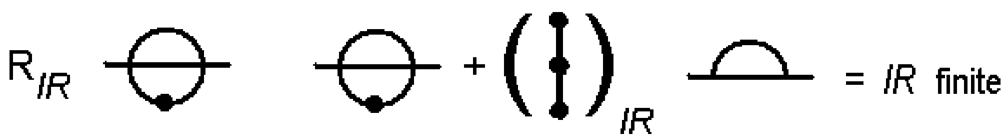


(21.0.54)

Suppose we want to compute the 2-loop contribution to the Z factor of the φ^4 vertex. We first nullify two external momenta because this simplifies the calculation

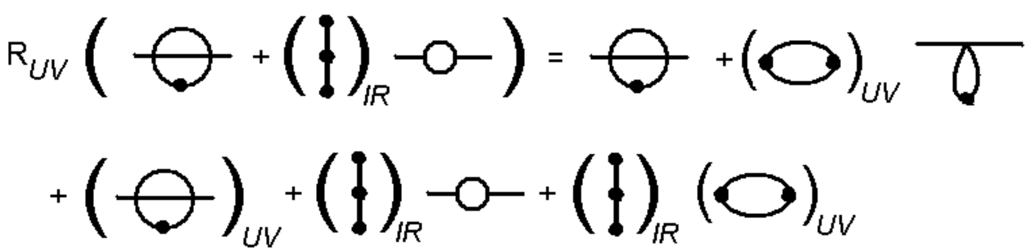
(21.0.55)

Next we subtract the IRD



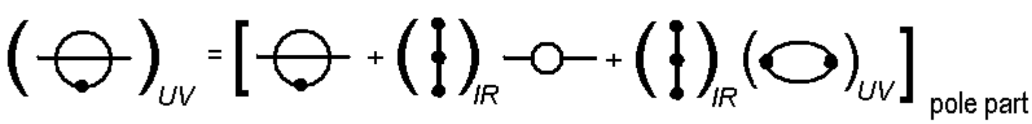
(21.0.56)

Finally we subtract the UVD



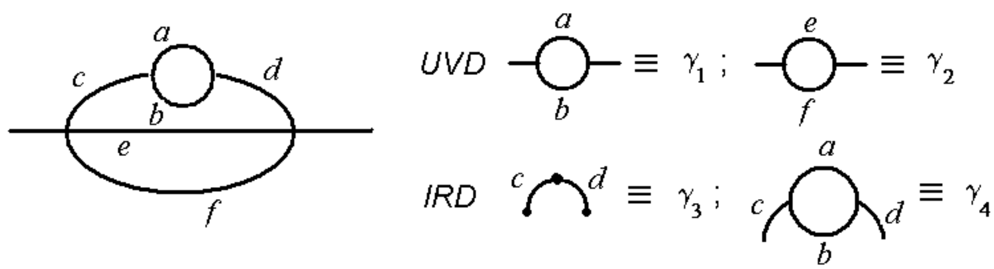
(21.0.57)

The tadpole graph vanishes, and the contribution to the Z factor follows then from



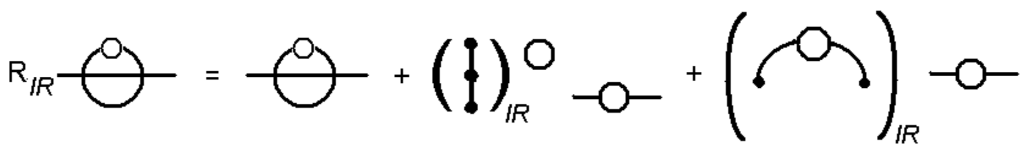
(21.0.58)

For fun we give one last example [10]. We consider the following 3-loop graph in massless $\lambda\varphi^4$ in $D = 4$



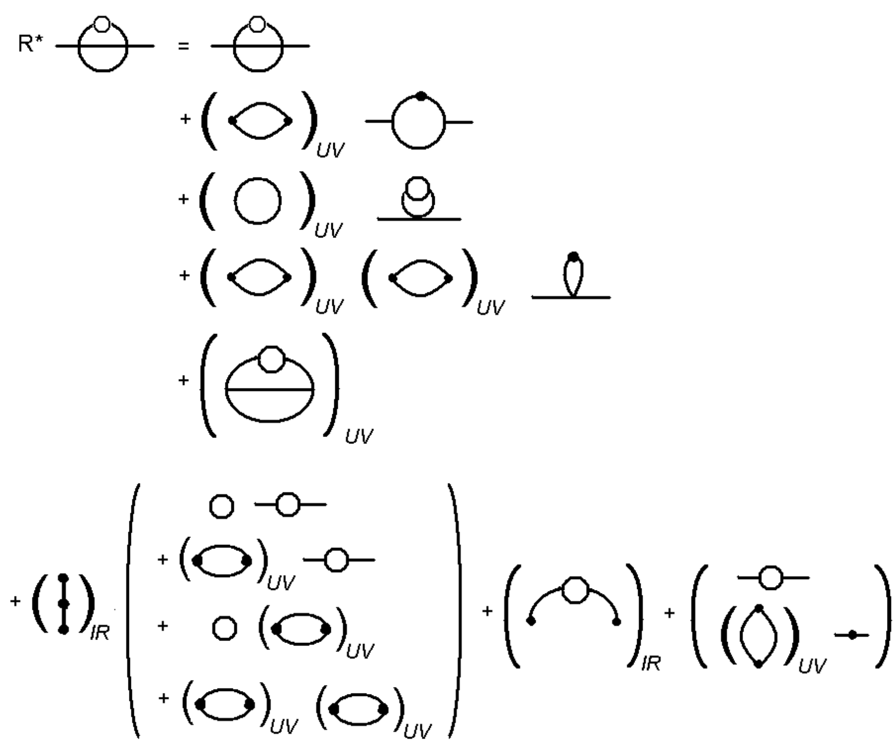
(21.0.59)

The infrared subtraction yields



(21.0.60)

Note the appearance of a disconnected graph. Next ultraviolet subtraction yields



(21.0.61)

The UVD are $UVD(\gamma_1) = UVD(\gamma_2) = \frac{-1}{(4\pi)^2\epsilon}$ with $\epsilon = n-4$, while the IRD are IRD $(\gamma_3) = \frac{2}{(4\pi)^2\epsilon}$ and $IRD(\gamma_4)$ is of the order $\frac{1}{\epsilon^2}$. The overall UVD of this graph is $P_G = \frac{-(1-\epsilon-\epsilon^2)}{3\epsilon^3(4\pi)^6}$ (where G denotes the original graph), and R^*G is UV and IR finite.

We end with final conclusions and comments

(i) Graphs in which all masses have been set to zero allow one to compute β functions in a much simpler way than keeping masses, but one introduces spurious IRD which one should subtract. We have explained the rules for subtracting IRD from Feynman graphs and for determining the final UVD which are relevant for β functions.

(ii) All UV counter terms and IR counter terms are polynomials in $\frac{1}{\epsilon}$. Tadpoles can be set to zero; even though they may contain UVD; in the total result for the UVD as given by the R^* scheme no UVD are lost.

(iii) One can write the counter terms as Feynman graphs with some propagators replaced by $\delta^D(k)$ or $\Box_k\delta^D(k)$. For example, in $D=2$ one has $R_{IR}\int\frac{d^2k}{k^2} = \int d^2k(\frac{1}{k^2} + \frac{\pi}{\epsilon}\delta^2(k))$. In this sense the IR counter terms are local in p space. In x-space they would be nonlocal, because infrared divergences deal with the large x behaviour.

(iv) To remove UVD one shrinks subgraphs, but to remove IRD one deletes subgraphs. The final UVD is the one one needs for β functions. For lower loops $R_{UV}R_{IR}$ is equal to $R_{IR}R_{UV}$, but for higher loops the order matters, and the correct order is $R^* = R_{IR}R_{UV}$. The order only matters if a graph contains a factor $\Box_p\delta(p)$.

(v) in the original BPHZ approach [4,11], one starts with $R_{UV}F = (1-t^F)\Pi_{H\epsilon\Phi}(1-t^H)F$ where H are all proper subgraphs of the Feynman diagram F which are superficially divergent (power-counting divergent), while t^F yields the overall divergence after all subdivergences have been subtracted. Furthermore, if $H \supset H'$ one should write $1-t^H$ to the left of $(1-t^{H'})$, but if H and H' are disjoint or overlapping, it does not matter in which order they appear. There exist refinements which show that one only needs subsets of subgraphs which are nonoverlapping ("forests"). One can then prove the forest formula $R_{UV}F = (1-t^F)\sum_i \Pi_{H\epsilon\phi_i}(-t^H)F$ where the ϕ_i are a forest which includes the empty set. In all examples above we have evaluated this forest formula both for UVD and IRD.

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Chapter 22

Parastatistics

It is well-known that if the interchange of two identical particles does not produce a new state, then the particles must satisfy either Bose-Einstein or Fermi-Dirac statistics[1] [1]. In particular, particles with half-integer spin satisfy Fermi-Dirac statistics, while particles with integer spin satisfy Bose-Einstein statistics [2]. In the former case, only one identical particle can be present in a given state, whereas in the latter case any number of identical particles can occupy a given state.

In the 1940's, one started considering the thermodynamics of models in which at most n_{max} identical particles can be present in a given state [3]. This led to the study of generalized types of statistics in quantum mechanics, called parastatistics [4]. In quantum field theory the first papers on parastatistics appeared in the early 1950's [5]. In time two classes of parastatistics were developed, parabose and parafermi statistics. They contain particles with ordinary Bose-Einstein and Fermi-Dirac statistics as special cases, and are consistent with such basic principles as special relativity and cluster decomposition. In parafermi statistics of degree p at most p identical particles can occupy one given state, but in parabose statistics there is no upper limit to the number of particles in a given state. Instead of (anti)commutation relations

[1]The argument goes as follows: if $\psi(\ldots,\xi_1,\ldots,\xi_2,\ldots) = e^{i\alpha}\psi(\ldots,\xi_2,\ldots,\xi_1,\ldots)$ then by repeating the interchange one obtains $e^{2i\alpha} = 1$ because for identical particles any interchange of particles gives the same phase factor. Hence $e^{i\alpha} = \pm 1$.

which are bilinear in creation and annihilation operator, in parastatistics the basic (anti)commutation relations are cubic in these variables (see (22.3.32)).

The reason we discuss parastatistics is partly due to morbid fascination with the exotic, but also because it has led to a crucial concept: color. Parafermi statistics with $p = 3$ for quarks was proposed by Greenberg [6] as a way to solve the spin-statistics problem for baryons which arose in the 1960's. Greenberg's solution is fully equivalent to the nowadays standard way of adding color to quarks, so this application of parastatistics has been very successful[2]. Ordinary Fermi statistics in the paraquark model is equivalent to the colorless sector of ordinary colored quarks; in particular parastatistics does allow a quark model for the baryon octet $\frac{1}{2}^+$ and the baryon decuplet 10^+ [8]. However, one cannot couple paraquarks to $SU(3)$ gluons [9]. Hence, although parastatistics has led to the concept of color, it is not a viable approach to particle physics.

We shall first discuss quantum mechanics with one parabose or one parafermi harmonic oscillator, next the case of f (for flavor) oscillators, and then jump to quantum field theory with second quantized fields whose creation and annihilation operators satisfy parastatistics. When we discuss one para-oscillator, we take the Hamiltonian for a harmonic oscillator, but this is clearly a special case. Therefore, when we discuss the general case of f oscillators we shall define parastatistics in a way which does not depend on the choice of Hamiltonian.

In parastatistics with one flavor one only requires that observables A satisfy the

[2] Another application which is nowadays considered somewhat peculiar concerns the neutrino theory of light. de Broglie proposed in 1932 that a photon is a bound state of a neutrino and an antineutrino, but later it was realized that the this cannot be reconciled with the helicity ± 1 of photons. Parastatistics of order 2, with an electron-neutrino and muon-neutrino bound to give a helicity -1 photon, avoids some of these pitfalls [7]. However, the existence of the tau lepton and its associated tau neutrino poses a problem: either there should be a fourth set of leptons and neutrinos which form with τ and ν_τ another system with $p = 2$, or the three known neutrinos should form a parafermionic system with $p = 3$.

equation of motion

$$i\hbar\frac{dA}{dt} = [A, H] \tag{22.0.1}$$

Practitioners of parastatistics sometimes argue that whatever modification of the existing theory one considers, one should always satisfy (22.0.1) because it yields Einstein's relation between energy and frequency. In a relativistic theory one should then also satisfy

$$i\hbar\frac{\partial A}{\partial x^j} = [A, P_j] \tag{22.0.2}$$

because it yields de Broglie's relation between momentum and wavelength. We shall not try to prove this; rather we accept that by definition parastatistics preserves (22.0.1), and then study the consequences. For the case of more than one flavor, one requires in addition that the commutation relations which yield a canonical transformation remain valid in the case of parastatistics [10]. Again we take this as a definition of parastatistics, rather than as a property to be derived. From these two requirements, the whole structure of parastatistics follows. For simplicity we set $\hbar = 1$ in what follows. We begin with bose-like oscillators.

1 One bose-like oscillator

Consider a bosonic harmonic oscillator. We assume that its Lagrangian and Hamiltonian have the same form as an ordinary quantum mechanics, namely $L = \frac{1}{2}\dot{q} - \frac{1}{2}q^2$ and

$$H = \frac{1}{2}(p^2 + q^2) = \frac{1}{2}(a^\dagger a + aa^\dagger); \quad a = \frac{1}{\sqrt{2}}(q + ip), \quad a^\dagger = \frac{1}{\sqrt{2}}(q - ip) \tag{22.1.3}$$

However, we do not as usual impose $[a, a^\dagger] = 1$ or $[p, q] = -i$, but rather derive the most general result for $[a, a^\dagger]$ which is compatible with the equations of motion (22.0.1). The latter read $\ddot{q} + q = 0$ in the Lagrangian approach, and defining as usual $p = \frac{\partial}{\partial \dot{q}}L$, we find the equations of motion in Hamiltonian form

$$i\dot{q} = [q, H] = ip; \qquad i\dot{p} = [p, H] = -iq \tag{22.1.4}$$

We require thus that the classical and quantum equations of motion are formally the same, and from this requirement we shall deduce the most general result for $[a, a^\dagger]$. This problem was resolved by Wigner in 1950 [4], and we shall recover his solution below.

In terms of a and $a^\dagger$ the equations of motion become

$$[a, H] = a; \qquad [a^\dagger, H] = -a^\dagger \tag{22.1.5}$$

The problem of parastatistics is to find the most general solution of this equation. Since H is hermitian and positive definite, its eigenvalues are positive, and since according to (22.1.5) $a^\dagger$ raises the eigenvalues while a lowers them by one unit, the spectrum is of the form $H = N_0 + n$, where $N_0 \geq 0$ is arbitrary, and $n = 0, 1, 2, \ldots$. Thus there is a lower bound for the energy. Denoting the ket eigenstates by $|n\rangle$, and the matrix elements of a and $a^\dagger$ by $a_{n,n+1} = \langle n|a|n+1\rangle$ and $a^\dagger_{n+1,n} = \langle n+1|a^\dagger|n\rangle$, one finds from $H = \frac{1}{2}(a^\dagger a + a a^\dagger)$ the fundamental relation $\frac{1}{2}|a_{n-1,n}|^2 + \frac{1}{2}|a_{n,n+1}|^2 = N_0 + n$ where $n = 0, 1, 2, \ldots$. Using that for $n = 0$ one has $a_{-1,0} = 0$, we can solve for the matrix elements of a and $a^\dagger$

$$a_{n,n+1} = a^\dagger_{n+1,n} = \begin{cases} \sqrt{2N_0 + n} & \text{for } n = \text{ even} \\ \sqrt{1+n} & \text{for } n = \text{ odd.} \end{cases} \tag{22.1.6}$$

(We can always redefine $|n+1\rangle$ if $|n\rangle$ is given, such that $a_{n,n+1}$ is real and positive). This is the general solution.

Note that

$$\langle n|[a, a^\dagger]|n'\rangle = \delta_{nn'} \begin{cases} 2N_0 & \text{for } n = \text{ even} \\ 2(1 - N_0) & \text{for } n = \text{ odd} \end{cases} \tag{22.1.7}$$

Only for $N_0 = 1/2$ does one find that $[a, a^\dagger]$ is a c-number, and $N_0 = 1/2$ correspond to ordinary statistics (the canonical commutation relation $[a, a^\dagger] = 1$). However, for other values of N_0, the matrix elements of $[a, a^\dagger]$ depend on n, and hence in these cases $[a, a^\dagger]$ is not a c-number. Instead, there exist polynomial relations between the operators which are n-independent. For example, for $N_0 = 1$ one finds that the following operator equation holds

$$aaa^\dagger - a^\dagger aa = 2a \quad \text{for } N_0 = 1 \tag{22.1.8}$$

which agrees with (22.3.32) for one flavor. This relation is weaker than Bose-Einstein statistics because if $[a, a^\dagger] = 1$ it is satisfied, but it also holds in the $J = 1$ case where $[a, a^\dagger]$ is not equal to one.

To gain more insight into the general solution we have obtained, one may introduce a set of 3 hermitian operators which are bilinear in a and $a^\dagger$

$$J_1 = \frac{1}{4}(aa + a^\dagger a^\dagger); \quad J_2 = \frac{i}{4}(aa - a^\dagger a^\dagger); \quad J_3 = \frac{1}{4}(aa^\dagger + a^\dagger a) = \frac{1}{2}H \tag{22.1.9}$$

Using the result for the matrix elements of a and $a^\dagger$, one may verify that they generate the noncompact Lie algebra of $SO(2,1) = Sp(2,R)$

$$[J_1, J_2] = -iJ_3; \quad [J_2, J_3] = iJ_1; \quad [J_3, J_1] = iJ_2. \tag{22.1.10}$$

(Of course, we could multiply J_1 and J_2 by i to obtain $SO(3)$, but then the new operators J_1 and J_2 would no longer be hermitian). We require that all states have positive norms, so that we need a unitary representation of $SO(2,1)$ in which J_3 is bounded from below (because $J_3 = \frac{1}{2}H$, and H is nonnegative). The general representation theory of $SO(2,1)$ is well-known [11], and the unitary representations with a lower bound on J_3 are parametrized by the value of the Casimir operator $C = J_3^2 - J_1^2 - J_2^2$, namely these representations are denoted by $D^+(\varphi < 0)$ and $C = \varphi(\varphi + 1)$. The states are denoted by $|n, \varphi\rangle$ and $J_3|n, \varphi\rangle = (-\varphi + n)|n, \varphi\rangle$. One can construct this representation of $SO(2,1)$ by defining $J_\pm = J_1 \pm iJ_2$ (so $J_+ = \frac{1}{2}a^\dagger a^\dagger$ and $J_- = \frac{1}{2}aa$). Then $J_\pm$ raise and lower the eigenvalues of J_3 by ± 1. A given representation $D^+(\varphi < 0)$ consists of a set of states for which $H = 2J_3 = -2\varphi, -2\varphi + 2, -2\varphi + 4, \ldots$. On the other hand, the states $|n\rangle$ which form a representation of a and $a^\dagger$ have eigenvalues $H = N_0, N_0 + 1, N_0 + 2 \ldots$. It is clear that $2\varphi = -N_0$ and that we must combine the two repesentations $D^+(\varphi = -\frac{1}{2}N_0)$ and $D^+(\varphi = -\frac{1}{2}N_0 - \frac{1}{2})$ to obtain the representation of a and $a^\dagger$.

The operator $N = H - N_0$ has eigenvalues $N = 0, 1, 2, \ldots$. Hence, N, a and $a^\dagger$ are the number operator and annihilation and creation operators, and because the spectrum is unbounded from above, we call this parabose statistics. We have seen that the different choices of parabose statistics are parameterized by a real continuous parameter $N_0 = -2\varphi \geq 0$. One calls $2N_0$ the order of the parabose statistics.

2 One fermi-like oscillator

Having analyzed the bose-like oscillator, it is straightforward but interesting to study the corresponding fermi-like oscillator. One could start from a real fermionic point particle $\psi(t)$ with action $S = \int \frac{i}{2}\psi\dot{\psi}dt$, and define $\pi = \frac{\partial}{\partial\dot{\psi}}S = -\frac{i}{2}\psi$. In the case of ordinary statistics one would at this point obtain a constraint $\phi = \pi + \frac{i}{2}\psi = 0$ which is second class since the Poisson bracket (or, rather, the anti-commutator because we prefer to work at the quantum level) is nonvanishing,

$$\left\{\pi + \frac{i}{2}\psi, \pi + \frac{i}{2}\psi\right\} = \frac{1}{i}\left(\frac{i}{2} + \frac{i}{2}\right) = 1 \tag{22.2.11}$$

The Dirac bracket would then be defined by

$$\{A, B\}_D = \{A, B\} - \{A, \phi\}\{\phi, \phi\}^{-1}\{\phi, B\} \tag{22.2.12}$$

and one would find, using $\{\phi, \psi\} = \frac{1}{i}$ and $\{\phi, \pi\} = \frac{1}{2}$, the following Dirac brackets

$$\begin{aligned} \{\psi, \psi\}_D &= 0 - \tfrac{1}{i}\tfrac{1}{i} = 1 \\ \{\psi, \pi\}_D &= \tfrac{1}{i} - \tfrac{1}{i}\tfrac{1}{2} = \tfrac{1}{2i} \\ \{\pi, \pi\}_D &= 0 - \tfrac{1}{2}\tfrac{1}{2} = -\tfrac{1}{4} \end{aligned} \tag{22.2.13}$$

By construction $\{A, \phi\}_D = 0$, and this explains the result for $\{\psi, \pi\}$ and $\{\pi, \pi\}$, given the result for $\{\psi, \psi\}$.

However, proceeding in this way, one would obtain the well known fact that the Hamiltonian of a real free fermionic point particle vanishes, $H = 0$. We therefore prefer to turn to a complex point particle with $L = i\psi^*\dot{\psi} - m\psi^*\psi$. Now there is no constraint; rather $\pi = -i\psi^*$, and the classical Hamiltonian is given by $H = \dot{\psi}\pi - L = m\psi^*\psi$. With the usual Fermi statistics one has then at the quantum level $\{-i\psi^\dagger, \psi\} = -i$, or $\{\psi, \psi^\dagger\} = 1$. Further, to treat ψ and $\psi^\dagger$ on equal footing, we antisymmetrize in ψ and $\psi^\dagger$, and in this way we arrive at the Hamiltonian for a fermionic oscillator

$$H = \frac{m}{2}(\psi^\dagger\psi - \psi\psi^\dagger) \tag{22.2.14}$$

In quantum mechanics it is customary to use the notation ψ and $\psi^\dagger$ for fermionic point particle, but in field theory one uses the notation a and $a^\dagger$ for the annihilation and creation operators of the modes of a fermionic field. We continue in this section with quantum mechanics and use ψ and $\psi^\dagger$ as variables.

We can now start the program of deriving parastatistics. The classical Euler-Lagrange equations of motion read

$$i\dot{\psi} - m\psi = 0, \qquad i\dot{\psi}^* + m\psi^* = 0 \tag{22.2.15}$$

and at the quantum level we impose the Heisenberg equation of motion (setting $m = 1$ for convenience)

$$i\dot{\psi} = [\psi, H] = \psi; \qquad i\dot{\psi}^\dagger = [\psi^\dagger, H] = -\psi^\dagger \tag{22.2.16}$$

We now start from this equation, and deduce the most general (anti)commutation relations between ψ and $\psi^\dagger$ which respect $[\psi, H] = \psi$ and $[\psi^\dagger, H] = -\psi^\dagger$.

Since H is hermitian, its eigenvalues are real, but they are not positive, as we shall show later. First we perform a similar analysis as for the bose-like case by introducing the following 3 hermitian operators.

$$J_1 = \frac{1}{2}(\psi + \psi^\dagger); \quad J_2 = \frac{i}{2}(\psi - \psi^\dagger); \quad J_3 = \frac{1}{2}(\psi^\dagger\psi - \psi\psi^\dagger) = H \tag{22.2.17}$$

(We do not introduce operators $J_1 \sim \psi\psi \pm \psi^\dagger\psi^\dagger$ because for ordinary Fermi statistics $\psi\psi = \psi^\dagger\psi^\dagger = 0$, and we want of course to include the ordinary case in our general approach). From the input $[\psi, H] = \psi$ and $[\psi^\dagger, H] = -\psi^\dagger$ one easily finds

$$[J_1, J_2] = iJ_3; \quad [J_2, J_3] = iJ_1; \quad [J_3, J_1] = iJ_2. \tag{22.2.18}$$

where the brackets [,] denote ordinary commutators. It is counter intuitive to use a commutator instead of anticommutator for $[J_1, J_2]$, but it is not wrong, and the result is rewarding: the J_k form the compact group $SO(3)$. The unitary representations are well-known from angular momentum theory: $D(J)$ with $J = 0, 1/2, 1, \ldots$. Thus $J_3 = H$ has eigenvalues $-J, -J+1, \ldots, +J$, and since this spectrum is bounded from below and from above, we call this parafermionic statistics. The lowest energy state has $H = -J$, and the number operator $N = H + J$ has $2J + 1$ eigenvalues $N = 0, 1, 2, \ldots, 2J$. One calls $2J$ the order of the Fermi parastatistics. From the usual matrix elements of $J_1 \pm iJ_2 = J_\pm$ in angular momentum theory one finds easily the matrix elements of ψ and $\psi^\dagger$

$$\psi_{n,n+1} = \psi^\dagger_{n+1,n} = \sqrt{(J+n+1)(J-n)}; \quad n = -J, \ldots, J \tag{22.2.19}$$

From these matrix elements, the (anti)commutation relations for ψ and $\psi^\dagger$ follow. For $J = 1/2$ one finds the 2-dimensional representation of angular momentum. We set $J_k = \frac{1}{2}\tau_k$, $J_- = \psi = \frac{1}{2}\tau_-$, and $J_+ = \psi^\dagger = \frac{1}{2}\tau_+$, and then $H = \frac{1}{4}(\tau_+\tau_- - \tau_-\tau_+) = \tau_3$ has eigenvalues $\pm 1/2$. This is the case of ordinary Fermi statistics

$$\{\psi, \psi^\dagger\} = 1; \quad \{\psi, \psi\} = \{\psi^\dagger, \psi^\dagger\} = 0 \qquad (J = 1/2) \tag{22.2.20}$$

The zero point energy of a parafermionic harmonic oscillator with quantum label J is

$$\langle H \rangle = -J \tag{22.2.21}$$

For $J = 1/2$ this is a well-known result.

For $J = 1$, one finds a more complicated result. One now finds

$$\psi\psi\psi = 0, \quad \psi\psi^\dagger\psi = 2\psi, \quad \psi\psi\psi^\dagger + \psi^\dagger\psi\psi = 2\psi \quad (J = 1) \tag{22.2.22}$$

The first relation has a clear interpretation: 3 times lowering a state in a triplet ($J = 1$) always yields zero. Note that the result $\psi^3 = 0$ is weaker than $\psi^2 = 0$, so it is a special case of one of the relations of ordinary Fermi statistics. The other two relations, however, are not even valid when ordinary Fermi statistics holds. For example, assuming that $\{\psi, \psi\} = 0$, the last relation would give $\psi = 0$, and assuming that $\{\psi, \psi^\dagger\} = 1$ and $\{\psi, \psi\} = 0$ the second relation would yield $\psi = 2\psi$. The general result in (22.3.32) for one flavor reduces to $2\psi\psi^\dagger\psi - \psi\psi\psi^\dagger - \psi^\dagger\psi\psi = 2\psi$ which holds if (22.2.22) holds and which can be rewriten as $[\psi, H] = \psi$.

3 Parastatistics for several flavors

We next consider $2f$ operators $a_1, \ldots, a_f$ and $a_1^\dagger, \ldots, a_f^\dagger$. (The label f stands for flavor). We require that each pair satisfies the parastatistics discussed in the previous two sections, but now we must settle the issue how oscillators of different flavors (anti)commute. For ordinary statistics one would define that for fermions all anticommutators $\{a_k \text{ or } a_k^\dagger, a_l \text{ or } a_l^\dagger\}$ vanish for $k \neq l$, while for bosons the corresponding commutators would vanish. So, one might propose that for the case of parastatistics the opposite should hold: different fermionic-like oscillators commute, and different bosonic-like oscillators anticommute. However, there are two arguments against this very simple proposal

(1) for one para-oscillator we had more structure than merely commutation or anticommutation relations, namely we obained rules which depend on a free parameter (N_0 and J, respectively)

(2) for ordinary statistics it is known that one gets the same physics whether fields with different flavors satisfy commutation or anticommutation relations.

We shall deduce the parastatistics relations by starting from ordinary statistics, and defining the group $SO(2f)$ for fermions and $Sp(2f, R)$ for bosons which leave the canonical (anti)commutation relations invariant. Then we shall impose the commutation relations between the generators of $SO(2f)$ and $Sp(2f, R)$ and a and $a^\dagger$ as the relations which define parastatistics. The canonical commutation relations themselves, having served their purpose, are dropped.

The canonical (anti)commutation relations for ordinary Fermi (upper signs) or Bose (lower signs) statistics read

$$[a_k, a_l^\dagger]_\pm = \delta_{kl}, \qquad [a_k, a_l]_\pm = 0, \qquad [a_k^\dagger, a_l^\dagger]_\pm = 0. \tag{22.3.23}$$

Consider the most general linear infinitesimal transformation, which mixes a's and $a^\dagger$'s and preserves the hermiticity conditions ${a'_k}^\dagger = (a'_k)^\dagger$

$$\begin{aligned} a'_k &= a_k - i \sum_{m=1}^{f} (\nu_{km} a_m + \lambda_{km} a_m^\dagger) \\ {a'_l}^\dagger &= a_l^\dagger + i \sum_{m=1}^{f} (\nu_{lm}^* a_m^\dagger + \lambda_{lm}^* a_m) \end{aligned} \tag{22.3.24}$$

This is a canonical transformation provided

$$\lambda_{kl} \pm \lambda_{lk} = 0; \qquad \nu_{kl}^* - \nu_{lk} = 0 \tag{22.3.25}$$

The matrix A defined by

$$\delta \begin{pmatrix} a \\ a^\dagger \end{pmatrix} = A \begin{pmatrix} a \\ a^\dagger \end{pmatrix}; \qquad A = \begin{pmatrix} -i\nu & -i\lambda \\ i\lambda^* & i\nu^* \end{pmatrix} \tag{22.3.26}$$

is an orthogonal matrix in the Fermi case since $SA + A^T S = 0$, with $S = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$, while it is a symplectic matrix in the Bose case since $\Omega A + A^T \Omega = 0$ with $\Omega = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$. On the real basis $(a_k + a_k^\dagger)$, $(a_k - a_k^\dagger)/i$ the corresponding matrix reads

$$A = \begin{pmatrix} Im(\nu + \lambda) & Re(\nu - \lambda) \\ -Re(\nu + \lambda) & Im(\nu - \lambda) \end{pmatrix} \tag{22.3.27}$$

Since $Re\nu$ is symmetric, $Im\nu$ is antisymmetric, while λ is antisymmetric/symmetric in the Fermi/Bose case, it is clear that A lies in the algebra of $SO(2f,R)$ in the fermionic case, while in the bosonic case A lies in $Sp(2f,R)$ because it is real and satisfies $\Omega A + A^T\Omega = 0$ with the same Ω.

The canonical transformations are generated by a unitary transformation U

$$a'_k = U^{-1}a_kU; \quad {a'_k}^{\dagger} = U^{-1}a_k^{\dagger}U \tag{22.3.28}$$

$$U = \exp\left[-i\sum_{m,n}\left(\nu_{mn}N_{mn} + \frac{1}{2}\lambda_{mn}L_{mn} + \frac{1}{2}\lambda^*_{nm}L^{\dagger}_{nm}\right)\right] \tag{22.3.29}$$

Unitarity of U requires that $N^{\dagger}_{km} = N_{mk}$ because ν_{km} is hermitian. The transformation rules for a_k agree with (22.3.24) provided the following commutators hold

$$\begin{aligned} [a_k, N_{mn}] &= \delta_{km}a_n \\ [a_k, L_{mn}] &= \delta_{km}a_n^{\dagger} \mp \delta_{kn}a_m^{\dagger} \\ \left[a_k, L^{\dagger}_{mn}\right] &= 0 \end{aligned} \tag{22.3.30}$$

The finite transformation rules obtained by exponentiating (22.3.24) from a group, hence the unitary matrices U should form the same group. In fact, we already showed that (22.3.24) is the fundamental representation of $SO(2f,R)$ or $Sp(2f,R)$ and we can thus determine the abstract anticommutation relations of the generators N_{mn}, L_{mn} and $L^{\dagger}_{mn}$. We can then construct an explicit representation of N_{mn}, L_{mn} and $L^{\dagger}_{mn}$ in terms of a_k and $a_k^{\dagger}$, because this should reproduce the fundamental representation in (22.3.24). One finds

$$\begin{aligned} N_{mn} &= \tfrac{1}{2}\left[a_m^{\dagger}, a_n\right]_{\mp} \\ L_{mn} &= \tfrac{1}{2}\left[a_m^{\dagger}, a_n^{\dagger}\right]_{\mp} \\ L^{\dagger}_{mn} &= \tfrac{1}{2}\left[a_n, a_m\right]_{\mp} \end{aligned} \tag{22.3.31}$$

We can now define parastatistics. For parastatistics the relations between a_k and $a_k^\dagger$ is given by (22.3.30) and (22.3.31). Hence parastatistics is defined by[3]

$$\begin{aligned} \left[a_k, \left[a_m^\dagger, a_n\right]_\mp\right] &= 2\delta_{km} a_n \\ \left[a_k, \left[a_m^\dagger, a_n^\dagger\right]_\mp\right] &= 2\delta_{km} a_n^\dagger \mp 2\delta_{kn} a_m^\dagger \\ \left[a_k, [a_m, a_n]_\mp\right] &= 0 \end{aligned} \tag{22.3.32}$$

where we recall that upper (lower) signs refer to the fermi (bose) case. We drop the canonical (anti)commutation relations. The commutation relations between N, L and $L^\dagger$ still hold because one can construct them from only (22.3.32). In fact the second relation follows from the first and the last relation in (22.3.32) if one uses the Jacobi identities $[[A, B]_\mp, C] + [[B, C]_\mp, A] + [[C, A]_\mp, B] = 0$. Moreover, we can derive the first relation from the requirement that an infinitesimal unitary transformation $a_k{}' = a_k + \alpha_{kl} a_l$ (with $\alpha*_{kl} = -\alpha_{lk}$) preserves the relations $[a_k, N_{ll}] = \delta_{kl} a_l$ [13].

For one fermion-like flavor we get our old result in (22.2.16) back: $\left[a, \frac{1}{2}(a^\dagger a - a a^\dagger)\right] = a$ and $\left[a^\dagger, \frac{1}{2}(a^\dagger a - a a^\dagger)\right] = -a^\dagger$. Thus for each flavor we have a spin which can take on any integer or half-integer nonnegative number. In principle the spins of different flavors can be different, but as we shall soon show, requiring a unique vacuum to exist in the theory all spins must be equal. Thus parastatistics for f fermion-like flavors is characterized by a unique spin. One calls $p = 2J$ the order of the parafermion statistics. For leptons we shall need parastatistics of order 2, while for quarks we shall need order 3.

For one boson-like flavor we also get our old results back. From (22.3.32) we now get $\left[a, \frac{1}{2}(a^\dagger a + a a^\dagger)\right] = a$ and this relation, together with the hermitian conjugate relation $\left[a^\dagger, \frac{1}{2}(a^\dagger a + a a^\dagger)\right] = -a^\dagger$ agrees with the definition of the boson-like oscillator (22.1.5). In addition (22.3.32) seems to yield more: $\left[a, a^\dagger a^\dagger\right] = 2a^\dagger$. However, this relation is equivalent to $[a^\dagger, H] = -a^\dagger$. Again the existence of a unique vacuum leads

[3] From (22.3.32) one may derive further relations such as $[a_k^\dagger, [a_m, a_n]_\mp] = \delta_{km} a_n - \delta_{kn} a_m$ [5]. These will not be needed in our discussion.

to the condition that all zero point energies $N_0^{(k)}$ for $1 \leq k \leq f$ are equal. One calls this common $p = 2N_0$ the order of the parabose statistics.

4 A unique vacuum

A useful set of operators are the Hamiltonians for each flavor

$$H_{kk} = \frac{1}{2}\left[a_k^\dagger, a_k\right]_\mp \tag{22.4.33}$$

One easily finds from (22.3.30)

$$\begin{aligned} [a_k, H_l] &= \delta_{kl} a_l \\ \left[a_k^\dagger, H_l\right] &= -\delta_{kl} a_l^\dagger \\ H_k^\dagger = H_k; \quad [H_k, H_l] &= 0 \end{aligned} \tag{22.4.34}$$

Consider next the number operators N_k which are the Hamiltonians minus zero point energies

$$N_k = \frac{1}{2}\left(a_k^\dagger a_k \mp a_k a_k^\dagger\right) \pm \genfrac{}{}{0pt}{}{J_k}{N_{0k}} \tag{22.4.35}$$

They take on values $N_k = 0, 1, 2, \ldots$ and commute with each other. For the fermion-like case the number of states is finite, $(2J_k + 1)$, but for the boson-like case it is infinite. Assume now that a unique vacuum $|0\rangle$ exists: $N_k|0\rangle = 0$ for all k, and $\langle 0|0\rangle = 1$. Since $|0\rangle$ is the lowest eigenstate of N_k, one has

$$a_k|0\rangle = 0 \tag{22.4.36}$$

as in ordinary statistics. It follows that

$$N_k|0\rangle = 0 = \mp\frac{1}{2} a_k a_k^\dagger|0\rangle \pm \genfrac{}{}{0pt}{}{J_k}{N_0^k} |0\rangle \tag{22.4.37}$$

hence

$$a_k a_k^\dagger|0\rangle = p|0\rangle, \quad p = 2J_k \quad \text{or} \quad p = 2N_0^k \tag{22.4.38}$$

The norm $||a_k^\dagger|0\rangle||^2$ is thus positive, as it should be.

One can show that $2N_0^k$ (and of course also $2J_k$) is a positive integer [12], called the order of the parastatistics. However, consider next the state

$$a_k a_l^\dagger |0\rangle \quad \text{for } k \neq l \tag{22.4.39}$$

If this state were nonzero, it would be an eigenvector of N_k with eigenvalue -1, but by definition N_k has only positive eigenvalues. (We subtracted a constant from each Hamiltonian H_k such that in the whole Hilbert space $N_k \geq 0$). Hence $a_k a_l^\dagger |0\rangle$ vanishes if k differs from l

$$a_k a_l^\dagger |0\rangle = \delta_{kl} \left(2J_k \text{ or } 2N_0^{(k)}\right) |0\rangle \tag{22.4.40}$$

Consider next the first defining relation in (22.3.32)

$$\left[a_k, \left[a_m^\dagger, a_n\right]_\mp\right] = 2\delta_{km} a_n \tag{22.4.41}$$

and its hermitian conjugate

$$\left[a_l^\dagger, \left[a_m^\dagger, a_n\right]_\mp\right] = -2\delta_{ln} a_m^\dagger \tag{22.4.42}$$

Multiplying the first relation by $a_l^\dagger$ from the right, and the second relation by a_k from the left, and adding, one finds

$$\left[a_k a_l^\dagger, \left[a_m^\dagger, a_n\right]_\mp\right] = 2\delta_{km} a_n a_l^\dagger - 2\delta_{ln} a_k a_m^\dagger \tag{22.4.43}$$

Acting with this operator on the vacuum, using (22.4.40), yields zero on the left-hand side, but the right-hand side yields $2\delta_{km}\delta_{nl}(p_n - p_k)$ where $p_k = 2J_k$ or $p_k = 2N_0^k$. Hence $p_n = p_k$.

This is a very strong and nice result: all flavors have the same zero point energies

$$a_k a_l^\dagger |0\rangle = p\delta_{kl}|0\rangle; \quad p = 2J \text{ or } p = 2N_0 \tag{22.4.44}$$

It also follows that all states in Fock space are obtained by acting with $a_k^\dagger$ only. To see this, note that if one finds the combination $a_m a_l^\dagger a_l^\dagger$ in a state, one can use (22.4.42) to move a_n to the right

$$a_l^\dagger a_m^\dagger a_n \mp a_l^\dagger a_n a_m^\dagger - a_m^\dagger a_n a_l^\dagger \pm a_n a_m^\dagger a_l^\dagger = -2\delta_{ln} a_m^\dagger \tag{22.4.45}$$

If one then uses (22.4.36) and (22.4.40) one can eliminate a_n. In this way one shows that states are given by products of $a_k^\dagger$ acting on $|0\rangle$.

5 The Green representation

The parastatistics relations are cubic in the operators a_k and $a_k^\dagger$, and are therefore complicated. However, they are equivalent to quadratic relations for other operators. Define

$$a_k = \sum_{\alpha=1}^{p} a_k^\alpha; \qquad a_k^\dagger = \sum_{\alpha=1}^{p} a_k^{\alpha\dagger} \tag{22.5.46}$$

where a_k^α satisfy what is called anomalous statistics and p is the order of the parastatistics. Anomalous statistics relations are the standard canonical (anti)commutation relations for operators of one flavor, but relations with the wrong sign for operators of different flavors. Hence the a_k^α and $a_k^{\alpha\dagger}$ satisfy the following quadratic relations

$$\begin{aligned} &\left[a_k^\alpha, a_l^{\alpha\dagger}\right]_\pm = \delta_{kl}; \quad [a_k^\alpha, a_l^\alpha]_\pm = 0; \quad \left[a_k^{\alpha\dagger}, a_l^{\alpha\dagger}\right]_\pm = 0 \\ &\left[a_k^\alpha, a_l^{\beta\dagger}\right]_\mp = \left[a_k^\alpha, a_l^\beta\right]_\mp = \left[a_k^{\alpha\dagger}, a_l^{\beta\dagger}\right]_\mp = 0 \text{ for } \alpha \neq \beta \end{aligned} \tag{22.5.47}$$

The crucial property which makes the Green representation so useful for parastatistics is the following theorem [5]

Theorem: anomalous statistics of a_k^α implies parastatistics for $a_k = \sum\limits_{\alpha=1}^{p} a_k^\alpha$.

The proof follows by straightforward substitution of $a_k = \sum\limits_{\alpha=1}^{p} a_k^\alpha$ into the parastatistics relations, and just using the anomalous (anti)commutators. For example, for parafermi statistics the first relation in (22.3.32) yields

$$\begin{aligned} &\sum_{\alpha,\beta,\gamma} [a_k^\alpha, [a^{\dagger\beta}_m, a_n^\gamma]] = \sum_{\alpha,\beta} [a_k^\alpha, [a^{\dagger\beta}_m, a_n^\beta]] \\ &= \sum_{\alpha,\beta} [a_k^\alpha, a^{\dagger\beta}_m a_n^\beta] - [a_k^\alpha, a_n^\beta a^{\dagger\beta}_m] \\ &= \sum_{\alpha,\beta} [a_k^\alpha, a^{\dagger\beta}_m] a_n^\beta + a^{\dagger\beta}_m [a_k^\alpha, a_n^\beta] - [a_k^\alpha, a_n^\beta] a^{\dagger\beta}_m - a_n^\beta [a_k^\alpha, a^{\dagger\beta}_m] \end{aligned}$$

$$
\begin{aligned}
&= \sum_\alpha [a_k^\alpha, {a^\dagger}_m^\alpha] a_n^\alpha + {a^\dagger}_m^\alpha [a_k^\alpha, a_n^\alpha] - [a_k^\alpha, a_n^\alpha] {a^\dagger}_m^\alpha - a_n^\alpha [a_k^\alpha, {a^\dagger}_m^\alpha] \\
&= \sum_\alpha \left(a_k^\alpha {a^\dagger}_m^\alpha a_n^\alpha + {a^\dagger}_m^\alpha a_k^\alpha a_n^\alpha \right) + \left(-a_k^\alpha a_n^\alpha {a^\dagger}_m^\alpha - a_n^\alpha a_k^\alpha {a^\dagger}_m^\alpha \right) \\
&+ \left(-{a^\dagger}_m^\alpha a_k^\alpha a_n^\alpha - {a^\dagger}_m^\alpha a_n^\alpha a_k^\alpha \right) + \left(a_n^\alpha a_k^\alpha {a^\dagger}_m^\alpha + a_n^\alpha {a^\dagger}_m^\alpha a_k^\alpha \right) \\
&= \sum_\alpha \delta_{km} a_n^\alpha + 0 + 0 + \delta_{km} a_n^\alpha \\
&= 2\delta_{km} a_n
\end{aligned}
\tag{22.5.48}
$$

The operators $N_k = N_{kk}$ where $N_{kl} = \frac{1}{2}\left[a_k^\dagger, a_l\right]_\mp$ take on a simple form in terms of the a_k^α operators

$$
N_k = \frac{1}{2} \sum_{\alpha,\beta=1}^{p} \left[a_k^{\alpha\dagger}, a_k^\beta\right]_\mp = \frac{1}{2} \sum_{\alpha=1}^{p} \left[a_k^{\alpha\dagger}, a_k^\alpha\right]_\mp = \sum_{\alpha=1}^{p} N_k^\alpha \tag{22.5.49}
$$

where $N_k^\alpha = \frac{1}{2}\left[a_k^{\alpha\dagger}, a_k^\alpha\right]_\mp$. Hence the Hamiltonian reads, after rescaling to regain the factors $\hbar\omega_k$,

$$
H = \sum_{k=1}^{p} \hbar\omega_k N_k = \sum_{k=1}^{p} \sum_{\alpha=1}^{p} \hbar\omega_k N_k^\alpha \tag{22.5.50}
$$

One can then define $a_k^\alpha|0\rangle = 0$ on the unique vacuum discussed before. Then $a_k|0\rangle = 0$ and also

$$
a_k a_l^\dagger |0\rangle = \left(\sum_{\alpha=1}^{p} a_k^\alpha\right)\left(\sum_{\beta=1}^{p} a_l^{\beta\dagger}\right)|0\rangle = \sum_{\alpha=1}^{p} a_k^\alpha a_l^{\alpha\dagger}|0\rangle = p\delta_{kl} \tag{22.5.51}
$$

Hence the whole structure of parastatistics is recovered.

Having come so far, one may from now on work in the Green representation. One may even feel that the derivation of parastatistics given in section 2 which was based on canonical transformations was unnecessarily complicated, and can be replaced by (22.5.46) and (22.5.47). In any case, we are now ready for applications.

6 Parastatistics and color

In the naive quark model without color, the spin $1/2^+$ octet (containing the proton, neutron, $\Sigma^+, \Sigma^0, \Sigma^-, \Lambda$ and Ξ^0, Ξ^- baryons) and the spin $3/2^+$ decuplet (containing the pion-nucleon resonances $\Delta^{++}, \Delta^{+}, \Delta^{0}, \Delta^{--}$, and other baryons, in particular the Ω^{-} isosinglet) are made from 3 quarks: the up quark, the down quark and the strange quark. Together they yield $16 + 40 = 56$ states. The Δ^{++} particle is made from 3 up-quarks, yielding a state which must be totally symmetric since it should yield spin 3/2. (All quarks are supposed to be in s-waves because the decuplet is the lowest-lying spin 3/2 multiplet). This violates Fermi-Dirac statistics for quarks. However, as observed by Greenberg [6], one can use parafermi statistics of order $p = 3$ to construct these 56 states as follows. One writes the quark creation operators $a^\dagger_\mu$ as a sum of quark operators $a^{\dagger\alpha}_\mu$ which satisfy "anomalous statistics" $a^\dagger_\mu = \sum_{\alpha=1}^{3} a^{\dagger\alpha}_\mu$. The subscript μ labels spin, $SU(3)$ quantum numbers, and momentum. Then

$$f^\dagger_{\lambda\mu\nu}|0\rangle = \{a^\dagger_\lambda, \{a^\dagger_\mu, a^\dagger_\nu\}\}|0\rangle = 4 \sum_{\substack{\alpha,\beta,\gamma=1 \\ \alpha\neq\beta\neq\gamma\neq\alpha}}^{3} a^{\alpha\dagger}_\lambda a^{\beta\dagger}_\mu a^{\gamma\dagger}_\nu |0\rangle \tag{22.6.52}$$

is totaly symmetric in λ, μ and ν, hence it contains 56 states. The 56 states $f^\dagger_{\lambda\mu\nu}|0\rangle$ are fermions because $\{f^\dagger_{\lambda\mu\nu}, a^\dagger_\rho\} = 0$, implying $\{f^\dagger_{\lambda\mu\nu}, f^\dagger_{\rho\sigma\tau}\} = 0$. One can not construct totally symmetric states with more than p parafermions. (Similarly, a totally anti-symmetric state $(a^\dagger_{\mu_1} a^\dagger_{\mu_2} \ldots a^\dagger_{\mu_N} - a^\dagger_{\mu_2} a^\dagger_{\mu_1} \ldots a^\dagger_{\mu_N} + \ldots)|0\rangle$ with more than p parabosons vanishes). For example, $(a^\dagger_\mu)^p|0\rangle = p!\alpha^{1\dagger}_\mu \ldots \alpha^{p\dagger}_\mu|0\rangle$, but $(a^\dagger_\mu)^{p+1}|0\rangle = 0$.

Mesons can be constructed from a parafermion quark and a parafermion antiquark of any order p as follows

$$b^\dagger_{\lambda\mu} = [a^\dagger_\lambda, \bar{a}^\dagger_\mu] = 2 \sum_{\alpha=1}^{p} a^{\alpha\dagger}_\lambda \bar{a}^{\alpha\dagger}_\mu \tag{22.6.53}$$

The operator $b^\dagger_{\lambda\mu}$ creates a boson since $[b^\dagger_{\lambda\mu}, a_\nu] = [b^\dagger_{\lambda\mu}, \bar{a}^\dagger_\sigma] = 0$ which implies $[b^\dagger_{\lambda\mu}, b^\dagger_{\nu\sigma}] = 0$. Thus for baryons the order of parastatistics is equal to the number of quarks in a baryon, $p = 3$, while for mesons any order of parastatistics is allowed [6].

It is clarifying to explicitly write down the states one can construct in a parafermion model for quarks with $p = 3$. The states $a^\dagger_\mu|0\rangle$ are constructed from quark creation operators which satisfy parafermi statistics. Similarly, $\bar{a_\mu}^\dagger|0\rangle$ are antiparaquarks with

$p = 3$. The states $\{a_\mu^\dagger, a_\nu^\dagger\}|0\rangle$ satisfy parabose statistics, while $[a_\mu^\dagger, a_\nu^\dagger]|0\rangle$ are ordinary bosons. Similarly $\{a_\mu^\dagger, \bar{a}_\mu^\dagger\}|0\rangle$ are parabosons, while $[a_\mu^\dagger, \bar{a}_\mu^\dagger]$ are bosons. Finally, $\{a_\mu^\dagger, \{a_\nu^\dagger, \bar{a}_\rho^\dagger\}\}|0\rangle$ and $\{a_\mu^\dagger, \{a_\nu^\dagger, \bar{a}_\rho^\dagger\}\}|0\rangle$ are ordinary fermions. If one uses parafermi statistics of rank 3 for quarks, each quark Green operator satisfies anomalous statistics. If the interaction between quarks and gauge fields are diagonal in the a_μ^α, and if anomalous quarks have the same charges 2/3 and $-1/3$ as the usual quarks, the net effect of the Green operators looks very similar to the notion of color. For example, in $\pi^0 \to 2\gamma$ the total decay rate in the pre-quark era was proportional to $1^2 - 0^2$ where 1^2 refers to a proton in the triangle and -0^2 to a neutron in the triangle. (The coupling $\vec{\pi}\bar{\psi}\gamma_5\vec{\tau}\psi$ leads to the matrix τ_3 for $\pi^0 \to 2\gamma$, which in turn yields the minus sign in $1^2 - 0^2$). For ordinary u, d quarks one gets $\left(\frac{2}{3}\right)^2 - \left(\frac{1}{3}\right)^2 = \frac{1}{3}$, but introducing color one gets the correct result: $3\left(\left(\frac{2}{3}\right)^2 - \left(\frac{1}{3}\right)^2\right) = 1^2 - 0^2$. The same result is obtained for the parastatistics realized in terms of anomalous quark operators.

For a textbook on parastatistics, see [14].

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