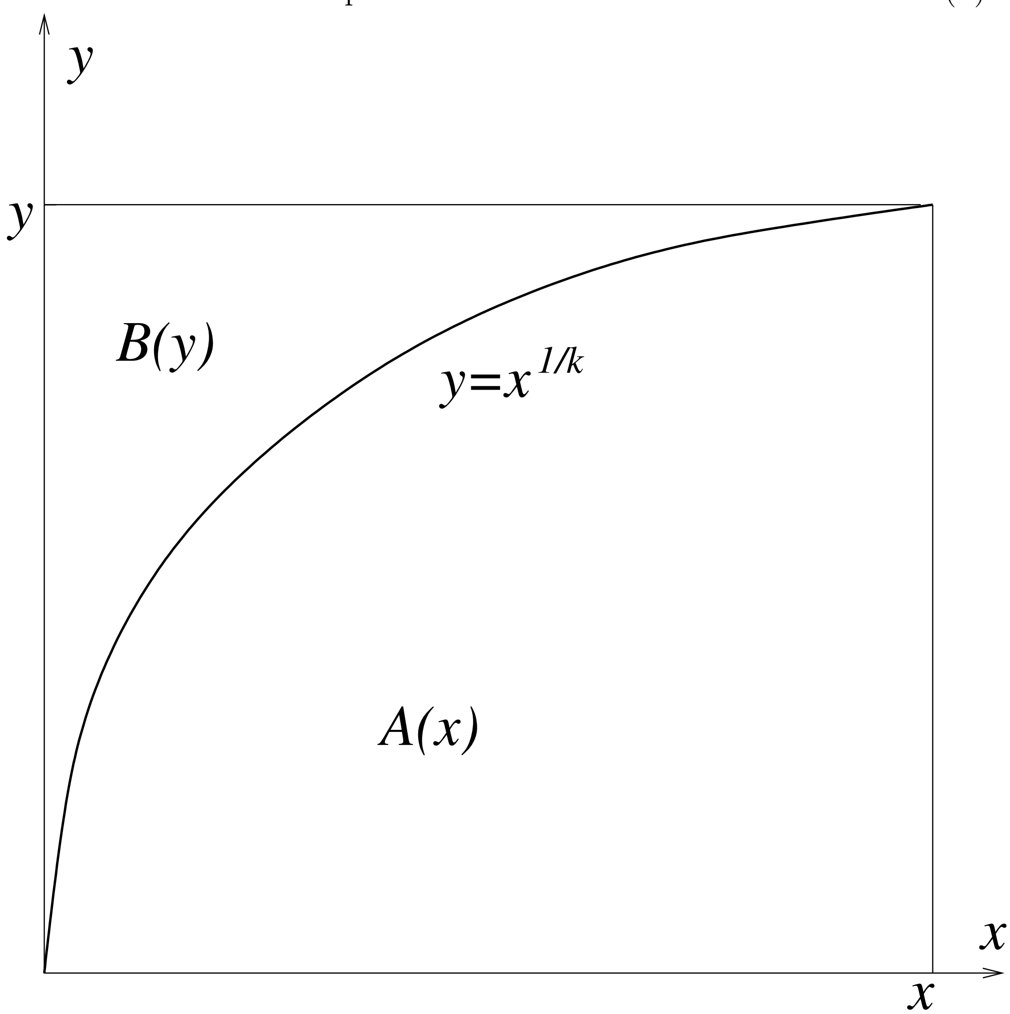
3 Differentiation (continued), Integration

Problem 3.1. Derive the formula for $(x^{1/k})'$ for a natural k, using implicit differentiation. Check if the result agrees with problem 2.1.

Problem 3.2. Derive the formula for $(f/g)'$ ("the quotient rule") in two ways: by using implicit differentiation and directly, by manipulating the difference quotient $((f(x)/g(x)) - (f(a)/g(a)))/(x - a)$. See the handout about differentiation from the last class if you are stuck.

Problem 3.3. Look at the pucture and derive a formula for the area $A(x)$



by observing that $A(x) + B(y(x)) = xy(x)$ and assuming that $B(y) = y^{k+1}/(k+1)$ in accordance with problem 1.4. Does the result agree with problem 1.5?

Problem 3.4. Differentiate some polynomials using the differentiation rules. Let $p(x) = p_0 + p_1x + p_2x^2 + ... + p_nx^n$ be a polynomial. Check that $p(0) = p_0$, $p'(0) = p_1$, that $p''(0) = 2p_2$,..., and finally $p^{(n)} = n!p_n$, where $n! = 1 * 2 * 3 * ... * n$. Conclude that any polynomial is the sum of its *Maclaurin series:*

$$p(x) = p(0) + p'(0)x + p''(0)/2! + ... + p^{(x)}(0)x^k/k! + ...$$

Note that the sum is in fact finite since $p^{(k)} = 0$ for $k > n$.

Now notice that any polynomial in x is also a polynomial of the same degree in $x - a$. Conclude that any polynomial is the sum of its *Taylor series:*

$$p(x) = p(a) + p'(a)(x - a) + p''(a)(x - a)^2/2! + ... + p^{(k)}(a)(x - a)^k/k! + ...$$

Problem 3.5. Antiderivatives and indefinite integrals

A function F is called *a primitive* or *an antiderivative* of another function f if f is the derivative of F, i.e. $F' = f$.

a) Check that if F is an antiderivative of f and C is any constant, then $F + C$ is also an antiderivative of f. Check that the converse is true for polynomials: any two primitives of the same polynomial differ by a constant. Conclude that all the primitives of a given polynomial are of the form $F + C$ where F is one of them and C is a constant.We will see later that this is true for a wider class of functions.

b) Learn how to antidifferentiate polynomials and other power functions.

c) Formulate Sums and Multiplier rules for antidifferentiation.

d) What about Leibniz?

e) What about Chain Rule?

f) By the *indefinite integral* of a given function f we mean the set all the antiderivatives of it, it is denoted by

$$\int f(x)dx, \text{ for example } \int xdx = x^2/2 + C$$

C is called *the integration constant.* Rewrite antidieffrentiation rules in terms of indefinite integrals.

g) $sin' = cos, cos' = -sin$. What are $\int sin(x)dx$ and $\int cos(x)dx$?

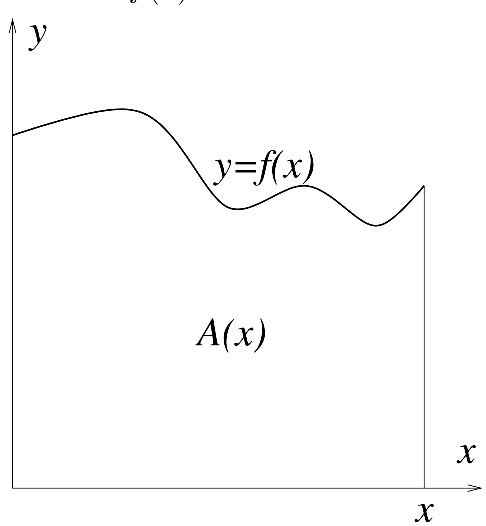
Problem 3.6. Free falls, energy conservation

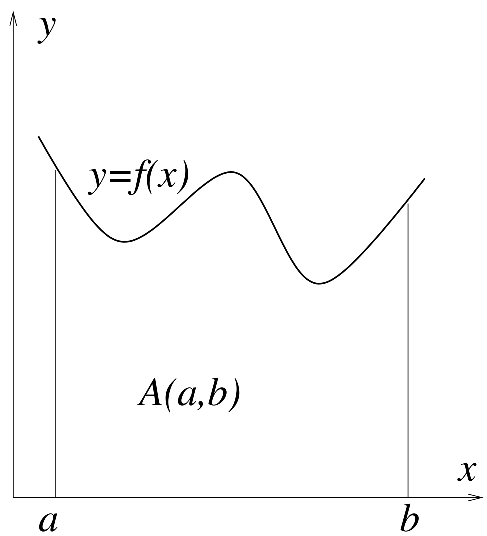
a) Starting with *Second Newton's Law* $F = ma$ where $a = y''$ and $F = -mg$, a is acceleration, g is the free fall acceleration and y is the distance measured vertically, derive the formula for the free fall of a stone, using antidifferentiation. Don't forget the integration constants! What is their meaning?

b) Check *the energy conservation* for a free falling stone, i.e. $mv^2/2+mgy = const$, i.e. the total energy = the kinetic energy + the potential energy is conserved ($v = y'$ is the velocity).

Problem 3.7. Definite integrals and Newton-Leibniz

We already came across Newton-Leibniz in problem 1.6. It says that if $A(x)$ if the area shown on the figure, then $A'(x) = f(x)$, i.e. $A(x)$ is a primitive of $f(x)$.





An equivalent formulation is that the area $A(a, b)$ on the figure can be calculated by the formula $A(a, b) = F(b) - F(a)$, where F is any primitive of f, it doesn't matter which one (why?). Of course the area under the x axis is counted with the $-$ sign, as well as the area in the case when $b < a$ The area $A(a, b)$ is called *the definite integral from a to b of f(x)* and denoted by

$$\int_a^b f(x)dx, \text{ i.e. } A(a,b) = \int_a^b f(x)dx = F(b) - F(a),$$

where F is any primitive of f.

a) Formulate the integration rules for definite integrals and try to see if they are consistent with what you know about area.

b) Show that definite integral is *additive*, i.e.

$$\int_a^c f = \int_a^b f + \int_b^c f$$

what is the geometrical meaning if it?

c) See how the *positivity* of the definite integral, i.e. the fact that $\int_a^b f \geq 0$ if $f \geq 0$ and $a \leq b$ is related to the positivity of the area and to the *Increasing Function Theorem* that says that functions with non-negative derivatives are non-decreasing. You can try to prove it for polynomials (think about how the sign of $f(x) - f(a)$ is related to the sign of f' when x and a are close).

We will take a harder look at these topics on July 31 (see the tentative schedule).

Surf the web to learn more about integration. See 2.3 of my lecture notes for more detailed explanations and pictures. Read the Feynman lecture #8 from the first handout. Visit http://www.calculus.net/ci2/practice/ if you are eager to practice, also take a look at homework #2 from Spring 2003 at http://world.std.com/~michaell/Homework/hw.html