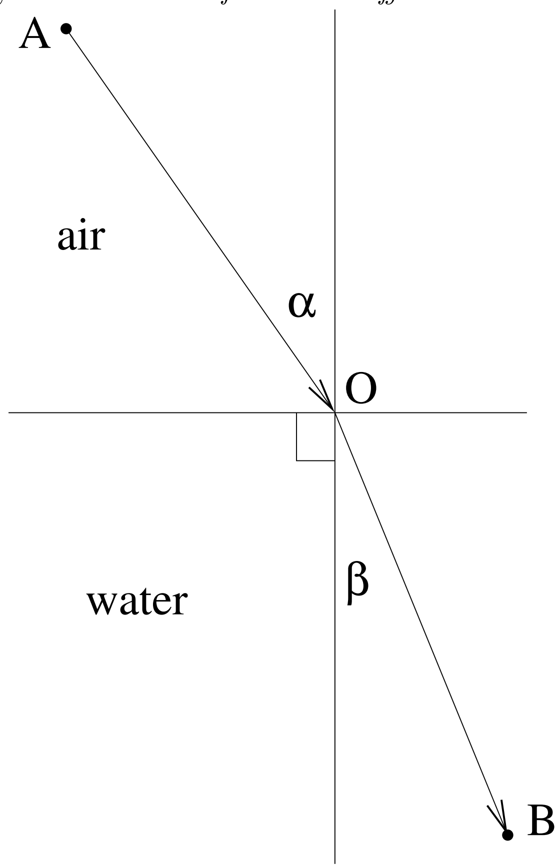
6 Minima and maxima

Problem 6.1. Fermat's principle and Snell's law

It was known already in antiquity that the light ray breaks when it enters water from air, and the tables of dependence of angles α and β where compiled as early as the second century A.D. However, it was only in 1621 when Willebrord Snell, a Dutch mathematician, discovered the formula $sin(\beta) = nsin(\alpha)$, where n is a constant, characteristic of water and air, the so-called *refraction coefficient*.



About 1650 Fermat came up with *the principle of least time*, that explained the phenomenon. It says that the light takes the path that requires the shortest possible time to get from point A to point B. He speculated that light travels in water slower than it travels in air, and that's why it takes the broken path. He found the calculations too difficult, and they were done by Leibniz in 1684.

Now derive the Snell's law from Fermat's principle, using what you know about minima and maxima and see how the refraction coefficient depends on the speed of light propagation in water and in air.

Try to understand why the swimming pools are deeper than they look.

It was reported by some mathematician that his dog also takes a broken path when he fetches a tennis ball tossed into the ocean, because he swims slower than he runs and wants to get to the ball as soon as possible (the link to this story is in my home page).

Problem 6.2. Show that a polygon of a given perimeter and a given number of sides that has the maximum possible area is equilateral, i.e. all sides of this polygon are of the same length.

Problem 6.3. Find the maximum of the function

$$H(p) = -plog_2(p) - (1-p)log_2(1-p),$$

assuming $0 \leq p \leq 1$ and $0log_2(0) = 0$.

Try to generalize to the experiment with more than 2 outcomes, i.e. find the maximum of $H(p_1, ..., p_n) = -p_1log_2(p_1) - ... - p_nlog_2(p_n)$, assuming $p_k \geq 0$ and $p_1 + ... + p_n = 1$.

Take a look at additional readings and problems # 6, especially fm-7.pdf and fm-8.pdf, where you can find some extra explanations, problems and examples.