# 8 Differential equations and their numerical solution

**Problem 8.1. Taylor's formula and finite differences**
Taylor's formula of degree $n$ with remainder is

$$f(x) = f(a) + f'(a)(x-a) + f''(a)(x-a)^2/2 + \ldots + f^{(n)}(a)(x-a)^n/n! + R_n(x),$$

where *the remainder*

$$R_n(x) = \frac{1}{n!}\int_a^x (x-t)^n f^{(n+1)}(t)dt$$

See page 116 in abih-integration.pdf in readings for pset4 for a derivation given by Johann Bernoulli.

**a)** Using Taylor's formula of degree 2 and assuming that $|f'''(x)| \leq M$, estimate the absolute value of the expression

$$\frac{f(x+h) - f(x-h)}{2h} - f'(x).$$

The result will show that $(f(x+h) - f(x-h))/2h$, that is called *the divided first central difference*, is a good approximation for $f'(x)$ when $h$ is small.

**b)** Using Taylor's formula of degree 3 and assuming that $|f''''(x)| \leq M$, estimate the absolute value of the expression

$$\frac{f(x+h) - 2f(x) + f(x-h)}{h^2} - f''(x).$$

The result will show that $(f(x+h) - 2f(x) + f(x-h))/h^2$, that is called *the divided second central difference*, is a good approximation for $f''(x)$.
Hint: the inequality $|\int_a^b f(x)dx| \leq \int_a^b |f(x)|dx$ will be handy.

**c)** In lecture # 9 (see flp-9.pdf from the additional readings for this problem set) Richard Feynman discusses a method to calculate the planetary motions. He says at the end that taking the time step 1000 times smaller will make the accuracy 1000000 times better (i.e. will reduce the numerical error 1000000 times). Do you understand why?

**Problem 8.2. Solving DEs by iterations**

**a)** Let $y(t) = sin(t)$. Show that $y(t)$ satisfies the differential equation $y'' = -y$ with the initial conditions $y(0) = 0, y'(0) = 1$. By integrating both parts of this equation twice and taking into account the initial contitions while doing so, derive the following integral equation:

$$y(t) = t - \int_0^t \left(\int_0^u y(v)dv\right) du$$

Now try to solve this equation by iterations, i.e. take $y_0(t) = 0$, then put

$$y_1(t) = t - \int_0^t \left( \int_0^u y_0(v)dv \right) du,$$

then

$$y_2(t) = t - \int_0^t \left( \int_0^u y_1(v)dv \right) du,$$

and so on. Watch the Maclaurin series for *sin* unfolding.

**b)** Repeat for *cos* instead of *sin*.