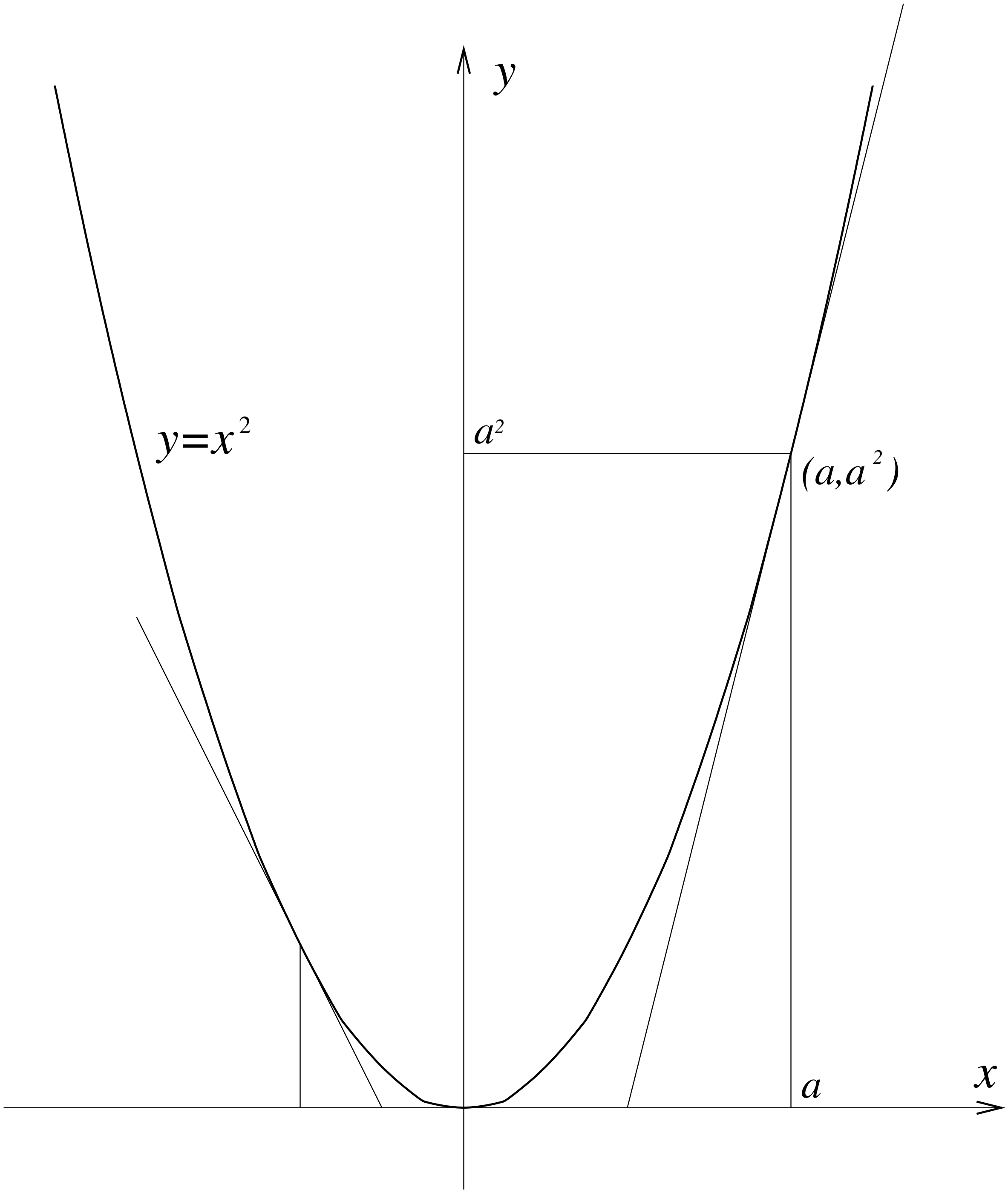
1 Areas, Sums and Tangents

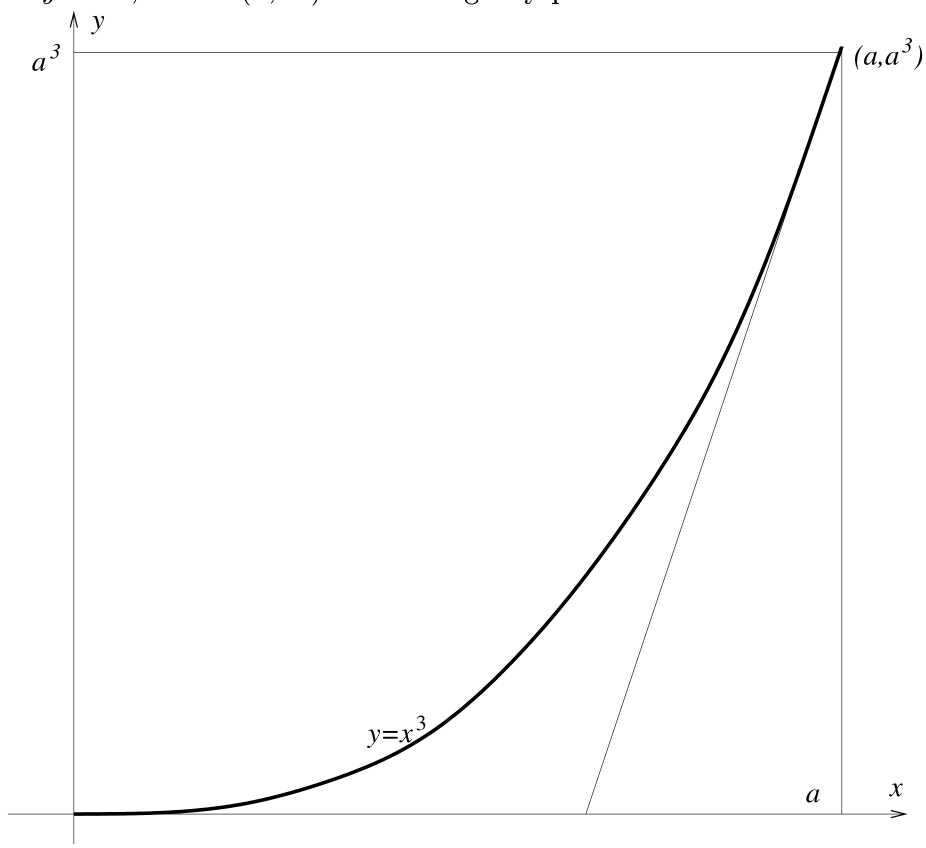
1.1 Solutions

Problem 1.1. Let $y = a^2 + b(x - a)$ be the equation of the tangent to our parabola $y = x^2$, where (a, a^2) is the tangency point.



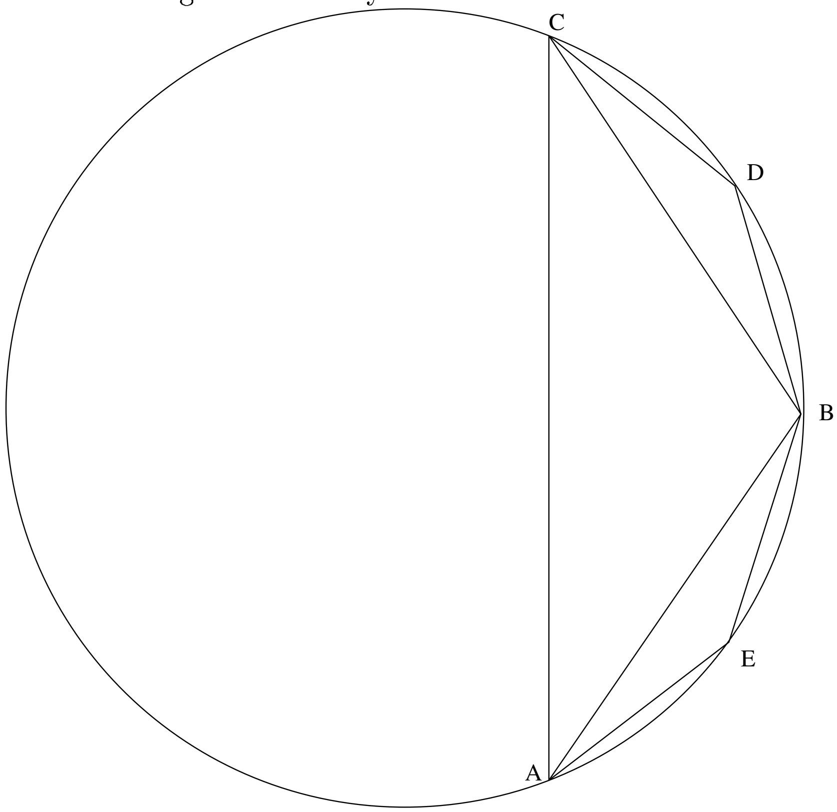
We want $x = a$ be a double root of the equation $x^2 - a^2 - b(x - a) = 0$. The equation can be rewritten as $(x - a)(x + a - b) = 0$, and $x = a$ is a double root if and only if $b = 2a$. So the slope of the tangent to our parabola $y = x^2$ at the point (a, a^2) is $2a$.

Problem 1.2. Let $y = a^3 + b(x - a)$ be the equation of the tangent to our cubic $y = x^3$, where (a, a^3) is the tangency point.



We want $x = a$ be a double root of the equation $x^3 - a^3 - b(x - a) = 0$. The equation can be rewritten as $(x - a)(x^2 + xa + a^2 - b) = 0$, and $x = a$ is a double root if and only if $b = 3a^2$. So the slope of the tangent to our cubic $y = x^3$ at the point (a, a^3) is $3a^2$.

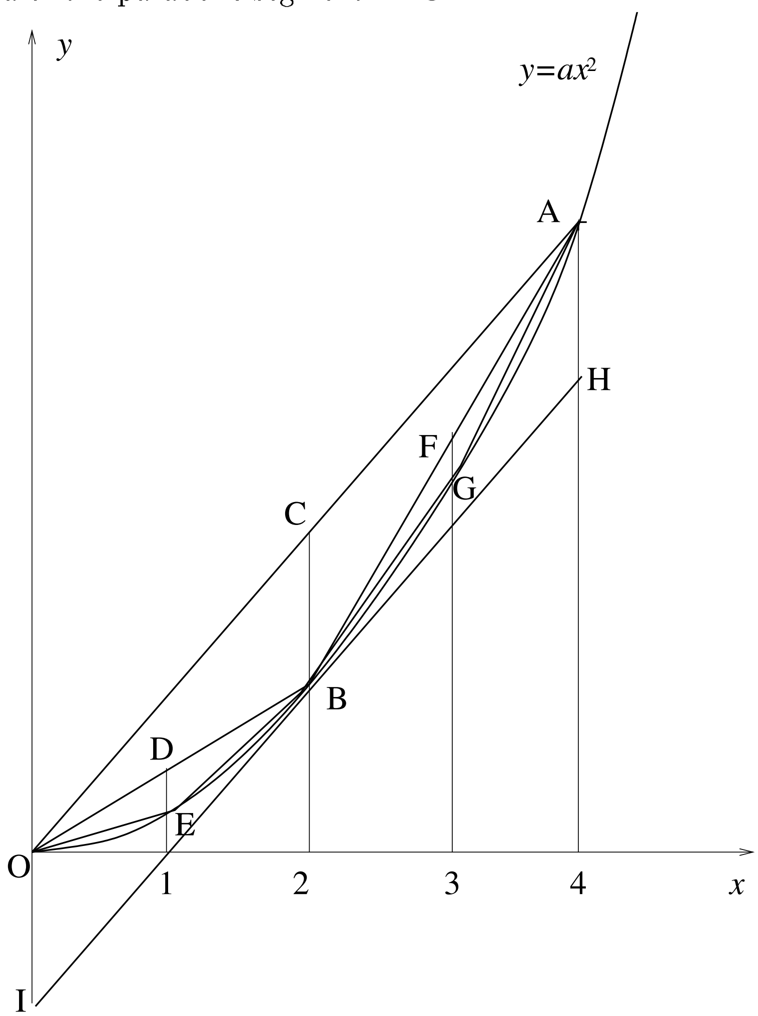
Problem 1.3. a) As I explained in the hints, the area of the isosceles triangle inscribed in a segment is always more than half the area of this segment.



In particular, the area of the triangle ABC $> 1/2$ the area of the segment ABC and the areas of the triangles CDB and AEB are more than half the areas of the corresponding segments. It follows that the area of the pentagon AEBDC is more than $3/4$ the area of the segment ABC. If we add 4 extra isosceles triangles incsribed into the segments corresponding to the arcs AE, EB, BD and DC, the total area of the resulting polygon will be more than $7/8$ the area of the segment ABC. Continuing in this manner, after N subdivisions we will get the polygon with the area greater than $(1 - 2^{-N})$ times the area of the segment ABC.

b) Since the area of the triangle ABO is half the area of the parallelogram IOAH and the area of that parallelogram is bigger than the area of the parabolic segment ABO (because it sits inside the parallelogram), the area of the triangle ABO is more than half the area of the segment ABO. By the same reasoning the area of each of the triangles OEB and BGA is bigger than half the areas of the segments they are inscribed in.

It follows that the area of the pentagon OAGBE is more than 3/4 the area of the parabolic segment ABO.



We can keep dividing the parabolic arcs in half along the x axis and adding more triangles to the polygon, appriximating the parablic segment ABO better and better. After N divisions the area of the polygon will be greater than $(1-2^{-N})$ the area of the segment ABO.

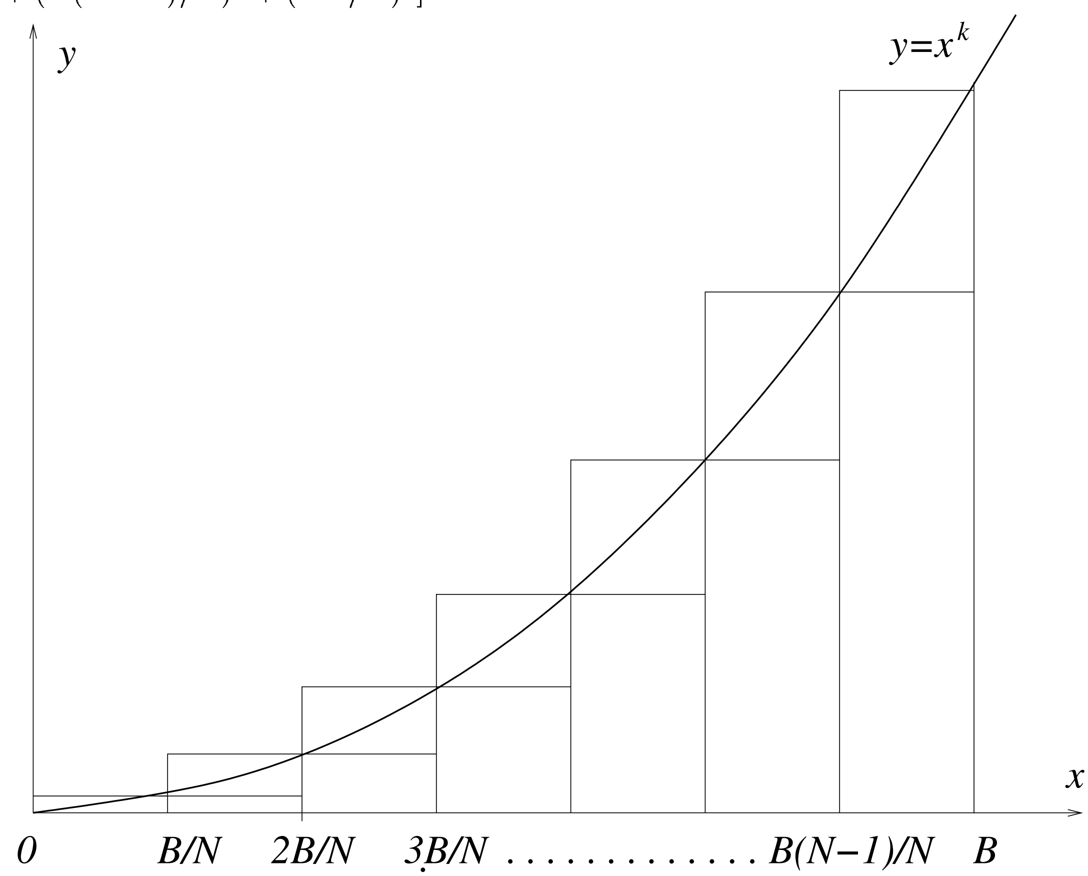
Now $|CB| = a2^2$ and $|ED| = |FG| = a1^2$, so area of the triangle ABO is $4|CB|/2 = 2a2^2$, the area of the triangle OEB + the area of of the triangle BGA is $4|DE|/2 = 2a1^2$, and the area of the pentagon AOEBG is $2a(2^2 + 1^2)$. Continuing bisecting and adding new triangles, we get the area of the parabolic segment as $2a(2^2+1^2+(1/2)^2+... = 8a(1+1/4+1/16+...)$.

To evaluate the infinite sum (a geometric series) in the parenthesis, let us denote it by S and notice that $(1-1/4)S = 1$, whence $S = 4/3$, and we finally get the area of the parabolic segment ABO as $2a4^2/3$.

Notice that it is 1/3 the area of the triangle OA4.

c) I hope you know what to do after you saw how to sum $1+1/4+1/16+....$ Anyhow, if $S(N) = 1+R+...+R^N$ then $(1-R)S(N) = 1-R^{N+1}$, so $S(N) = (1-R^{N+1})/(1-R)$. By the same trick $S = 1+R+R^2+... = 1/(1-R)$. Notice that $S(N)$ approximates S fo big N only if $|R| < 1$, while the formula $1/(1-R)$ makes sense for any $R \neq 1$.

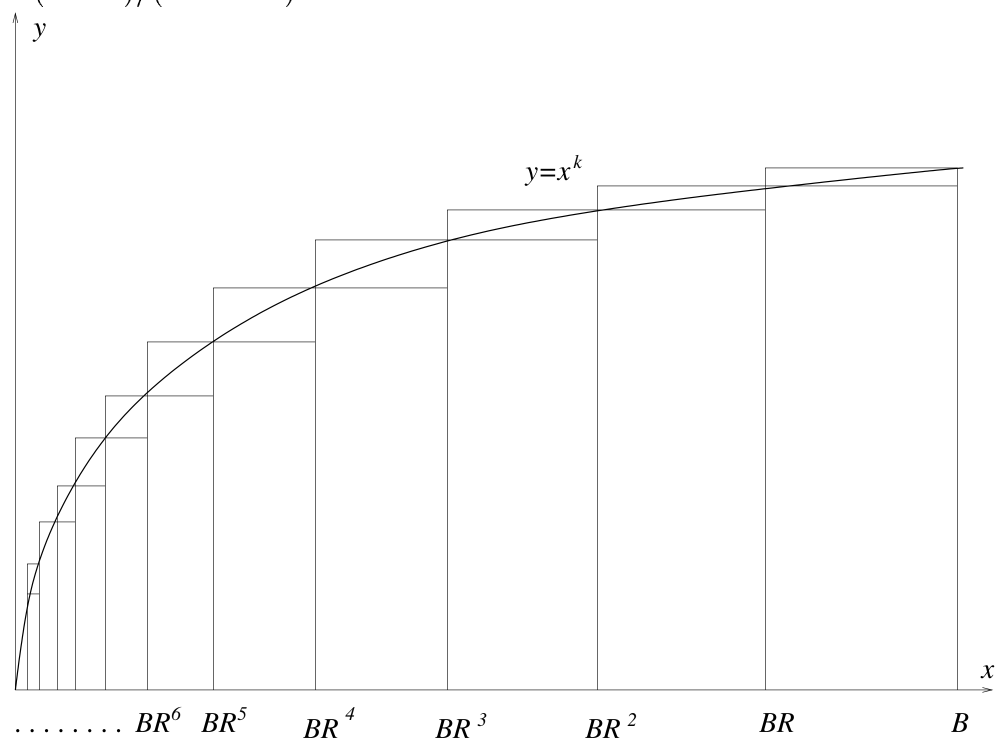
Problem 1.4. The area under the upper staircase is $B/N[(B/N)^k+(2B/N)^k+ ... + (B(N-1)/N)^k + (BN/N)^k]$



$= B^{k+1}[1+2^k+3^k+...+N^k]/N^{k+1}$. To evaluate the area $A_k(B)$ under our curve we have to evaluate this formula for $N = \infty$. To do that we need a formula for the sum $S_k(N) = 1+2^k+...+N^k$. There are several ways to get it, some of them are explained in the handouts (see “Induction and Mathematical Induction” and “Recursion” by George Polya). For example, you may realize that $S_k(N)$ should be a polynomial of degree $k+1$ in N, then you can calculate $S_k(N)$ for $k+2$ different values of N and then find the polynomial (it is defined uniquely, as a polynomial of degree $k+1$). Then you can use mathematical induction to show that the formula works for every N.

Actually, to figure out $A_k(B)$ we don't need the whole formula for $S_k(N)$, only the leading coefficient of that formula counts, and it is easy to guess. Here is how. Assuming $S_k(N) = aN^{k+1} + ...$, we get $N^k = S_k(N) - S_k(N-1) = (k+1)aN^k + ...$, where ... denotes the terms of lower degree in N, so we must have $a = 1/(k+1)$. Now $n^{k+1}/(k+1) - (n-1)^{k+1}/(k+1) - n^k = ...$, and summing up for n from 1 to N we can see that $|S_k(N) - N^{k+1}/(k+1)| \leq$ {something that depends only on k}N^k. It only remains to multiply both sides of this inequality by B^{k+1}/N^{k+1} to see that $A_k(B) = B^{k+1}/(k+1)$. Notice the agreement of this formula with problem 2b.

Problem 1.5. The area under the upper staircase is $B(1-R)B^k + BR(1-R)(BR)^k + ... = B^{k+1}(1-R)(1 + R^{k+1} + (R^{k+1})^2 + ... + (R^{k+1})^n + ...) = B^{k+1}(1-R)/(1-R^{k+1})$



The area under the lower staircase is R^k times the area the upper staircase, and these areas are close to each other when R is close to 1. To get the area under the curve we have to evaluate these expressions for $R = 1$, i.e. we have to make sense of $B^{k+1}(1-R)/(1-R^{k+1})$ for $R = 1$. It is easy when k is a non-negative integer, then the denominator factors as $(1-R)(1+R+...+R^k)$, and the whole expression evaluates to $B^{k+1}/(k+1)$ in agreement with problem 4. This formula holds for any $k > -1$ as well.

Problem 1.6. I explained it in the class, see sections 1.2 and 2.1 of my lecture notes (online) for more details. In problem 1 we wanted $x = a$ to be a double root of the equation $x^2 - a^2 - b(x - a) = 0$, i.e. we wanted $(x - a)^2$ to be a factor in $x^2 - a^2 - b(x - a) = 0$. This happens only if $(x^2 - a^2)/(x - a) = b$ when $x = a$.

Likewise, in problem 2 we wanted $(x - a)^2$ to be a factor in $x^3 - a^3 - b(x - a)$. This happens only if $(x^3 - a^3)/(x - a) = b$ when $x = a$.

Division of polynomials and its connection with roots are explained in section 1.2 of my lecture notes.