Rethinking Calculus

BY

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Playing with Formulas

Differentiating Polynomials

How to make sense of $\dfrac{x^2-a^2}{x-a}$ for $x=a$?

Of course, we just factor the numerator

and cancel $x-a$, so we get

$$(x^2)' = \frac{x^2-a^2}{x-a}\,|_{x=a} = \frac{(x+a)(x-a)}{x-a}|_{x=a} = 2x\,,$$

and now we can differentiate x^2.

With a bit more work we get

$$(x^3)' = 3x^2, (x^4)' = 4x^3, \ldots, (x^n)' = n\,x^{n-1}$$

This trick will work for any polynomial $f(x)$

because $x-a$ divides $f(x)-f(a)$, so

$$f'(x) = \frac{f(x)-f(a)}{x-a}|_{x=a}$$

We don't have to divide polynomials because of...

Differentiation Rules

- $(f+g)' = f' + g'$
- $(kf)' = kf'$ for any constant k
- $(fg)' = f'g + fg'$
- $(f(g(x))' = f'(g(x))g'(x)$

Demonstrating these rules for polynomials is a matter of simple algebra of course.

Roots

How to make sense of $\frac{\sqrt{x}-\sqrt{a}}{x-a}$ for $x=a$?

It's the same problem that we started with, turned upside down, so we know what to do.

$$\frac{\sqrt{x}-\sqrt{a}}{x-a}|_{x=a} = \frac{\sqrt{x}-\sqrt{a}}{(\sqrt{x}-\sqrt{a})(\sqrt{x}+\sqrt{a})}|_{x=a}$$

so we get $(\sqrt{x})' = \frac{1}{\sqrt{x}+\sqrt{a}}|_{x=a} = \frac{1}{2\sqrt{x}}$

It's clear now that $(\sqrt[n]{x})' = \frac{1}{n(\sqrt[n]{x})^{n-1}}$

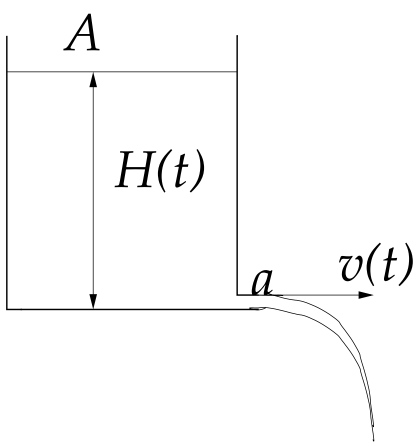
(powers upside down, again)

Implicit Differentiation, Quotients

Another way to derive the formula for $(\sqrt[n]{x})'$ is to rewrite $y=\sqrt[n]{x}$ as $y^n=x$, to differentiate this equation to get $ny^{n-1}y'=1$ and to solve for y'. This trick, called *implicit differentiation*, makes it easy to get $(x^{m/n})'$, $(u/v)'$ and even y' if $y^7+y+x=0$, when we are at a loss to derive a formula for y itself.

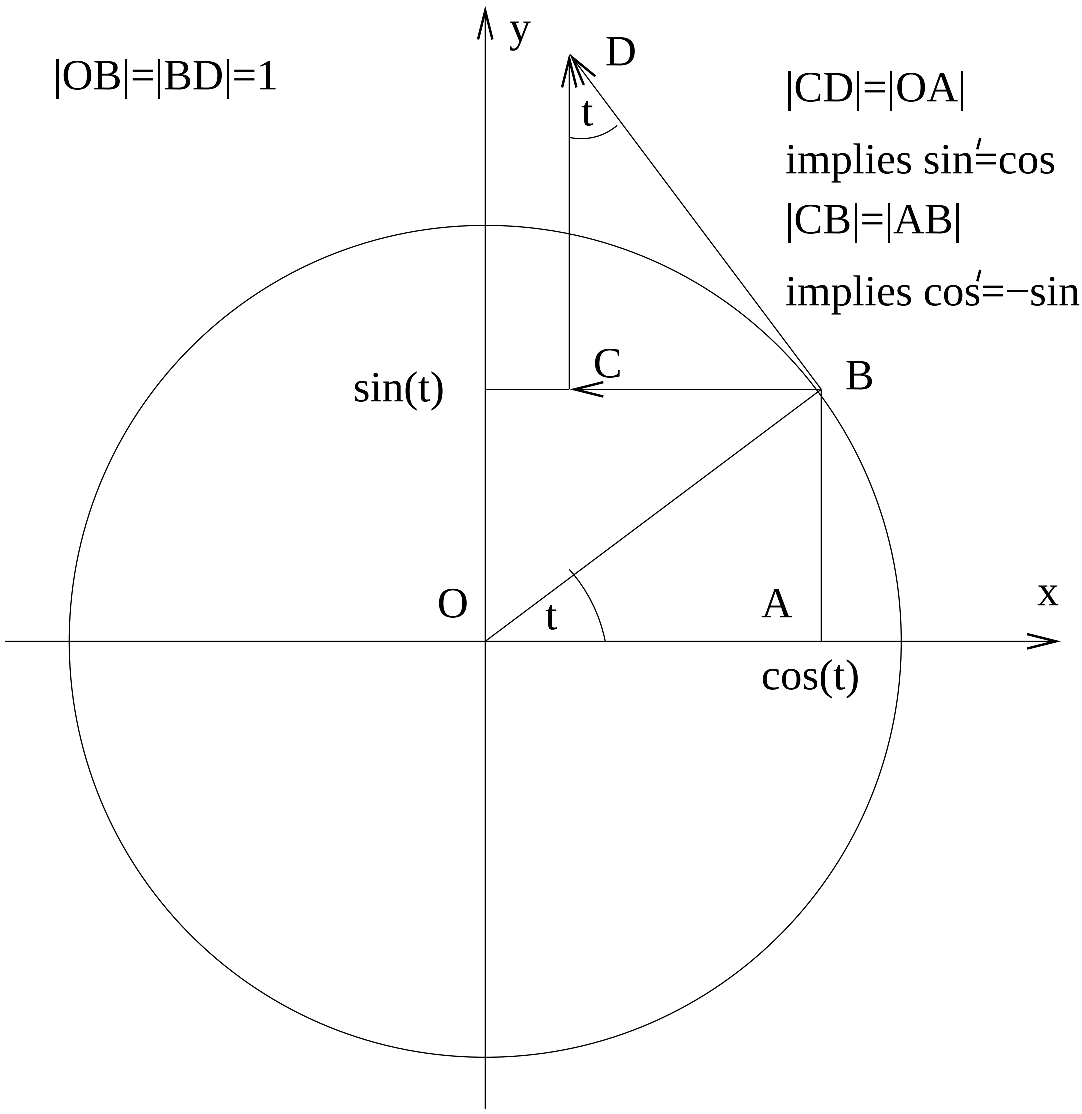
We are stretching it a bit here, of course, by assuming that y' is defined, but it turnes out O.K. if we don't have to divide by zero, as the *implicit function theorem* says.

An Application: a Holy Bucket.



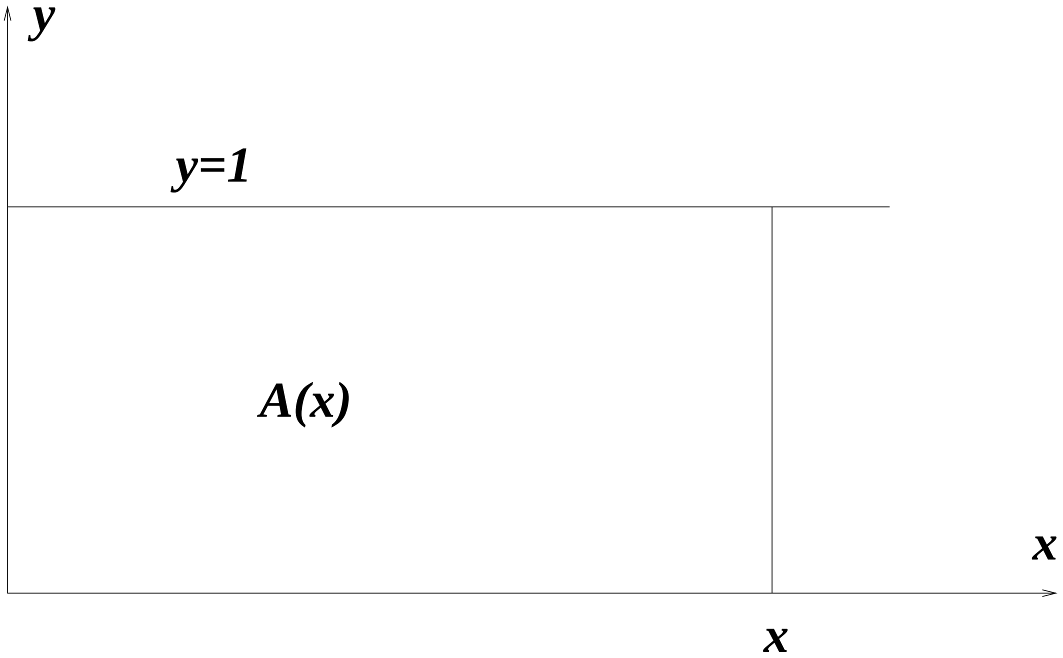
From energy conservation $v=\sqrt{2gH}$, from incompressibility $AH'=-eav$, where e is the *efflux* coefficient.

sin and cos

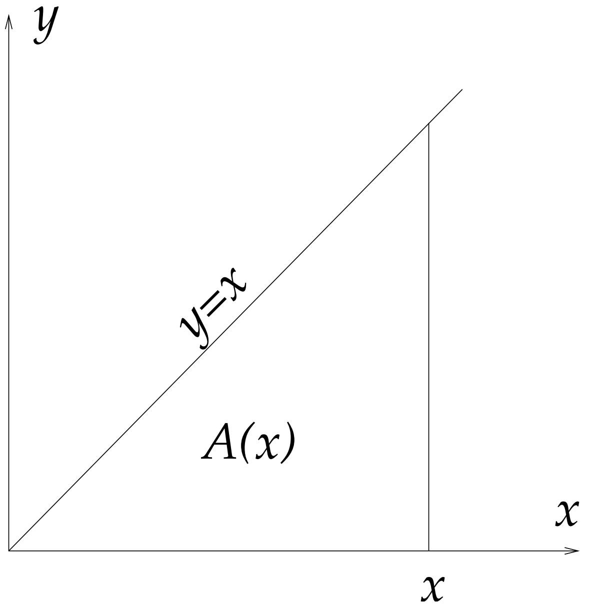


|OB|=|BD|=1
y
D
t
|CD|=|OA|
implies sin′=cos
|CB|=|AB|
implies cos′=−sin
C
B
sin(t)
O
t
A
x
cos(t)

Areas, Newton-Leibniz by example

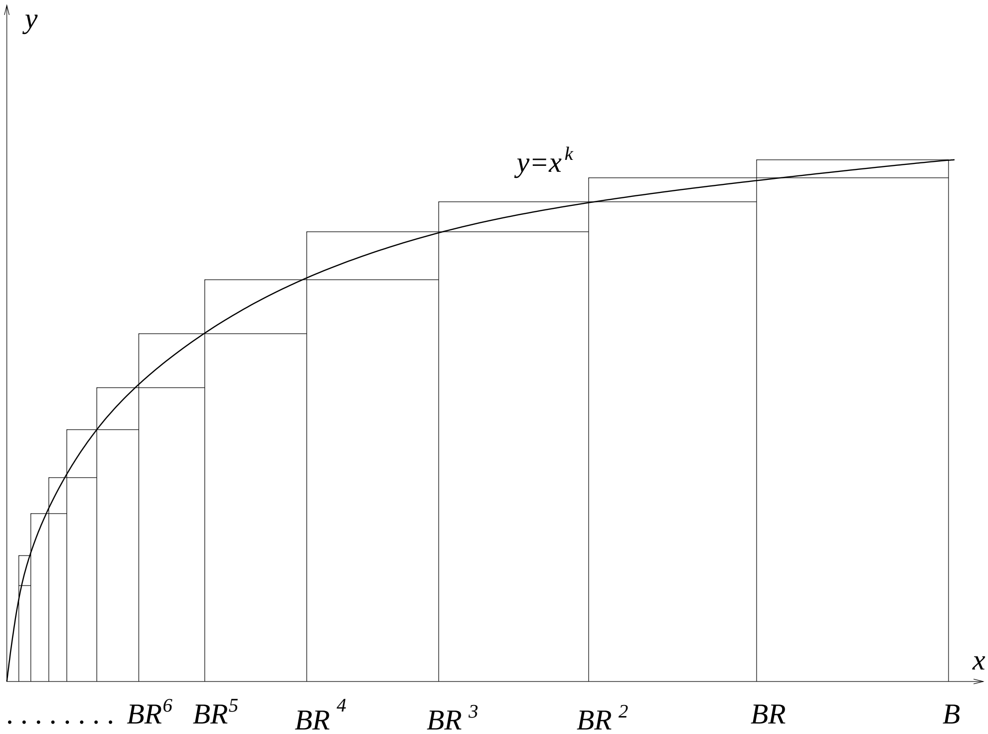


$A(x) = x$, so $A'(x) = 1$.



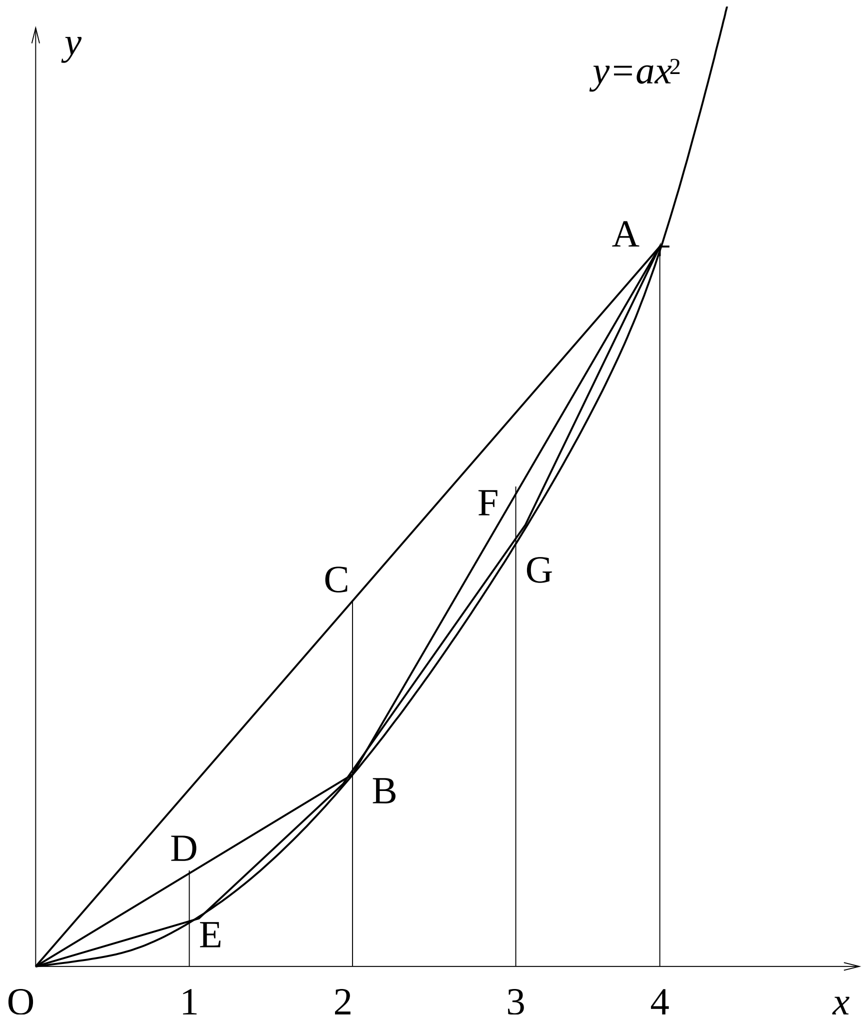
$A(x) = x^2/2$, so $A'(x) = x$.

What about the other powers? Fermat's idea:

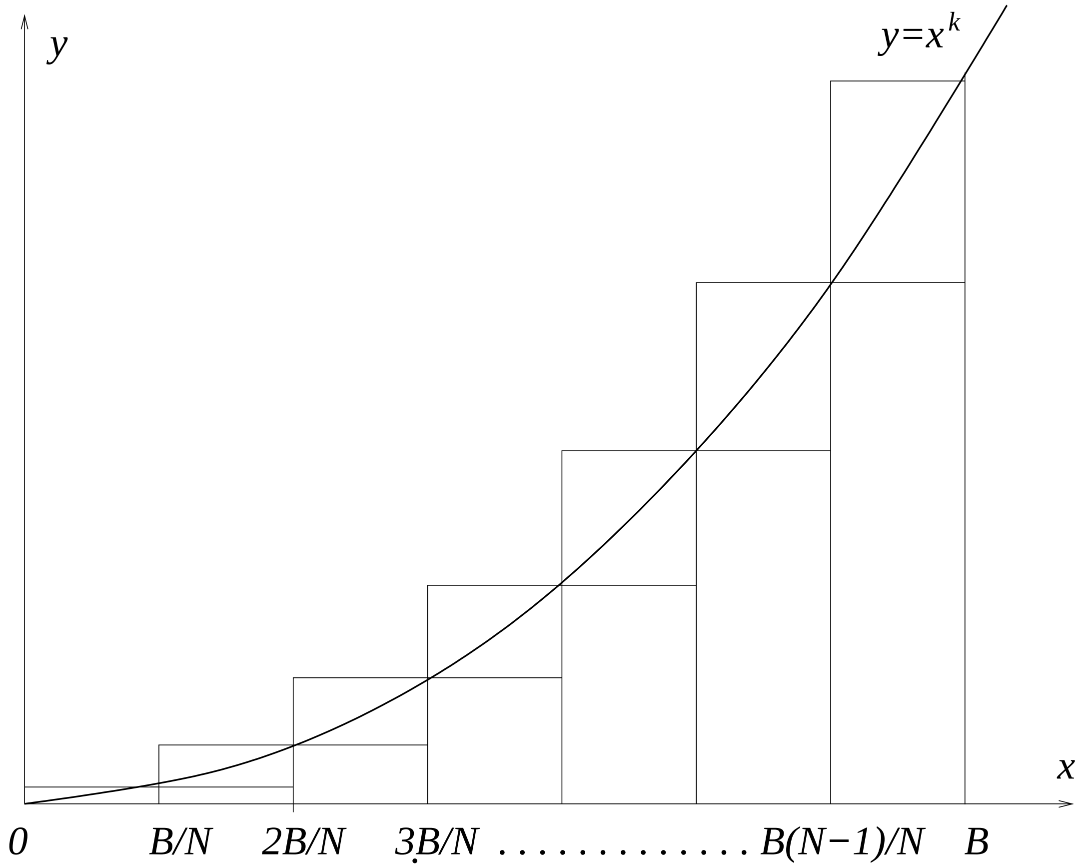


$$A(B) = \frac{B^{k+1}}{k+1} \text{ , so } A'(B) = B^k$$

Archimedes approach:



Uniform grid approach:



Antiderivatives and integrals

$$F' = f \Leftrightarrow \int f(x)\,dx = F(x) + C$$

$$\int x^k dx = x^{k+1}/(k+1) + C \text{ for } k \neq -1,$$

$$\int cos = sin + C, \int sin = -cos + C, \text{ etc.}$$

$$\int_a^b f(x)dx = F(b) - F(a),\ f = F'$$

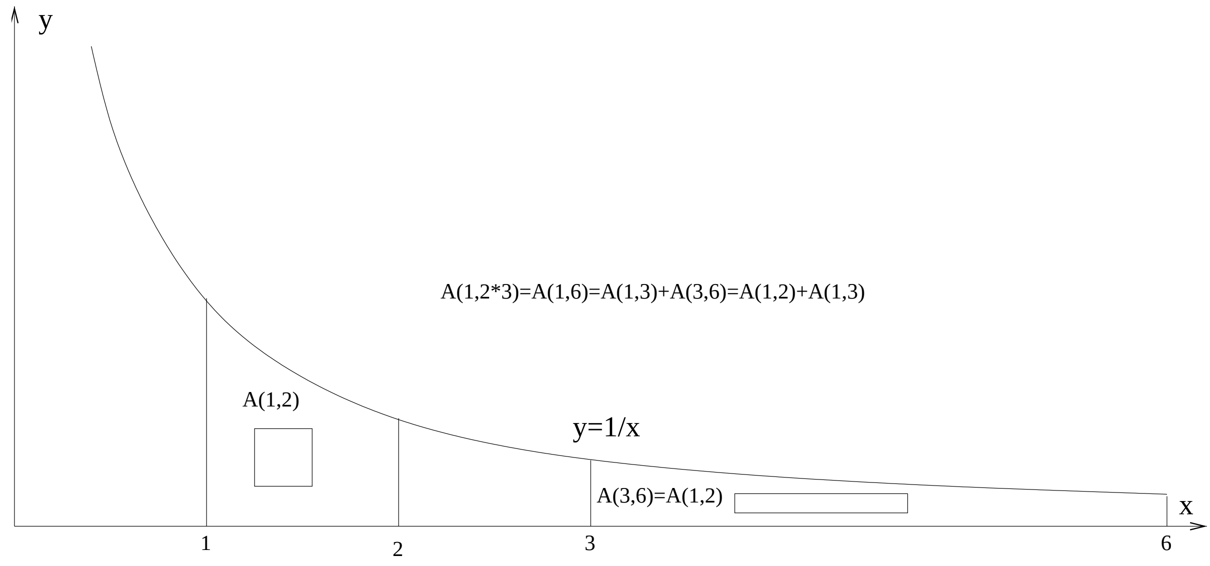
Integration rules, positivity, additivity.

Now what about $\int dx/x$?

$$\int_a^b dx/x = \frac{b^0 - a^0}{-1+1} = \frac{0}{0},$$

and we meet our old friend again.

But geometrically speaking, the area under $1/x$ makes sense, we just have to figure out what it is. To do it, we just look at the picture...



...and see that it is some sort of a logarithm. It is called the natural logarithm, so

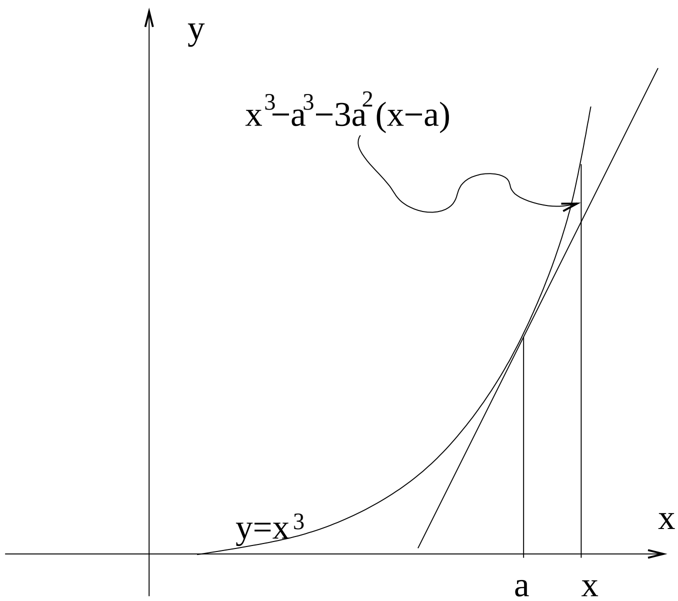
$$\int_a^b dx/x = ln(b) - ln(a), \int dx/x = ln(x) + C,$$

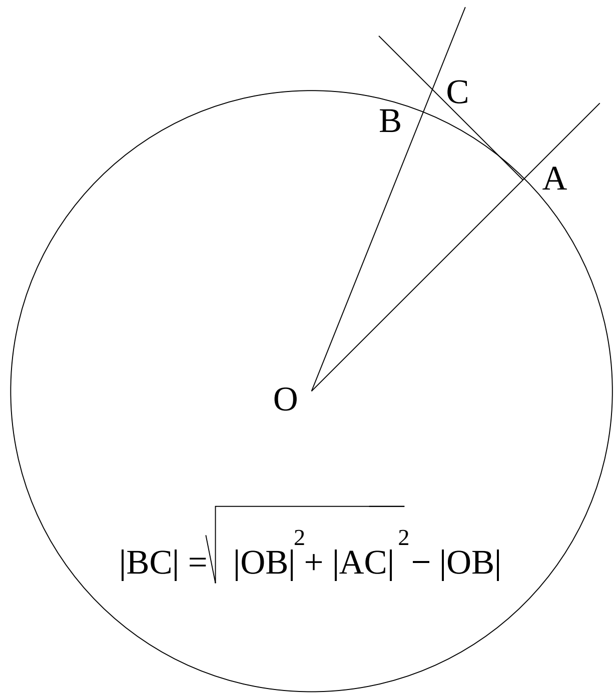
and $(e^x)' = e^x$ (by implicit differentiation).

Playing with Inequalities

Why a tangent looks like a tangent

After examining a few examples





we arrive at the estimate

$$\left| f(x) - f(a) - f'(a)(x-a) \right| \leqslant K(x-a)^2$$

and call f ULD (uniformly Lipschitz differentiable).

It follows that

$$\left| \frac{f(x) - f(a)}{x-a} - f'(a) \right| \leqslant K|x-a|,$$

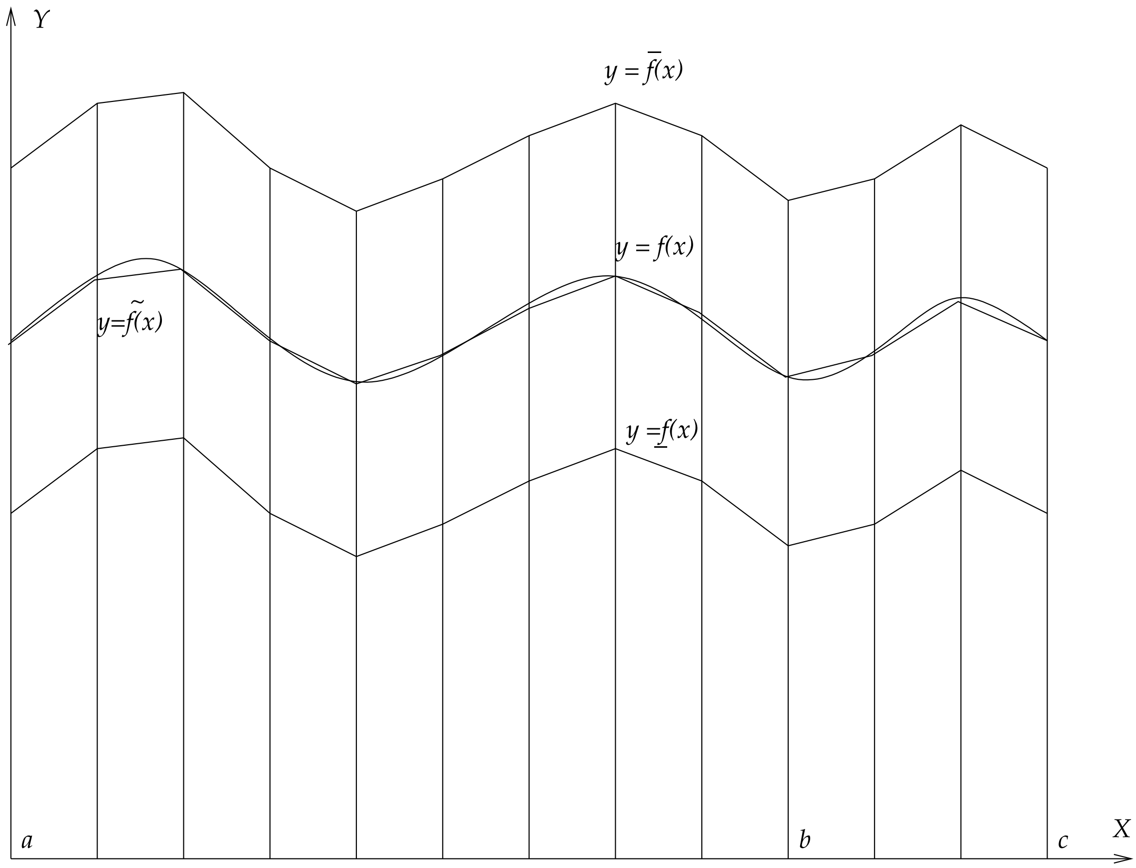
and we conclude that $|f'(x) - f'(a)| \leqslant 2K|x-a|$,
i.e. f' is Lipschitz.

Increasing function theorem

$$f' \geqslant 0 \text{ and } A \leqslant B \Rightarrow f(A) \leqslant f(B)$$

We first assume that $f' \geqslant c > 0$ and look at the estimate defining ULD. We see that
$0 \leqslant x - a \leqslant c/K \Rightarrow f(a) \leqslant f(x)$,
and therefore $f(A) \leqslant f(B)$ because we can get from A to B by taking steps shorter than c/K (according to Archimedes).
Now for $f' \geqslant 0$ $(f + cx)' = f' + c \geqslant c$,
and we can conclude that
$f(B) - f(A) \geqslant -c(B - A)$ for any $c > 0$,
and therefore $f(A) \leqslant f(B)$. Q.E.D.

Integrability of Lipschitz functions and Newton-Leibniz



We can pick piecewise $-$ linear $\bar{f}$ and $\underline{f}$,

$\underline{f} \leqslant f \leqslant \bar{f}$ and $\bar{f} - \underline{f} \leqslant 4Lh$, where h is

the mesh size and L is the Lipschitz constant for f. Then the inequality

$$\int_a^b \underline{f} \leqslant \int_a^b f \leqslant \int_a^b \bar{f}$$

will define $\int_a^b f$ uniquely. Positivity, additivity follow from these ineqialities.

We can now prove Newton-Leibniz
by integrating the inequality
$|f(u)-f(b)| \leqslant L|u-b|$ from b to x to get

$$|\int^{x} f(u)du - \int^{b} f(u)du - f(b)(x-b)| \leqslant$$
$$\leqslant \int_{b}^{x} |f(u)-f(b)|du \leqslant L\int_{b}^{x} |u-b|du = \frac{L}{2}(x-b)^2,$$

which means exactly that $\int^{x} f(u)du$ is a ULD function of x and $\frac{d}{dx}\int^{x} f(u)du = f(x)$.

Differentiation as factoring

For polynomials differentiation can be viewed as factoring $f(x) - f(a) = Q(x, a)(x - a)$, of polynomials in 2 variables, x and a, $f'(x) = Q(x, x)$. Similarly, uniform Lipschitz differentiation can be viewed as factoring in the class of Lipschitz functions of 2 variables. Indeed, when Q is Lipschitz,

$$|f(x) - f(a) - Q(a, a)(x - a)| =$$
$$|Q(x, a) - Q(a, a)| \cdot |x - a| \leqslant L|x - a|^2$$

In the other direction, assuming f ULD, and defining $Q(x, a) = (f(x) - f(a))/(x - a)$ for $x \neq a$ and $Q(a, a) = f'(a)$, we can see that

$$Q(x, a) = \int_0^1 f'(a + t(x - a))dt, \text{ and}$$

therefore is Lipschitz in x since f' is, but Q is symmetric, so it's Lipschitz in both variables. An "elementary" proof that does not use integrals is also available, but is a bit more complicated. See the latest version of my article at http://www.mathfoolery.com/Article/simpcalc-v1.pdf

Interestingly, we can fit the classical differentiation theory into this algebraic scheme too. We just require $Q(x, a)$ to be continuous in x at $x = a$.

Other moduli of continuity

When the Lipschitz condition is too restrictive, for example, to treat $x^{3/2}$ as differentiable, we can relax it, i.e., replace our basic estimate with

$$|f(x) - f(a) - f'(a)(x - a)| \leqslant K|x - a|m(|x - a|),$$

where m is some modulus of continuity, i.e., continuous at $0, m(0) = 0$, increasing and subadditive i.e., $m(x + y) \leqslant m(x) + m(y)$, for example, $|.|^{\alpha}$, with $0 < \alpha < 1$.

The whole theory remains true, with some obvious modifications. Now, for any uniformly continuous function f defined on a closed finite interval there is a modulus of continuity m such that

$$|f(x) - f(a)| \leqslant m(|x - a|).$$

Therefore, we don't miss any of the classical theory of continuously differentiable functions.

Many variables

Similar to the case of 1 variable, we define the derivative by uniform inequality.

$$|f(x+h) - f(x) - f'(x)h| \leqslant K|h|m(|h|) \tag{1}$$

where $|.|$ is some norm.

Automatic continuity of the derivative

The automatic continuity of the derivative still holds, but the proof is a bit more complicated. Here is the idea. We can get from the point x to the point $x+h+k$ either directly or go to $x+h$ first and then to $x+h+k$ the total increment of f should be the same. Now consider the approximations of these increments by the differentials.

$$|f(x+h+k) - f(x+h) - f'(x+h)k| \leqslant K|k|m(|k|)$$

$$\begin{aligned} |-f(x+h+k) + f(x) + f'(x)(h+k)| \\ \leqslant K|h+k|m(|h+k|) \end{aligned}$$

"Adding" these estimates and (1) together, and using the triangle inequality and linearity of f', we conclude that

$$\begin{aligned} |(f'(x) - f'(x+h))k| \leqslant K(|h|m(|h|) + |k|m(|k|) \\ + |h+k|m(|h+k|)) \leqslant 6Km(|h|)|k| \end{aligned}$$

when $|k| = |h|$ and m is increasing and subadditive. So $|f'(x+h) - f'(x)| \leqslant 6Km(|h|)$, and we are done.

Differentiation as factoring

The trouble in many variables is that we can no longer divide by $x-a$ to define the difference quotient. But the idea survives. In one direction, the factoring $f(x)-f(a)=P(x,a)(x-a)$ with $m-$continuous P implies differentiability, since

$$|f(x)-f(a)-P(a,a)(x-a)| = |(P(x,a)-P(a,a))(x-a)| \leqslant Lm(|x-a|)|x-a|.$$

In the opposite direction, we can define the difference quotient $Q(x,a)$ as the average of f' over the segment $[a,x]$, i.e.,

$$Q(x,a)=\int_0^1 f'(a+t(x-a))dt$$

and observe that the argument we used to show continuity of Q in case of one variable still applies.

Conclusions

- Differentiation can be treated as factoring in a certain class of functions. This idea is more general and conceptually simpler than the classical approach. It is also closer to modern mathematics.
- This point of view makes it possible to do and use calculus independently of its classical foundations, i.e. real numbers, continuity and limits, but in a mathematically rigorous way, starting with simple examples.
- The proofs in this streamlined approach are so simple that they can be done by the students as problem sets. See a modest example at my web page at http://www.mathfoolery.com /Problem_sets/hw.html
- Calculus of specific classes of functions, i.e. Lipschitz, Holder etc. is more relevant to the practical applications, i.e., numerical analysis.
- Calculus is not carved in stone, it is still alive and growing.
- Mathematics is the art of problem solving, not a dry set of formal rules.

Many thanks for listening to me

See my preprint "You can simplify calculus"
(by Michael Livshits) at arxiv.org for more details.
Check my home page at www.mathfoolery.com
and click on My Calculus Project

Let us start the true calculus reform by rethinking
the subject and making it more understandable.

The slides for this talk are available online at
http://www.mathfoolery.com/talk-2010.pdf

More questions? Comments? Remarks?