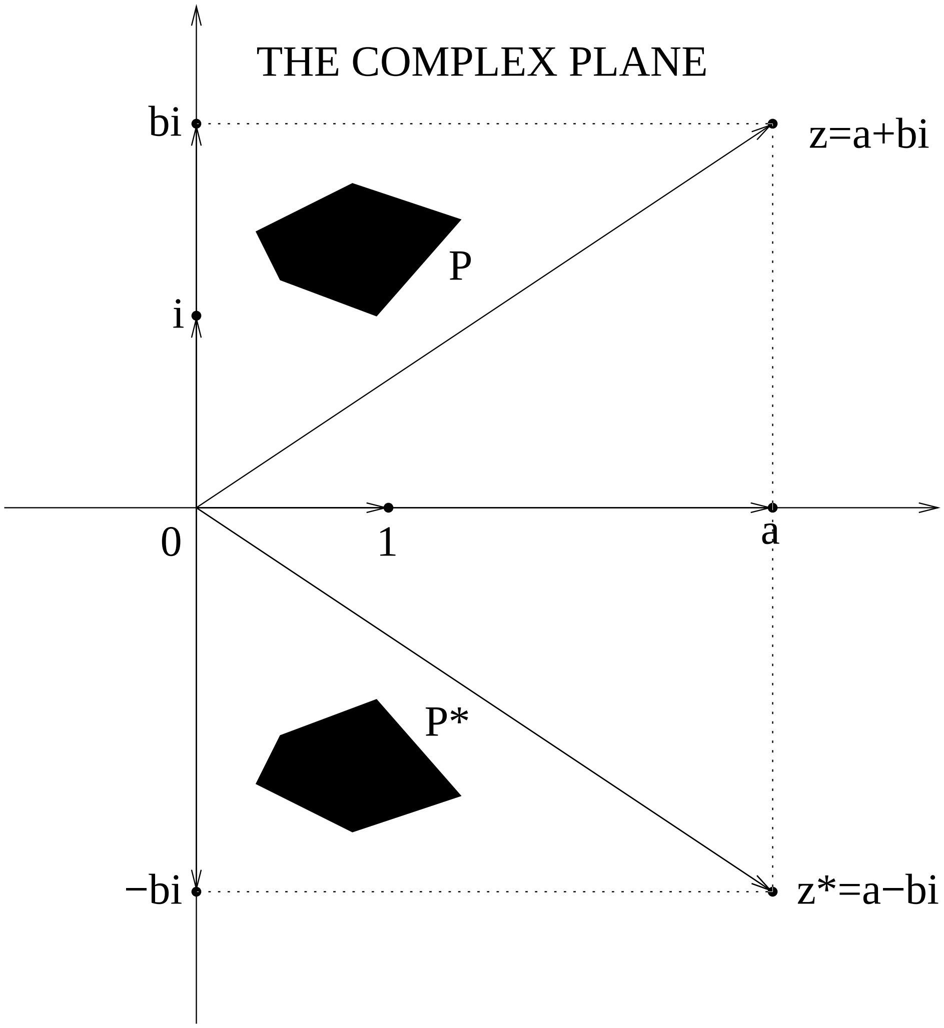
4.1 Complex numbers, rotations and trigs

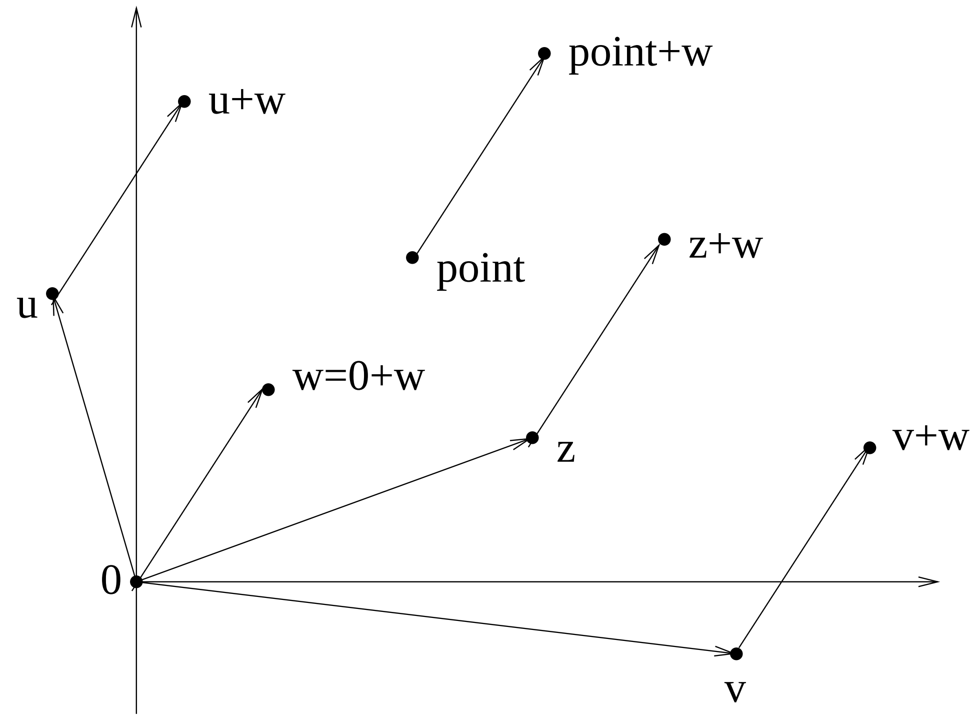
Complex numbers can be viewed as poins on the Cartesian coordinate plane, or as vectors with their tails at the origin, according to the picture.



We say that the complex numbers $z^* = a - bi$ and $z = a + bi$ are *conjugate* to each other. They correspond to the points that are reflections of each other in the horizontal coordinate axis. It's clear that $(z^*)^* = z$. The notation $\overline{z}$ instead of z^* is also popular. Geometrically speaking, the transformation sending z to z^* is the reflection in the horizontal coordinate axis. The points on this axis correspond to the real numbers and the points on the vertical coordinate axis correspond to the purely imaginary numbers, i.e. the numbers of the form bi with real b. We call a the *real part* of z and write $b = Re(z)$, we also call b the *imaginary part* of z and write $b = Im(z)$. It is also clear that the addition of vectors corresponds to the addition of the corresponding complex numbers because $(a+bi)+(c+di) = (a+c)+(b+d)i$.

Problem 4.1.1. Express $Re(z)$ and $Im(z)$ in terms of z and z^*.

Problem 4.1.2. Let w be a complex number. Convince yourself that the geometric transformation that sends any complex number z to $z + w$ is the translation of the complex plane by the vector w (look at the picture).



Now we know how to add complex numbers, how to conjugate them and what these two operations mean geometrically. It's time to start doing multiplication and to discover how it is related to the rotations and rescaling. Mutiplying a complex number by a real number we only have to multiply both real and imaginary part or this complex number by that real number, $c(a - bi) = ca + cbi$. To multiply 2 complex numbers we only have to take into account that $i^2 = -1$.

Problem 4.1.3. Multiplication of complex numbers, the modulus

a) Derive the formula: $(a + bi)(c + di) = (ac - bd) + (ac + bd)i$.

b) Check that $wz = zw$, $(zw)^* = z^*w^*$ and $z(v + w) = zv + zw$

c) Show that the number zz^* is real and > 0 for $z \neq 0$. The square root of this number is called the *modulus* or the *absolute value* of z and is denoted by $|z|$, so $|z|^2 = zz^*$. Can you figure out how to divide complex numbers?

Problem 4.1.4. The modulus and the distance

a) Verify that $|z|$ is the distance to the point z from 0 on the complex plane (remember the good old Pythagoras!), that agrees with the notion of the absolute value for the reals.

b) Show that $|z - w|$ is the distance between the correcponding points on the complex plane.

c) Let w be a fixed complex number and $R > 0$. What kind of figure is formed by all the zs such that $|z - w| = R$? What is R? What is w?

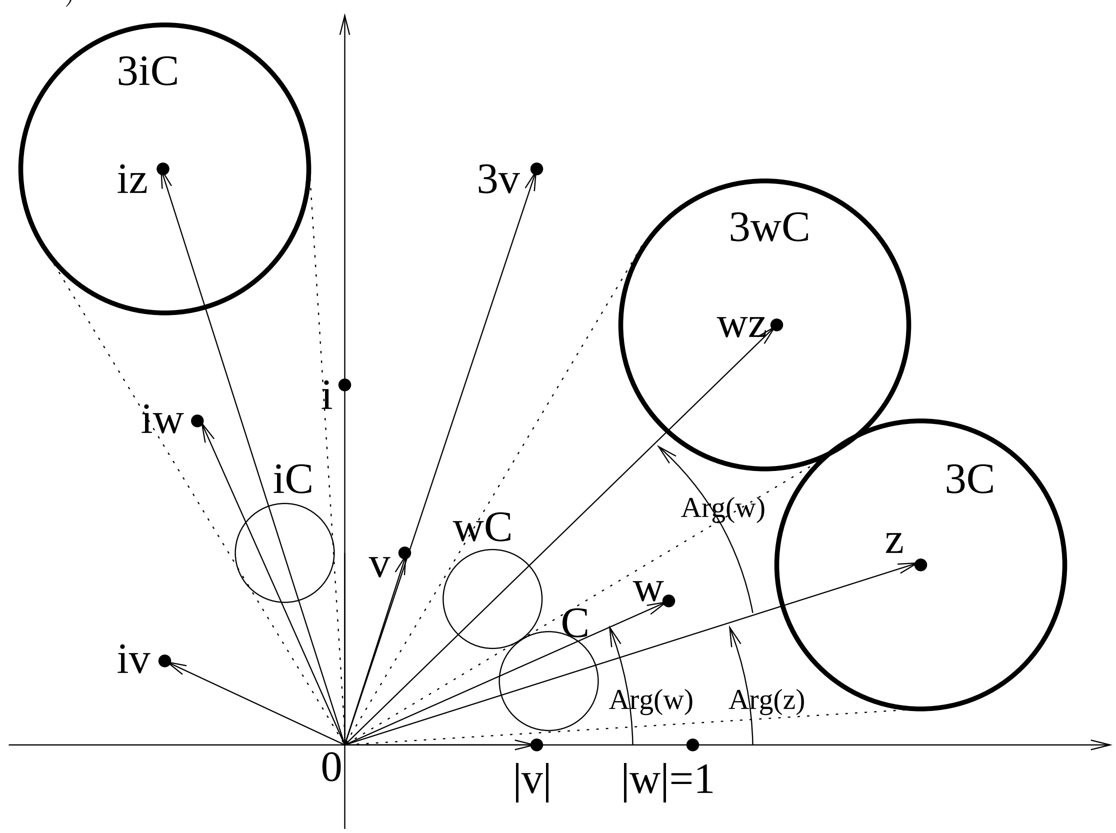
Problem 4.1.5. Multiplication, stretching and rotation

a) Check that $|wz| = |w||z|$, what is the angle between v and iv?

b) Let $|w| = 1$. Show that the transformation of the complex plane sending z to wz preserves the distance and doesn't move the origin. Since this transformation preserves the orientation (for example, it moves a positive frame $(1, i)$ into a positive frame (w, iw)), we conclude that it is the rotation of the plane around the origin by the angle $Arg(w)$ (see the picture).

c) Convince youself that $Arg(wz) = Arg(w) + Arg(z)$ (see the picture).

d) Let a be a real positive. Then the transformation $z \mapsto az$ dilates any figure on the plane by factor a and doesn't move the origin, it is a *homothety* of the complex plane at the origin. Some examples for $a = 3$ are shown on the picture (of course for $a < 1$ we will have a contraction instead of a dilation).



e) By representing any complex number w as a product of its modulus $|w|$ and a number of modulus 1, i.e., $w = |w|(w/|w|)$ conclude that the transformation $z \mapsto wz$ is a combination of the rotation by the angle $Arg(w)$ followed by the dilation by $|w|$, as you may remember from the applet at http://www.math.utah.edu/˜palais/bob/Rotation/Rotation.htm

We can see now that many familiar notions of plane geometry can be expressed naturally in terms of complex numbers.

Problem 4.1.6. Trigonometry and complex exponents

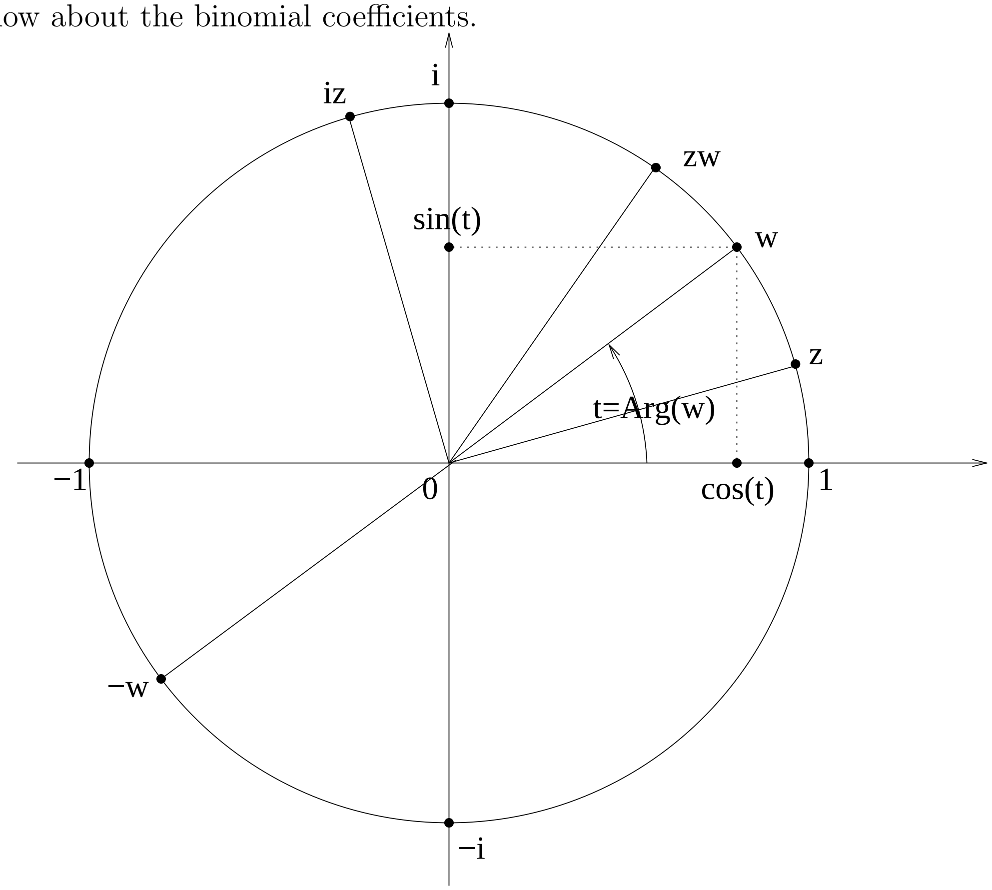
Now, armed with our understanding of complex numbers, we can take a fresh look at trigonometry.

a) See from the pictire that for any any complex number w of absolute value $|w| = 1$ and argument $Arg(w) = t$ we have $Im(w) = sin(t)$ and $Re(w) = cos(t)$, or, to put it differently, if $w = u + vi$ and $u^2 + v^2 = 1$ then $sin(t) = v$ and $cos(t) = u$ where $t = Arg(w)$.

b) We know that when we multiply 2 complex numbers their arguments add and their absolute values multiply. Now, assuming $|z| = 1$, $z = a+bi$ and $Arg(z) = s$ derive the formulas for $cos(s+t)$ and $sin(s+t)$ from the formula for multiplication of complex numbers in terms of their real and imaginary parts (see problem 4.1.3 a).

c) Try to understand why the formula $e^{it} = cos(t) + isin(t)$ by Euler is consistent with what you know by now about trigs and exponents.

d) By observing that $cos(nt) + isin(nt) = (cos(t) + isin(t))^n$ derive the formulas for *sin* and *cos* of multiple angles, say, for $n = 2$, 3 and 4. For an arbitrary n it's called *de Moivre's* formula and you can write it too if you know about the binomial coefficients.



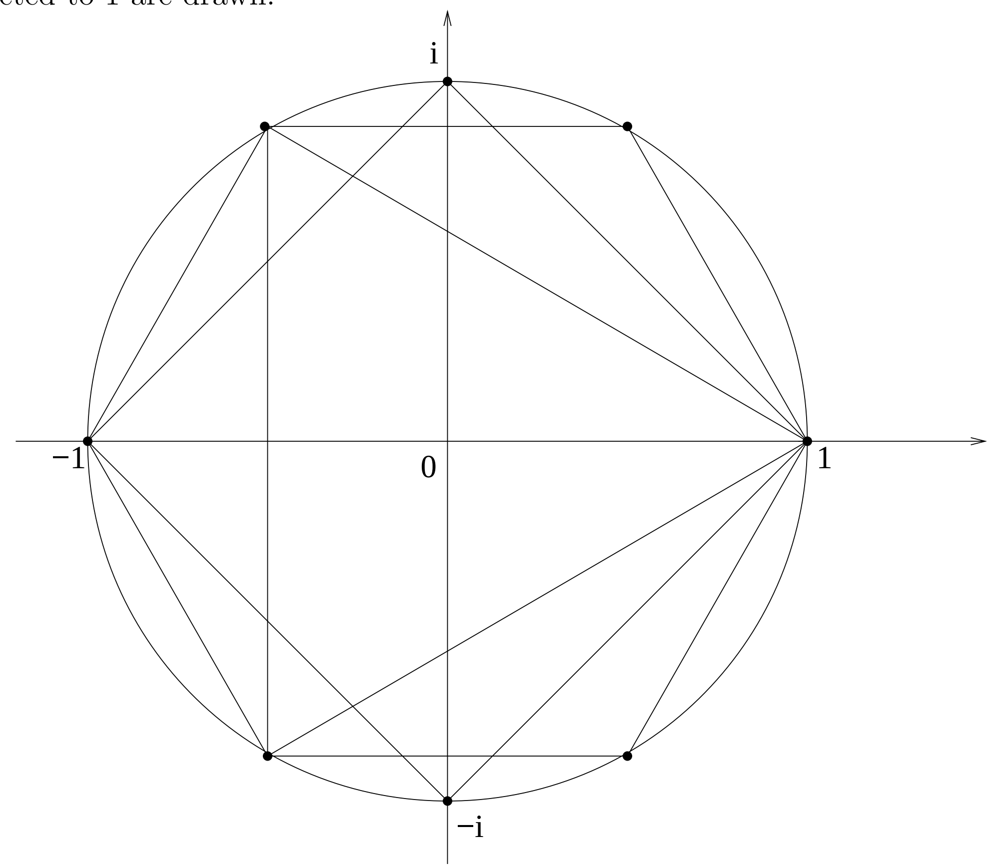
e) What kind of points on the complex plane correspond to the complex numbers $e^{2\pi ik/n}$ for $k = 1, 2, \ldots, n$? Calculate $|e^{a+bi}|$.

f) Derive the formulas for sin' and cos' from $(e^z)' = e^z$ and the chain rule.

g) Try to see why the notation $ln(z) = ln|z| + iArg(z)$ makes sense.

h) Check out a cool lecture by Richard Feynman related to this material at `ftp://hssp06.mathfoolery.org/HSSP_scans` (file flp-1-22.pdf).

i) On the diagram below, a regular triangle, a square and a regular hexagon, all inscribed in the unit circle, together with their diagonals connected to 1 are drawn.



For any of these polygons, calculate the product of the lengths of those sides and diagonals that contain the point 1. Do you see any relation between this number (the product) and the number of the sides of the polygon? Generalize to the regular n-gon and try to prove this relation. Hint: **e)**, **4.1.4 b)** and the fact that any *monic*, (i.e., with the leading coefficient 1) polynomial $p(z)$ of degree n with the roots $z_1, \ldots, z_n$ can be written as the product $(z - z_1) \ldots (z - z_n)$.