5 Increasing Function Theorem

Problem 5.1. Let $p(x) = p_0 + p_1 x + p_2 x^2 + ... + p_n x^n$ be a polynomial with with $p_1 > 0$. Show that for $a < b$, a and b being small enough, $p(a) < p(b)$, i.e. $p(x)$ is an *increasing* function for x small enough.

How small a and b should be to make sure that $p(a) < p(b)$ for $a < b$?

Problem 5.2. Notice that in the previous problem $p_1 = p'(0)$. Now show that if $p'(c) > 0$ then $p(x)$ is increasing when x is close to c.

How close x should be to c to make sure that $p(x)$ is increasing?

Problem 5.3. Assume that $p'(x) > k$ for any x between a and b, $a < b$ again and $k > 0$ is a constant. Show that $p(a) < p(b)$.

Problem 5.4. Try to show that the conclusion from the previous problem still holds when $k = 0$.

Problem 5.5. Replace all the strict inequalities in the previous problem with non-strict ones (i.e. $<$ with $\leq$ and $>$ with $\geq$) and show that the result ($p(a) \leq p(b)$) still holds. This is called *Increasing Function Theorem* for polynomials.

See section 2.5 of the lecture notes for help if you are stuck, read it anyway when you are done with this problem set.